# On $k$-Planar Graphs Without Short Cycles

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## Abstract

We study the impact of forbidding short cycles to the edge density of $k$-planar graphs; a *$k$-planar* graph is one that can be drawn in the plane with at most $k$ crossings per edge. Specifically, we consider three settings, according to which the forbidden substructures are 3-cycles, 4-cycles or both of them (i.e., girth $\geq 5$). For all three settings and all $k \in \{1, 2, 3\}$, we present lower and upper bounds on the maximum number of edges in any $k$-planar graph on $n$ vertices. Our bounds are of the form $c\,n$, for some explicit constant $c$ that depends on $k$ and on the setting. For general $k \geq 4$ our bounds are of the form $c\sqrt{k}n$, for some explicit constant $c$. These results are obtained by leveraging different techniques, such as the discharging method, the recently introduced density formula for non-planar graphs, and new upper bounds for the crossing number of 2– and 3-planar graphs in combination with corresponding lower bounds based on the Crossing Lemma.

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## 1 Introduction

"What is the minimum and maximum number of edges?" is one of the most fundamental questions one can ask about a finite family of graphs. In some cases the question is easy to answer; for instance, for the class of all graphs on $n$ vertices the answer is even trivial. Another such family is the one of planar graphs. More precisely, for planar graphs on $n$ vertices we know from Euler's Formula that they have at most $3n - 6$ edges. Furthermore,


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every planar graph on $n$ vertices can be augmented (by adding edges) to a maximal planar graph with exactly $3n-6$ edges. Important advances have recently been made for non-planar graphs in the context of graph drawing beyond-planarity [15]. But often an answer is much harder to come by. Specifically, there exist graph classes that are relevant in Graph Drawing where exact bounds on their edge density are difficult to derive.

The family of graphs that can be embedded on the Euclidean plane with at most $k$ crossings per edge, called *$k$-planar*, is a notable example. Tight bounds on the edge density of these graphs, for small values of $k$, are crucial as they lead to improvements on the well-known Crossing Lemma [3]. This was first observed by Pach and Tóth [22], who back in 1997 presented one of the early improvements of the Crossing Lemma by introducing tight bounds on the edge density of 1- and 2-planar graphs. Since then, only two improvements emerged; one by Pach, Radoičić, Tardos, and Tóth [17, 18] in 2004 and one by Ackerman [1] in 2019, both by introducing corresponding bounds on the edge density of 3- and 4-planar graphs, respectively. On the other hand, it is worth noting that these progressive refinements on the Crossing Lemma led to corresponding improvements also on the upper bound on the edge density of general $k$-planar graphs with the best one being currently $3.81\sqrt{k}n$ due to Ackerman [1]. To the best of our knowledge, for 5-planar graphs a tight bound is missing from the literature, even though it would yield further improvements both on the Crossing Lemma and on the upper bound of the edge density of general $k$-planar graphs. Variants of the Crossing Lemma have also been proposed for specific classes of graphs, e.g., bipartite graphs [5, 12, 13].

In this work, we continue the study of this line of research focusing on special classes of graphs; in particular, on graphs not containing some fixed, so-called *forbidden substructures*. We consider three settings, according to which the forbidden substructures are 3-cycles ($C_3$-free), 4-cycles ($C_4$-free) or both of them (girth $\geq 5$). For each of these settings, the problem of finding edge density bounds has been studied both in general and assuming planarity. In particular, while $C_3$-free $n$-vertex graphs may have $\Theta(n^2)$ edges, $C_4$-free graphs and graphs of girth 5 have at most $O(n^{\frac{3}{2}})$ edges; see e.g. [14, 23]. For $C_3$-free planar graphs and planar graphs of girth 5, one can easily derive upper bounds on their edge density using Euler's Formula; see, e.g., Table 1. For $C_4$-free planar graphs, Dowden [11] proved that every such graph has at most $\frac{15}{7}(n-2)$ edges, and that this bound is best possible. For $k$-planar graphs, Pach, Spencer and Tóth [19, 20] provided a lower bound on the crossing number of $C_4$-free $k$-planar graphs, which can be used to obtain an asymptotic upper bound of $O(\sqrt[3]{k}n)$ on the edge density of such graphs with $n$ vertices. Another related research branch focuses on bipartite graphs (that avoid all odd-length cycles). For this setting, Angelini, Bekos, Kaufmann, Pfister, and Ueckerdt [5] have proposed lower and upper bounds on the edge density of several classes of graphs beyond-planarity, including 1- and 2-planar graphs.

## Our contribution

We study the class of $k$-planar graphs in the absence of 3-cycles, 4-cycles and both of them. Our results are summarized as follows:

- For each of the aforementioned settings, we present lower and upper bounds on the maximum number of edges of $k$-planar graphs with $n$ vertices when $k \in \{1, 2, 3\}$. Our findings are summarized in Table 1.
- We next use these bounds to derive corresponding lower bounds on the crossing numbers of the graphs that avoid the forbidden patterns studied. For a summary refer to Table 2.
- We use the two-way dependency between edge density and Crossing Lemma to derive new bounds on the edge density of $k$-planar graphs for values of $k$ greater than 3.

**Table 1** Maximum number of edges in $k$-planar graph classes, ignoring additive constants; results from the literature are shown in blue square brackets, results form this paper are shown in red angle brackets, bounds without a citation are derived from Euler's formula. The lower bound for 2-planar $C_4$-free graphs trivially holds for 3-planar $C_4$-free graphs.

| | unrestricted | | $C_3$-free | | $C_4$-free | | Girth 5 | |
|---|---|---|---|---|---|---|---|---|
| $k$ | lower | upper | lower | upper | lower | upper | lower | upper |
| 0 | $3n$ | $3n$ | $2n$ | $2n$ | $\frac{15n}{7}$ [11] | $\frac{15n}{7}$ [11] | $\frac{5n}{3}$ | $\frac{5n}{3}$ |
| 1 | $4n$ [8] | $4n$ [8] | $3n$ [10] | $3n$ ⟨5⟩ | $2.4n$ ⟨7⟩ | $2.5n$ ⟨6⟩ | $\frac{13n}{6}$ ⟨9⟩ | $2.4n$ ⟨8⟩ |
| 2 | $5n$ [22] | $5n$ [22] | $3.5n$ [5] | $4n$ ⟨10⟩ | $2.5n$ ⟨14⟩ | $3.93n$ ⟨11⟩ | $\frac{16n}{7}$ ⟨17⟩ | $3.597n$ ⟨15⟩ |
| 3 | $5.5n$ [18] | $5.5n$ [18] | $4n$ [5] | $5.12n$ ⟨18⟩ | – | $4.933n$ ⟨20⟩ | $2.5n$ ⟨22⟩ | $4.516n$ ⟨21⟩ |
| $k$ | $\Omega(\sqrt{k})n$ [22] | $3.81\sqrt{k}n$ [1] | | $3.19\sqrt{k}n$ ⟨19⟩ | | $3.016\sqrt{k}n$ ⟨12⟩ <br> $O(\sqrt[3]{k})n$ [20] | | $2.642\sqrt{k}n$ ⟨16⟩ <br> $O(\sqrt[3]{k})n$ [20] |

**Table 2** Bounds on the crossing numbers, ignoring additive constants; hold for sufficiently large $m$.

| | unrestricted | | $C_3$-free | $C_4$-free | Girth 5 |
|---|---|---|---|---|---|
| Graph class | lower | upper | lower | lower | lower |
| 2-planar | | $\frac{10n}{3}$ ⟨3⟩ | | | |
| 3-planar | | $\frac{33n}{5}$ ⟨4⟩ | | | |
| general | $0.034\frac{m^3}{n^2}$ [1] | | $0.049\frac{m^3}{n^2}$ ⟨18⟩ | $0.054\frac{m^3}{n^2}$ ⟨11⟩ | $0.071\frac{m^3}{n^2}$ ⟨15⟩ |

To obtain the above results, we leverage different techniques from the literature, such as the discharging method, the recently introduced density formula for non-planar graphs [16], and new upper bounds for the crossing number of 2– and 3-planar graphs (Theorems 3 and 4) in combination with corresponding lower bounds based on the Crossing Lemma.

## 2 Preliminary Techniques and Tools

In this section, we describe techniques that we use in our proofs, namely, the discharging method [1, 2] (Section 2.1) and a method derived from a well-known probabilistic proof [3] of the Crossing Lemma (Section 2.2), which we formalise in the following. This section is concluded with two theorems of independent interest providing upper bounds on the number of crossings of (general) 2- and 3-planar graphs (Section 2.3).

### 2.1 The Discharging Method

In some of our proofs, we employ the discharging method [1, 2], which is summarised as follows. Consider a biconnected graph $G = (V, E)$ on $|V| = n$ vertices drawn in $\mathbb{R}^2$ and its planarization $G' = (V', E')$, where at every crossing both edges are subdivided using a new vertex of degree four. We denote the set of faces of $G'$ by $F'$ and call them *cells*. For a face $f \in F'$ we denote by $\mathcal{V}(f)$ and $\mathcal{V}'(f)$ the set of vertices from $V$ and $V'$, respectively, that appear on the boundary $\partial f$ of $f$. Furthermore, let $|f| = |\mathcal{V}'(f)|$ denote the *size* of $f$.

To each face $f \in F'$ we assign a charge $\mathrm{ch}(f) = |\mathcal{V}(f)| + |f| - 4$. Using Euler's formula $|V'| - |E'| + |F'| = 2$, it is not difficult to check (see [1]) that $\sum_{f \in F'} \mathrm{ch}(f) = 4n - 8$.

We then distribute these charges so as to collect a discharge of at least $\alpha$, for some $\alpha > 0$, for every pair $(v, f) \in V \times F'$ such that $v \in \mathcal{V}(f)$.

Then $4n - 8 = \sum_{f \in F'} \mathrm{ch}(f) \geq \sum_{v \in V} \alpha \deg_G(v) = 2\alpha|E|$ which implies

$$m = |E| \leq \frac{2}{\alpha}(n-2)\,. \tag{1}$$

The main challenge when applying this discharging method is to manage the redistribution of charges so that every vertex receives its due, for $\alpha$ as large as possible. As a natural first attempt, we may have each $f \in F'$ discharge $\alpha$ to each $v \in \mathcal{V}(f)$. This leaves $f$ with a *remaining* charge of

$$\mathrm{ch}^-(f) = \mathrm{ch}(f) - \alpha|\mathcal{V}(f)| = (1-\alpha)|\mathcal{V}(f)| + |f| - 4\,. \tag{2}$$

If $\mathrm{ch}^-(f) \geq 0$, for all $f \in F'$, then we are done. However, in general, we may have $\mathrm{ch}^-(f) < 0$, for some $f \in F'$. In such a case we have to find some other face(s) that have a surplus of remaining charge they can send to $f$.

## 2.2 The Crossing Lemma

We can obtain upper bounds on the density also using the Crossing Lemma [4]. As a basis, we need both an upper and a lower bound for the crossing number in terms of the number of vertices and edges. Upper bounds are discussed in Section 2.3. In this section we derive a lower bound using the Crossing Lemma, along the lines of its well-known probabilistic proof [3, Chapter 40].

▶ **Theorem 1.** *Let $\mathcal{X}$ be a hereditary[1] graph family and $a, b \in \mathbb{R}$ such that for every $H \in \mathcal{X}$ with $\nu$ vertices and $\mu$ edges we have $\mathrm{cr}(H) \geq a\mu - b\nu$. Then for every graph $G \in \mathcal{X}$ with $n$ vertices and $m$ edges with $2am \geq 3bn$ we have*

$$\mathrm{cr}(G) \geq \frac{4a^3}{27b^2} \cdot \frac{m^3}{n^2}\,.$$

**Proof.** Let $\Gamma$ be a minimum-crossing drawing of $G$. We take a random induced subgraph $G_p = (V_p, E_p)$ of $G$ by selecting every vertex independently at random with probability $p$ and consider the drawing $\Gamma_p$ of $G_p$ defined by $\Gamma$. Then any such graph $G_p$ is in $\mathcal{X}$, and so the lower bound on $\mathrm{cr}(G_p)$ from above holds for $G_p$ and thus also in expectation:

$$\mathsf{E}(\mathrm{cr}(\Gamma_p)) \geq a \cdot \mathsf{E}(E_p) - b \cdot \mathsf{E}(V_p)\,.$$

We have $\mathsf{E}(V_p) = pn$ and $\mathsf{E}(E_p) = p^2 m$. Furthermore, note that $\Gamma$ is a minimum-crossing drawing of $G$ and, therefore, no pair of adjacent edges crosses. Thus, for a crossing to be present in $\Gamma_p$, all four endpoints of the crossing edge pair need to be selected. Therefore, we have $\mathsf{E}(\mathrm{cr}(\Gamma_p)) = p^4 \mathrm{cr}(\Gamma) = p^4 \mathrm{cr}(G)$. Putting everything together yields

$$\mathrm{cr}(G) \geq \frac{am}{p^2} - \frac{bn}{p^3}\,. \tag{3}$$

The function on the right hand side of the above inequality has its unique maximum at $p = \frac{3bn}{2am}$. Setting $p = \frac{3bn}{2am}$ to (3) yields:

$$\mathrm{cr}(G) \geq \frac{4a^3}{27b^2} \cdot \frac{m^3}{n^2}\,. \tag{4}$$

As a sanity check, we need $p \leq 1$. So the bound holds for $2am \geq 3bn$. ◀

[1] Closed under taking induced subgraphs.

The simple observation that one can remove relatively few edges from a $k$-planar graph to obtain a $(k-1)$-planar graph allows to lift density bounds for $i$-planar graphs, with $i < k$, to bounds for $k$-planar graphs. By iteratively removing edges from the graph and a drawing of it with maximum number of crossings, we can show the following. (The proof can be found in the full version of this paper [6].)

▶ **Theorem 2.** *Let $\mathcal{X}$ be a monotone*[2] *graph family, let $k$ be a positive integer, and let $\mu_i(n)$ be an upper bound on the number of edges for every $i$-planar graph from $\mathcal{X}$ on $n$ vertices, for $0 \le i \le k-1$. Then for every $G \in \mathcal{X}$ with $n \ge 4$ vertices and $m$ edges we have*

$$\mathrm{cr}(G) \ge km - \sum_{i=0}^{k-1} \mu_i(n)\,.$$

### 2.3 Upper Bounds on the Crossing Number of 2- and 3-planar graphs

The Crossing Lemma provides us with pretty good lower bounds for crossing numbers. As a complement, we also need corresponding upper bounds. For a $k$-planar graph $G$, we have a trivial bound of $\mathrm{cr}(G) \le km/2$. So if $G$ is 2-planar, then $\mathrm{cr}(G) \le m \le 5n-10$. But we can do better, as the following theorem demonstrates.

▶ **Theorem 3.** *Every 2-planar graph on $n \ge 2$ vertices can be drawn with at most $(10n-20)/3$ crossings.*

**Proof.** Let $G = (V, E)$ be a 2-planar graph on $n$ vertices, and let $\Gamma$ be any 2-plane drawing of $G$ with a minimum number of crossings (among all 2-plane drawings of $G$). We allow multiple edges between the same pair of vertices in $\Gamma$, but no loops nor homotopic edge pairs (that is, for each pair $e_1, e_2$ of edges between the same two vertices, neither of the two parts of the plane bounded by the simple closed curve $e_1 \cup e_2$ is empty). Without loss of generality we assume that $\Gamma$ is maximal 2-plane, that is, adding any edge to $\Gamma$ results in a graph that is not 2-plane anymore. We may assume that adjacent edges do not cross in $\Gamma$ [18, Lemma 1.1]. We claim that a 1/3-fraction of the edges in $\Gamma$ is uncrossed.

Let us first argue how the claim implies the statement of the theorem. Denote by $x$ the number of edges that have at least one crossing in $\Gamma$. The number $\gamma$ of crossings in $\Gamma$ is upper bounded by $2x/2 = x$ because every edge has at most two crossings and every crossing is formed by exactly two edges. Every 2-planar graph on $n \ge 3$ vertices has at most $5n-10$ edges [21, 22], and this bound also holds for 2-plane multigraphs without loops or parallel homotopic edges [7]. It follows that $\gamma \le x \le \frac{2}{3}(5n-10) = (10n-20)/3$.

So it remains to prove the claim. Consider a vertex $v$ and denote by $X(v)$ the set of edges incident to $v$ that have at least one crossing in $\Gamma$. Let $e \in X(v)$, let $c$ denote the crossing of $e$ closest to $v$, let $e^-$ denote the part of $e$ between $v$ and $c$, and let $\chi(e)$ denote the edge that crosses $e$ at $c$. As $\chi(e)$ has at most two crossings, at least one of the two curves that form $\chi(e) \setminus c$ is uncrossed. Pick such a curve and denote it by $\chi(e)^-$. The curve $\chi(e)^-$ has two endpoints, one of which is $c$ and the other is a vertex of $G$, which we denote by $\psi(e)$. As adjacent edges do not cross in $\Gamma$, we have $\psi(e) \ne v$. By closely following $e^-$ and $\chi(e)^-$ we can draw a curve between $v$ and $\psi(e)$ in $\Gamma$ that does not cross any edge of $\Gamma$. Thus, by the maximality of $\Gamma$ we conclude that $v\psi(e)$ is an edge in $\Gamma$, and it is uncrossed because $\Gamma$ is crossing-minimal by assumption. In this way, we find an uncrossed edge $\eta(e) = v\psi(e)$ for

[2] Closed under taking subgraphs and disjoint unions.

each $e \in X(v)$. Different edges $e \neq f$ in $X(v)$ may yield the same edge $\eta(e) = \eta(f)$. But in this case by construction $\eta(e) = \eta(f)$ is homotopic to both $e^- \cup \chi(e)^-$ and $f^- \cup \chi(f)^-$, that is, the simple closed curve $e^- \cup \chi(e)^- \cup f^- \cup \chi(f)^-$ bounds a face in $\Gamma \setminus \eta(e)$. It follows that there is no other edge $g \in X(v) \setminus \{e, f\}$ for which $\eta(g) = \eta(e)$, that is, for every uncrossed edge $u$ incident to $v$ in $\Gamma$ we have $|\eta^{-1}(u) \cap X(v)| \leq 2$. Therefore, at least a 1/3-fraction of the edges incident to $v$ in $\Gamma$ is uncrossed. As this holds for every vertex $v$, it also holds globally, which completes the proof of the claim and of the theorem. ◀

In a similar fashion, we can obtain an improved upper bound for 3-planar graphs, as the following theorem demonstrates. The proof can be found in the full version of this paper [6]. We remark that the argument used in the proof of Theorem 4 does not work for larger $k > 3$.

▶ **Theorem 4.** *Every 3-planar graph on $n \geq 2$ vertices can be drawn with at most $(33n-66)/5$ crossings.*

## 3 1-planar graphs

In this section we focus on 1-planar graphs and we present lower and upper bounds on their edge density assuming that they are either $C_3$-free (Section 3.1) or $C_4$-free (Section 3.2) or of girth 5 (Section 3.3).

### 3.1 C$_3$-free 1-planar graphs

We start with the case of $C_3$-free 1-planar graphs, where we can derive an upper bound of $3(n-2)$ on their edge density (see Theorem 5); for a matching lower bound (up to a small additive constant) refer to [10].

▶ **Theorem 5.** *Every $C_3$-free 1-planar graph with $n \geq 4$ vertices has at most $3(n-2)$ edges.*

**Proof.** We derive the upper bound by an application of the recently introduced edge-density formula for non-planar graphs [16] given as follows:

$$|E| \leq t\,(|V|-2) - \sum_{c \in \mathcal{C}} \left( \frac{t-1}{4} ||c|| - t \right) - |\mathcal{X}|, \tag{5}$$

where $\mathcal{C}$ and $\mathcal{X}$ denote the sets of cells and crossings, respectively. By setting $t = 3$ to (5), one gets $|E| \leq 3(n-2) + \frac{1}{2}|\mathcal{C}_5| - \frac{1}{2}|\mathcal{C}_6| - \ldots - |\mathcal{X}|$, where $C_i$ denotes the set of cells of size $i$ with the size of a cell being the number of vertices and edge-segments on its boundary. Since each crossing is incident to at most two cells of size 5 (as otherwise a $C_3$ is inevitably formed), it follows that $\frac{1}{2}|\mathcal{C}_5| \leq |\mathcal{X}|$, which by the formula given above implies that $|E| \leq 3(n-2)$. ◀

### 3.2 C$_4$-free 1-planar graphs

We continue with the case of $C_4$-free 1-planar graphs. As in the case of $C_3$-free 1-planar graphs, we can again derive an upper bound of $3(n-2)$ for the edge-density using the density formula of (5), since each crossing is incident to at most two cells of size 5 (as otherwise a $C_4$ is formed). In the following theorem, we present an improved upper bound.

▶ **Theorem 6.** *Every $C_4$-free 1-planar graph with $n \geq 4$ vertices has at most $\frac{5}{2}(n-2)$ edges.*

**Proof.** We apply the discharging method with $\alpha = 4/5$ so that the statement follows by (1). By (2) we have

$$\mathrm{ch}^-(f) = \frac{1}{5}|\mathcal{V}(f)| + |f| - 4\,. \tag{6}$$

In particular, we have $\mathrm{ch}^-(f) > 0$ for all faces with at least four edge segments on the boundary. It remains to handle triangles.

As the graph $G$ is 1-planar, every edge of $G'$ is incident to at least one vertex in $V$. It follows that

$$|\mathcal{V}(f)| \geq \lceil |f|/2 \rceil\,, \tag{7}$$

for each $f \in F'$. So every triangle $f \in F'$ has either three vertices in $V$ and $\mathrm{ch}^-(f) = -2/5$ (type-1) or two vertices in $V$ and one vertex in $V' \setminus V$ with $\mathrm{ch}^-(f) = -3/5$ (type-2).

We will argue how to make up for the deficits at triangles by transferring charges from neighboring faces.

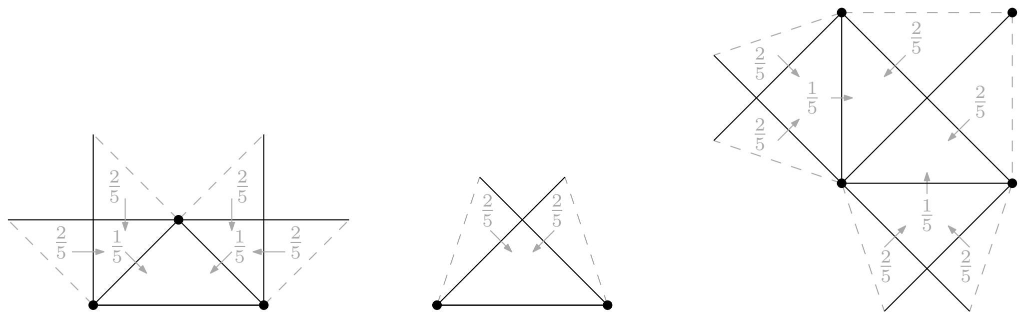
First, let us discuss faces of size at least five. So consider $f \in F'$ with $|f| \geq 5$, and let $k$ denote the number of triangles adjacent to $f$ in the dual of $G'$. Then for any vertex $v \in \mathcal{V}'(f) \setminus \mathcal{V}(f)$, at most one of the two edges incident to $v$ along $\partial f$ can be incident to a triangle of $F'$ (because otherwise the two edges of $G$ that cross at $v$ induce a $C_4$). Thus,

$$k \leq |\mathcal{V}(f)| + \frac{|f| - |\mathcal{V}(f)|}{2} = \frac{|f| + |\mathcal{V}(f)|}{2}\,.$$

Together with (6) we obtain

$$\mathrm{ch}^-(f) = \frac{1}{5}|\mathcal{V}(f)| + |f| - 4 = \frac{|f| + |\mathcal{V}(f)|}{5} + \frac{4}{5}|f| - 4 \geq \frac{2}{5}k\,,$$

which shows that $f$ can send a charge of $2/5$ to every adjacent triangle.



**(a)** type-1. **(b)** type-2. **(c)** type-2.

**Figure 1** Triangles in the planarization of $C_4$-free 1-planar graphs.

Next, consider a face $f$ with $|f| = 4$. Combining (2) and (7) we obtain $\mathrm{ch}^-(f) = |\mathcal{V}(f)|/5 \geq 2/5$. We claim that $f$ can send a charge of $2/5$ to every triangle that is adjacent to $f$ via an edge of $E' \setminus E$ and a charge of $1/5$ to every triangle that is adjacent to $f$ via an edge of $E$. To see this, let us consider the three different types of quadrangles in $F'$. By (7) we have $|\mathcal{V}(f)| \geq 2$.

If $|\mathcal{V}(f)| = 2$, then there is at most one triangle adjacent to $f$ because any two triangles adjacent to $f$ induce a $C_4$. So in this case $f$ can send a charge of $2/5$ to every adjacent triangle.

If $|\mathcal{V}(f)| = 3$, then any triangle adjacent to $f$ via an edge of $E' \setminus E$ induces a $C_4$ in $G$. Thus, there exist at most two triangles adjacent to $f$ and every such triangle is adjacent via an edge of $E$. So in this case $f$ can send a charge of $1/5$ to every adjacent triangle.

Finally, if $|\mathcal{V}(f)| = 4$, then every triangle adjacent to $f$ is adjacent via an edge of $E$. As $\mathrm{ch}^-(f) = \frac{4}{5}$, also in this case $f$ can send a charge of $1/5$ to every adjacent triangle. This completes the proof of our claim.

So let us consider the incoming charges at triangles. For a type-1 triangle $f$, neither of the adjacent faces is a type-1 triangle because such a pair would induce a $C_4$ in $G$. If at least two adjacent faces are type-2 triangles, then for each such triangle $g$, neither of the other ($\neq f$) two faces adjacent to $g$ are triangles because together with $f$ and $g$ they would induce a $C_4$. It follows that $g$ receives a charge of $2 \cdot 2/5 = 4/5$ from its two other ($\neq f$) neighbors, see Figure 1a. As $\mathrm{ch}^-(g) = -3/5$, the remaining charge of $1/5$ can be passed on to $f$. Then $f$ receives a charge of $2 \cdot 1/5 = 2/5 = -\mathrm{ch}^-(f)$ overall. Otherwise, at least two of the three faces adjacent to $f$ have size at least four. Each passes a charge of $1/5$ across the joint edge, which is in $E$, to $f$. So the deficit of $\mathrm{ch}^-(f) = -2/5$ is covered in this case as well.

It remains to consider type-2 triangles. Let $f$ be a type-2 triangle, and consider the two faces $g_1, g_2$ that are adjacent to $f$ via an edge of $E' \setminus E$. If both $g_1$ and $g_2$ are triangles, then they induce a $C_4$ in $G$, in contradiction to $G$ being $C_4$-free. If both $g_1$ and $g_2$ have size at least four, then $f$ receives a charge of $2 \cdot 2/5 = 4/5$ from them, which covers $\mathrm{ch}^-(f) = -3/5$ and even leaves room to sent a charge of $1/5$ across its third edge, which is in $E$, see Figure 1b.

Hence, we may assume that without loss of generality $g_1$ is a type-2 triangle and $|g_2| \geq 4$. The third face $g_3 \notin \{g_1, g_2\}$ adjacent to $f$ is not a type-1 triangle because then $g_3$ together with $g_1$ would induce a $C_4$ in $G$. If $g_3$ is a type-2 triangle, then neither of its two other ($\neq f$) neighbors is a triangle because together with $f$ and $g_1$ there would be a $C_4$ in $G$. Therefore, we are in the case discussed above, where $g_3$ receives a charge of $4/5$ from its neighbors and passes on $1/5$ to $f$. Otherwise, we have $|g_3| \geq 4$ and thus $g_3$ sends a charge of $1/5$ to $f$ across the joint edge, which is in $E$. Together with the charge of $2/5$ that $f$ receives from $g_2$ via the joint edge, which is in $E' \setminus E$, this suffices to cover $\mathrm{ch}^-(f) = -3/5$, see Figure 1c. ◀

▶ **Theorem 7.** *For every sufficiently large $n$, there exists a $C_4$-free 1-planar graph on $n$ vertices with $2.4n - O(1)$ edges.*

**Proof sketch.** It can be observed that the construction in Figure 2a is a 1-plane drawing and achieves the required number of edges. To show that it is $C_4$-free, we first observe that all (red) degree four vertices behave symmetrically and all (black) degree five vertices behave symmetrically. Further, all neighbors of degree four vertices have degree five, thus any cycle must contain a degree five vertex. We consider an arbitrary degree five vertex $b$ and show that each neighbor of a vertex in $N(b)$ (circled in Figure 2b) has one unique neighbor in $N(b)$ (indicated by arrows in Figure 2b); see Figure 2b. Thus, the construction is $C_4$-free. ◀

## 3.3 1-planar graphs of girth 5

▶ **Theorem 8.** *Every 1-planar graph of girth 5 on $n$ vertices has at most $2.4n$ edges.*

**Proof.** We go through the proof of the 1-planar $C_4$-free case, and note that the arrow case as well as the type-1 triangles do not occur. If we choose $\alpha = \frac{5}{6}$, the type-2 triangles have a negative charge of $-\frac{4}{6}$, and can get charges of $\frac{2}{6}$ from their immediate neighboring cells which are of size at least 4. Note that for the case that those neighboring cells are of size 4, they have only one type-2 triangle by the $C_4$-freeness property, which suffices to provide

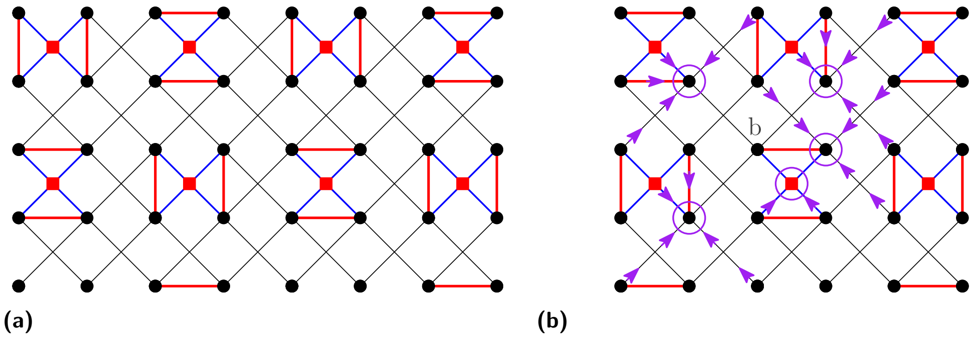



**Figure 2** (a) A dense $C_4$-free 1-plane graph. (b) The neighborhood of a vertex $b$.

enough charge. If a neighboring cell $c$ has size 5, then, by 1-planarity, it shares with at least one neighbor a planar edge, through which it does not have to contribute charge. Since the remaining charge of $c$ is at least $3 \cdot \frac{1}{6} + 5 - 4 = 1.5$, it can contribute to four neighbours $\frac{2}{6}$ charge each. If a neighboring cell $c$ is of size larger than 5, then its remaining charge is at least $\frac{1}{6}|\mathcal{V}(c)| + |c| - 4 \geq |c| - 3 \geq \frac{2}{6}|c|$, and therefore there is enough charge to provide $\frac{2}{6}$ charge to every neighboring type-2 triangle. This immediately gives that an $n$-vertex 1-planar graph of girth 5 has at most $\frac{12}{5}n = 2.4n$ edges. ◀

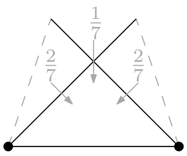


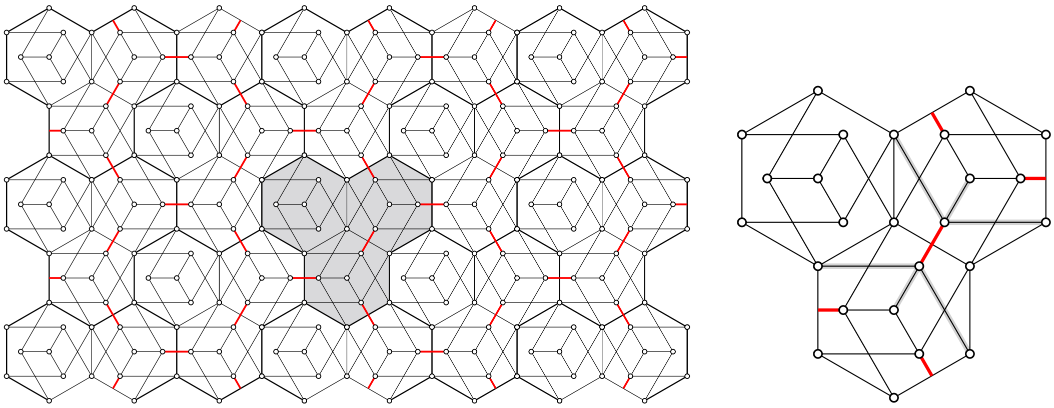

**Figure 3** Triangles in the planarization of 1-planar graphs of girth 5.

▶ **Theorem 9.** *For every sufficiently large $n$, there exists a 1-planar graph of girth 5 on $n$ vertices with $(2 + \frac{1}{6})n - O(1) \approx 2.167n - O(1)$ edges.*

**Proof sketch.** It can be observed that the construction in Figure 4a is a 1-plane drawing and has the required number of edges. To show that it has girth 5, we first observe the subgraph within each hexagonal tile has girth 5. It is a Petersen graph, which is known to have girth 5. It follows that every $C_3$ or $C_4$, if any, uses vertices from at least two different tiles. Second, we argue that no $C_3$ or $C_4$ uses a red edge. The neighbors of the two endpoints of a red edge are at pairwise distance at least two; see Figure 4b. As red edges are the only edges that cross tile boundaries and boundary edges are shared among adjacent tiles, it follows that every $C_3$ or $C_4$, if any, uses at least two nonadjacent vertices $u, v$ on the boundary $\partial T$ of a tile $T$ and exactly one vertex $z$ in the interior of $T$. Then $u$ and $v$ are at distance three along $\partial T$ and thus do not form a $C_3$. Further, the tile $T$ is the unique common tile of $u$ and $v$, so there is no common neighbor of $u$ and $v$ outside of $T$. As $z$ is the only common neighbor of $u$ and $v$ inside $T$, it follows that there is no $C_4$ through $u, v, z$. ◀

## 4 2-planar graphs

In this section, we focus on 2-planar graphs and we present bounds on their edge density assuming that they are $C_3$-free (Section 4.1) or $C_4$-free (Section 4.2) or of girth 5 (Section 4.3).



(a) (b)

**Figure 4** (a) A 1-planar graph of girth 5 with about $2.167n$ edges (Theorem 9). The construction consists of repeated triplets of hexagonal tiles (bordered by thick edges, also shown in (b)).

## 4.1 $C_3$-free 2-planar graphs

For the maximum edge density of $C_3$-free 2-planar graphs, we can derive an upper bound of $4(n-2)$ (see Theorem 10); for a lower bound of $3.5(n-2)$ refer to [5].

▶ **Theorem 10.** *$C_3$-free 2-planar graphs with $n$ vertices have at most $4(n-2)$ edges.*

**Proof.** To derive the upper bound, we apply the discharging method with $\alpha = \frac{1}{2}$ so that the statement follows by (1). By (2) we have

$$\mathrm{ch}^-(f) = \frac{1}{2}|\mathcal{V}(f)| + |f| - 4\,. \tag{8}$$

In particular, we have $\mathrm{ch}^-(f) \geq 0$ for all faces with at least four edges on the boundary. It remains to handle triangles. Since we consider $C_3$-free graphs, we distinguish between three types of triangles; those with 0, 1 and 2 vertices on their boundaries and it is not difficult to observe that the latter ones have zero charge, while the former ones have charge $-1$ and $-\frac{1}{2}$, respectively.

For each triangle $f$ with zero or one vertices on its boundary, our strategy is to transfer at least $\frac{1}{4}$ and at most $\frac{1}{3}$ units of charge from the cells neighboring $f$. If $f$ has no vertices on its boundary, then we will transfer $\frac{1}{3}$ units of charge from each neighboring cell. Otherwise, we will transfer $\frac{1}{4}$ units of charge each from two neighboring cells of $f$; see Figure 5.

Assume first that $|\mathcal{V}(f)| = 0$; see Figure 5a. Since $f$ is triangular, it follows that $f$ is formed by three mutually crossing edges. Our strategy is to transfer $\frac{1}{3}$ units of charge from each cell neighboring $f$. Since $f$ neighbors three such cells, this is enough to bring the remaining charge of $f$ from $-1$ to 0. Let $c$ be a neighboring cell of $f$. It follows that $|c| \geq 4$ with two vertices on its boundary. Thus its remaining charge is:

$$\frac{1}{2}|\mathcal{V}(c)| + |c| - 4 \geq 1 - 4 + |c| = |c| - 3$$

This implies that if $|c| \geq 5$, then the remaining charge of $c$ is at least 2, in which case $c$ can transfer $\frac{1}{3}$ units of charge to $f$ and its remaining charge will be enough to distributed to the rest of its neighboring cells. For the second case, we assume $|c| = 4$. Since $c$ has two vertices that appear consecutively on its boundary, it follows that one of the sides that bound $c$ is

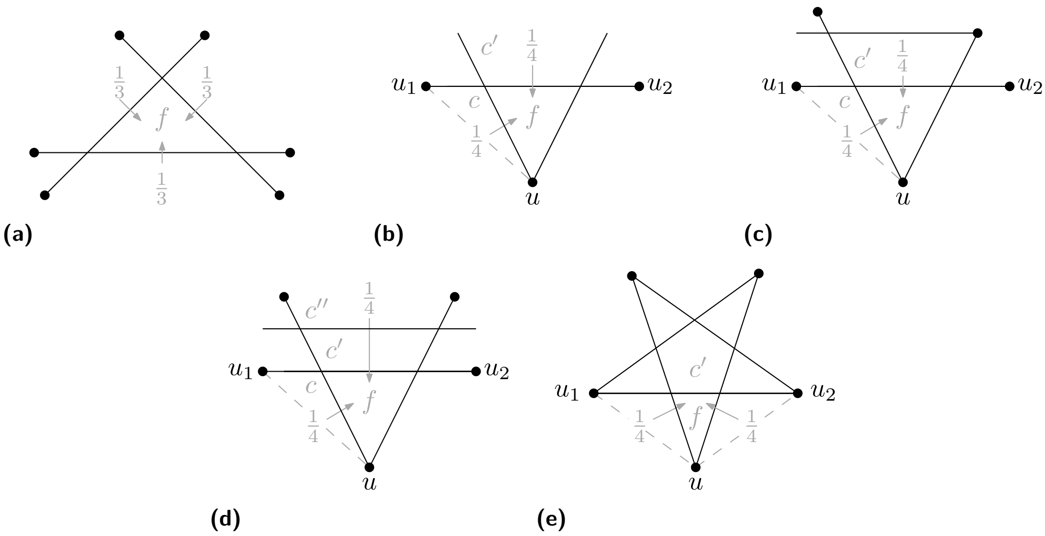



**Figure 5** Triangles in the planarization of $C_3$-free 2-planar graphs.

crossing free. Denote this side by $e$ and let $c'$ be the cell on the other side of $e$. It follows that $c'$ is neither a triangle with zero vertices nor a triangle with one vertex on its boundary. Hence, there is no need to transfer charge from $c$ to $c'$ according to our strategy. It follows that there are at most 3 neighboring cells that $c$ may have to transfer charge to. Hence, $c$ can transfer $\frac{1}{3}$ units of charge to $f$ and its remaining charge will be enough to distributed to the rest of its neighboring cells, if needed.

To complete the proof of the theorem, we next consider the case in which $|\mathcal{V}(f)| = 1$; see Figure 5b. Let $u$ be the vertex on the boundary of $f$ and let $(u_1, u_2)$ be the edge with one of its segments on the boundary of $f$. Let $c_1$ and $c_2$ be the two neighboring cells of $f$ that share the two sides of $f$ incident to its vertex. Since we consider $C_3$-free graphs, it follows that $(u, u_1)$ and $(u, u_2)$ cannot be both in the graph. Assume that $(u, u_i)$ with $i \in \{1, 2\}$ is not part of the graph. Then, the corresponding cell $c \in \{c_1, c_2\}$ neighboring $f$ and having vertex $u$ and $u_i$ on its boundary has size at least 4, which means that its remaining charge is at least:

$$\frac{1}{2}|\mathcal{V}(c)| + |c| - 4 \geq 1 - 4 + |c| = |c| - 3 \geq \frac{1}{4}|c|$$

Hence, we can safely transfer $\frac{1}{4}$ units of charge from $c$ to $f$, since the remaining charge of $c$ would be enough for being distributed to the remaining cells neighboring $c$, if needed. This implies that if both $(u, u_1)$ and $(u, u_2)$ are not in the graph, then each of the cells $c_1$ and $c_2$ can transfer $\frac{1}{4}$ units of charge to $f$ and then we are done. So, in the rest we can assume that this is not the case.

Let $c'$ be the face neighboring $f$ that is on the other side of the edge $(u_1, u_2)$. If $c'$ has at least two vertices on its boundary, then as above we transfer $\frac{1}{4}$ units of charge from $c'$ to $f$ and the remaining charge of $c'$ would be enough for being distributed to the remaining cells neighboring $c'$, if needed. So, it remains to consider the cases in which $c'$ has either no or one vertex on its boundary.

Assume first that $c'$ has one vertex on its boundary, that is, $\mathcal{V}(c') = 1$. Then:

$$\frac{1}{2}|\mathcal{V}(c')| + |c'| - 4 \geq \frac{1}{2} - 4 + |c'| = |c'| - 3.5$$

If $c'$ is such that $|c'| \geq 5$, then $|c'| - 3.5 \geq \frac{1}{4}|c'|$ holds and as above we can safely transfer $\frac{1}{4}$ units of charge from $c'$ to $f$. So, it remains to argue for the cases in which $|c'| \in \{3, 4\}$. First, we observe that $|c'| \neq 3$, as otherwise the two edges incident to $u$ bounding $f$ would form a pair of parallel edges. Hence, we may assume that $|c'| = 4$; see Figure 5c. Since $c'$ has one vertex on its boundary, its remaining charge is $\frac{1}{2}$. In this case, we argue that at most two neighboring cells, namely, $f$ and another one, may need additional charge from $c'$. In particular, the two cells neighboring $c'$ that have the vertex of $c'$ on their boundary do not need additional charge, since none of them can be a triangle with zero or one vertex on its boundary. This means that we can safely transfer $\frac{1}{4}$ units of charge from $c'$ to $f$, as desired.

To complete the case analysis, we need to consider the case that $c'$ has no vertex on its boundary. In this case, the remaining charge of $c'$ is $|c'| - 4$. If $|c'| \geq 6$, then the remaining charge of $c'$ is at least 2, which implies that $\frac{1}{4}$ units of charge can be safely transferred to $f$ and the remaining charge of $c'$ will be enough for being distributed to the rest of the cells neighboring $c'$, if needed. So, we may assume that $|c'| \in \{3, 4, 5\}$. First, we observe that $|c'| \neq 3$, as otherwise the two edges incident to $u$ bounding $f$ would form a pair of crossing edges, which is not possible in simple drawings. Hence, $|c'| \in \{4, 5\}$. If $|c'| = 4$, then its remaining charge is 0 and clearly it cannot transfer charge to $f$. In this case, we consider the cell $c''$ neighboring $c'$, which does not share a crossing point with $f$; see Figure 5d. It follows that $|c''| \geq 4$ and $c''$ has two vertices on its boundary. Since $c'$ does not require a transfer of charge, we transfer $\frac{1}{4}$ units of charge from $c''$ to $f$ and as in the first case of the proof the remaining charge of $c''$ for being distributed to the rest of the cells neighboring $c''$.

To complete the proof of the case $|\mathcal{V}(f)| = 1$, consider now the case $|c'| = 5$. In this case, the remaining charge of $c'$ is 1 and this is enough to contribute a $\frac{1}{4}$ to at most four neighboring cells. Hence, we may assume that $c'$ has to transfer $\frac{1}{4}$ units of charge to exactly five neighboring cells; see Figure 5e. In this case, it follows that none of the edges $(u, u_1)$ and $(u, u_2)$ is part of the graph (as otherwise there is a $C_3$; a contradiction). However, we have assumed that one of these edges belongs to the graph. ◀

## 4.2 $C_4$-free 2-planar graphs

We continue with the case of $C_4$-free 2-planar graphs, deriving an upper bound $3.929n$ on their maximum edge density (Theorem 11); for a lower bound of $2.5n - O(1)$ refer to Theorem 14.

▶ **Theorem 11.** *Every $C_4$-free 2-planar graph on $n \geq 2$ vertices has at most*

$$\sqrt[3]{\frac{190,125}{3,136}}n < 3.929n$$

*edges.*

**Proof.** Let $G$ be a $C_4$-free graph with $n$ vertices and $m$ edges, and let $\Gamma$ be a minimum-crossing drawing of $G$. Then Theorem 2 in combination with the upper bound of $\frac{15}{7}(n-2)$ by Dowden [11] regarding the edge density of $C_4$-free planar graphs and Theorem 6 yields:

$$\text{cr}(G) \geq 2m - \frac{5}{2}n - \frac{15}{7}n = 2m - \frac{65}{14}n\,.$$

By applying Theorem 1 for $a = 2$ and $b = \frac{65}{14}$, we obtain the following lower bound on the number of crossings of $G$ when $m \geq \frac{195n}{56} \approx 3.482n$.

$$\text{cr}(G) \geq \frac{6,272m^3}{114,075n^2}\,. \tag{9}$$

Assume now that $G$ is additionally 2-planar. Then by Theorem 3, we obtain $\mathrm{cr}(G) \leq \frac{10n}{3}$. Hence, by (9) we have:

$$\frac{6,272m^3}{114,075n^2} \leq \frac{10}{3}n \iff m^3 \leq \frac{190,125}{3,136}n^3 \,. \qquad \blacktriangleleft$$

▶ **Corollary 12.** *Every $C_4$-free $k$-planar graph on $n \geq 2$ vertices and $m \geq 3.483n$ edges has at most*

$$\sqrt{\frac{114,075}{12,544}} \cdot \sqrt{k} \cdot n < 3.016\sqrt{k}n$$

*edges.*

**Proof.** Let $G$ be a $C_4$-free $k$-planar graph with $n$ vertices and $m \geq 3.483n$ edges. By (9), we know a lower bound on its number of crossings, namely,

$$\mathrm{cr}(G) \geq \frac{6,272m^3}{114,075n^2} \,.$$

On the other hand, since $G$ is $k$-planar, it holds $\frac{km}{2} \geq cr(G)$. Combining those, we get:

$$\frac{12,544m^2}{114,075n^2} \leq k \iff m \leq \sqrt{\frac{114,075}{12,544}kn} \,. \qquad \blacktriangleleft$$

▶ Remark 13. An asymptotically better bound of $\Theta(\sqrt[3]{k}n)$ edges, which however holds for significantly denser graphs only, can be obtained by combining an improved crossing lemma for $C_4$-free graphs by Pach, Spencer, and Tóth [20, Theorem 3.1] with the trivial upper bound of at most $km/2$ crossings for $k$-planar graphs.

▶ **Theorem 14.** *For every sufficiently large $n$, there exists a $C_4$-free 2-planar graph on $n$ vertices with $2.5n - O(1)$ edges.*

**Proof sketch.** The construction is illustrated in Figure 6. The proof that it fulfills the claimed properties follows the same ideas as the proof of Theorem 7. The proof can be found in the full version of this paper [6]. ◀

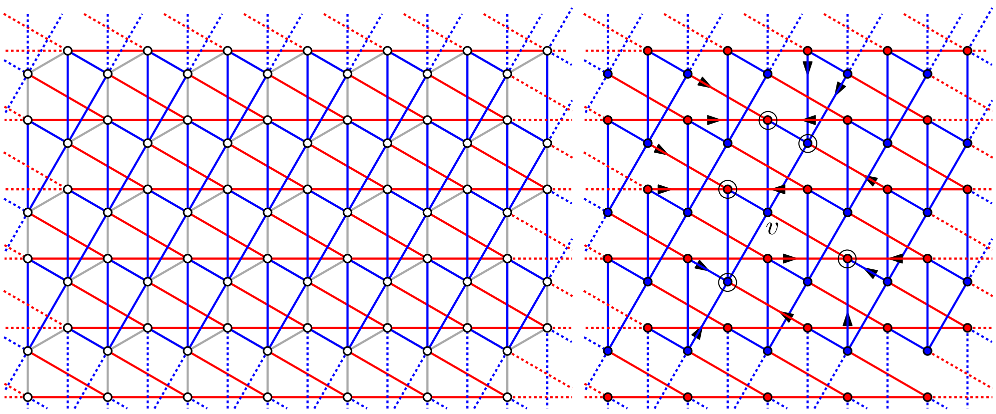



**Figure 6** A 2-plane graph with $\approx 2.5n$ edges, shown red and blue. Gray shows the grid only.

## 4.3 2-planar graphs of girth 5

We conclude Section 4 with the case of 2-planar graphs of girth 5.

▶ **Theorem 15.** *Every 2-planar graph of girth 5 on $n$ vertices has at most*

$$\sqrt[3]{\frac{11,163}{240}}n < 3.597n$$

*edges.*

**Proof.** Let $G$ be a graph of girth 5 with $n$ vertices and $m$ edges. As a consequence of Euler's Formula, every planar graph of girth $g$ on $n \geq 3$ vertices has at most $g(n-2)/(g-2)$ edges. Plugging this together with Theorem 8 into Theorem 2 we get

$$\operatorname{cr}(G) \geq 2m - \frac{5}{3}n - \frac{12}{5}n = 2m - \frac{61}{15}n\,.$$

By applying Theorem 1 for $a = 2$ and $b = \frac{61}{15}$, we obtain the following lower bound on the number of crossings of $G$ when $m \geq \frac{61n}{20}$.

$$\operatorname{cr}(G) \geq \frac{800m^3}{11,163n^2}\,. \tag{10}$$

Assume now that $G$ is additionally 2-planar. Then by Theorem 3, we obtain $\operatorname{cr}(G) \leq \frac{10n}{3}$. Hence, by (10) we have:

$$\frac{800m^3}{11,163n^2} \leq \frac{10}{3}n \iff m^3 \leq \frac{11,163}{240}n^3\,.$$ ◀

The next corollary follows from (10) of the proof of Theorem 15; its proof is analogous to the one of Corollary 12.

▶ **Corollary 16.** *Every $k$-planar graph of girth 5 on $n \geq 2$ vertices and $m \geq 3.05n$ edges has at most*

$$\sqrt{\frac{11,163}{1,600}} \cdot \sqrt{k} \cdot n < 2.642\sqrt{k}n$$

*edges.*

**Proof.** Let $G$ be a $k$-planar graph of girth 5 with $n$ vertices and $m \geq 3.05n$ edges. By (10), we know a lower bound on its number of crossings, namely,

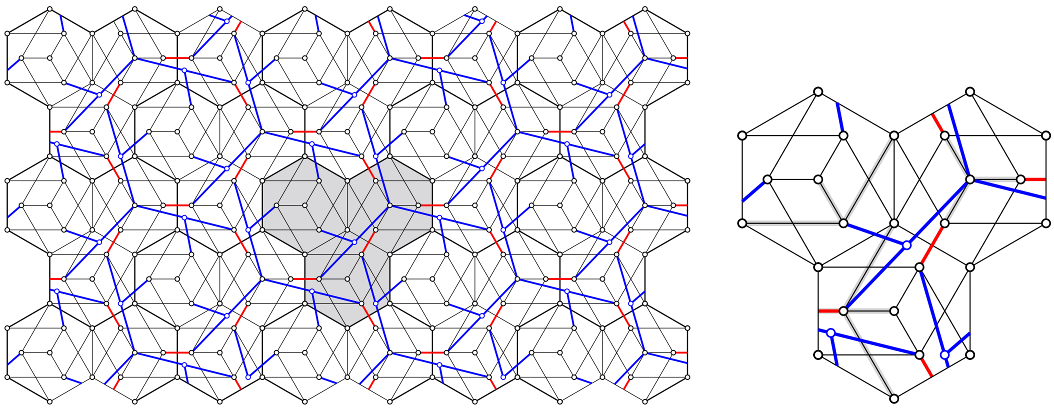
$$\operatorname{cr}(G) \geq \frac{800m^3}{11,163n^2}\,.$$

On the other hand, since $G$ is $k$-planar, it holds $\frac{km}{2} \geq \operatorname{cr}(G)$. Combining those, we get:

$$\frac{1,600m^2}{11,163n^2} \leq k \iff m \leq \sqrt{\frac{11,163}{1,600}kn}\,.$$ ◀

▶ **Theorem 17.** *For every sufficiently large $n$, there exists a 2-planar graph of girth 5 on $n$ vertices with $(2+\frac{2}{7})n - O(1) \approx 2.286n - O(1)$ edges.*

**Proof sketch.** The construction in Figure 7 is an extension of the construction in Figure 4 that achieves the claimed properties. The proof of those properties follows the same ideas as the proof of Theorem 9. For details refer to the full version of this paper [6]. ◀



(a) (b)

**Figure 7** Illustration for the proof of Theorem 17. The construction consists of repeated triplets of hexagonal tiles (bordered by thick edges, also shown in (b)).

## 5 3-planar graphs

This section is devoted to 3-planar graphs and is structured analogously to Section 4. For space reasons, the proofs of the upper bounds have been deferred to only the full version of this paper [6], as their proofs are very similar to the corresponding ones of Section 4.

### 5.1 $C_3$-free 3-planar graphs

▶ **Theorem 18.** *Every $C_3$-free 3-planar graph on $n \geq 2$ vertices has at most*

$$\sqrt[3]{\frac{2,673}{20}}n < 5.113n$$

*edges.*

▶ **Corollary 19.** *Every $C_3$-free $k$-planar graph on $n \geq 2$ vertices and $m \geq \frac{9}{2}n$ edges has at most*

$$\sqrt{\frac{81}{8}} \cdot \sqrt{k} \cdot n < 3.182\sqrt{k}n$$

*edges.*

### 5.2 $C_4$-free 3-planar graphs

▶ **Theorem 20.** *Every $C_4$-free 3-planar graph on $n \geq 2$ vertices has at most*

$$\sqrt[3]{\frac{3,764,475}{31,360}}n < 4.933n$$

*edges.*

▶ **Theorem 21.** *Every 3-planar graph of girth 5 on $n$ vertices has at most*

$$\sqrt[3]{\frac{368,379}{4,000}}n < 4.516n$$

*edges.*

▶ **Theorem 22.** *For every sufficiently large $n$, there are 3-planar graphs of girth 5 with $2.5n - O(1)$ edges.*

## 6 Conclusions and open problems

In this work, we continued an active research branch in Graph Drawing seeking for new edge density bounds for $k$-planar graphs that avoid certain forbidden substructures, namely, cycles of length 3 or 4 or both of them. For each of these settings, our focus was on $k$-planar graphs, with $k \in \{1, 2, 3\}$, as well as on general $k$. Several open problems have been triggered:

- The first one is the obvious one, that is, to close the gaps between the lower and the upper bounds reported in Table 1. We believe that this is a challenging open problem.
- In particular, it seems to us that the lower bounds for 2- and 3-planar can be improved.
- Note that there is a lot of empty space to fill in Table 2 where we did not find any reasonably good bounds.
- Another promising research direction is to study the edge density of $k$-planar graphs that are either $C_k$ free for $k > 4$ or are of girth $r$ with $r > 5$.
- Even though we focused on $k$-planar graphs, we believe that extending the study to other beyond-planar graph classes is a challenging research direction that worth to follow.
- On the algorithmic side, the recognition problem is of interest; in particular, assuming optimality. A concrete question here, e.g., is whether the problem of recognizing if a graph is optimal $C_3$-free 1-planar can be done in polynomial time. Recall that in the general setting this problem can be solved in linear time [9].

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