Parameterized Algorithms for Beyond-Planar Crossing Numbers

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Abstract

Beyond-planar graph classes are usually defined via forbidden configurations or patterns in a drawing. In this paper, we formalize these concepts on a combinatorial level and show that, for any fixed family $\mathcal{F}$ of crossing patterns, deciding whether a given graph G admits a drawing that avoids all patterns in $\mathcal{F}$ and that has at most c crossings is FPT w.r.t. c. In particular, we show that for any fixed k, deciding whether a graph is k-planar, k-quasi-planar, fan-crossing, fan-crossing-free or min-k-planar, respectively, is FPT with respect to the corresponding beyond-planar crossing number.

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1 Introduction

In practice, many graphs and networks are non-planar. Since the presence of crossings in a drawing has been identified as a major influence on the readability and also for its fundamental theoretical importance (see e.g. [38, 40, 39]), the crossing number of graphs, i.e., the minimum number of crossings one can achieve in a topological drawing of a graph, has been the subject of intensive research. It is one of the classical results of Garey and Johnson [24] that computing the crossing number is NP-complete, and this holds even under severe restrictions. Concerning approximation results, though some promising progress has been made, e.g., for graphs of bounded genus [27] and, most recently, for graphs of bounded degree [20], finding a constant-factor approximation currently seems elusive. On the other hand, Grohe [25] showed that the crossing number of a graph can be computed in FPT time, i.e., it can be decided whether a given graph G admits a drawing with at most c crossings in time $f(c)n^{O(1)}$ where f is a computable function and n denotes the size of the input. The polynomial dependency on the input is quadratic for Grohe's algorithm and was later improved to linear by Kawarabayashi and Reed [32]. Recent years have seen a couple of extensions, particularly of Grohe's approach. Pelsmajer et al. [36] showed that the approach can be extended to show that the odd crossing number is FPT. Very recently Hamm and Hliněný [26] showed that the problem of extending a partial drawing with the minimum number of crossings is FPT with respect to the achievable number of crossings.

More recent studies on the perception of drawings have somewhat refined the impact of crossings on readability [30, 33]. Rather than simply minimizing their number, there are several factors that influence the readability of drawings with crossings. Aside from geometric aspects, such as crossing angles, it has been identified as particularly relevant that the drawn graphs are sparse and close to planar in the sense that the crossings are well-distributed [29]. This has lead to the research field of *beyond-planarity*, which is concerned

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with the study of graph classes that are defined by the existence of a drawing where certain bad configurations of crossings are avoided. The goal of beyond-planarity is to extend results concerning combinatorics, algorithms and geometric representations known for planar graphs to larger, more general classes of so-called beyond-planar graphs. A prime example are 1-planar graphs that have a drawing where each edge is involved in at most one crossing. By now there is a vast amount of literature that deals with various aspects of beyond-planarity; see e.g. [29, 22, 9]. Unfortunately, even the concept of 1-planarity is quite complicated. While efficient algorithms are known for testing whether a given graph is outer 1-planar [28, 3], testing whether a given graph G is 1-planar is NP-complete [34]. Also recognizing k-planar graphs is NP-hard for every $k \geq 1$ [44] and similar hardness results are known for several beyond-planar graph classes, e.g., k-gap-planar graphs [4, 5], fan-planar graphs and fan-crossing-free graphs [15]. Efficient algorithms are known only for few and rather special cases such as recognizing optimal 1-planar graphs [13], triangulated 1-planar graphs [12, 17] or optimal 2-planar graphs [23].

In light of these hardness results it is natural to consider FPT approaches for recognizing beyond-planar graphs. Bannister et al. [6] showed that testing 1-planarity is FPT when parameterized by the vertex cover number, treedepth or cyclomatic number, but is NP-complete even for graphs of bounded bandwidth, which in particular rules out FPT algorithms with respect to, e.g., pathwidth. Therefore the usefulness of structural parameters for recognizing beyond-planar graphs seems to be fairly restricted. One of the arguably most natural parameters to consider in the context of beyond-planarity is the distance of the input graph G from being planar. There are multiple ways to measure this; the *skewness* of G, denoted by $\text{skew}(G)$ is the minimum number of edges one can remove from G to turn it into a planar graph. The crossing number of G, denoted by $\text{cross}(G)$ is the minimum number of crossings in a topological drawing of G. Finally, given a beyond-planar drawing style $\mathcal{D}$, such as 1-planar drawings, one may consider the *beyond-planar crossing number* of a graph G, i.e., the minimum number $\text{cross}_{\mathcal{D}}(G)$ of crossings in any planar drawing of G according to style $\mathcal{D}$. For example, the 1-planar crossing number $\text{cross}_{1-\text{planar}}(G)$ is the minimum number of crossings in any 1-planar drawing of G and it is ∞ if no such drawing exists. Clearly, for any drawing style $\mathcal{D}$ it holds that $\text{skew}(G) \leq \text{cross}(G) \leq \text{cross}_{\mathcal{D}}(G)$. It turns out that, for most beyond-planar graph classes the skewness and the crossing number are not a suitable parameter. By a well-known result of Cabello and Mohar [16] testing 1-planarity is NP-complete even for input graphs G with $\text{skew}(G) = 1$ and $\text{cross}(G) \leq 10$. In other words, testing 1-planarity is paraNP-hard parameterized by both the skewness and the crossing number.

Beyond-planar crossing numbers can deviate significantly from the crossing number [19, 45]. Hence it is natural to ask whether recognizing beyond-planar graphs is FPT with respect to the beyond-planar crossing number. To mention a specific example, for the class of 1-planar graphs: is it possible to test whether an input graph G admits a 1-planar drawing with at most c crossings in FPT time with respect to c? Indeed, this is the case: the recent work of Hamm and Hliněný [26] shows that given a graph G and a partial drawing of a subgraph $H \subseteq G$, it can be decided in FPT time whether the given partial drawing can be completed into a 1-planar drawing of G with at most c crossings. In particular, if we assume that the partial drawing is empty, their result shows that 1-planarity is FPT by the 1-planar crossing number. The purpose of this paper is to understand which other beyond-planar graph classes admit an analogous result. As it turns out, Grohe's approach [25] for the crossing number, which is also behind the algorithm of Hamm and Hliněný [26] is fairly flexible and seems applicable to several notions of beyond-planarity. However, rather than crafting individual proofs for different classes of beyond-planar graphs, we seek to establish a meta theorem that allows to obtain such results for a wide variety of beyond-planar graph classes.

Large parts of the literature on beyond-planar graphs mention that beyond-planar graph classes are defined by the existence of a drawing that avoids certain crossing configurations or crossing patterns. In this sense, a beyond-planar graph class is defined by a set of forbidden crossing patterns. Interestingly, the idea of what a crossing configuration or pattern is precisely and what it means that a drawing contains such a pattern have not been formalized in the literature. In most papers, the forbidden configuration of crossings is given in a figure but it is usually followed by a more precise verbal definition, which then serves as a basis for theoretical considerations; see e.g. [2, 22, 31].

Contribution. In this paper, we present an attempt at formalizing the idea of forbidden patterns of crossings, where crossing patterns are modeled as small graphs and we seek a drawing whose planarization avoids these graphs. We note that there are various options to design such notions; we believe that our formalization provides an interesting trade-off: it is sufficiently general to express several known beyond-planar graph classes with a small number of patterns and on the other hand is amenable to algorithmic techniques for computing drawings that avoid a fixed list of patterns and have at most c crossings in FPT time. Our notion of patterns is simple and yet powerful enough that our results apply to the following notions of beyond-planar graphs: (simple) k-planar, (simple) k-quasi-planar, (simple) fan-crossing, (simple) fan-crossing free, (simple) min-k-planar.

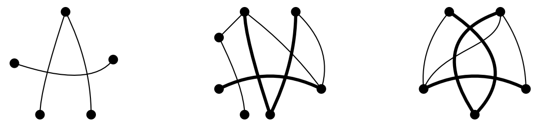
A set $\mathcal{F}$ of forbidden crossing patterns defines a beyond-planar drawing style $\mathcal{D}(\mathcal{F})$, namely the drawings that avoid all patterns in $\mathcal{F}$. Our main result is the following meta theorem, which states that the beyond-planar crossing number $\text{cross}_{\mathcal{D}(\mathcal{F})}$ is FPT with respect to its natural parameterization.

▶ **Theorem 1.** *For any fixed set $\mathcal{F}$ of crossing patterns, the problem of testing whether a given graph G admits a drawing with at most c crossings that avoids all patterns in $\mathcal{F}$ is FPT with respect to c.*

In particular, since our notion of patterns is sufficiently general, this yields, among others, FPT results for the beyond-planar crossing number for several beyond-planar graph classes.

▶ **Corollary 2.** *The beyond-planar crossing number is FPT for the following notions of beyond-planarity and any fixed k: k-planar, k-quasi-planar, fan-crossing, fan-crossing free, min-k-planar.*

Outline. After discussing preliminaries in Section 2, we develop our notion of crossing patterns in Section 3. The next two sections together describe an FPT algorithm for testing the existence of a drawing with at most c crossings that avoids a fixed set $\mathcal{F}$ of crossing patterns. It is based on the approach of Grohe [25] for the crossing number and works in the same two phases. In the first phase (Section 4), we bound the treewidth of the graph in terms of the beyond-planar crossing number, while the second phase solves the problem on graphs of bounded treewidth via Courcelle's theorem [21]. Section 6 presents some generalizations that allow us to cover additional beyond-planar graph classes such as simultaneous planarity and 2-layer drawings with various restrictions. We summarize our findings and discuss open questions in Section 7. Lemmas marked with ($\star$) are omitted due to space restrictions.



(*a*) (*b*) (*c*)

Figure 1 (*a*) Forbidden configuration in a fan-crossing-free drawing. (*b*) – (*c*) Two drawings containing the forbidden configuration.

2 Preliminaries

A *drawing* Γ of a graph $G = (V, E)$ maps every vertex $v \in V$ to a point $\Gamma(v) \in \mathbb{R}^2$ and every edge $uv \in E$ to an open simple Jordan arc with endpoints $\Gamma(u)$ and $\Gamma(v)$ that does not pass through any $\Gamma(w)$ for $w \in V \setminus \{u, v\}$. A crossing is a common interior point of two edges. A graph is *planar* if it can be drawn in the plane without edge crossings. We say that a drawing Γ is *simple* if two edges cross at most once and no two adjacent edges cross.

Let Γ be a drawing of a graph G. The *planarization* of Γ is the drawing Γ' we obtain from Γ by replacing every crossing point by a crossing vertex. In this paper we do not allow self-crossings. For graphs considered in this paper we allow multi-edges and self-loops.

3 Combinatorial Description of Drawings and Crossing Patterns

Beyond-planar graph classes are usually defined via forbidden configurations or patterns in a drawing. Our goal is to formalize this concept on a combinatorial level. As an example, consider the class of *fan-crossing-free graphs*, which contains all graphs that admit a drawing that does not contain two adjacent edges that both cross a third edge. The corresponding forbidden pattern given in the literature (see e.g. [22]) is shown in Fig. 1(a). Clearly the drawing shown in Fig 1(b) contains the forbidden pattern. Similarly, also the drawing shown in Fig 1(c) contains an edge that is crossed by two edges e, f sharing an endpoint and is thus not fan-crossing-free. However, its planarization does not contain a subgraph isomorphic to the planarization of the forbidden pattern. The presence of the pattern is obscured by the additional crossing between the adjacent edges in the drawing. It is therefore natural to require for a drawing D to contain a given pattern P, that crossings that appear within the pattern P must appear also within an occurrence of P in D, whereas crossings that are absent in a pattern P but are present in the drawing can be ignored and do not help to avoid the pattern P.

Now consider 1-planar graphs. A natural forbidden pattern in this graph class is the one shown in Fig. 2(a). However, there are many more (see Fig. 2(b)–(d) for examples), as some of the involved edges may share an endpoint or edges may cross each other multiple times. This leads to the observation that, in many crossing patterns, e.g., for k-planarity, the crossings are of vital importance, whereas the vertices are often not. In fact, often drawing styles only specify the presence of crossings on some edges but are agnostic to whether the endpoints of the involved edges are identical or not and they therefore rather only require that there exists a part of some edge that is involved in certain crossings. To be able to express this, our patterns allow to contain *subdivision vertices* of degree 1, which signify that the corresponding edge of the pattern may be mapped to a part of an edge. In particular, the class of 1-planar graphs is defined by the absence of the single pattern shown in Fig. 2(e).

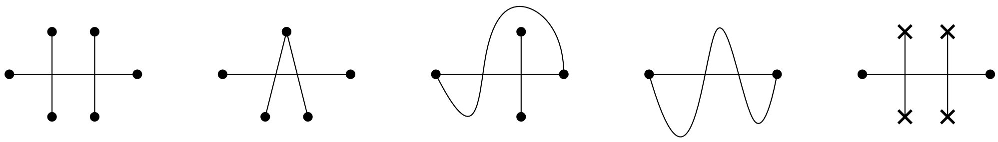


Figure 2 Forbidden patterns in a 1-planar drawing. Crosses represent subdivision vertices.

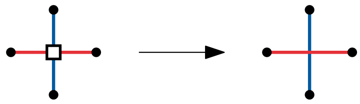


Figure 3 Illustrations of the smoothing operation at a crossing vertex.

▶ **Definition 3.** *A* crossing pattern *is a graph* $P = (V_P, E_P)$ *with* $V_P = R \cup C \cup S$ *such that* (i) *each* subdivision vertex *in* S *has degree* 1*,* (ii) *each* crossing vertex *in* C *has degree* 4 *and its incidences are partitioned into two sets of size* 2*, and* (iii) *each* real vertex *in* R *and each subdivision vertex in* S *is adjacent to at least one crossing vertex.*

We note that condition *(ii)* allows us to uniquely identify the two edge segments that form a crossing. In our figures, we usually indicate the pairing by a planar edge-colored drawing where the incidences that are part of the same set in the partition are drawn oppositely in the rotation around the crossing vertex and have the same color. Condition (iii) is a technical condition that we will need later. It is however, a natural assumption, as it ensures that our patterns indeed express conditions on crossing configurations in drawings rather than structural conditions on the input graph.

Next, we need to define when a drawing Δ of a graph G contains a crossing pattern P. A simple idea would be to consider the planarization of Δ and to require that it contains a subgraph isomorphic to P, where crossing and real vertices of P are mapped to crossing and real vertices of the planarization, respectively. This is, however, not sufficient. In addition we want to be able to remove parts of an edge, where if the removed part does not start at a real vertex, we place a subdivision vertex. Finally, as explained above, it is crucial that we can ignore crossings that are present in the drawing but that are not relevant for forming the pattern. To formalize this, we define the concept of a drawing representation.

▶ **Definition 4.** *A* drawing representation *is a graph* $D = (V_R, E_R)$ *whose vertex set* $V_R = R \cup C \cup S$ *consists of* real vertices *in* R*,* crossing vertices *in* C *and* subdivision vertices *in* S *such that* (i) *each subdivision vertex has degree at most* 2*,* (ii) *each crossing vertex has degree* 4 *and its edge incidences are partitioned into two sets of size 2.*

Two drawing representations are *isomorphic* if there is an isomorphism between them that maps real vertices to real vertices, crossing vertices to crossing vertices and subdivision vertices to subdivision vertices. Observe that for any drawing Δ of a graph G, the planarization D of Δ is a drawing representation, where the partition of the incidences at each crossing vertex c is defined such that incidences that are opposite in the rotation around the crossing that corresponds to c are in the same set of the partition.

Let c be a crossing vertex in a drawing representation. To be able to ignore crossings, we define the operation of *smoothing* c as (i) inserting edges between those neighbors of c that are endpoints of edges whose incidences are in the same set of the partition for c and (ii) deleting c and its incident edges; see Fig. 3. In addition we allow the removal of isolated

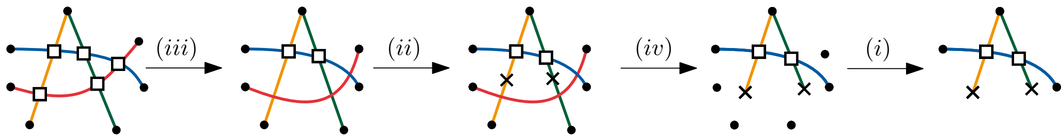


Figure 4 Illustration of the operations allowed on a drawing representation.

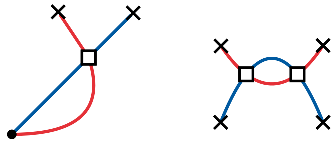


Figure 5 Forbidden crossing patterns in simple drawings.

vertices, the addition of subdivision vertices on edges of the drawing representation and the removal of edges. To keep the degree of crossing vertices at 4, we restrict the last operation to edges that are not incident to a crossing vertex.

▶ **Definition 5.** *Let D, D' be drawing representations. Then D* contains *D' if and only if a drawing representation isomorphic to D' can be obtained from D by a sequence of the following operations*

(i) *deleting an isolated vertex,*
(ii) *subdividing an edge by introducing a subdivison vertex,*
(iii) *smoothing a crossing vertex,*
(iv) *deleting an edge that is not incident to a crossing vertex.*

Figure 4 shows an example of a sequence of such operations. We note that, while the planarization of a drawing of a graph G constitutes a drawing representation that is planar, the smoothing operation may turn such a representation into a non-planar graph. Additionally, observe that a crossing pattern as defined above is a drawing representation. In particular, we have now formally defined when a drawing representation contains a crossing pattern. We define that a drawing Δ of a graph G *contains* a crossing pattern P if and only if the planarization of Δ (considered as a drawing representation) contains P. If Δ does not contain P, we also say that Δ *avoids* P.

Note that a drawing of a graph G is simple if and only if it avoids the patterns shown in Fig. 5. The following lemma shows that several widely known beyond-planar drawing styles are captured by our notion of forbidden crossing patterns. For a definition of the mentioned graph-classes we refer to [35, 18, 14, 1, 7, 8]. Let P_k denote the crossing pattern consisting of an edge between two real vertices that is subdivided by $k+1$ crossing vertices that are all adjacent to two private subdivision neighbors; see Fig. 6(a) for $k = 2$.

▶ **Lemma 6** ($\star$). *Let Δ be a drawing of a graph G. Then Δ is*

(i) *k-planar if and only if it avoids P_k.*
(ii) *fan-crossing free if and only if it avoids the pattern in Fig. 6(b).*
(iii) *fan-crossing if and only if it avoids the pattern in Fig. 6(c).*
(iv) *quasi-planar if and only if it avoids the pattern in Fig. 6(d).*
(v) *min-1-planar if and only if it avoids the pattern in Fig. 6(e).*

Observe that also the number of forbidden crossing patterns in a k-quasi-planar or min-k-planar drawing is bounded by $f(k)$ for a computable function f, since the number of orders in which k edges can cross each other is bounded. Hence also min-k-planarity and k-quasi-planarity can be expressed by a finite set $\mathcal{F}$ of forbidden patterns for any fixed k.

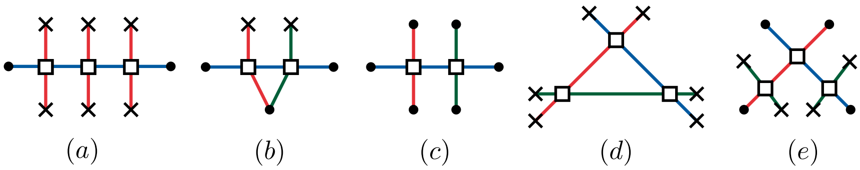


Figure 6 Examples of crossing patterns.

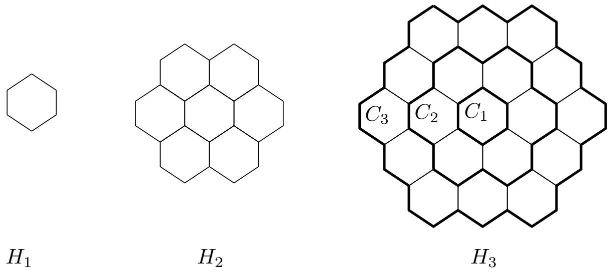


Figure 7 The hexagonal grids H_1, H_2, H_3 and the principal cycles of H_3.

4 Bounding Treewidth

Let $G = (V, E)$ be a graph for the rest of this section and let $F \subseteq E$. For a fixed family $\mathcal{F}$ of crossing patterns, we say that G has an $(F, \mathcal{F}, c)$-*good drawing* if there exists a drawing of G that (i) does not contain any pattern in $\mathcal{F}$, (ii) has at most c crossings and (iii) none of the crossings involve an edge from F. Let $c \geq 1$ be fixed for the rest of this section. The goal of the following two sections is to give an algorithm that solves the following problem in quadratic time. For a fixed family $\mathcal{F}$ of crossing patterns, given a graph G, an integer c and a subset $F \subseteq E$, the question is whether there exists an $(F, \mathcal{F}, c)$-good drawing of G. Similarly to the algorithm given by Grohe [25] solving the generalized c-crossing number problem in quadratic time, our algorithm works in two phases. First the size of the input graph is reduced iteratively until we get a graph with treewidth bounded by a constant only depending on c. In the second phase the algorithm solves the problem on this graph of bounded treewidth.

To bound the treewidth of our input graph we use Grohe's approach of an irrelevant-vertex strategy. However, several modifications are necessary, as Grohe only counts crossings, whereas we additionally have to ensure the absence of certain crossing patterns. In particular, Lemmas 10–12 are corresponding adaptations of Lemmas 5–7 in Grohe's paper [25].

4.1 Fundamentals

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A *topological embedding* of G into H is a mapping h that maps every vertex $v \in V$ to a vertex in $h(v) \in V'$ and every edge $e \in E$ to a path $h(e)$ in H such that for all $v \neq w \in V$ we have $h(v) \neq h(w)$, for all $e \neq f \in E$ the paths $h(e)$ and $h(f)$ have at most their endpoints in common and for every edge $uv \in E$ the endpoints of $h(e)$ are $h(u)$ and $h(v)$, and $h(w) \notin h(e)$ for all $w \in V \setminus \{u, v\}$.

We denote the *hexagonal grid* with radius r by H_r; see Figure 7. We number the r concentric cycles $C_1, \dots, C_r$ from the interior to the exterior. Let G be a graph with subgraph H. An *H-component* of G is either a connected component C of $G - H$ together with all edges between C and H and their endpoints in H, or an edge $uv \in E(G) \setminus E(H)$ such that $u, v \in H$, together with its endpoints; see Figure 8. We call the vertices in

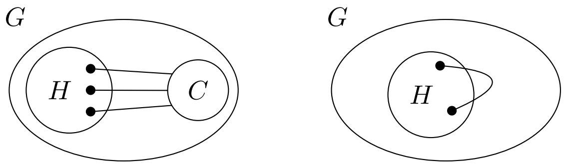


Figure 8 Illustrations of H-components.

the intersection of an H-component with H the *vertices of attachment* of the component. Let $h : H_r \to G$ be a topological embedding. We call the subgraph $h(H_r \setminus C_r)$ the *interior* of $h(H_r)$. An $h(H_r)$-component is *proper*, if it has at least one vertex of attachment in the interior of $h(H_r)$. The topological embedding h is *flat* if the union of $h(H_r)$ with all its proper components is planar.

For a fixed family $\mathcal{F}$ of crossing patterns, the $\mathcal{F}$*-crossing* number of G is the minimum number of crossings over all drawings of G that do not contain a pattern in $\mathcal{F}$. The following theorem is due to Thomassen [43] who stated the result for the genus of a graph rather than its $\mathcal{F}$-crossing number. However, observe that the $\mathcal{F}$-crossing number of a graph is an upper bound for its genus and thus our modification of the theorem holds.

▶ **Theorem 7** ([43]). *For all $c, r \geq 1$ there is an $s \geq 1$ such that the following holds. If a graph G has $\mathcal{F}$-crossing number at most c and $h\colon H_s \to G$ is a topological embedding, then there is a subgrid $H_r \subseteq H_s$ such that the restriction $h|_{H_r}$ of h to H_r is flat.*

The following version of the well-known *Excluded Grid Theorem* will be a useful tool.

▶ **Theorem 8** ([25, 41, 11, 37]). *Let $s \geq 1$. Then there is a $w \geq 1$ and a linear-time algorithm that, given a graph G, either (correctly) recognizes that the treewidth of G is at most w or computes a topological embedding $h : H_s \to G$.*

4.2 The Algorithm

Let $G = (V, E)$ be a graph and let Δ be a drawing of G in the plane. Let γ be a Jordan curve in the plane that intersects Δ only in vertices; we call such a curve a *boundary curve*. Let G_I be the graph formed by the vertices and edges in the interior of γ and the vertices on γ. Let $G_I + \gamma$ denote the graph we obtain from G_I by adding an edge between two vertices u, v if u, v lie on γ and are connected via a subcurve of γ that does not contain other vertices of G_I as interior points. Assume there exists a crossing-free drawing Γ of $G_I + \gamma$ whose outer face is γ. We denote the drawing obtained from Δ by adding γ, replacing its interior with Γ and then removing γ by $\Delta_{\gamma \to \Gamma}$; see Figure 9. A drawing style $\mathcal{D}$ is *patch-closed* if, whenever $\Delta \in \mathcal{D}$, then also $\Delta_{\gamma \to \Gamma} \in \mathcal{D}$ for any boundary curve γ and any crossing-free drawing Γ of $G_I + \gamma$.

▶ **Lemma 9** ($\star$). *Every drawing style $\mathcal{D}$ that is defined by the absence of a finite set of crossing patterns and the requirement that certain edges must not be involved in any crossing, is patch-closed.*

For the rest of this section let c be fixed and let $r := 4c + 3$. This choice becomes relevant in the proof of Lemma 11. Moreover, let s be sufficiently large, such that for every graph of $\mathcal{F}$-crossing number at most c and every topological embedding $h : H_s \to G$, there is a subgrid $H_r \subseteq H_s$ such that the restriction $h|_{H_r}$ of h to H_r is flat by Theorem 7. By Theorem 8 there is a w and a linear-time algorithm that, given a graph G of treewidth greater than w, computes a topological embedding $h\colon H_s \to G$. Note that w depends on s, which depends on r, which ultimately depends on c. Therefore w depends only on c.

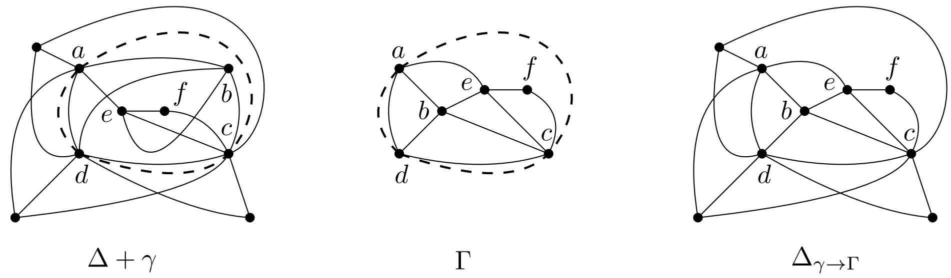


Figure 9 Illustration of patch-closeness. The dashed curve represents γ.

▶ **Lemma 10** ($\star$). *Let $\mathcal{F}$ be a finite family of crossing patterns and let w, r be fixed as above. There is a linear-time algorithm that, given a graph G, either recognizes that the $\mathcal{F}$-crossing number of G is greater than c, or recognizes that the treewidth of G is at most w, or computes a flat topological embedding $h\colon H_r \to G$.*

By applying this lemma, we either learn that G has treewidth at most w, or that the $\mathcal{F}$-crossing number of G is greater than c, or we obtain a flat embedding $h\colon H_r \to G$. In the first two cases we are done, as we can either move to the second phase or reject the instance immediately. Hence we now assume that we obtain a flat embedding $h\colon H_r \to G$. We denote the subgrid of H_r bounded by the second principal cycle C_2 as H^2. The *kernel* of h is the subgraph of G consisting of $h(H^2)$ and all $h(H_r)$-components whose vertices of attachment are all in $h(H^2)$. The *boundary of the kernel* is the subgraph of G consisting of $h(C_2)$ and all $h(H_r)$-components whose vertices of attachment are all in $h(C_2)$. The subgraph containing the kernel without its boundary is called the *interior of the kernel*.

▶ **Lemma 11.** *Let G be a graph, $F \subseteq E(G)$ and let Δ be an $(F, \mathcal{F}, c)$-good drawing of G. Let $h\colon H_r \to G$ be a flat topological embedding. Then there is an $(F, \mathcal{F}, c)$-good drawing Δ' of G such that none of the edges of the kernel of h is involved in any crossing of Δ'.*

Proof. For $1 \le i \le r-1$, let B_i be the subgraph of G consisting of $h(C_i)$ and all $h(H_r)$-components whose vertices of attachment are all in $h(C_i)$. Moreover, let R_i be the subgraph of G consisting of the images of the i-th and the $(i+1)$-th principal cycle $h(C_i)$, $h(C_{i+1})$ and the images of all edges in H_r connecting these two cycles. Observe that for i, j with $1 \le i < j-1 \le r-2$, the graphs $R_i \cup B_i$ and $R_j \cup B_j$ are disjoint. Note that at most two edges are involved in any crossing and recall that $r = 4c+3$. For each crossing that involves two edges e, f where e belongs to B_i and f to B_j, we either have $i = j$ or $|i-j| = 1$. Thus, at most $2c$ distinct B_i's contain a crossed edge and by the pigeonhole-principle, there is an i_0 with $2 \le i_0 \le r-1$ such that none of the edges in $B_{i_0} \cup R_{i_0}$ is involved in any crossing in Δ.

Then, there is a Jordan curve γ around $B_{i_0} \cup R_{i_0}$ that intersects Δ only in vertices of $B_{i_0} \cup R_{i_0}$ such that all edges of $B_{i_0} \cup R_{i_0}$ lie in the interior of γ. Let G_I be the graph consisting of the vertices and edges in the interior of γ. Since h is flat, there exists a planar drawing Γ of $G_I + \gamma$. Moreover, since $(F, \mathcal{F}, c)$-good drawings are patch-closed by Lemma 9, also $\Delta' = \Delta_{\gamma \to \Gamma}$ is $(F, \mathcal{F}, c)$-good. ◀

▶ **Lemma 12.** *There is a linear-time algorithm that, given a graph G and an edge set $F \subseteq E(G)$, either recognizes that the $\mathcal{F}$-crossing number is greater than c or recognizes that the treewidth of G is at most w or computes a graph G' and an edge set $F' \subseteq E(G')$ with $|V(G')| < |V(G)|$ such that G has an $(F, \mathcal{F}, c)$-good drawing if and only if G' has an $(F', \mathcal{F}, c)$-good drawing.*

Proof. We apply the algorithm of Lemma 10. If the $\mathcal{F}$-crossing number of G is greater than c or the treewidth of G is at most w we are done. Otherwise the algorithm returns a flat topological embedding $h : H_r \to G$. Let K be the kernel of h, I its interior, and B its boundary. Let G' be the graph obtained from G by contracting the connected subgraph I to a single vertex v_I. Let F' be the subset of F that is not contracted in G' together with the set of all edges of B and all edges incident to the new vertex v_I. Observe that G' and F' can be computed in linear time and $|V(G')| < |V(G)|$. It remains to show that G has an $(F, \mathcal{F}, c)$-good drawing if and only if G' has an $(F', \mathcal{F}, c)$-good drawing.

First assume that G has an $(F, \mathcal{F}, c)$-good drawing Δ. By Lemma 11 there exists an $(F, \mathcal{F}, c)$-good drawing Δ' such that none of the edges of the kernel of h is involved in any crossing of Δ'. Let Γ be the drawing of G' we obtain from Δ' by contracting the kernel of h into a single vertex v_I. Clearly, all edges of F' are uncrossed in Δ'. Note that since Δ' does not contain any pattern in $\mathcal{F}$, if Γ contains such a pattern $P \in \mathcal{F}$, then the occurrence of P must contain v_I. However, since all edges incident to v_I are crossing-free in Γ, v_I cannot be contained in an occurrence of a crossing pattern by property *(iii)* of crossing patterns.

Conversely, assume that G' has an $(F', \mathcal{F}, c)$-good drawing Γ. By the choice of F', the wheel induced by the vertices of B and v_I is crossing-free in Γ. Since H is flat, there exists a planar drawing of the kernel with the vertices of B on the outer face and in the same order as they appear in Γ. Hence the planar graph I can be embedded into a small neighborhood of $\Gamma(v_I)$. Again, this does not introduce new crossing patterns, since all edges incident to a vertex in I are uncrossed and no edge in F is involved in any crossing in the resulting drawing of G. Thus we obtain an $(F, \mathcal{F}, c)$-good drawing of G. ◀

By repeatedly applying the algorithm from Lemma 12 to an input graph G, we either eventually obtain an equivalent instance whose $\mathcal{F}$-crossing number exceeds c, or we arrive at an equivalent instance that has bounded treewidth. In the former case, we can reject the instance, in the latter case we move on to Phase 2.

▶ **Corollary 13.** *There is a quadratic-time algorithm that, given a graph G, either recognizes that the $\mathcal{F}$-crossing number of G is greater than c or computes a graph G' and an edge set $F' \subseteq E(G)$ such that the treewidth of G' is at most w and G has an $(F, \mathcal{F}, c)$-good drawing if and only if G' has an $(F', \mathcal{F}, c)$-good drawing.*

5 Bounded Treewidth and Crossings

Now we describe Phase 2 of our algorithm. If in Phase 1 we do not find out that the $\mathcal{F}$-crossing number of G is greater than c, we end up with a graph G' of treewidth at most w, where w depends only on c, and an edge set $F' \subseteq E$ such that G has an $(F, \mathcal{F}, c)$-good drawing if and only if G' has an $(F', \mathcal{F}, c)$-good drawing. Now we use Courcelle's Theorem to show that we can decide in quadratic time whether G' has an $(F', \mathcal{F}, c)$-good drawing. To this end, we describe an MSO_2-formula $\varphi_{\mathcal{F}}(F)$ such that $G \vDash \varphi_{\mathcal{F}}(F)$ if and only if G has an $(F, \mathcal{F}, c)$-good drawing. As drawings of graphs cannot be expressed in MSO_2, we rather express this as the existence of a suitable planarization of G, or in fact rather a drawing representation that is planar.

We use the following primitive formulas as basic building blocks. The formula *conn* returns whether a vertex set $X \subseteq V$ is connected by edges from a set $E' \subseteq E$, *deg0, deg1, deg2* express that a given vertex x has degree 0, 1 or 2, respectively, in the subgraph formed by the edges in the set E', *disjoint* expresses whether two edge sets or vertex sets are disjoint, $minor_H$ expresses whether G contains a fixed graph H as a minor and *planar* expresses whether G is planar.

Let $G = (V, E)$ be a graph and let $F \subseteq E$ be a set of edges that must not receive a crossing. Assume we want to find a drawing representation with at most c crossings. Our first step is to construct from G a new graph G^+ by subdividing each edge in $E \setminus F$ with c *crossing dummies*. The idea is that a planarization of a drawing of G with at most c crossings can be described by identifying at most c pairs of crossing dummies of G^+ so that the resulting graph is planar. For the treatment of crossing patterns, we need to be able to place subdivision vertices on the edges of such a planarization. To enable this, we further modify G^+ into a graph $G^\star$ where we subdivide each edge that is incident to a crossing dummy by two *subdivision dummies*. The vertex set of $G^\star$ is $V \cup C \cup S$, where V contains the vertices of G, C contains the crossing dummies and S contains the subdivision dummies.

Expressing a Drawing Representation. The formula $\varphi_{\mathcal{F}}$ we construct applies to the graph $G^\star$ and receives as input the vertex sets V, C, S and $2c$ free variables $x_1, \dots, x_c$, $y_1, \dots, y_c$. The idea is that their values shall be vertices from C and the intended meaning is that the crossing dummy x_i is identified with the crossing dummy y_i to form an actual crossing. By smoothing all subdivision dummies and all crossing dummies that were not identified with another crossing dummy we can consider the result as a drawing representation, whose real vertices are the vertices of G and whose crossing vertices are formed by pairs of crossing dummies that were identified. It therefore makes sense to require that the values of the x_i are pairwise distinct, with the single exception that we allow $x_i = y_i$ to allow the use of fewer than c crossings. We express these requirements with the following formula.

$$\text{distinctCrossings} = \bigwedge_{i=1}^{c} \bigwedge_{j \neq i} (x_i \neq x_j) \wedge (x_i \neq y_j) \wedge (y_i \neq y_j)$$

As we work with $G^\star$ rather than the original graph G, paths whose interior vertices are all subdivision or crossing dummies play a particular role as such paths correspond to a part of an edge of G. The following predicate expresses that an edge set E' forms a path between two vertices x and y, whose interior vertices, if any, are subdivision or crossing dummies.

$$\begin{aligned}\text{path}(E', x, y) = &\exists_{I \subseteq C \cup S} \operatorname{conn}(I, E') \wedge \forall_{v \in I} \operatorname{deg2}(v, E') \wedge \\ &\forall_{v \in V} (v \notin I \wedge v \neq x \wedge v \neq y \rightarrow \operatorname{deg0}(v)) \wedge \operatorname{deg1}(x, E') \wedge \operatorname{deg1}(y, E')\end{aligned}$$

Using this we can easily formulate the requirement of having no self-crossings.

$$\text{noSelfCrossings} = \bigwedge_{i=1}^{c} \forall_{E' \subseteq E} \neg \operatorname{path}(E', x_i, y_i)$$

Constraining the Drawing Representation. Let $G^\times$ denote the graph obtained from $G^\star$ by identifying x_i with y_i for $i \in \{1, \dots, c\}$. In what follows we want to express properties of $G^\times$ in an MSO_2-formula. However, our formula does not have access to the graph $G^\times$, instead we have to express all conditions in terms of the original graph $G^\star$. A vertex $v \in V(G^\times)$ is incident to an edge $e \in E(G^\times)$ if and only if v is incident to e in $G^\star$ or if e is incident to a vertex v' that is identified with v in $G^\times$. This incidence relation in $G^\times$ can be expressed as follows.

$$\operatorname{inc}^\times(v, e) = \operatorname{inc}(v, e) \vee \left(\bigvee_{i=1}^{c} (\operatorname{inc}(x_i, e) \wedge v = y_i) \vee (\operatorname{inc}(y_i, e) \wedge v = x_i) \right)$$

Now any formula φ that expresses a graph property can be transformed into a formula $\varphi^\times$ that expresses that $G^\times$ has the same property by replacing each occurrence of inc in φ by $\text{inc}^\times$ and replacing any comparison of vertices $a = b$ by the expression $a = b \vee \bigvee_{i=1}^{c} ((x_i = a \wedge y_i = b) \vee (x_i = b \wedge y_i = a))$. In particular, $\text{planar}^\times$ expresses that the graph $G^\times$ is planar.

Avoiding Patterns. Assume $x_1, \ldots, x_c$ and $y_1, \ldots, y_c$ are chosen such that they satisfy $\text{distinctCrossings} \wedge \text{noSelfCrossings} \wedge \text{planar}^\times$. Then smoothing the subdivision dummies and the unused crossing dummies defines a planarization of a drawing of G with at most c crossings and hence a drawing representation of such a drawing. Checking whether this representation contains a fixed crossing pattern P would possibly require placing subdivision vertices. However, it is never necessary to place more than two subdivision vertices on an edge of the planarization, and therefore by construction $G^\star$ contains already a sufficient number of subdivision dummies that can be used as subdivision vertices. We also do not remove edges and vertices but rather specify the vertices and edges that we keep. Moreover, by working in the graph $G^\star$, all crossings are automatically smoothed and we rather express explicitly, which crossings we want to keep.

Let $P = (R_P \cup C_P \cup S_P, E_P)$ be a fixed crossing pattern. A *chain* in P is a path p that connects two vertices from $R_P \cup S_P$, whose interior vertices are in C_P, and for each interior vertex c of p the incidences with the two edges in p are in the same set of the partition of incidences of c. Note that there is a unique set of chains that cover P exactly, see Fig. 6, which we call the *chain cover* of P.

To express the presence of the pattern P, we choose a subdivision dummy in $G^\star$ for each subdivision vertex of P, a real vertex in $G^\star$ for each real vertex of P, as well as, for each chain c in the chain cover of P, a crossing dummy in $G^\star$ for each crossing vertex of p and a set of edges in $G^\star$. Observe that for each crossing vertex we thus choose two crossing dummies, one for each of the two chains of the chain cover that contain it. We then express that the edge sets chosen for the chains are pairwise disjoint and for each chain p the corresponding chosen edge set forms a path in $G^\star$ that connects the two vertices chosen for the endpoints of p, does not contain real vertices in its interior and visits the crossing dummies chosen for its crossing vertices in the correct order. Finally, we express that all crossings of P are actually present by specifying for each crossing vertex c of P that the two chains p_1, p_2 of the chain cover that contain c have chosen a pair of crossing dummies for c that is identified by the choices of the x_is and y_is.

▶ **Lemma 14** ($\star$). *For every crossing pattern P there exists an MSO_2-formula φ_P that determines for the graph $G^\star$ whether the graph $G^\times$ defined by identifying the crossing dummies x_i and y_i for $i \in \{1, \ldots, c\}$ contains P. The size of φ_P depends only on c and the size of P.*

Using the formula $\varphi_{\mathcal{F}} = \text{planar}^\times \wedge \bigwedge_{P \in \mathcal{F}} \neg\varphi_P$ it then follows that G admits an $(F, \mathcal{F}, c)$-good drawing if and only if $G^\star \models \varphi_{\mathcal{F}}$. Observe that, since $G^\star$ is obtained from G by subdividing edges, it has bounded treewidth. Moreover, the size of $\varphi_{\mathcal{F}}$ depends only on the number of crossings c and the size of the patters in $\mathcal{F}$. Using Courcelle's theorem, we obtain the following result, which together with Corollary 13 implies our main result Theorem 1.

▶ **Theorem 15.** *Let $\mathcal{F}$ be a family of crossing patterns and let c be an integer. Let $G = (V, E)$ be a graph of treewidth at most c and let $F \subseteq E$. Then there is an algorithm that decides in quadratic FPT-time whether G admits an $(F, \mathcal{F}, c)$-good drawing.*

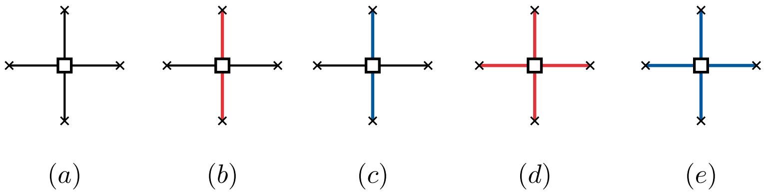


Figure 10 Forbidden crossing patterns for SEFE.

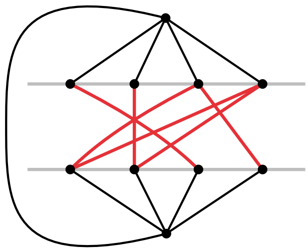


Figure 11 Sketch of the reduction for the 2-layered setting.

6 Extensions and Further Applications

Our notion of drawing representations and crossing patterns can easily be extended to graphs whose edges are colored with a fixed number of colors. In this setting, two drawing representations are isomorphic if there is an isomorphism between them that for every color c maps c-colored edges to c-colored edges. As Phase 1 of our algorithm works irrespective of such a coloring and the MSO_2 formula from Phase 2 can be straightforwardly adapted to take into account the colors of the edges, Theorem 1 also holds for colored graphs and colored crossing patterns.

This enables several additional applications. For a graph whose edges are red and black, we can express that black edges must not be crossed by forbidding the crossing patterns in Figure 10$(a),(b)$. By employing the reduction sketched in Figure 11 and further patterns on the red edges, this allows to compute the beyond-planar crossing number of 2-layered drawings in FPT time for all beyond-planar graph classes that can be expressed in terms of our crossing patterns.

▶ **Corollary 16.** *For any fixed set $\mathcal{F}$ of crossing patterns testing whether a given bipartite graph admits a 2-layer drawing with at most c crossings that avoids all patterns in $\mathcal{F}$ is FPT with respect to c.*

Considering graphs whose edges are black, red, or blue, and allowing only crossings between red and blue edges by forbidding the patterns in Fig. 10$(a)-(e)$ allows to model the famous SEFE problem [10, 42].

▶ **Corollary 17.** *Testing whether a pair of graphs (G_1, G_2) admits a SEFE with at most c crossings is FPT with respect to c.*

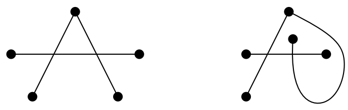


Figure 12 Two drawings that both contain the pattern shown in Fig. 6(b).

7 Conclusion

We introduced a combinatorial definition of a crossing pattern and showed that for every beyond-planar graph class $\mathcal{C}$ that is defined by the absence of a finite number of such patterns, deciding whether a graph is in $\mathcal{C}$ is FPT with respect to the corresponding beyond-planar crossing number. Our results are applicable to many beyond-planar graph classes, the ones we mentioned in the paper are only an extract.

However, since our current definition of crossing patterns does not take topological aspects into account, it is not able to distinguish the two drawings shown in Fig. 12. Recall that fan-planar graphs are defined by the absence of the pattern shown in Fig. 6(c) and the configuration in Fig. 12(b). Thus an interesting open question is, whether our notion of crossing patterns can be adapted such that also graph classes that rely on topological properties such as fan-planar or geometric k-planar graphs can be handled.

Another open question is whether bounding the treewidth also works for a definition of crossing patterns without condition (iii); i.e., if we do not require that real and subdivision vertices are adjacent to crossing vertices.

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