Improving the Crossing Lemma by Characterizing Dense 2-Planar and 3-Planar Graphs

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Abstract

The classical Crossing Lemma by Ajtai et al. and Leighton from 1982 gave an important lower bound of $c\frac{m^3}{n^2}$ for the number of crossings in any drawing of a given graph of n vertices and m edges. The original value was $c = 1/100$, which then has gradually been improved. Here, the bounds for the density of k-planar graphs played a central role. Our new insight is that for $k = 2, 3$ the k-planar graphs have substantially fewer edges if specific local configurations that occur in drawings of k-planar graphs of maximum density are forbidden. Therefore, we are able to derive better bounds for the crossing number $\text{cr}(G)$ of a given graph G. In particular, we achieve a bound of $\text{cr}(G) \geq \frac{73}{18}m - \frac{305}{18}(n-2)$ for the range of $5n < m \leq 6n$, while our second bound $\text{cr}(G) \geq 5m - \frac{407}{18}(n-2)$ is even stronger for larger $m > 6n$.

For $m > 6.79n$, we finally apply the standard probabilistic proof from the BOOK and obtain an improved constant of $c > 1/27.61$ in the Crossing Lemma. Note that the previous constant was $1/29$. Although this improvement is not too impressive, we consider our technique as an important new tool, which might be helpful in various other applications.

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1 Introduction

The classical Crossing Lemma by Ajtai et al.[4] and Leighton [10] has been considerably improved constant-wise from $\frac{1}{100}$ in many subsequent works [3, 11, 13] and for many variants [16], such as bipartite graphs [5], graphs of bounded girth [12], multigraphs [9, 14], etc. Székely [17] gave an collection of applications of the Crossing Lemma in discrete geometry.

The gradual improvement of the above mentioned constant has been mainly done by using the linear bounds for the number of edges for planar, 1-planar, 2-planar, etc. graphs. k-planar graphs have a drawing where each edge is crossed at most k times. Density bounds for k-planar n-vertex graphs have been subject to intensive research in the past. While planar graphs have at most $3n - 6$ edges, the best known upper bounds for 1-planar, 2-planar and 3-planar graphs are $4n - 8$ [18], $5n - 10$ [13] and $5.5n - 11.5$ [7] respectively; for the corresponding non-simple versions the bounds might slightly differ [7]. They have been directly applied for better bounds for the crossing lemma. The current best constant of $\frac{1}{29}$ uses even the bound for 4-planar graphs [1], which is $6n - 12$.

We will perform a more refined analysis by considering drawings that are in some sense between k-planar and $k+1$-planar drawings for $k = 1, 2$. In their paper from 2006 [11], Pach, Radoicic, Tardos and Tóth used a similar approach to improve the corresponding constant of the Crossing Lemma. They considered the density of 1-planar drawings with a fixed number of crossing-free triangles, a class of drawings between planar and 1-planar in general.

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A similar road has been taken in the paper [2] about simple quasi-planar graphs. While the general density bound here is $6.5n$, the authors consider drawings without triangular cells that have no vertex on the boundary. For such a more general class, a bound of $7n$ can be derived. This bound has not been applied for the Crossing Lemma, though. We will apply such a refined look to 2- and 3-planar drawings: It turns out that either we can prove much smaller bounds for the edge density than provided by the upper bounds of the corresponding k-planar classes (which is per se good for the Crossing Lemma) or we can characterize the drawing in a very good way, which simplifies the way of counting the crossings.

The idea has been motivated by some results in the literature. (Non-simple) optimal 2-planar and 3-planar graphs have been characterized [7], and there is very limited flexibility for the structure of such graphs. We know that with much less restrictions on the drawings, the limits of the maximum density for some superclasses for 1-planar and 2-planar graphs are still roughly at the same value. Examples for this effect are the min-1-planar and min-2-planar graphs [8] as superclasses of 1-planar and 2-planar graphs, as well as gap-planar graphs as a superclasse of 2-planar graphs [6].

To use the concept of k-planarity for various values of k, we planned to specify at which point between k- and $k+1$-planarity the density is changing. This turned out to be difficult, and hence we go the other way around and forbid local configurations that have to occur in optimal k-planar drawings. That leads to nice insights on the density bounds and surprising results. Note that all our results hold for non-simple graphs and non-simple drawings.

2 Definitions and Notation

A *drawing* or *topological graph* D is a graph drawn in the plane such that the vertices are pairwise distinct points and the edges are represented as Jordan arcs connecting the corresponding endpoints. We assume simplicity in the sense that edges do not overlap other vertices in the interior. Two edges might cross, but we do not allow that more than two edges cross at a single point. We also assume that two edges have only a finite number of common interior points and no two edges meet tangentially. Remark that we will consider not necessarily simple drawings, i.e., we will allow non-homotopic multiple edges as well as adjacent crossing edges, while loops are forbidden. Since we mostly assume that the number of crossings will be minimal, there will be no empty lenses, i.e., empty regions having a boundary that is being defined by two edges; c.f. Proposition 10.

The *crossing number* $\mathrm{cr}(D)$ is defined to be the total number of crossing points in D. For an abstract graph G, the *crossing number* $\mathrm{cr}(G)$ is the minimum value of $\mathrm{cr}(D)$ over all drawings D with D is a drawing of G. A drawing D is *k-planar* if no edge is crossed more than k times. A graph G is *k-planar* if it has a k-planar drawing.

Forbidden configurations. We now define three forbidden configurations that play a key role: A *full k-planar p-gon* F_p^k can be described by a p-cycle C_p of planar edges with no other vertices inside, which is then greedily extended by a maximal number of edges to be placed inside that are as short as possible observing this subgraph is still k-planar. To finally arrive at F_p^k, we delete the planar cycle C_p at the boundary. In this way, we define a *full 2-planar pentagon* F_5^2 to be the graph $K_5 - C_5$ drawn in the way described above (see Figure 1a). Similarly, we can define full 2-planar hexagons F_6^2 and full 3-planar hexagons F_6^3 as specific drawings of subgraphs of $K_6 - C_6$. More precisely, a *full 2-planar hexagon* consists of the six short, i.e., 2-hop edges inside a planar C_6 (see Figure 1b). A *full 3-planar hexagon* consists of all possible 2-hop and two 3-hop edges inside a planar C_6 (see Figure 1c).

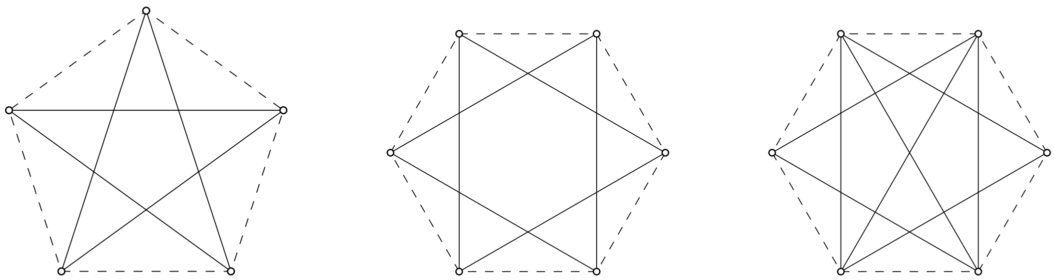


Figure 1 (a) A full 2-planar pentagon F_5^2, (b) a full 2-planar hexagon F_6^2 and (c) a full 3-planar hexagon F_3^6 with their boundaries (dashed).

Clearly, a configuration F_p^k may be crossed by some other edges. But for full 2-planar pentagons and full 2-planar hexagons, this cannot happen in the case of 2-planar drawings, which motivates to define the planar 5-cycle resp. 6-cycle surrounding them as their *boundary* (even if not all of its edges may exist in a drawing). This implies that, for 2-planar drawings, full 2-planar pentagons and hexagons are edge-disjoint (while they may have common boundary edges). Similarly, in the case of 3-planarity and full 3-planar hexagons, the cycle surrounding them consists of uncrossed edges if there are no empty lenses. With this in mind, we analogously define the *boundary* of a full 3-planar hexagon, and observe that these configurations are edge-disjoint for 3-planar drawings.

Using the definitions above, we are able to state our main results in the next section.

3 Results

In this section, we present our results. The proofs of Theorem 1 and Theorem 3 use the discharging method and can be found in Section 4.

▶ **Theorem 1.** *Any graph G with $n \geq 3$ vertices that admits a 2-planar F_5^2-free drawing has at most $4.5(n-2)$ edges. If the drawing is also F_6^2-free, then G has at most $\frac{13}{3}(n-2)$ edges.*

Counting the number of edges in a drawing consisting of $0.5(n-2)$ full 2-planar hexagons, we see that the first of the two bounds is tight. For the second bound, we refer to a pentagonalization of the plane, where four edges have been added within each pentagon.

▶ **Corollary 2.** *For every 2-planar drawing of any graph with $n \geq 3$ vertices and $\frac{13}{3}(n-2)+x$ edges for $x \in [0, \frac{2}{3}(n-2)]$, the number of F_5^2 and F_6^2 configurations is at least x.*

Note that G cannot be 2-planar for $x > \frac{2}{3}(n-2)$ by the corresponding density bound.

Proof. Assume that drawing D is a 2-planar drawing of a graph with n vertices and $\frac{13}{3}(n-2)+x$ edges such that the number of F_5^2 or F_6^2 configurations is $y < x$. We can destroy those configurations by removing one edge from each F_5^2 and F_6^2. Hence, we still have more than $\frac{13}{3}(n-2)$ edges, which is a contradiction to Theorem 1. ◀

This implies that drawings of optimal 2-planar graphs consist of $\frac{2}{3}(n-2)$ full 2-planar pentagons, a fact that has been already known [7]. Similar results hold for 3-planar drawings.

▶ **Theorem 3.** *Any graph with $n \geq 3$ vertices that admits a 3-planar F_6^3-free drawing has at most $5(n-2)$ edges.*

This bound is tight, which one can see by considering optimal 2-planar graphs.

The next corollary allows us to characterize drawings of dense 3-planar graphs very well. This extends the characterization of optimal 3-planar graphs, which must have a drawing consisting of $\frac{1}{2}(n-2)$ F_6^3 configurations and their boundaries [7].

▶ **Corollary 4.** *For ervery 3-planar drawing of any graph with $n \geq 3$ vertices and $5(n-2)+x$ edges for $x \in [0, 0.5(n-2)]$, the number of F_6^3 configurations is at least x.*

Note that G cannot be 3-planar for $x > 0.5(n-2)$ by the corresponding density bound.

Proof. Analogously to the proof of Corollary 2, we assume that there is a 3-planar drawing D of a graph with n vertices and $5(n-2)+x$ edges such that the number of F_5^2 or F_6^2 configurations is $y < x$. Those configurations can be destroyed by removing one edge from each F_6^3, hence we still have more than $5(n-2)$ edges, which is a contradiction to Theorem 3. ◀

A consequence of this is a new upper bound for the edge density of simple 3-planar graphs, i.e., the case where multi-edges are forbidden. Note that the best known bound before was $5.5n - 11.5$ edges [7] and there exist examples with $5.5n - 15$ edges [11].

▶ **Corollary 5.** *There are no 3-planar graphs on $n \geq 3$ vertices with $5.5n - 11.5$ edges. Therefore, any simple 3-planar graph on $n \geq 3$ vertices has at most $5.5n - 12$ edges.*

Proof. Assume that there exists a (not necessarily simple) 3-planar graph G with $5.5n - 11.5$ edges. Then, by Corollary 4, we would find in any 3-planar drawing D of G at least $0.5(n-2) - 0.5$ full 3-planar hexagons. Let $\mathcal{H}$ be any triangulation on the set of vertices that includes all the boundaries of all F_6^3 configurations in D. As F_6^3 configurations consist of four triangles, only $2(n-2) - 4(0.5(n-2) - 0.5) = 2$ triangles in $\mathcal{H}$ do not belong to an F_6^3.

Now we count the edges. Starting with the edges of $\mathcal{H}$, each F_6^3 consists of five additional edges. The other two triangles may contain one additional edge, which gives in total at most $3(n-2) + 2.5(n-2) - 2.5 + 1 = 5.5n - 12.5$ edges, contradicting the assumed density. ◀

From Theorem 1 and Theorem 3 we can also derive new lower bounds for the number of crossings in a graph. The proof can be found in Section 5.

▶ **Theorem 6.** *Let G be a graph with $n > 2$ vertices and m edges. Then*
(a) $\mathrm{cr}(G) \geq \frac{73}{18}m - \frac{305}{18}(n-2)$,
(b) $\mathrm{cr}(G) \geq 5m - \frac{407}{18}(n-2)$.

A slightly weaker bound than in (a) of $\mathrm{cr}(G) \geq 4m - \frac{50}{3}(n-2)$ can be derived with a significantly shorter proof by only applying Theorem 3; we point this out in the proof.

That improves the best known results for $m > 5(n-2)$, which are $\mathrm{cr}(G) \geq 4m - \frac{103}{6}(n-2)$ [11] respectively $\mathrm{cr}(G) \geq 5m - \frac{139}{6}(n-2)$ [1]. Theorem 6 implies directly a better constant in the Crossing Lemma.

▶ **Theorem 7.** *Let G be a graph with n vertices and m edges. Then $\mathrm{cr}(G) \geq \frac{6000}{165649}\frac{m^3}{n^2} - \frac{218351}{165649}n > \frac{1}{27.61}\frac{m^3}{n^2} - 1.32n$. If $m \geq 6.79n > \frac{407}{60}n$, then $\mathrm{cr}(G) \geq \frac{6000}{165649}\frac{m^3}{n^2} > \frac{1}{27.61}\frac{m^3}{n^2}$.*

Proof. Let G be a graph with n vertices and m edges. For the case $m \geq \frac{407}{60}n$, we construct a random subgraph G' by selecting every vertex of G independently with probability $p = \frac{407}{60}n/m \leq 1$. We denote the number of edges and vertices in G' by m' and n'. By Theorem 6 and linearity of expectation, we obtain $\mathbb{E}[\mathrm{cr}(G')] \geq 5\mathbb{E}[m'] - \frac{407}{18}\mathbb{E}[n']$. We replace $\mathbb{E}[n'] = pn$, $\mathbb{E}[m'] = p^2 m$ and $\mathbb{E}[\mathrm{cr}(G')] = p^4\,\mathrm{cr}(G)$, and get

$$\mathrm{cr}(G) \geq \frac{5m}{p^2} - \frac{407n}{18p^3} = \frac{6000}{165649}\frac{m^3}{n^2}.$$

For the case $m < \frac{407}{60}n$ we compare the bound $\mathrm{cr}(G) \geq \frac{6000}{165649}\frac{m^3}{n^2} - \frac{218351}{165649}n$ with the corresponding best known linear bounds $\mathrm{cr}(G) \geq m - 3(n-2)$, $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$ [11] and Theorem 6. ◀

One direct application of the improved Crossing Lemma is a new bound on the edge density for k-planar graphs.

▶ **Corollary 8.** *For $k \geq 2$, any simple k-planar graph with n vertices has at most $3.72\sqrt{k}n$ edges.*

Proof. As in [13], the new bound for k-planar graphs can be derived directly from the new Crossing Lemma and the fact that each edge can be crossed at most k times:

$$\frac{1}{27.61}\frac{m^3}{n^2} \leq \mathrm{cr}(G) \leq km/2,$$

which then leads to $m \leq \sqrt{13.805k}n \leq 3.72\sqrt{k}n$. ◀

The best previous constant in the bound was 3.81.

4 Proof of Theorems 1 and 3

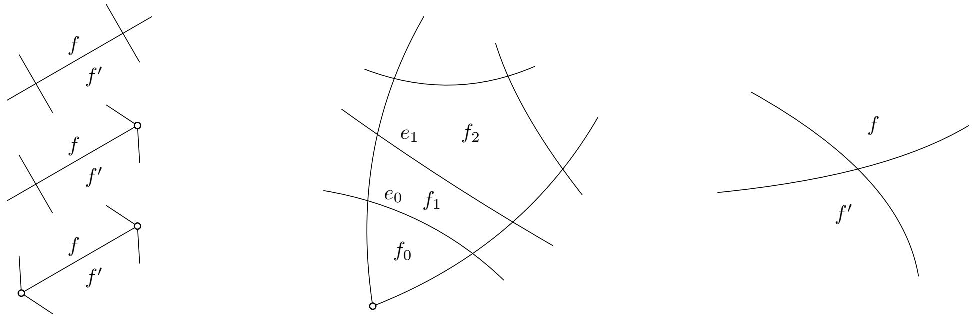
In this section, we give the proofs of the two central theorems of our paper. First, we will introduce some necessary concepts, we basically adopted the notation by Ackerman [1].

Notation. We interpret a drawing D as a plane map $M(D) = (V', E')$ whose vertices V' are either vertices $V(D)$ of D or crossing points of D. An edge e in E' connects two vertices of V', i.e., it is a crossing-free segment of an edge of D, which we denote by $\overline{e}$. We call an edge of E' an *r-edge*, if $r \in \{0, 1, 2\}$ of its endpoints are vertices of D. For a vertex $v \in V(D)$, we write $\deg(v)$ for its *degree*. The degree of a crossing is always four.

Let F' be the set of faces of $M(D)$. For a face $f \in F'$, we write $|f|$ for the number of edges in E' that are incident to f. Similarly, $|V(f)|$ denotes the number of (real) vertices of D that are incident to f. Note that we will assume 2-connectivity, hence the boundary of every face is a simple cycle and we avoid double-counting of the vertices. A face with $|f| = s$ is called a *s-gon*. In the cases of $s = 3, 4, 5, 6, 7$ we write instead *triangle, quadrilateral, pentagon, hexagon* and *heptagon*. If we want to denote that $|V(f)| = r$ and $|f| = s$, we write *r-s-gon* and use this wording also for 2-triangles, 0-quadrilaterals, etc. for simplicity. If we only want to specify for a face that $|V(f)| = r$, then we call it an r-face.

Further, we need some definitions for relations between faces in F'. Two faces are *r-neighbors* if they share an r-edge. Let now be $e_0 \in E'$ a 0-edge of a face $f_0 \in F'$ and $f_1 \in F'$ the 0-neighbor of f_0 at e_0. For $i \geq 1$, if $f_i \in F'$ is a 0-quadrilateral, then let be $f_{i+1} \in F'$ the 0-neighbor of f_i at the edge e_i opposite to f_{i-1}. The face f_i, for which i is maximal, is called the *wedge-neighbor* of f_0 at e_0. Since D is 3-planar, we have $i \leq 3$. Notice the alternative definition of a wedge-neighbor by Ackerman [1]. Finally, we define two faces $f, f' \in F'$ to be *vertex-neighbors*, if f and f' share a crossing-vertex c, but not an edge in E' incident to c. See Figure 2 for an illustration of the defined terms.

Preliminaries for the proofs. We prove both theorems by induction. This will allow us, as in [1], to study only 2-connected drawings (see Proposition 9). For $n = 3$, independently from the forbidden configurations, there are at most three non-homotopic edges in any drawing and therefore both theorems hold. If $n > 3$ and there is a vertex $v \in G$ with $\deg(v) \leq 4$, then the theorems follow after removing v by induction.



(a) **(b)** **(c)**

Figure 2 Illustrations of the defined neighborhood-relations. (a) From top to bottom: The faces f and f' are 0-neighbors, 1-neighbors, 2-neighbors resp. (b) The 0-pentagon f_2 is the wedge-neighbor of the 1-triangle f_0 at its edge e_0. (c) The faces f and f' are vertex-neighbors.

▶ **Proposition 9.** *If D is not 2-connected, then Theorem 1 and Theorem 3 are true.*

Proof. The argument follows the lines of [1]. To argue for the different scenarios of Theorem 1 and Theorem 3 at the same time, let $a(n-2)$ for $a \in \{\frac{13}{3}, 4.5, 5\}$ be an upper bound on the number of edges, which we want to prove. Assume that there is a vertex $x \in E'$ such that $M(D) \setminus \{x\}$ is not connected. Then x is either a vertex or a crossing of D.

If x is a vertex of D, then $D \setminus \{x\}$ is not connected, so let $D_1, ..., D_k$ be the connected components of $D \setminus \{x\}$. Let further D' be the drawing induced by $V(D_1) \cup \{x\}$ and D'' the drawing induced by $V(D_2) \cup ... \cup V(D_k) \cup \{x\}$. Let $|V(D')| = n'$, $|V(D'')| = n''$ and observe $n' + n'' = n + 1$. Since every vertex has at least degree four, $4 < n', n'' < n$ holds. By induction, it follows $m \leq (an' - 2a) + (an'' - 2a) = a(n+1) - 4a < a(n-2)$.

Assume now that x is a crossing of D. Let $\hat{D}$ be the drawing obtained by replacing x by a vertex. This increases the number of vertices by one and the number of edges by two. Let $D_1, ..., D_k$ be the connected components of $\hat{D} \setminus \{x\}$. Again, let D' be the drawing induced by $V(D_1) \cup \{x\}$ and D'' the drawing induced by $V(D_2) \cup ... \cup V(D_k) \cup \{x\}$. For $|V(D')| = n'$, $|V(D'')| = n''$ we observe $4 < n', n'' < n$. By induction, we get $m \leq (an' - 2a) + (an'' - 2a) - 2 = a(n+2) - 4a - 2 < a(n-2)$. ◀

Therefore we will always assume that D is 2-connected. As both theorems consider upper bounds for the number of edges for the specific graph classes, we also assume that we consider graphs G that are edge-maximum for the specific class of graphs, and for such graphs a corresponding drawing D that is crossing-minimum. These assumptions will enable us to conduct a focused analysis of the bounds for the number of edges.

▶ **Proposition 10.** *Let D be a drawing that is either (1) 2-planar F_5^2-free or (2) 2-planar F_5^2-free and F_6^2-free or (3) 3-planar F_6^3-free and maximally-dense-crossing-minimal under this restriction. Then the following properties hold:*

(a) *There are no empty lenses.*

(b) *For all faces $f \in F'$ we have $|f| \geq 3$.*

(c) *The wedge-neighbor of a 0-triangle or a 1-triangle is a face $f \in F'$ with $|f| \geq 4$ that is not a 0-quadrilateral.*

(d) *If there are two vertices $u, v \in V(D)$ on the boundary of a face $f \in F'$, then the edge uv is part of the boundary of f. Therefore every face $f \in F'$ with $|V(f)| > 2$ is a 3-triangle.*

Proof.

(a) Since there are no two homotopic edges, there are no empty lenses with two vertices. Any other empty lens can be destroyed by swapping the segments of the edges of D that define the empty lens (without creating one of the forbidden configurations). This reduces the number of crossings contradicting that D is crossing-minimal.

(b) Loops and self-intersecting edges are forbidden, so there is no face $f \in F'$ with $|f| = 1$. Every face $f \in F'$ with $|f| = 2$ is an empty lens, which does not appear in D by (a).

(c) Let f be an arbitrary face. By definition, the face f is never a 0-quadrilateral. If $|f| = 3$, then this would imply an empty lens.

(d) For an arbitrary face f, assume that no edge $e = uv$ exists on the boundary of f. Therefore, we may insert e contradicting that D is maximally dense. By this, we cannot create one of the three forbidden configurations F_5^2, F_6^2 and F_6^3, since they do not contain planar edges. This does not create homotopic edges as every other edge $e' = uv$ homotopic to e would have been already on the boundary of f or would have formed an empty lens with an edge of the boundary of f contradicting (a).
Assume now that a face f with $|V(f)| > 2$ exists that is not a 3-triangle. Then we find three vertices in $V(D)$ on the boundary of f, which do not all appear next to each other. We introduce a new edge between two of them, contradicting the maximality of D. ◀

In the following, we will use the *discharging method*. See [1, 2, 8, 15] for similar applications of this technique. We define a *charging function* $\mathrm{ch} : F' \to \mathbb{R}$ that assigns an *initial charge* of

$$\mathrm{ch}(f) = |f| + |V(f)| - 4 \tag{1}$$

to every face $f \in F'$. It is known that for the total charge $\sum_{f \in F'} \mathrm{ch}(f) = 4n - 8$ holds (refer to [2] for details). The challenge now is to redistribute the charge so that in the end every face $f \in F'$ has a charge of $\mathrm{ch}'(\cdot)$ that satisfies $\mathrm{ch}'(f) \geq \alpha |V(f)|$ for a suitable $\alpha > 0$, while the total charge does not change. From this and the observation that $\sum_{f \in F'} |V(f)| = \sum_{v \in V(D)} \deg(v) = 2m$ holds, we can derive an upper bound of

$$m \leq \frac{2}{\alpha}(n-2) \tag{2}$$

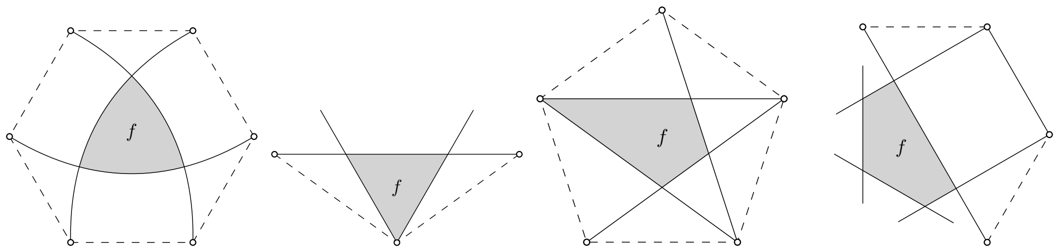
on the number of edges. For a given α and a face f with charge c, we say that $|c - \alpha|V(f)||$ is the *demand* of f, if $c - \alpha|V(f)|$ is negative, otherwise we call it the *excess* of f. If f has no demand, then we also say that f is satisfied.

4.1 Proof and Discharging for Theorem 1

▶ **Theorem 1.** *Any graph G with $n \geq 3$ vertices that admits a 2-planar F_5^2-free drawing has at most $4.5(n-2)$ edges. If the drawing is also F_6^2-free, then G has at most $\frac{13}{3}(n-2)$ edges.*

Proof. We start with the bound of $\frac{13}{3}(n-2)$. Let D be a 2-planar, F_5^2-free and F_6^2-free drawing that is maximally-dense-crossing-minimal. Assign to every face $f \in F'$ the initial charge $\mathrm{ch}(f)$ according to Equation (1). The initial charges are distributed in the following way:

- Step 1: Each 0-triangle receives $\frac{1}{3}$ charge from each of its wedge-neighbors.
- Step 2: Each 1-triangle receives $\frac{1}{26}$ charge from both 1-neighbors.
- Step 3: Each 1-triangle receives $\frac{5}{13}$ charge from its wedge-neighbor.
- Step 4: Each 2-quadrilateral contributes its excess to its wedge-neighbor.
- Step 5: For each 2-triangle f, let $\mathcal{C}(f)$ be the inclusion-minimal planar cycle of D enclosing f (i.e. the planar cycle that does not contain other planar edges). Then f distributes its excess equally over those faces that lie inside $\mathcal{C}(f)$ and have a demand.



(a) (b) (c) (d)

Figure 3 Discharging for Theorem 1. Planar edges that exist by Proposition 10 are dashed.

Denote the charges after the i-th step by $\mathrm{ch}_i(\cdot)$. With this, we have $\mathrm{ch}'(\cdot) = \mathrm{ch}_5(\cdot)$.

▶ **Proposition 11.** *For all faces $f \in F'$, we have* $\mathrm{ch}'(f) \geq \frac{6}{13}|V(f)|$.

Proof. We analyze the final charge $\mathrm{ch}'(\cdot)$ for all faces. Note that a face contributes through each edge of its boundary in Step 1-3 at most once and the only contributing faces in Step 1 are 2-quadrilaterals (see Figure 3a) and in Step 2 2-triangles (see Figure 3b). Also $\mathrm{ch}_3(f) \geq \frac{6}{13}|V(f)|$ already implies $\mathrm{ch}'(f) \geq \frac{6}{13}|V(f)|$. Because of Proposition 10 there are only 3-triangles and faces f with $|f| \geq 3$ and $|V(f)| \leq 2$.

- *f is a 0-triangle.* Then f receives in Step 1 in $3 \cdot \frac{1}{3}$ charge and never contributes charge. Therefore $\mathrm{ch}_3(f) = -1 + 1 = 0 \geq \frac{6}{13} \cdot 0$.
- *f is a 1-triangle.* Then f receives in Step 2 $2 \cdot \frac{1}{26}$ charge, in Step 3 $\frac{5}{13}$ charge and never contributes charge. Therefore $\mathrm{ch}_3(f) = 0 + \frac{6}{13} \geq \frac{6}{13} \cdot 1$.
- *f is a 2-triangle.* Then f starts with 1 charge and contributes in Step 2 at most $2 \cdot \frac{1}{26}$ charge. Therefore $\mathrm{ch}_3(f) \geq 1 - \frac{1}{13} = \frac{12}{13} \geq \frac{6}{13} \cdot 2$.
- *f is a 3-triangle.* Then f never receives or contributes charge. Thus $\mathrm{ch}_3(f) = 2 \geq \frac{6}{13} \cdot 3$.
- *f is a 0-quadrilateral.* Then f starts with 0 charge and never receives or contributes charge as it cannot be the wedge-neighbor of another face. Therefore $\mathrm{ch}_3(f) = 0 \geq \frac{6}{13} \cdot 0$.
- *f is a 1-quadrilateral.* Then f starts with 1 charge. If f contributes in Step 3 to less than two 1-triangles, we have $\mathrm{ch}_3(f) \geq 1 - \frac{5}{13} = \frac{8}{13} \geq \frac{6}{13} \cdot 1$. Otherwise, we know that f is bounded by a 5-cycle of planar edges (Figure 3c). Here, charges do not change in Step 4, but we can find $\frac{3}{13}$ charge from the excesses of 2-triangles in this 5-cycle and move that to f in Step 5. Therefore, we have $\mathrm{ch}'(f) = 1 - 2 \cdot \frac{5}{13} + \frac{3}{13} = \frac{6}{13} \geq \frac{6}{13} \cdot 1$.
- *f is a 2-quadrilateral.* Then f has one wedge-neighbor, to which it contributes either $\frac{1}{3}$ charge in Step 1 or $\frac{5}{13}$ charge in Step 3. So we have $\mathrm{ch}_3(f) \geq 2 - \frac{5}{13} = \frac{21}{13} \geq \frac{6}{13} \cdot 2$
- *f is a 0-pentagon.* Note that all wedge-neighbors of f are 1-triangles or 2-quadrilaterals, as otherwise there would be an edge with three crossings or a face with two real vertices that are not connected by an edge. If f contributes to five 1-triangles in Step 3, then we would have an F_5^2 configuration, which is forbidden. Otherwise, at least one 2-quadrilateral contributes its excess of $\frac{14}{13}$ to f in Step 4 (see Figure 3d). Therefore we have $\mathrm{ch}_4(f) \geq 1 + \frac{14}{13} - 4 \cdot \frac{5}{13} = \frac{7}{13} \geq \frac{6}{13} \cdot 0$.
- *f is a 1-pentagon or a 2-pentagon resp.* Then f contributes to at most three or two 1-triangles resp. in Step 3. Therefore, we have $\mathrm{ch}_3(f) \geq 2 - 3 \cdot \frac{5}{13} = \frac{11}{13} \geq \frac{6}{13} \cdot 1$ resp. $\mathrm{ch}_3(f) \geq 3 - 2 \cdot \frac{5}{13} = \frac{29}{13} \geq \frac{6}{13} \cdot 2$.
- *f is a 0-hexagon.* If f contributes to six 1-triangles in Step 3, then we would have an F_6^2 configuration, which is forbidden. Otherwise, we have $\mathrm{ch}_3(f) \geq 2 - 5 \cdot \frac{5}{13} = \frac{1}{13} \geq \frac{6}{13} \cdot 0$.
- *f is a 1-hexagon resp. 2-hexagon.* Then f contributes to at most four resp. three 1-triangles in Step 3 and we have $\mathrm{ch}_3(f) \geq 3 - 4 \cdot \frac{5}{13} = \frac{19}{13} \geq \frac{6}{13} \cdot 2$.

- *f is a face with $|f| \geq 7$.* Then f may contribute charge to at most $|f|$ wedge-neighbors in Step 3. Therefore $\mathrm{ch}_3(f) \geq |f| + |V(f)| - 4 - \frac{5}{13} \cdot |f| \geq \frac{8}{13} \cdot 7 + |V(f)| - 4 \geq \frac{6}{13}|V(f)|$.

Therefore, all faces $f \in F'$ are satisfied, which proves the proposition. ◀

Combining Proposition 11 and Equation (2), $m \leq 2 \cdot \frac{13}{6}(n-2)$ is implied, as claimed.

For drawings, where F_6^2 configurations are allowed, we can use similar discharging steps to prove the bound of $4.5(n-2)$ on the number of edges. Here we set $\alpha = \frac{4}{9}$, and therefore 1-triangles can receive $\frac{1}{18}$ charge from both its 1-neighbors each in Step 2 without creating a demand for any 2-triangles. Therefore, faces have to contribute in Step 3 only $\frac{1}{3}$ charge to satisfy all 1-triangles. Now let f be a 0-hexagon that is the wedge-neighbor of six 1-triangles. Starting with 2 charge, it contributes at most $6 \cdot \frac{1}{3}$ in Step 3, and therefore ends with $0 \geq \frac{4}{9} \cdot 0$ charge. For all other faces we still have enough charge with the same analysis as above.

Therefore, there exists a function $\mathrm{ch}'(\cdot)$ satisfying $\mathrm{ch}'(f) \geq \frac{4}{9}|V(f)|$ for all $f \in F'$, while the total amount of charge is still $4n - 8$. By Equation (2) we get $m \leq 2 \cdot \frac{9}{4}(n-2)$. ◀

4.2 Proof and Discharging for Theorem 3

▶ **Theorem 3.** *Any graph with $n \geq 3$ vertices that admits a 3-planar F_6^3-free drawing has at most $5(n-2)$ edges.*

Proof. Let D be a 3-planar F_6^3-free drawing that is maximally-dense-crossing-minimal. As in the proof of Theorem 1, we assign the initial charges $\mathrm{ch}(f)$ to the faces of $M(D)$ and redistribute them to achieve a function $\mathrm{ch}'(\cdot)$. The discharging takes place in seven steps:

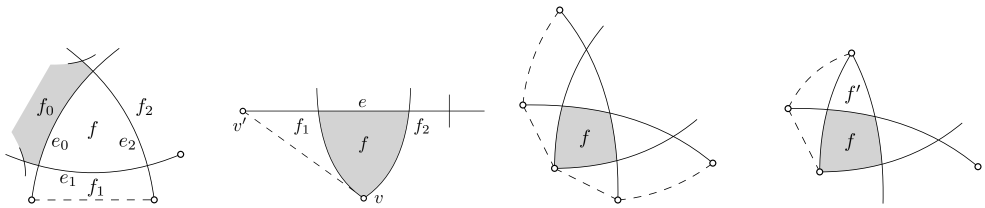
- Step 1: Each 0-triangle receives 1 charge from each 0-neighbor that is a 2-quadrilateral.
- Step 2: Each 0-triangle with a demand receives $\frac{1}{3}$ charge from all wedge-neighbors.
- Step 3: Each 2-triangle distributes its excess equally over all 1-neighbors that are 1-triangles.
- Step 4: Each 1-triangle receives its demand from its wedge-neighbor.
- Step 5: Each face distributes its excess equally over the wedge-neighbors that are 0-pentagons, but at most 0.3 to each of them, and keeps the rest.
- Step 6: Each face distributes its excess equally over all vertex-neighbors that are 0-quadrilaterals or 0-pentagons. The 0-quadrilaterals distribute this charge equally over their 0-neighbors that have a demand.
- Step 7: For each face f, let $\mathcal{C}(f)$ be the inclusion-minimal planar cycle of D enclosing f (i.e. the planar cycle that does not contain other planar edges). Then f distributes its excess equally over those faces that lie inside $\mathcal{C}(f)$ and have a demand.

Again, we denote by $\mathrm{ch}_i(\cdot)$ the charges after the i-th step and by $\mathrm{ch}'(\cdot)$ the final charges. Our goal is to show $\mathrm{ch}'(f) \geq 0.4|V(f)|$ for all faces $f \in F'$. Note that this is already implied by $\mathrm{ch}_4(f) \geq 0.4|V(f)|$, as in Step 5-7 faces contribute only their excesses. We structure the proof into several propositions, collecting statements about the discharging steps.

▶ **Proposition 12.** *After Step 2, 0-triangles are and remain satisfied.*

Proof. Let f be a 0-triangle. We have $\mathrm{ch}(f) = -1$. If f receives in Step 1 charge, then $\mathrm{ch}_1(f) = 0$. Otherwise, f receives $3 \cdot \frac{1}{3}$ charge in Step 2, so $\mathrm{ch}_2(f) = 0$. 0-triangles do not contribute charge in Step 3-4, since they are not wedge-neighbors of 1-triangles. Therefore, $\mathrm{ch}'(f) \geq 0.4 \cdot 0$ holds. ◀

▶ **Proposition 13.** *In Step 1-2, 0-faces contribute no charge.*



(a) **(b)** **(c)** **(d)**

Figure 4 Illustrations for the proofs of Proposition 13, Proposition 14 and Proposition 15.

Proof. No faces except 2-quadrilaterals contribute charge in Step 1, so we consider only Step 2. Assume that a 0-face f_0 contributes charge to a 0-triangle f in Step 2, and f_0 and f are therefore 0-neighbors at an edge e_0. Let e_1, e_2 be the other edges of f and f_1, f_2 the 0-neighbors at these edges (see Figure 4a). Since f_0 is a 0-face, it is incident to two crossings each with $\overline{e}_1$ and $\overline{e}_2$ and these edges also cross each other at f. Therefore $\overline{e}_1$ and $\overline{e}_2$ have already three crossings and end at f_1 resp. f_2. The edge $\overline{e}_0$ ends also at one of f_1 or f_2, as otherwise it would have four crossings. W.l.o.g. e_0 ends at f_1 and by Proposition 10 f_1 is a 2-quadrilateral. Hence, $\mathrm{ch}_1(f) \geq 0.4 \cdot |V(f)|$, contradicting that f receives charge later. ◀

▶ **Proposition 14.** *After Step 3, 1-triangles have a demand of at most* 0.3 *charge.*

Proof. Let f be a 1-triangle with the real vertex v and the 0-edge e. Let further f_1, f_2 be the 1-neighbors of f (see Figure 4b). Then $\overline{e}$ ends at one of f_1 and f_2, as otherwise it would have more than three crossings. W.l.o.g. let f_1 be that face with the vertex v' to which $\overline{e}$ is incident. Then by Proposition 10 the edge vv' exists and f_1 is a 2-triangle. Therefore, f_1 starts with 1 charge and has an initial excess of 0.2. Thus, f receives 0.1 charge in Step 3. We have $\mathrm{ch}_3(f) = 0.1$, which is equivalent to a demand of 0.3. ◀

▶ **Proposition 15.** *After Step 4, all 1-quadrilaterals are satisfied.*

Proof. Let f be a 1-quadrilateral. We have $\mathrm{ch}(f) = 1$ and f contributes charge only in Step 2 and Step 4. If f contributes to at most one wedge-neighbor or to two 1-triangles, then $\mathrm{ch}_4(f) \geq 1 - 0.6 \geq 1 \cdot 0.4$. Otherwise, f contributes either to two wedge-neighbors that are both 0-triangles or to one 0-triangle and one 1-triangle. In the first case, both 0-triangles are already satisfied after Step 1, as they have wedge-neighbors that are 2-quadrilaterals (see Figure 4c). In the second case, f contributes not more than 0.2 charge to the 1-triangle f', because one of its 1-neighbors is a 2-triangle contributing its excess of 0.2 charge only to f' in Step 3 (see Figure 4d). Therefore, we have $\mathrm{ch}_4(f) \geq 1 - \frac{1}{3} - 0.2 \geq 1 \cdot 0.4$. ◀

▶ **Proposition 16.** *After Step 4, all faces are and remain satisfied that are not 0-pentagons that are the wedge-neighbor of four or five 1-triangles.*

Proof. Note again that for a face f the charge $\mathrm{ch}_4(f) \geq 0.4|V(f)|$ implies already that it has no demand in Step 5-7, since faces only there contribute their excesses.

To see that 0-triangles and 1-quadrilaterals are satisfied, we refer to Propositions 12 and 15. 1-triangles are satisfied by definition of Step 4. Remember that only 3-triangles and r-s-gons with $r \leq 2, s \geq 3$ can exist by Proposition 10. Now we discuss the other cases:

- *f is a 2-triangle.* We start with $\mathrm{ch}(f) = 1$. As only wedge-neighbors contribute in Step 1-2 and Step 4 and f cannot be a wedge-neighbor of another face, the only critical step is Step 3. Here, f contributes in total at most its excess of 0.2 charge, and therefore $\mathrm{ch}_4(f) \geq 1 - 0.2 \geq 2 \cdot 0.4$.

- *f is a 3-triangle.* We start with $\mathrm{ch}(f) = 2$ and f never contributes charge. It follows that $\mathrm{ch}_4(f) = 2 \geq 3 \cdot 0.4$ holds.
- *f is a 0-quadrilateral.* Again, f never contributes charge, and therefore $\mathrm{ch}(f) = \mathrm{ch}_4(f) = 0 \geq 0 \cdot 0.4$ holds.
- *f is a 2-quadrilateral.* We start with $\mathrm{ch}(f) = 2$. Note that f contributes only once as it has only one wedge-neighbor, and therefore we have $\mathrm{ch}_4(f) \geq 2 - 1 \geq 2 \cdot 0.4$.
- *f is a 0-pentagon with at most three wedge-neighbors that are 1-triangles.* We have $\mathrm{ch}(f) = 1$ and f contributes only to three faces. With Proposition 13 and Proposition 14 $\mathrm{ch}_4(f) \geq 1 - 3 \cdot 0.3 \geq 0 \cdot 0.4$ follows.
- *f is a 1-pentagon or a 2-pentagon.* f starts with $\mathrm{ch}(f) \geq 2$ and we have $\mathrm{ch}_4(f) \geq 2 - 3 \cdot \frac{1}{3} \geq 2 \cdot 0.4$.
- *f is a face with $|f| \geq 6$.* Then f may contribute to at most $|f|$ wedge-neighbors charge. Therefore, we have $\mathrm{ch}_4(f) \geq |f| + |V(f)| - 4 - \frac{1}{3} \cdot |f| \geq |V(f)|$. ◀

It remains to prove that 0-pentagons with four or five wedge-neighbors that are 1-triangles have at least zero charge after Step 7. We show this by the following four propositions, which we only state here; the proofs can be found in Appendix A.

▶ **Proposition 17.** *In Step 5, each 0-pentagon receives* 0.3 *charge from all wedge-neighbors that are not 1-triangles, 0-triangles or 0-pentagons.*

▶ **Proposition 18.** *In Step 6, each 1-face and 2-face f with $|f| \geq 5$ and each 0-face f with $|f| \geq 7$ contributes at least* 0.4 *charge to the vertex-neighbors that are 0-quadrilaterals or 0-pentagons.*

▶ **Proposition 19.** *After Step 7, all 0-pentagons that are the wedge-neighbor of four 1-triangles are satisfied.*

▶ **Proposition 20.** *After Step 7, all 0-pentagons that are the wedge-neighbor of five 1-triangles are satisfied.*

By Propositions 16, 19 and 20 $\mathrm{ch}'(f) \geq 0.4 \cdot |V(f)|$ holds for all faces $f \in F'$. Since charge is only moved, its total amount is still $4n - 8$ and Equation (2) implies $m \leq \frac{2}{0.4}(n-2)$. ◀

5 Proof of Theorem 6

In this section, we present the proof of Theorem 6 that shows how to use the earlier stated observations and theorems and leads to a better bound for the Crossing Lemma.

▶ **Theorem 6.** *Let G be a graph with $n > 2$ vertices and m edges. Then*
(a) $\mathrm{cr}(G) \geq \frac{73}{18}m - \frac{305}{18}(n-2)$,
(b) $\mathrm{cr}(G) \geq 5m - \frac{407}{18}(n-2)$.

Proof. We start proving the bound in (a). If $m \leq 5(n-2)$, then the bound follows from the linear bound $\mathrm{cr}(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2)$ [11]. So assume $m > 5(n-2)$ and let D be a crossing-minimal drawing of G. From D, we iteratively remove the edge with the most crossings until $5(n-2)$ edges are left. In particular, as long as the maximum number of crossings is three, we always remove an edge from an F_6^3 configuration. By Theorem 3, we stop latest, when there are no F_6^3 configurations. By this process, edges are iteratively deleted until we reach $5(n-2)$ edges, as following:

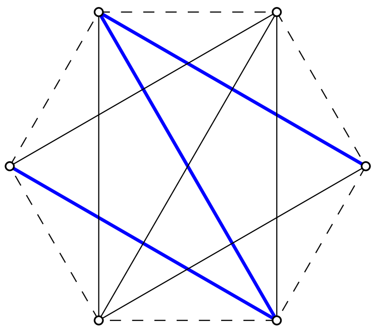


Figure 5 Three independent edges (blue) with three crossings in an F_6^3 configuration. In D_3 one of them is already deleted, in D_{3-} also the other two.

- m_{5+} edges with five or more crossings – denote the resulting drawing by D_4,
- then m_4 edges with four crossings – denote the resulting drawing by D_3 and the set of edges deleted in this step by E_4,
- then m_3 edges with three crossings from F_6^3 configurations – denote the resulting drawing by D_{3-}.

Note that m_4 or m_3 could be zero in the case that we reached $5(n-2)$ already during step (1) or (2). Afterwards we have m_3 edge-disjoint F_6^3 configurations with a missing edge in D_{3-}. So we are able to find $2m_3$ more independent edges with three crossings and delete them (see Figure 5). Continue the deletion process by still removing the edge with the most crossings until this edge no longer has three or more crossings; we denote the number of these deleted edges by m_{3-}. Call the achieved drawing D_2. By applying the linear bound from [11] again, we have

$$\begin{aligned}\operatorname{cr}(G) &\geq [5m_{5+} + 4m_4 + 3m_3] + [2 \cdot 3m_3 + 3m_{3-}] + \left[\frac{7}{3}(5(n-2) - 2m_3 - m_{3-}) - \frac{25}{3}(n-2)\right] \\ &= 5m_{5+} + 4m_4 + \frac{13}{3}m_3 + \frac{2}{3}m_{3-} + \frac{10}{3}(n-2). \end{aligned} \tag{3}$$

As all values are non-negative, it is not hard to see that this is at least

$$\geq 4(m_{5+} + m_4 + m_3 + 5(n-2)) - \frac{50}{3}(n-2) = 4m - \frac{50}{3}(n-2).$$

For the better bound of $\operatorname{cr}(G) \geq \frac{73}{18}m - \frac{305}{18}(n-2)$ we have to elaborate on the value m_4, as there was no slack in the last inequality.

As a preparation, we first consider the structure of D_2. Let c_{pent} be the number of F_5^2 configurations and c_{hex} the number of F_6^2 configurations in D_2. Let further E_0 be the set of crossing-free edges on the boundary of the forbidden configurations in D_3 resp. D_2 that do not exist in D_2, and therefore may be added. We denote $|E_0| = m_0$ and state the following; the proof is in Appendix B.

▶ **Proposition 21.** *With the notation above,* $c_{pent} + c_{hex} \geq \frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0$.

Next, we show how to limit the number of the edges of E_4, i.e., the deleted edges that were accounted with four crossings in D_4. For that, we introduce a triangulation $\mathcal{H}$ on the set of the n vertices of D_4 that contains (1) the boundary of every F_6^3 configuration in D_3, (2) the boundary of every F_5^2 and F_6^2 configuration in D_2, (3) every edge in E_4 that lies completely outside of these forbidden configurations. This definition is refined in the proof of the next proposition. Note that it is always possible to achieve such a triangulation $\mathcal{H}$, because edges of E_4 cannot cross each other.

▶ **Proposition 22.** *Let $\mathcal{H}' \subseteq \mathcal{H}$ be the set of triangles that do not belong to the forbidden configurations and let $c_\triangle = |\mathcal{H}'|$. Then $m_4 \leq m_3 + c_{hex} + 4m_0 + 4c_\triangle$.*

The proof can be found in Appendix B. Combining the results, we can finish the first part of the proof. Proposition 21 implies

$$c_\triangle \leq 2(n-2) - 4m_3 - \left[\frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0\right] \cdot 3 - c_{hex} = 3m_{3-} - 3m_0 - c_{hex},$$

because the total number of triangles is $2(n-2)$ and a pentagon resp. hexagon contains three resp. four triangles. Together with Proposition 22, this gives

$$m_4 \leq m_3 + c_{hex} + 4m_0 + 4(3m_{3-} - 3m_0 - c_{hex}) \leq 5m_3 + 12m_{3-}.$$

Multiplying this term by $\frac{1}{18}$ and adding it to Equation (3), we get as desired

$$\begin{aligned} \operatorname{cr}(G) &\geq 5m_{5+} + 4m_4 + \frac{13}{3}m_3 + \frac{2}{3}m_{3-} + \frac{10}{3}(n-2) + \frac{m_4 - (5m_3 + 12m_{3-})}{18} \\ &\geq \frac{73}{18}(m_{5+} + m_4 + m_3) + \frac{10}{3}(n-2) \\ &= \frac{73}{18}(m_{5+} + m_4 + m_3 + 5(n-2)) - \frac{305}{18}(n-2). \end{aligned}$$

For the bound in (b) see the following: If $m \leq 6(n-2)$, then we can apply the bound of (a). So let be $m > 6(n-2)$. Iteratively delete the edge with the most crossings in a crossing-minimal drawing D until $6(n-2)$ edges are left; these edges have at least five crossings, as the density of 4-planar graphs is $\leq 6(n-2)$ [1]. With the bound in (a), this implies

$$\operatorname{cr}(G) \geq 5(m - 6(n-2)) + \frac{73}{18} \cdot 6(n-2) - \frac{305}{18}(n-2) = 5m - \frac{407}{18}(n-2). \quad ◀$$

6 Discussion

We have improved the leading constant of the lower bound for the crossing number of a given graph G. Although this improvement does not seem to be too impressive at first sight, we worked out some interesting observations for drawings with a limited number of crossings per edge. This leads to further improvements, conjectures and suggestions for future research.

In particular, we have improved for $m > 5(n-2)$ the lower bound of the crossing number, unfortunately we did not reach tightness. We confirm the conjecture by [11] that $\operatorname{cr}(G) \geq \frac{25}{6}m - \frac{35}{2}(n-2)$ holds and highlight that this bound would follow from our proofs, if we were able to show a slightly stronger statement in Proposition 22, namely $m_4 \leq m_3 + c_{hex} + 4m_0 + \frac{4}{3}c_\triangle$. The corresponding upper bound can be obtained by a construction where the plane subgraph consists only of pentagonal and hexagonal faces [11].

Applying our technique to 4-planar drawings might show that these drawings without full hexagons F_6^4 have density $\leq 5.5(n-2)$. This would provide a characterization of optimal 4-planar graphs, which is a well-known open problem. Further, we can look at 5-planar graphs, a class that has been considered as too complex for actual research. Just applying Corollary 8 improves the current known density bound from $8.52n$ to $8.32n$.

It seems to be worthwhile to apply the idea to bipartite graphs to obtain improvements of the Crossing Lemma. Here, the corresponding linear bound $\operatorname{cr}(G) \geq 3m - \frac{17}{2}n + 19$ used in the current proof in [5] is not tight.

Furthermore, we have indicated a way how to obtain the exact density bound of optimal simple 3-planar graphs. Note that we only did one step in this direction.

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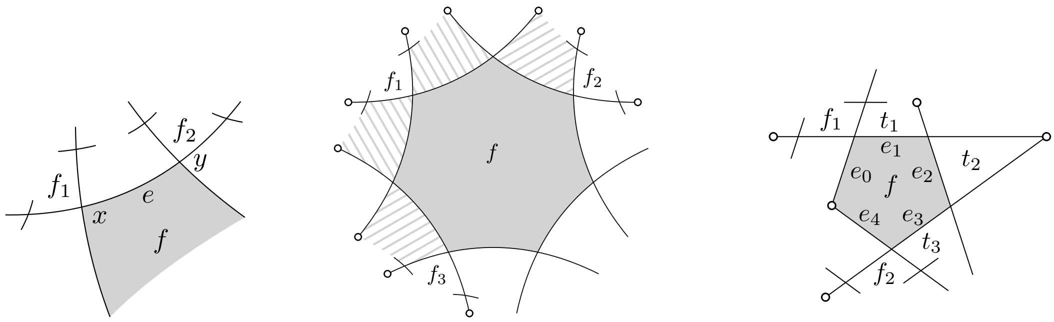


(a) (b) (c)

Figure 6 (a) No face contributes to two consecutive vertex-neighbors in Step 6. (b) If a 0-heptagon f contributes to three vertex-neighbors f_1, f_2, f_3 in Step 6, then it contributes not to all its seven wedge-neighbors in Step 1-5. (c) A 1-pentagon f contributing to all three wedge-neighbors in Step 1-5 and to two vertex-neighbors in Step 6 leads to a contradiction.

A Details for Section 4

▶ **Proposition 17.** *In Step 5, each 0-pentagon receives* 0.3 *charge from all wedge-neighbors that are not 1-triangles, 0-triangles or 0-pentagons.*

Proof. Note that in the calculations of Proposition 16, we assumed for all faces that are possibly a wedge-neighbor of a 0-pentagon except 0-triangles, 1-triangles, 1-quadrilaterals and 0-pentagons that they give at least 0.3 charge to all wedge-neighbors. If such a face f has a wedge-neighbor that is a 0-pentagon, then it did not contribute charge to it in Step 1-4, and therefore has 0.3 charge left for it in Step 5. The only critical case is a 1-quadrilateral f with a 0-pentagon and a 0-triangle f' as wedge-neighbors. Observe that in this case $\mathrm{ch}_1(f') = 0$ already, because there is a 2-quadrilateral next to f' as in Figure 4c, and therefore $\mathrm{ch}_4(f) = 1$. Thus, f can contribute 0.3 charge to the 0-pentagon. ◀

▶ **Proposition 18.** *In Step 6, each 1-face and 2-face f with $|f| \geq 5$ and each 0-face f with $|f| \geq 7$ contributes at least* 0.4 *charge to the vertex-neighbors that are 0-quadrilaterals or 0-pentagons.*

Proof. Let e be a 0-edge of a face f incident to the crossings x and y. Let f_1 be the vertex-neighbor of f at x and f_2 the vertex-neighbor at y. If f_1 and f_2 are 0-faces, then $\overline{e}$ has more than three crossings, a contradiction. Therefore, no face can contribute charge through two consecutive crossings on its boundary in Step 6. For a face f, this implies that it can contribute to at most $\left\lfloor \frac{|f|}{2} \right\rfloor$ vertex-neighbors in this step.

Now we distinguish different cases for the face f that might contribute to vertex-neighbors. We start with the case that f is a 0-face. Here, after Step 5, f has an excess of

$$\mathrm{ch}_5(f) \geq |f| - 4 - |f| \cdot 0.3 = 0.7|f| - 4,$$

which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ for $|f| \geq 8$ and therefore enough. If f is a 0-heptagon, then, by the inequality above, there is enough charge if f contributes to at most two vertex-neighbors in Step 6. So assume that it contributes to three vertex-neighbors. It is not hard to see that there are wedge-neighbors of f with $|f| \geq 4$ and $|V(f)| \geq 1$ (see Figure 6a), so f contributes in Step 1-5 to at most six faces. Now we have $\mathrm{ch}_5(f) \geq 3 - 6 \cdot 0.3 = 1.2$, which is sufficient to give three vertex-neighbors 0.4 charge each in Step 6.

Let f now be a 1-face. Then f has an excess of

$$\mathrm{ch}_5(f) - 0.4 \geq |f| + 1 - 4 - (|f| - 2) \cdot \frac{1}{3} - 0.4 = \frac{2}{3}|f| - 3.4 + \frac{2}{3},$$

which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ if $|f| \geq 6$. If f is a 1-pentagon, then, by the inequality above, there is enough charge if f contributes to at most one vertex-neighbor in Step 6. So assume the opposite, i.e., f contributes to two vertex-neighbors f_1, f_2 in Step 6. If f contributes in Step 1-5 to only one or two wedge-neighbors, then its excess after Step 5 is at least $1.6 - \frac{2}{3}$ charge and therefore sufficient, so assume this is not the case either.

Walking along the boundary of f, let e_0 be a 1-edge of f, let e_1, e_2, e_3 be the 0-edges and let e_4 be the other 1-edge of f. Let further t_i be the wedge-neighbor of f at e_i for $i \in \{1, 2, 3\}$. W.l.o.g. f_1 lies at the crossing of e_0, and therefore the face t_2 is a 1-triangle, as otherwise f would not contribute charge to t_2 in Step 1-5 (see Figure 6b). Therefore, f_2 lies at the crossing of e_4. Note that $\overline{e}_2$ ends at t_1 or t_3, say w.l.o.g. at t_1. But then $|t_1| \geq 4$ and $|V(t_1)| \geq 1$, so f does not contribute charge to t_1 in Step 1-5, a contradiction to our assumption. Therefore, 1-pentagons can contribute 0.4 charge to the desired vertex-neighbors.

The last case is that f is a 2-face. Then f has an excess of

$$\mathrm{ch}_5(f) - 0.8 \geq |f| + 2 - 4 - (|f| - 3) \cdot \frac{1}{3} - 0.8 \geq \frac{2}{4}|f| - 1.8,$$

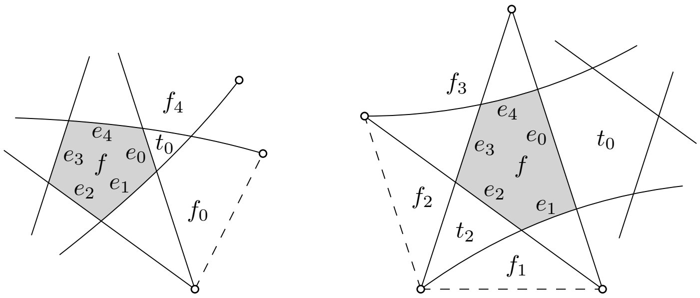
which is at least $0.4 \cdot \left\lfloor \frac{|f|}{2} \right\rfloor$ for $|f| \geq 5$. ◀

▶ **Proposition 19.** *After Step 7, all 0-pentagons that are the wedge-neighbor of four 1-triangles are satisfied.*

Proof. We introduce a notation for the edges and faces at a 0-pentagon f. Let $e_i, i \in \{0, ..., 4\}$ be the edges forming the boundary of f, so that e_i and $e_{(i+1 \bmod 5)}$ have a crossing at f. Further we denote by t_i the wedge-neighbor of f at e_i and by f_i the vertex-neighbor of f at the crossing of e_i and $e_{(i+1 \bmod 5)}$.

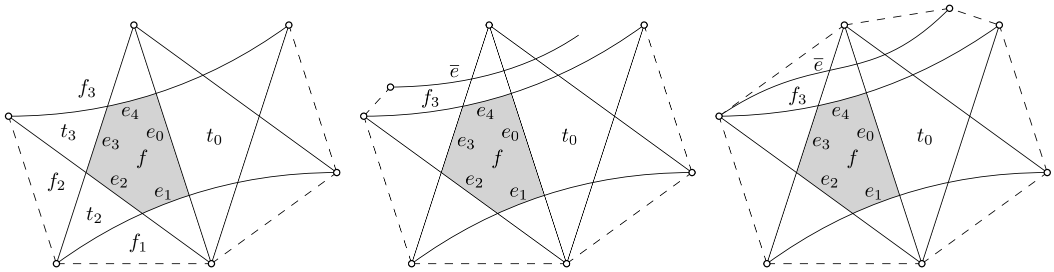
Let f be a 0-pentagon with four wedge-neighbors that are 1-triangles. So we have $\mathrm{ch}_4(f) \geq 1 - 4 \cdot 0.4 = -0.2$. Let w.l.o.g. t_0 be the wedge-neighbor of f that is not a 1-triangle. If t_0 is not a 0-triangle or 0-pentagon, then it contributes, by Proposition 17, 0.3 charge to f in Step 5 and f is satisfied. Otherwise, distinguish between the type of the face t_0.

- *Case 1: t_0 is a 0-triangle.* Observe that $\overline{e}_1$ and $\overline{e}_4$ already have three crossings and $\overline{e}_0$ two crossings. Therefore, $\overline{e}_0$ ends at f_0 or f_4, say w.l.o.g. f_0, so f_0 is a 2-quadrilateral (see Figure 7a). Then f_0 has an excess of 0.2 after Step 5, as it only contributes in Step 1 charge. Note that the vertex-neighbors of f_0 are f and f_4. Since f_4 is not a 0-face, f_0 contributes its excess of 0.2 charge in Step 6 only to f, and therefore $\mathrm{ch}_6(f) \geq 0$.
- *Case 2: t_0 is a 0-pentagon.* Again, $\overline{e}_1$ and $\overline{e}_4$ already have three crossings and $\overline{e}_0$ two crossings. Therefore, f_2 is a 2-triangle and also one of f_1 and f_3, say w.l.o.g. f_1 (see Figure 7b). So we have $\mathrm{ch}_3(t_2) = -0.2$, and therefore $\mathrm{ch}_4(f) \geq 1 - 3 \cdot 0.3 - 0.2 \geq -0.1$. Note that the only face besides f that may receive charge from t_0 in Step 5 is f_4. Therefore, we distinguish two cases:
  - *Case 2.1: f_4 is a 0-pentagon.* If less than three wedge-neighbors of t_0 are 1-triangles, then $\mathrm{ch}_4(t_0) \geq 0.4$ and f receives enough charge in Step 5. If three wedge-neighbors of t_0 are 1-triangles, then the 1-triangle at the vertex at which $\overline{e}_1$ ends has -0.2 charge after Step 3, as it lies between two 2-triangles (see Figure 7c). Therefore, we have $\mathrm{ch}_4(t_0) = 1 - 2 \cdot 0.3 - 0.2 = 0.2$ and f can receive a half of it in Step 5, which is enough.



(a) (b) (c)

Figure 7 Illustrations for the proof of Proposition 19 Case 1 and 2.1.



(a) (b) (c)

Figure 8 Illustrations for the proof of Proposition 19 Case 2.2.

- *Case 2.2: f_4 is not a 0-pentagon.* If $\text{ch}_4(t_0) \geq 0.1$, then f receives its missing charge already in Step 5. So assume the opposite, which implies that four wedge-neighbors of t_0 are 1-triangles (see Figure 8a).
  If now f_3 is a 2-triangle, then $\text{ch}_3(t_3) = -0.2$ and we have $\text{ch}(f) = 1 - 2 \cdot 0.3 - 2 \cdot 0.2 = 0$ and f never has a demand. Otherwise, there is an edge $\overline{e}$ crossing $\overline{e}_3$ at f_3 that has already three crossings, and therefore f_3 is either a 2-quadrilateral or a 1-triangle. In the first case, f_3 has an excess of at least 0.2 and only contributes it to f in Step 6. In the second case, we have a planar cycle of length seven, in which all faces except f are satisfied after Step 6. Here, f receives its demand in Step 7 from a 2-quadrilateral that is a vertex-neighbor of t_0 (Figure 8c). In all cases f is satisfied after Step 6. ◀

▶ **Proposition 20.** *After Step 7, all 0-pentagons that are the wedge-neighbor of five 1-triangles are satisfied.*

Proof. We continue to use the notation introduced in the proof of Proposition 19. Let f be a 0-pentagon with five wedge-neighbors that are 1-triangles. We distinguish the number of 0-neighbors of f that are 0-quadrilaterals. Note that at most two such faces can exist next to a 0-pentagon.

- *Case 1: No 0-neighbor of f is a 0-quadrilateral.* Then all five vertex-neighbors are 2-triangles (see Figure 9a) and we have $\text{ch}'(f) = \text{ch}_4(f) = 1 - 0.5 \cdot 2 = 0$.
- *Case 2: Exactly one 0-neighbor of f is a 0-quadrilateral.* Assume w.l.o.g. that this 0-quadrilateral lies at e_0. We consider the wedge-neighbors t_2 and t_3 of f (see Figure 9b). Observe that $\text{ch}_3(t_2) = \text{ch}_3(t_3) = -0.2$, and therefore $\text{ch}_4(f) \geq 1 - 3 \cdot 0.3 - 2 \cdot 0.2 = -0.3$. Note further that f_0 and f_4 cannot be 0-faces, and therefore, by Proposition 18, f receives the missing charge in Step 6 if $|f_0|$ or $|f_4|$ is at least five. The same holds if one of f_0

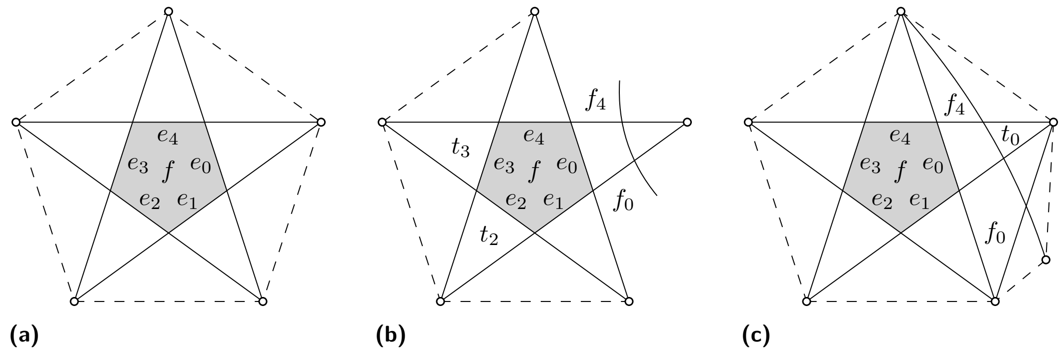


Figure 9 Illustrations for the proof of Proposition 20 Cases 1 and 2.

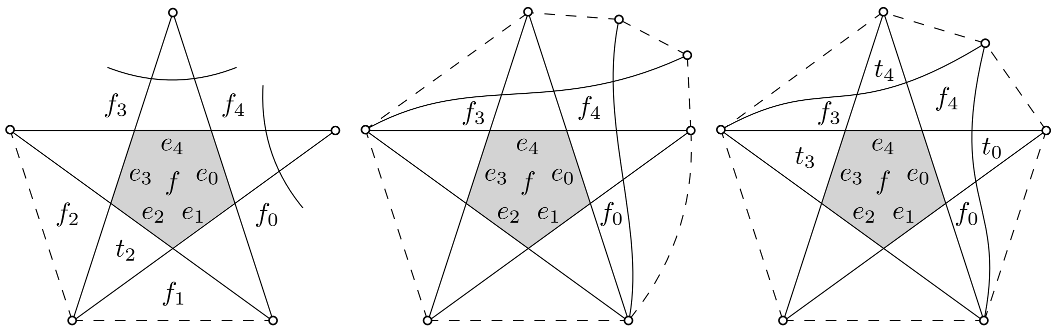
and f_4 is a 2-quadrilateral or both are 1-quadrilaterals, because in this case there is an excess of at least 0.3 charge after Step 5, which is only contributed to f (their other vertex-neighbor is t_0, which does not receive charge in Step 5, see Figure 9b).

Further f_1 and f_2 cannot be 2-triangles or 3-triangles. If both are 1-triangles, then there would be homotopic multi-edges, which is not allowed. So the last case to consider is when one of them – w.l.o.g. f_4 – is a 1-triangle and the other – therefore f_0 – is a 1-quadrilateral. If f_4 is the only wedge-neighbor, to which f_0 contributes in Step 1-5, then it contributes its excess of 0.3 charge to f in Step 6 and f is satisfied. Otherwise, the second wedge-neighbor of f_0 is also a 1-triangle and we have a planar cycle of length six (see Figure 9c). Here, f_0 contributes 0.1 charge to f in Step 6 and the 1-neighbor of f_0 that is a 2-triangle can contribute its excess of 0.2 to f in Step 7. Therefore, we have $\mathrm{ch}'(f) \geq 0$.

- *Case 3: Exactly two 0-neighbors of f are 0-quadrilaterals.* W.l.o.g. one 0-quadrilateral is at e_0. If the other 0-quadrilateral would be at e_2 (resp. e_3), then e_1 (resp. e_4) would have four crossings. Therefore, we can assume w.l.o.g. that the second 0-quadrilateral is at e_4. Here, we have $\mathrm{ch}_3(t_2) = -0.2$ as f_1 and f_2 are 2-triangles, thus $\mathrm{ch}_4(f) \geq 1 - 4 \cdot 0.3 - 0.2 = -0.4$ (see Figure 10a).

  We distinguish the type of the vertex-neighbor f_4. Note that $|f_4| \geq 4$ and f_4 cannot be a 2-quadrilateral. If f_4 is not a 0-quadrilateral, 1-quadrilateral, 0-pentagon or 0-hexagon, then, by Proposition 18, f_4 contributes 0.4 charge to f in Step 6, and therefore f is satisfied. The other cases are more complex, but they all have in common that if one of f_0 and f_3 is a 2-quadrilateral, then it has an excess of at least $2 - 2 \cdot 0.4 - 0.3 = 0.9$ charge after Step 5 and this is enough to ensure $\mathrm{ch}_6(f) \geq 0$.

  - *Case 3.1: f_4 is a 0-quadrilateral.* Then the only case to consider is that f_3 and f_4 are 1-triangles. This directly implies a planar cycle of length seven (see Figure 10b). Here, we make use of the second part of Step 6 and have two 2-quadrilaterals contributing 0.9 charge each to the 0-neighbors of f at e_0 and e_4, which then is moved to f. Therefore, f is satisfied after Step 6.
  - *Case 3.2: f_4 is a 1-quadrilateral.* Then t_0 and t_4 receive at least 0.2 charge in Step 3, and therefore we have $\mathrm{ch}_4(f) \geq 1 - 2 \cdot 0.3 - 3 \cdot 0.2 = -0.2$. If now f_4 contributes to less than two 1-triangles in Step 4, f receives from f_4 enough charge in Step 6. Otherwise, f_0 and f_3 are 1-triangles, implying a planar cycle of length six (Figure 10c). Here, t_0 and t_4 have two 1-neighbors that are 2-triangles and $\mathrm{ch}_3(t_0) = \mathrm{ch}_3(t_4) = -0.1$ holds. Therefore, f contributes only $2 \cdot 0.1 + 0.2 + 2 \cdot 0.3 = 1$ charge and f never has a demand.



(a) **(b)** **(c)**

Figure 10 Illustrations for the proof of Proposition 20 Case 3, 3.1 and 3.2.

- *Case 3.3: f_4 is a 0-pentagon.* We introduce some new notation for f_4 and its wedge-neighbors, likewise for the 0-pentagon f itself: Let $\tilde{f} := f_4$, $\tilde{e}_0$ the edge-segment of $\overline{e}_4$ at $\tilde{f}$, $\tilde{e}_1$ the edge-segment of $\overline{e}_0$ at $\tilde{f}$ and so on (see Figure 11a). Analogously, we denote by $\tilde{t}_i$ the wedge-neighbor of $\tilde{f}$ at $\tilde{e}_i$ and by $\tilde{f}_i$ the vertex-neighbor at the crossing of $\tilde{e}_i$ and $\tilde{e}_{(i+1 \mod 5)}$. Note that $\tilde{t}_0 = f_0$ and $\tilde{t}_1 = f_3$ are 1-triangles or 2-quadrilaterals and, as pointed out above, we only have to consider the case that both are 1-triangles.

  Observe that f is the only vertex-neighbor of f_4 that may receive charge from f_4 in Step 6, as all its other vertex-neighbors cannot be 0-faces. Distinguish the number of 1-triangles that are wedge-neighbors of f_4. Note that the wedge-neighbors of f_4 can never be 0-triangles or 0-pentagons, so, by Proposition 17, they contribute 0.3 charge to f_4 if they are not 1-triangles. If three or less wedge-neighbors of f_4 are 1-triangles, then $\mathrm{ch}_5(f) \geq 1 - 3 \cdot 0.3 + 2 \cdot 0.3 \geq 0.7$, which then is contributed to f in Step 6 implying $\mathrm{ch}_6(f) \geq 0$. If all five wedge-neighbors of f_4 are 1-triangles, then we have the F_6^3 configuration, which is forbidden. So the case remains that four wedge-neighbors of f_4 are 1-triangles. Here, $\mathrm{ch}_5(f_4) \geq 1 - 4 \cdot 0.3 + 0.3 = 0.1$ holds and this charge is contributed to f in Step 6, so there is only 0.3 charge missing for f.

  By symmetry, $\tilde{t}_2$ is w.l.o.g. a 1-triangle. If $\tilde{t}_3$ is the wedge-neighbor of $\tilde{f}$ that is not a 1-triangle, then it must be 2-quadrilateral and this implies a planar cycle of length seven, in which f is the only face with a demand after Step 6 (see Figure 11b). The 2-quadrilateral $\tilde{t}_3$ has an excess of 0.9 charge after Step 6 and contributes it in Step 7 to f. Therefore, f is satisfied.

  So assume now that $\tilde{t}_3$ is a 1-triangle and $\tilde{t}_4$ is the wedge-neighbor that is not a 1-triangle (see Figure 11c). Note that $\tilde{t}_4$ is not a 0-face. So for all cases, except that $\tilde{t}_4$ is a 1-quadrilateral or 2-quadrilateral, Proposition 18 guarantees that $\tilde{t}_4$ contributes in Step 6 0.4 charge to all its vertex-neighbors. In particular, the 0-neighbor of f at e_0 receives 0.4 charge and gives it completely to f. Thus, in this case, f is satisfied.

  If $\tilde{t}_4$ is a 2-quadrilateral, then we have $\mathrm{ch}_5(t_4) = 0.9$ and it contributes in the same way enough charge to f via the 0-neighbor of f at e_0. This works also if $\tilde{t}_4$ is a 1-quadrilateral contributing to only one wedge-neighbor (namely $\tilde{f}$) in Step 1-5.

  In the last case where $\tilde{t}_4$ is a 1-quadrilateral and contributes to $\tilde{f}$ and another wedge-neighbor in Step 1-5, this second wedge-neighbor is $\tilde{f}_3$ and must be a 1-triangle. This implies a planar cycle of length seven (see Figure 11d). In this case, $\tilde{t}_4$ and its 1-neighbor that is a 2-triangle have an excess of 0.1 resp. 0.2 charge after Step 5 and contribute it to f in Step 6 and Step 7. Therefore, f is satisfied.

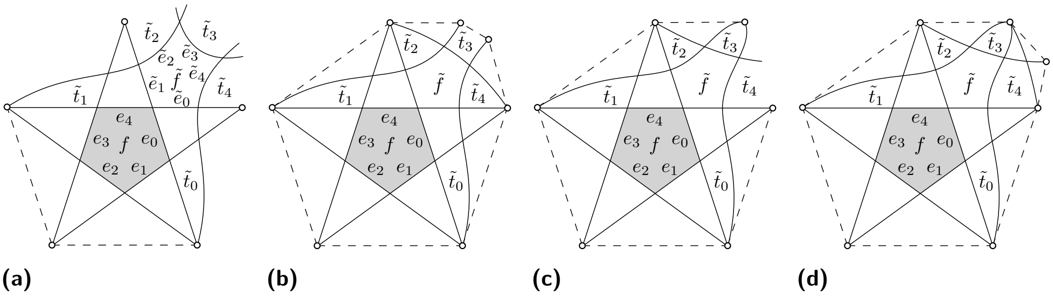


Figure 11 Illustrations for Case 3.3 in the proof of Proposition 20.

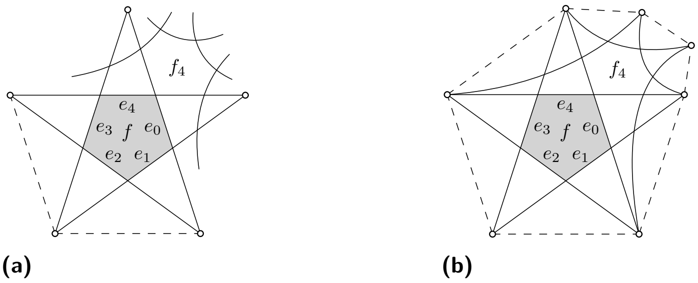


Figure 12 Illustrations for Case 3.4 in the proof of Proposition 20.

- *Case 3.4: f_4 is a 0-hexagon.* Note that no wedge-neighbor of f_4 can be a 0-face, so f_4 contributes no charge in Step 1-3 and Step 5 (see Figure 12a). If at most four wedge-neighbors of f_4 are 1-triangles, then $\text{ch}_5(f_4) \geq 2 - 4 \cdot 0.3 = 0.8$ holds by Proposition 17. In this case, there is at most one other vertex-neighbor of f_4 besides f that can be a 0-quadrilateral or 0-pentagon and f_4 can contribute to both 0.4 charge in Step 6. That is enough to satisfy f.
  If five wedge-neighbors of f_4 are 1-triangles, then no vertex-neighbor of f_4 except f is a 0-face. Therefore, f receives the excess of f_4 in Step 6, which is at least $2 - 5 \cdot 0.3 = 0.5$. So again f is satisfied.
  Assume now that all six wedge-neighbors of f_4 are 1-triangles (see Figure 12b). Then two of them have a demand of only 0.2 after Step 3 as they have two 1-neighbors that are 2-triangles. Therefore, $\text{ch}_5(f_4) \geq 2 - 4 \cdot 0.3 - 2 \cdot 0.2 = 0.4$. Here, f is the only face to which f_4 contributes in Step 6 and we have $\text{ch}_6(f) \geq 0$. ◀

B Details for Section 5

▶ **Proposition 21.** *With the notation above, $c_{pent} + c_{hex} \geq \frac{2}{3}(n-2) - \frac{4}{3}m_3 - m_{3-} + m_0$.*

Proof. Insert the m_0 missing planar edges to D_2 at the boundaries of the forbidden configurations. Further add a vertex v and five edges in every F_6^3 configuration from D_3 as shown in Figure 13. More precisely, notice that in D_2 three edges have been deleted from each F_6^3 configuration. Those three edges form a path consisting of a 2-hop edge, a 3-hop edge and a second 2-hop edge. Only one 3-hop edge e still exists and it is crossing-free in D_2. We arbitrarily choose a side of e and place the new vertex v close to e at this side. We realize the five new edges by connecting v to the two vertices of the configuration that are on same side of e, further to the two endpoints of e, and to one of the two endpoints on the opposite side of e. We do not create new forbidden configurations by this operation, thus the number of F_5^2 and F_6^2 configurations in D_2 does not change.

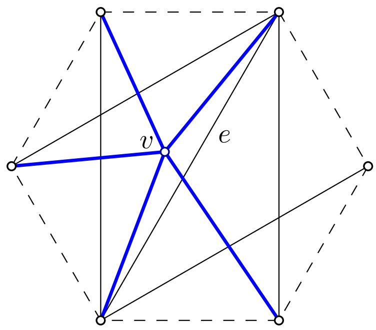


Figure 13 Illustration for the proof of Proposition 21. We augment each F_6^3 configuration after the deletion of the three blue edges in Figure 5 by one vertex and five edges for the drawing $\tilde{D}$.

As a next step, we remove one edge from each F_5^2 and F_6^2 configuration in D_2 and call this drawing $\tilde{D}$. Remark that $\tilde{D}$ is 2-planar, F_5^2-free, F_6^2-free and has $5(n-2)-2m_3-m_{3-}+m_0+5m_3-(c_{pent}+c_{hex})$ edges on $n+m_3$ vertices.

Assume we have fewer F_5^2 and F_6^2 configurations in D_2 than stated in the proposition. Then $\tilde{D}$ would have more than

$$5(n-2)-2m_3-m_{3-}+m_0+5m_3-\left[\frac{2}{3}(n-2)-\frac{4}{3}m_3-m_{3-}+m_0\right]=\frac{13}{3}(n-2+m_3)$$

edges, which contradicts the statement of Theorem 1 for $\tilde{D}$. ◀

▶ **Proposition 22.** *Let $\mathcal{H}' \subseteq \mathcal{H}$ be the set of triangles that do not belong to the forbidden configurations and let $c_\triangle = |\mathcal{H}'|$. Then $m_4 \leq m_3 + c_{hex} + 4m_0 + 4c_\triangle$.*

Proof. Our strategy is to account for every edge in E_4 an unique F_6^2 or an unique F_6^3 configuration (there are $m_3 + c_{hex}$ of them) (Case 1) or to account for four non-assigned E_4 edges either an edge that might be inserted in D_2 in a planar way (Case 2) or a triangle in $\mathcal{H}'$ (Cases 3 and 4). For the assignment, we use the fact that edges of E_4 do not cross each other and D_4 is 4-planar.

1. $e \in E_4$ *lies completely in one of the forbidden configurations.* This can only be the case in an F_6^2 or F_6^3 configuration as all five edges of an F_5^2 configuration still exist in D_2. In each F_6^2 or F_6^3 configuration all 2-hops exist in D_3. Therefore, e is a 3-hop and crosses the other 3-hops inside the hexagon, which therefore cannot be in E_4. So e is the only edge in E_4 inside the forbidden configuration and can be assigned to it.
2. $e \in E_4$ *starts in a forbidden configuration P and ends in another one, say P'.* Let uu' be the edge on the boundary of P that e crosses. We will assign the edge e to uu' and argue that $uu' \in E_0$. Let e_1, e_2 and e_1', e_2' resp. be the 2-hop edges of P and P' that enclose the edge uu' (see Figure 14a). Each of these four edges is crossed at least twice by edges belonging to the same forbidden configurations P or P'. Edge e crosses at least two of those four edges. And since those edges must not be crossed more than four times, there are at most four edges of E_4 that will be assigned to the same boundary edge uu'. Note also that $uu' \in E_0$, as otherwise e has at least five crossings (two each in the forbidden configurations and one with uu'). Therefore, at most four edges of E_4 will be assigned to $uu' \in E_0$.
3. $e \in E_4$ *is completely outside of any forbidden configuration.* By the properties of triangulation $\mathcal{H}$, e is an edge of a triangle of the triangulation $\mathcal{H}'$. We assign e to that triangle. By this, at most three such edges of E_4 belong to the same triangle in $\mathcal{H}'$.

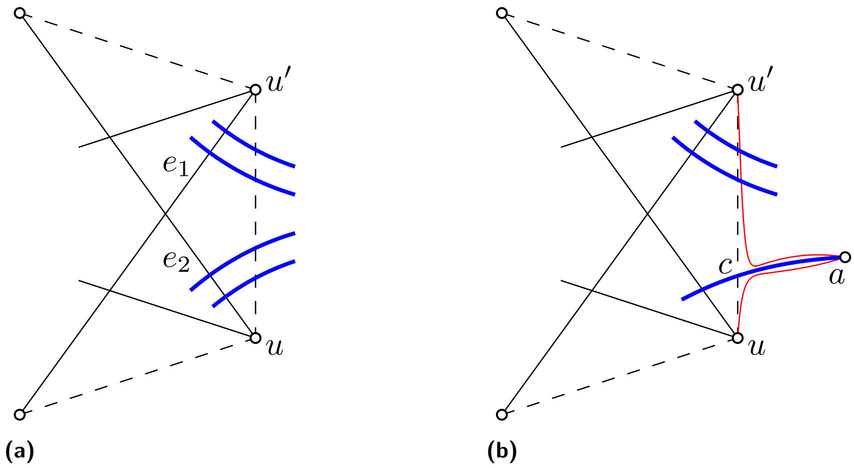


Figure 14 (a) At most four edges of E_4 (blue) can leave a forbidden configuration P through the same edge of its boundary, as otherwise one of the 2-hops e_1, e_2 of P has more than four crossings. (b) For an edge uu' on the boundary of a forbidden configuration P that is crossed by edges in E_4 (blue), the neighboring triangle in $\mathcal{H}'$ is defined by the edges ua and au' (red).

4. *$e \in E_4$ lies partially in the faces of $\mathcal{H}'$ and a forbidden configuration.* This is the remaining case. Let uu' be a boundary edge of the forbidden configuration P that is crossed by edge $e \in E_4$. To define the triangle t, which is adjacent to the boundary edge uu', we consider the crossing edge, say $e = ab$ that has the crossing c with uu' that is closest to u. Consider the two segments (a, c) and (c, b) of e, such that (a, c) is completely outside of the forbidden configuration P. We define the edges of the triangle t, which is adjacent to uu', to be the edge that closely follows the two segments (u, c) and (c, a); as the third edge of t, we take the edge that closely follows the two segments (a, c) and (c, u'), see Figure 14b. Note that, by the choice of the triangulation, Case 4 can only occur on one of the edges of the triangle t, here the edge uu' (an edge in E_4 that enters t through another edge and ends at u or u' would cross e, a contradiction to the fact that edges of E_4 do not cross each other). We distinguish two cases:
   - *uu' is crossed by at most two edges of E_4.* We assign those edges to the triangle t. In the extreme case, we might have two more edges from Case 3 being assigned to t. Thus, not more than four edges are assigned to t in total.
   - *uu' is crossed more often.* As in Case 2, we observe that there are at most four edges crossing uu'. We claim that neither the edge ua nor $u'a$ can be in E_4. For that, observe that uu' is crossed already at least three times (by assumption) and e at least two times (in the forbidden configuration P). Every edge crossing ua or ua' must cross either uu' or e and they cannot have four additional crossings in total since D_4 is 4-planar.

   This implies that it is sufficient to assign the edges in E_4 that cross uu' to the triangle t. ◀