Flips in Colorful Triangulations

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Abstract

The associahedron is the graph $\mathcal{G}_N$ that has as nodes all triangulations of a convex N-gon, and an edge between any two triangulations that differ in a flip operation. A *flip* removes an edge shared by two triangles and replaces it by the other diagonal of the resulting 4-gon. In this paper, we consider a large collection of induced subgraphs of $\mathcal{G}_N$ obtained by Ramsey-type colorability properties. Specifically, coloring the points of the N-gon red and blue alternatingly, we consider only *colorful* triangulations, namely triangulations in which every triangle has points in both colors, i.e., monochromatic triangles are forbidden. The resulting induced subgraph of $\mathcal{G}_N$ on colorful triangulations is denoted by $\mathcal{F}_N$. We prove that $\mathcal{F}_N$ has a Hamilton cycle for all $N \geq 8$, resolving a problem raised by Sagan, i.e., all colorful triangulations on N points can be listed so that any two cyclically consecutive triangulations differ in a flip. In fact, we prove that for an arbitrary fixed coloring pattern of the N points with at least 10 changes of color, the resulting subgraph of $\mathcal{G}_N$ on colorful triangulations (for that coloring pattern) admits a Hamilton cycle. We also provide an efficient algorithm for computing a Hamilton path in $\mathcal{F}_N$ that runs in time $\mathcal{O}(1)$ on average per generated node. This algorithm is based on a new and algorithmic construction of a tree rotation Gray code for listing all n-vertex k-ary trees that runs in time $\mathcal{O}(k)$ on average per generated tree.

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1 Introduction

The *associahedron* is a polytope of fundamental interest and importance [6, 14, 17], as it lies at the heart of many recent developments in algebraic combinatorics and discrete geometry; see [21] and the references therein. In this paper we are specifically interested in its combinatorial structure, namely the graph of its skeleton; see Figure 1. This graph, which we denote by $\mathcal{G}_N$, has as nodes all triangulations of a convex N-gon ($N \geq 3$), and an edge between any two triangulations that differ in a *flip* operation, which consists of removing an edge shared by two triangles and replacing it by the other diagonal of the resulting 4-gon. The graph $\mathcal{G}_N$ is isomorphic to the graph that has as nodes all binary trees with $N-2$ vertices, and an edge between any two trees that differ in a tree rotation. Each binary tree arises as the geometric dual of a triangulation, with the root given by "looking through" a fixed outer edge, and flips translate to tree rotations under this bijection; see Figure 2.

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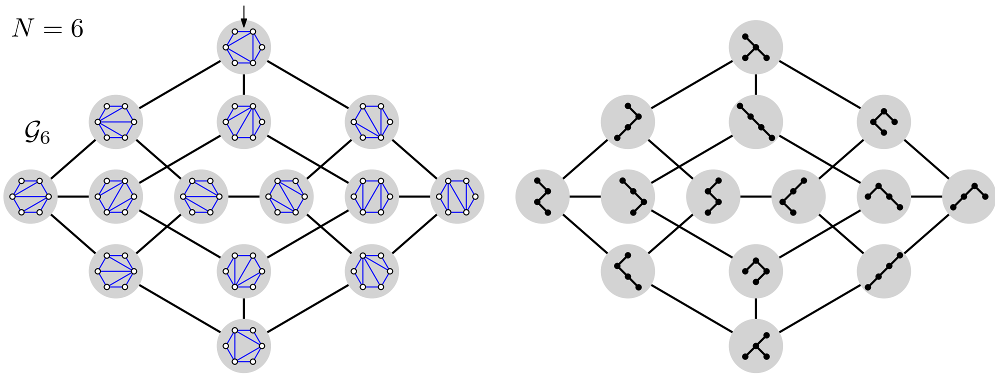

Figure 1 The graph of the 3-dimensional associahedron. The top edge of each triangulation is the outer edge that determines the root of the corresponding binary tree (see little arrow).

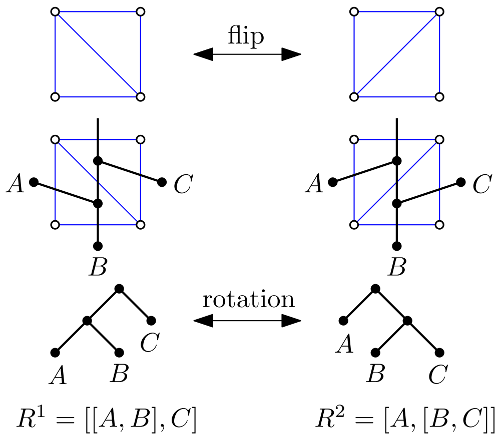

Figure 2 Correspondence between flips in triangulations (top) and rotations in binary trees (bottom).

Properties of the graph $\mathcal{G}_N$ have been the subject of extensive investigations in the literature. Most prominently, the diameter of $\mathcal{G}_N$ was shown to be $2N - 10$ for all $N > 12$ [22, 24]. Furthermore, the graph $\mathcal{G}_N$ is regular with degree $N - 3$, and this number is also its connectivity [16]. The chromatic number of $\mathcal{G}_N$ is at most $\mathcal{O}(\log N)$ [3, 10], while the best known lower bound is only 4.

Another fundamental graph property that we focus on in this paper is Hamiltonicity. To this end, Lucas [18] first proved that $\mathcal{G}_N$ admits a Hamilton cycle for $N \geq 5$, and a short proof was given by Hurtado and Noy [16]. A Hamilton path in $\mathcal{G}_N$ can be computed efficiently and yields a Gray code ordering of all binary trees by rotations [19]. This algorithm is a special case of the more general Hartung-Hoang-Mütze-Williams permutation language framework [12, 13, 20, 5].

In this paper, we consider a large collection of induced subgraphs of $\mathcal{G}_N$ obtained by Ramsey-type colorability properties. This line of inquiry was initiated by Sagan [23], following a sequence of problems posed by Propp on a mailing list in 2003. Specifically, we label the points of the convex N-gon by $1, \ldots, N$ in counterclockwise order, and we color them red (`r`) and blue (`b`) alternatingly. It follows that point i is colored red if i is odd and blue if i is even. For even N, any two neighboring points have opposite colors, whereas for odd N this property is violated for the first and last point, which are both red.

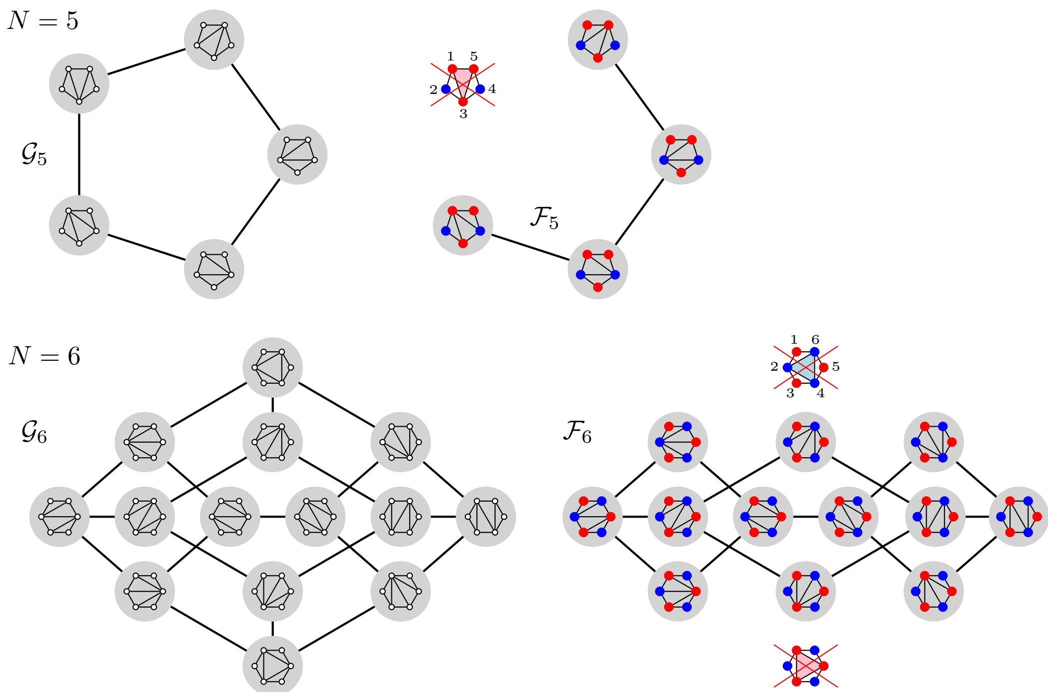

Figure 3 Induced subgraphs of the associahedron $\mathcal{G}_5$ (top left) and $\mathcal{G}_6$ (bottom left) obtained for the coloring sequence rbrb$\cdots$ by forbidding monochromatic triangles. The triangulations with monochromatic triangles are still shown, but they are not part of the graphs $\mathcal{F}_5$ and $\mathcal{F}_6$ (top right and bottom right, respectively) and hence crossed out.

We say that a triangulation is *colorful* if every triangle has points of both colors, i.e., no triangles in which all three points have the same color. We write $\mathcal{F}_N$ for the subgraph of $\mathcal{G}_N$ induced by all colorful triangulations. In other words, $\mathcal{F}_N$ is obtained from $\mathcal{G}_N$ by deleting all triangulations that have a monochromatic triangle; see Figure 3.

1.1 Sagan's problem and its generalization

Sagan [23] proved that $\mathcal{F}_N$ is a connected graph, and he asked [personal communication] whether $\mathcal{F}_N$ admits a Hamilton path or cycle. Looking at the first two interesting instances $N = 5$ and $N = 6$ in Figure 3, we note that $\mathcal{F}_5$ has a Hamilton path, but no cycle, and $\mathcal{F}_6$ has no Hamilton path and hence no cycle either. Furthermore, $\mathcal{F}_7$ admits a Hamilton path (see Figure 12), but no Hamilton cycle, which seems rather curious (cf. Theorem 5 below). We prove the following result.

▶ **Theorem 1.** *For any $N \geq 8$, the graph $\mathcal{F}_N$ has a Hamilton cycle.*

The resolution of Sagan's question immediately gives rise to the following more general problem: We consider an arbitrary sequence α of coloring the points $1, \ldots, N$ red or blue, and let $\mathcal{F}_\alpha$ be the corresponding induced subgraph of $\mathcal{G}_N$ obtained by forbidding monochromatic triangles. For which sequences α does $\mathcal{F}_\alpha$ admit a Hamilton path or cycle?

Formally, a *coloring sequence* is a sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of even length $\ell \geq 2$ with $\alpha_i \geq 1$ for $i = 1, \ldots, \ell$, and it encodes the coloring pattern

$$\mathtt{r}^{\alpha_1}\mathtt{b}^{\alpha_2}\mathtt{r}^{\alpha_3}\mathtt{b}^{\alpha_4}\cdots\mathtt{r}^{\alpha_{\ell-1}}\mathtt{b}^{\alpha_\ell} \tag{1}$$

for the points $1, \dots, N$, where $N = \sum_{i=1}^{\ell} \alpha_i$, and $\mathtt{r}^{\alpha_i}$ and $\mathtt{b}^{\alpha_j}$ denote α_i-fold and α_j-fold repetition of red and blue, respectively. In words, the first α_1 many points are colored red, the next α_2 many points are colored blue, the next α_3 many points are colored red etc. Clearly, the special cases considered by Sagan are $\alpha_1 = \alpha_2 = \cdots = \alpha_\ell = 1$ for even $N = \ell$, or $\alpha_1 = 2$ and $\alpha_2 = \cdots = \alpha_\ell = 1$ for odd $N = \ell + 1$, respectively (in the second case, the two consecutive points of the same color are 1 and 2 instead of 1 and N as before, but this is only a cyclic shift of indices). We let $\mathcal{F}_\alpha$ be the induced subgraph of $\mathcal{G}_N$ induced by the colorful triangulations with coloring sequence α.

We provide the following generalization of Theorem 1 before. Specifically, our next theorem applies to all coloring patterns with at least 10 changes of colors.

▶ **Theorem 2.** *For any coloring sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of (even) length $\ell \geq 10$, the graph $\mathcal{F}_\alpha$ has a Hamilton cycle.*

Note that there are 2^{N-2} different coloring sequences satisfying the conditions of the theorem, i.e., there are exponentially many subgraphs of the associahedron to which Theorem 2 applies. This also shows that the associahedron has cycles of many different lengths.

In view of the last theorem, it remains to consider short coloring sequences, i.e., sequences of length $\ell \leq 8$. We offer three simple observations in this regime. We first consider the easiest case $\ell = 2$, i.e., the coloring sequence has the form $\alpha = (a, b)$. The resulting graph $\mathcal{F}_\alpha$ for $\alpha = (4, 4)$ is shown in Figure 4. Another way to think about such a triangulation is as a triangulation of the so-called double-chain, where each triangle has to touch both chains. We observe that the number of colorful triangulations in this case is $\binom{N-2}{a-1} = \binom{N-2}{b-1}$ where $N := a + b$. Moreover, these triangulations are in bijection with bitstrings of length $N - 2$ with $a - 1$ many 0s and $b - 1$ many 1s, so-called *$(a-1, b-1)$-combinations*. This bijection is defined as follows; see Figure 4: Given a triangulation, we consider a ray separating the red from the blue points, and we record the types of triangles intersected by this ray one after the other, specifically we record a 1-bit or 0-bit if the majority color of the three triangle points is red and blue, respectively. We see that flips in the triangulations correspond to adjacent transpositions in the corresponding bitstrings. In the following, we use the generic

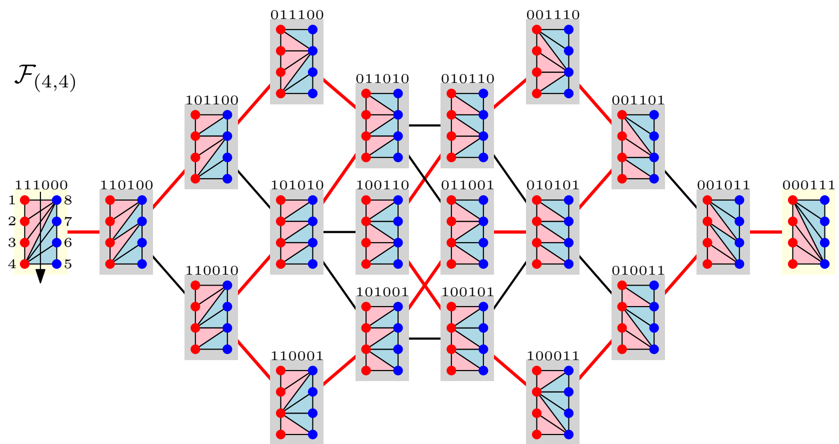

Figure 4 Flip graph of colorful triangulations for the coloring sequence $\alpha = (4, 4)$ ($\mathtt{rrrrbbbb}$), which is isomorphic to the flip graph of (3, 3)-combinations under adjacent transpositions. The black arrow in the leftmost triangulation is the ray that separates red from blue points, and the combination is obtained by reading the triangle types that intersect this ray from top to bottom (red=1, blue=0). The nodes of degree 1 and a Hamilton path in the flip graph are highlighted.

term *flip graph* for any graph that has as nodes a set of combinatorial objects, and an edge between any two objects that differ in a certain change operation. From what we said before, it follows that $\mathcal{F}_{(a,b)}$ is isomorphic to the flip graph of $(a-1,b-1)$-combinations under adjacent transpositions. Applying known results from [4, 9] thus yields the following theorem.

▶ **Theorem 3.** *For integers $a,b \geq 1$ with $a+b \geq 3$, the graph $\mathcal{F}_{(a,b)}$ is isomorphic to the flip graph of $(a-1,b-1)$-combinations under adjacent transpositions. Consequently, $\mathcal{F}_{(a,b)}$ has a Hamilton path if and only if $a \in \{1,2\}$, or $b \in \{1,2\}$, or a and b are both even. Furthermore, if $a,b \geq 2$, then $\mathcal{F}_{(a,b)}$ has no Hamilton cycle.*

The reason for the non-existence of a Hamilton cycle is that $\mathcal{F}_{(a,b)}$ has two nodes of degree 1, corresponding to the combinations $1^{a-1}0^{b-1}$ and $0^{b-1}1^{a-1}$; see Figure 4.

The next result is a simple observation for the special case of coloring sequences of length $\ell = 4$ with exactly two non-consecutive blue points; see Figure 5.

▶ **Theorem 4.** *For integers $a,b \geq 1$, the graph $\mathcal{F}_{(a,1,b,1)}$ is isomorphic to an $a \times b$ rectangular grid with one pending edge attached to each node. Consequently, it does not have a Hamilton path unless $a \cdot b \leq 2$.*

The nodes of degree 1 are the triangulations in which the two blue points are not connected by an edge, in which case the only possible flip restores this edge between them.

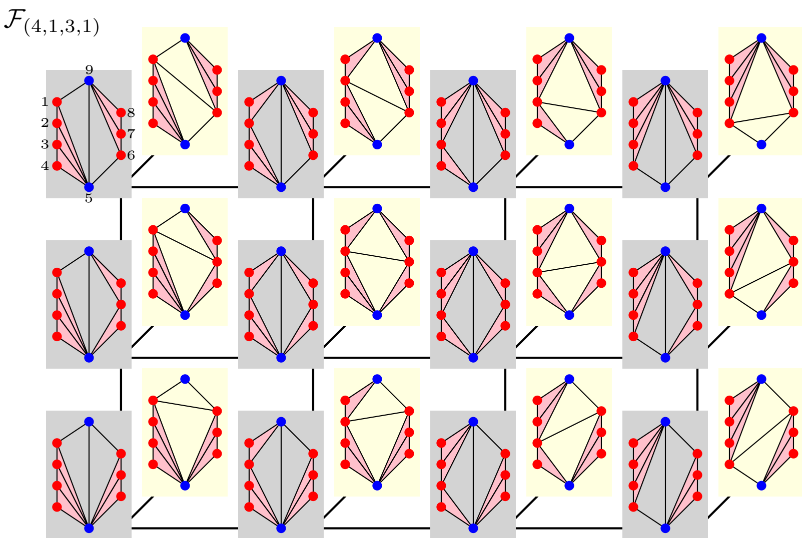

Figure 5 Illustration of Theorem 4 for the coloring sequence $\alpha = (4,1,3,1)$ (`rrrrbrrrb`). The nodes of degree 1 in the flip graph are highlighted.

The last result is for coloring sequences of length $\ell = 6$ and yields an infinite family of natural flip graphs that admit a Hamilton path but no Hamilton cycle, despite the fact that they have minimum degree 2; see Figure 6.

▶ **Theorem 5.** *For $\alpha = (a,1,1,1,1,1)$, the graph $\mathcal{F}_\alpha$ has no Hamilton cycle if $a \geq 1$, but a Hamilton path unless $a \in \{1,3\}$.*

1.2 Algorithmic questions and higher arity

We also provide an algorithmic version of Theorem 1.

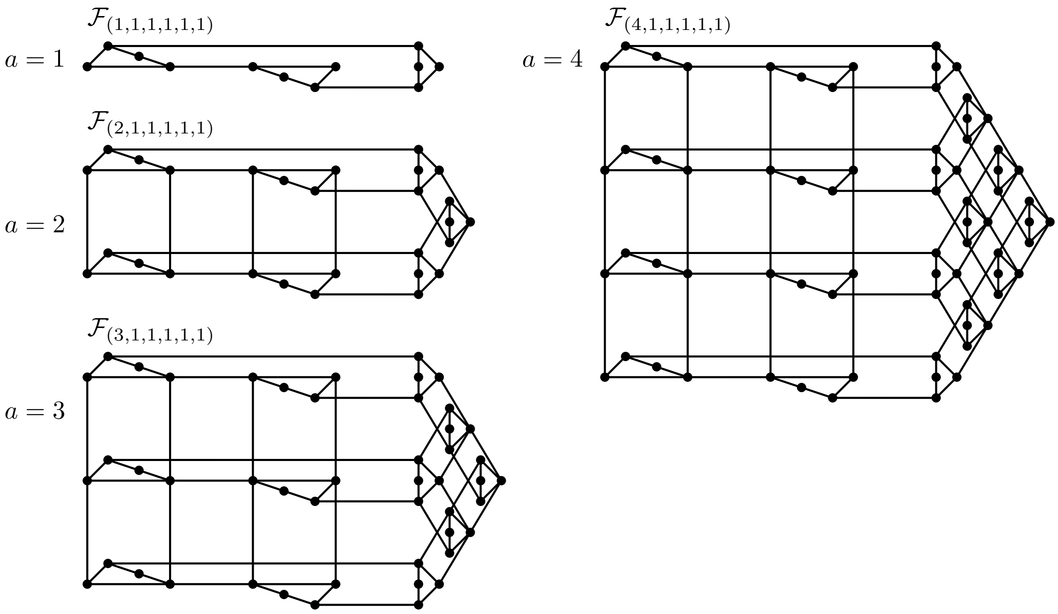

Figure 6 Illustration of the family of graphs $\mathcal{F}_{(a,1,1,1,1,1)}$.

▶ **Theorem 6.** *For any $N \geq 8$, a Hamilton path in the graph $\mathcal{F}_N$ can be computed in time $\mathcal{O}(1)$ on average per node.*

The initialization time and memory requirement for this algorithm are $\mathcal{O}(N)$.

Our construction of a Hamilton path/cycle in $\mathcal{F}_N$ relies on a Gray code ordering of ternary trees by rotations. We first describe this setup, generalizing our earlier definitions about triangulations and binary trees; see Figures 7 and 8 for illustration. Let $k \geq 2$ and $n \geq 1$ be integers, and let $N := (k-1)n + 2$. We consider a *dissection* of a convex N-gon into n many $(k+1)$-gons. A *flip* operation removes an edge shared by two $(k+1)$-gons and replaces it by one of the other $k-1$ possible diagonals of the resulting $2k$-gon. Dissections of an N-gon into $(k+1)$-gons are in bijection with k-ary trees with n vertices. Each k-ary trees arises as the geometric dual of a dissection into $(k+1)$-gons, with the root given by "looking through" the outer edge $1N$, and flips translate to tree rotations under this bijection.

We denote the corresponding flip graph of dissections of an N-gon into $(k+1)$-gons by $\mathcal{G}_{N,k+1}$. The associahedron is the special case $k = 2$, i.e., the graph $\mathcal{G}_{N,3} = \mathcal{G}_N$. By what we said before, the graph $\mathcal{G}_{N,k+1}$ is isomorphic to the rotation graph of k-ary trees with n vertices, where $N = (k-1)n + 2$. Huemer, Hurtado, and Pfeifle [15] first proved that $\mathcal{G}_{N,k+1}$ has a Hamilton cycle for all $k \geq 3$, which combined with the results of Hurtado and Noy [16] for the case $k = 2$ (binary trees) yields the following theorem.

▶ **Theorem 7** ([16] for $k = 2$; [15] for $k \geq 3$)**.** *For any $k \geq 2$, $n \geq \max\{2, 5-k\}$ and $N := (k-1)n + 2$, the graph $\mathcal{G}_{N,k+1}$ has a Hamilton cycle.*

The proof from [15] for the case $k \geq 3$ does not generalize the simple inductive construction of a Hamilton path/cycle in the associahedron (the case $k = 2$) described in [16], and it imposes substantial difficulties when translating it to an efficient algorithm. Consequently, we provide a unified and simplified proof for Theorem 7, valid for all $k \geq 2$, which can be turned into an efficient algorithm. This result generalizes the efficient algorithm for computing a Hamilton path in the associahedron provided by Lucas, Roelants van Baronaigien, and Ruskey [19].

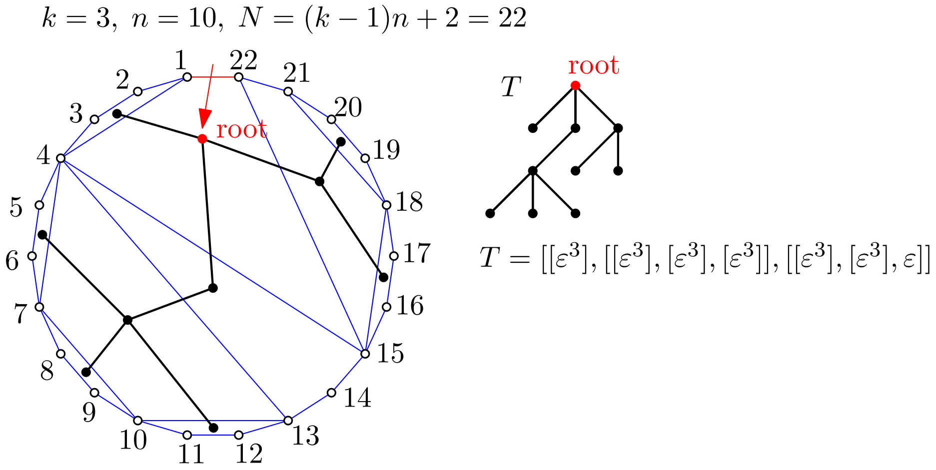

Figure 7 Bijection between dissections of an N-gon into $(k+1)$-gons and k-ary trees, illustrated for the case $k = 3$.

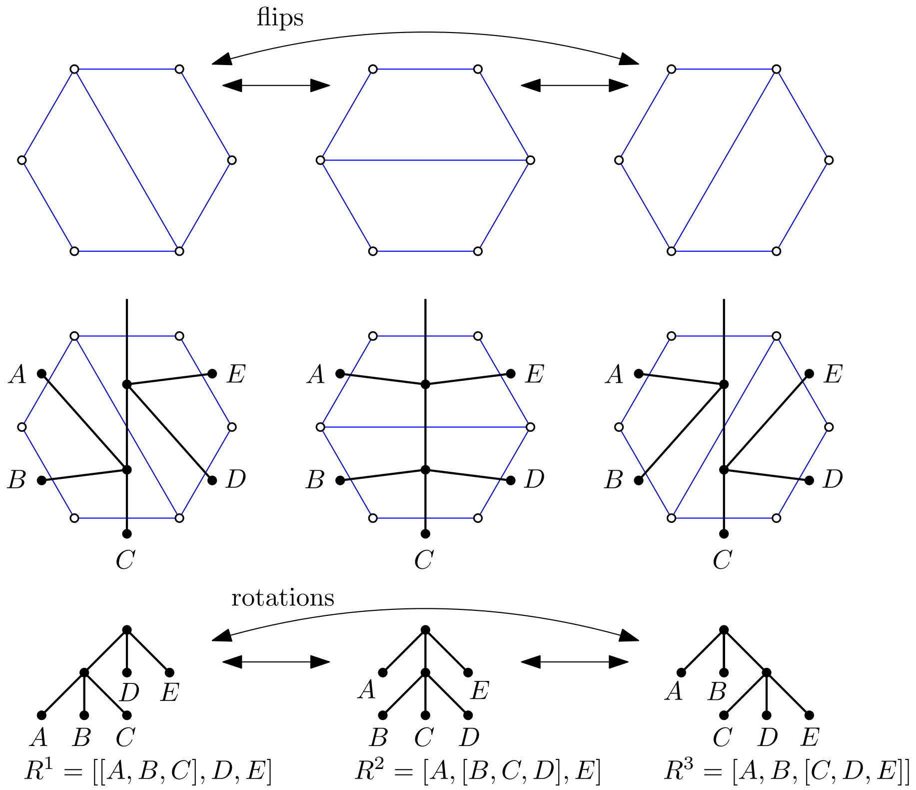

Figure 8 Correspondence between flips in quadrangulations (top) and rotations in ternary trees (bottom); cf. Figure 2.

▶ **Theorem 8.** *For any $k \geq 2$, $n \geq \max\{2, 5-k\}$ and $N := (k-1)n + 2$, a Hamilton path in $\mathcal{G}_{N,k+1}$ can be computed in time $\mathcal{O}(k)$ on average per node.*

The initialization time and memory requirement for this algorithm are $\mathcal{O}(kn)$.

We implemented the algorithms mentioned in Theorems 6 and 8 in C++, and made the code available for download and experimentation on the Combinatorial Object Server [7].

1.3 Three colors

We now consider colorings of the points $1, \ldots, N$ with more than two colors. To start with, we color the points in counterclockwise order alternatingly red (r), blue (b) and green (g), and we consider triangulations in which every triangle has points of all three colors, i.e., one point of each color. This setting has also been considered by Sagan [23]. Note that flips of a single diagonal as before are not valid operations anymore (in the sense that the flip graph would not have any edges), so we consider a modified flip operation instead which consists of a particular sequence of 4 flips. Specifically, a *twist* "rotates" a triangle that is surrounded by three triangles, i.e., the inner triangle is removed, creating an empty 6-gon, and the triangle is inserted the other way; see Figure 9. We write $\mathcal{H}_N$ for the flip graph of colorful triangulations under twists; see Figure 10.

▶ **Theorem 9.** *For any N that is a multiple of 3, the graph $\mathcal{H}_N$ is connected.*

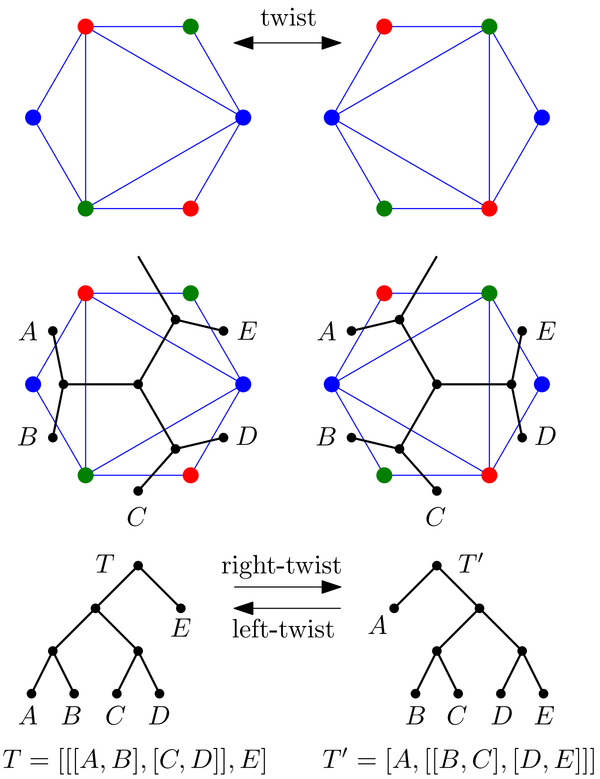

Figure 9 Twist operation and the corresponding binary trees.

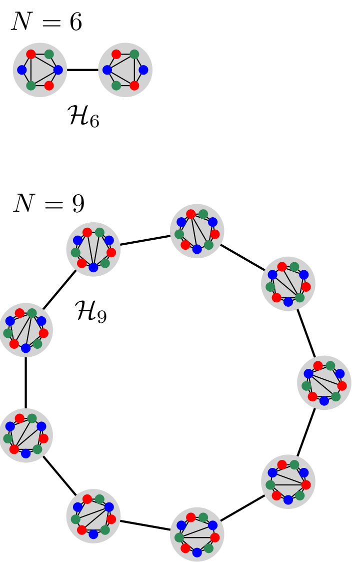

Figure 10 Fflip graphs $\mathcal{H}_6$ and $\mathcal{H}_9$.

1.4 Outline of this paper

In this extended abstract we focus on proving Theorems 1 and 2. Before providing the proofs in Section 3, we collect a few definitions and auxiliary results in Section 2. The proofs of all other results can be found in the preprint [1]. We conclude with some open questions in Section 4.

2 Preliminaries

2.1 String operations

For any string x and any integer $k \geq 0$, we write x^k for the k-fold concatenation of x. Given any sequence $x = (x_1, \ldots, x_\ell)$, we write $\mathrm{rev}(x) := (x_\ell, x_{\ell-1}, \ldots, x_1)$ for the reversed sequence.

2.2 Dissections and trees

For integers $k \geq 2$ and $n \geq 1$, let $N := (k-1)n + 2$. We write $\mathcal{D}_{N,k+1}$ for the set of all dissections of a convex N-gon into $(k+1)$-gons. In particular, $\mathcal{D}_{N,3}$ are triangulations of a convex N-gon. We write $\mathcal{T}_{n,k}$ for the set of all k-ary trees with n vertices, and $t_{n,k} := |\mathcal{T}_{n,k}|$. Both objects are counted by the *k-Catalan numbers* (OEIS sequence A062993), i.e., we have

$$|\mathcal{D}_{N,k+1}| = |\mathcal{T}_{n,k}| = t_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

We also define $t'_{n,3} := \sum_{i=0}^{n} t_{i,3} \cdot t_{n-i,3}$ as the number of pairs of ternary trees with n vertices in total (OEIS A006013). We have the explicit formula

$$t'_{n,3} = \frac{1}{n+1}\binom{3n+1}{n}.$$

2.3 Colorful triangulations

For any coloring sequence α, we write $\mathcal{C}_\alpha$ for the set of colorful triangulations with coloring pattern defined in (1). By these definitions, $\mathcal{F}_\alpha$ is the subgraph of $\mathcal{G}_N$ induced by the triangulations in $\mathcal{C}_\alpha$. Sagan's question concerned the special case $\alpha := 1^N$ for even N and $\alpha := (2, 1^{N-2})$ for odd N, and for those particular coloring sequences α we simply write $\mathcal{C}_N = \mathcal{C}_\alpha$ and $\mathcal{F}_N = \mathcal{F}_\alpha$. Sagan proved the following.

▶ **Theorem 10** ([23, Thm. 2.1])**.** *For any $q \geq 1$ we have*

$$|\mathcal{C}_N| = \begin{cases} 2^q \cdot t_{q,3} = \frac{2^q}{2q+1}\binom{3q}{q} & \text{if } N = 2q+2, \\ 2^q \cdot t'_{q,3} = \frac{2^q}{q+1}\binom{3q+1}{q} & \text{if } N = 2q+3. \end{cases}$$

The two sequences in this theorem are OEIS A153231 and A369510, respectively.

2.4 Graphs

For a graph G, we write $\Delta(G)$ for its maximum degree. Also, we write $G \simeq H$ for two graphs G and H that are isomorphic.

For any integer $d \geq 1$, the *d-dimensional hypercube Q_d* is the graph that has as vertices all bitstrings of length d, and an edge between any two strings that differ in a single bit.

▶ **Lemma 11** ([8])**.** *For any $d \geq 2$ and any set E of at most $2d-3$ edges in Q_d that together form vertex-disjoint paths, there is a Hamilton cycle that contains all edges of E.*

For integers $a \geq 1$ and $d \geq 1$ we define $S(a, d)$ as the set of all a-tuples of non-decreasing integers from the set $\{1, \ldots, d\}$, i.e., $S(a, d) = \{(j_1, \ldots, j_a) \mid 1 \leq j_1 \leq j_2 \leq \cdots \leq j_a \leq d\}$. Furthermore, we let $G(a, d)$ be the graph with vertex set $S(a, d)$ and edges between any two a-tuples that differ in a single entry by ± 1; see Figure 11.

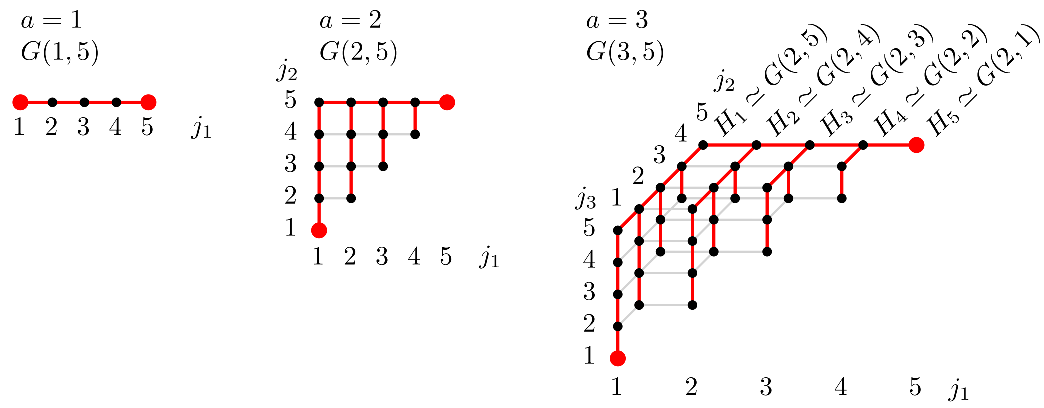

Figure 11 Illustration of the graph $G(a, d)$ and Lemma 12. The spanning trees and the extremal vertices are highlighted.

▶ **Lemma 12.** *For any $a \geq 1$ and $d \geq 1$, the graph $G(a, d)$ has a spanning tree T with $\Delta(T) \leq 3$.*

We refer to the vertices 1^a and d^a as *extremal* vertices, and note that they have degree 1 in $G(a, d)$, unless $a = d = 1$, in which case the graph is a single vertex having degree 0.

Proof. We argue by induction on a and d. For $a = 1$ and any $d \geq 1$, the graph $G(a, d)$ is the path on d vertices, so the claim is trivially true. For the induction step let $a \geq 2$. We split $G(a, d)$ into subgraphs H_i for $i = 1, \ldots, d$ where H_i contains all vertices in which the first coordinate equals i. Note that $H_i \simeq G(a-1, d-(i-1))$ for all $i = 1, \ldots, d$, in particular $H_d \simeq G(a-1, 1)$ is a single vertex. By induction, H_i has a spanning tree T_i with $\Delta(T_i) \leq 3$ for all $i = 1, \ldots, d$. Furthermore, the two extremal vertices have degree 1 in T_i for $i = 1, \ldots, d-1$ and degree 0 in T_d. We join the trees T_i to a single spanning tree T of $G(a, d)$ by adding the edges $\big((i, d^{a-1}), (i+1, d^{a-1})\big)$ for $i = 1, \ldots, d-1$ between their extremal vertices. ◀

3 Colorful triangulations

In this section we consider the setting of colorful triangulations introduced by Sagan, with the goal of proving Theorems 1 and 2. ri

3.1 Alternating colors

We first assume that the number N of points is even and the coloring sequence is $\alpha = 1^N$, i.e., the coloring pattern along the points $1, \ldots, N$ is $\mathtt{rbrb} \cdots \mathtt{rb} = (\mathtt{rb})^{N/2}$. Recall that $\mathcal{C}_N$ denotes the set of all colorful triangulations with this coloring sequence.

Let $T \in \mathcal{C}_N$ be a colorful triangulation. We say that an edge of T is *monochromatic* if both endpoints have the same color, and we say that it is *colorful* if both endpoints have distinct colors. We observe the following:

(i) Every triangle of T has exactly one monochromatic edge.
(ii) Every monochromatic edge of T is an inner edge.

Consequently, if we remove from T all monochromatic edges, keeping only the colorful ones, then the resulting dissection $r(T)$ is a quadrangulation on the point set. Indeed, by (i) every triangle is destroyed, and by (i)+(ii) destroying a triangle creates a quadrangle. While

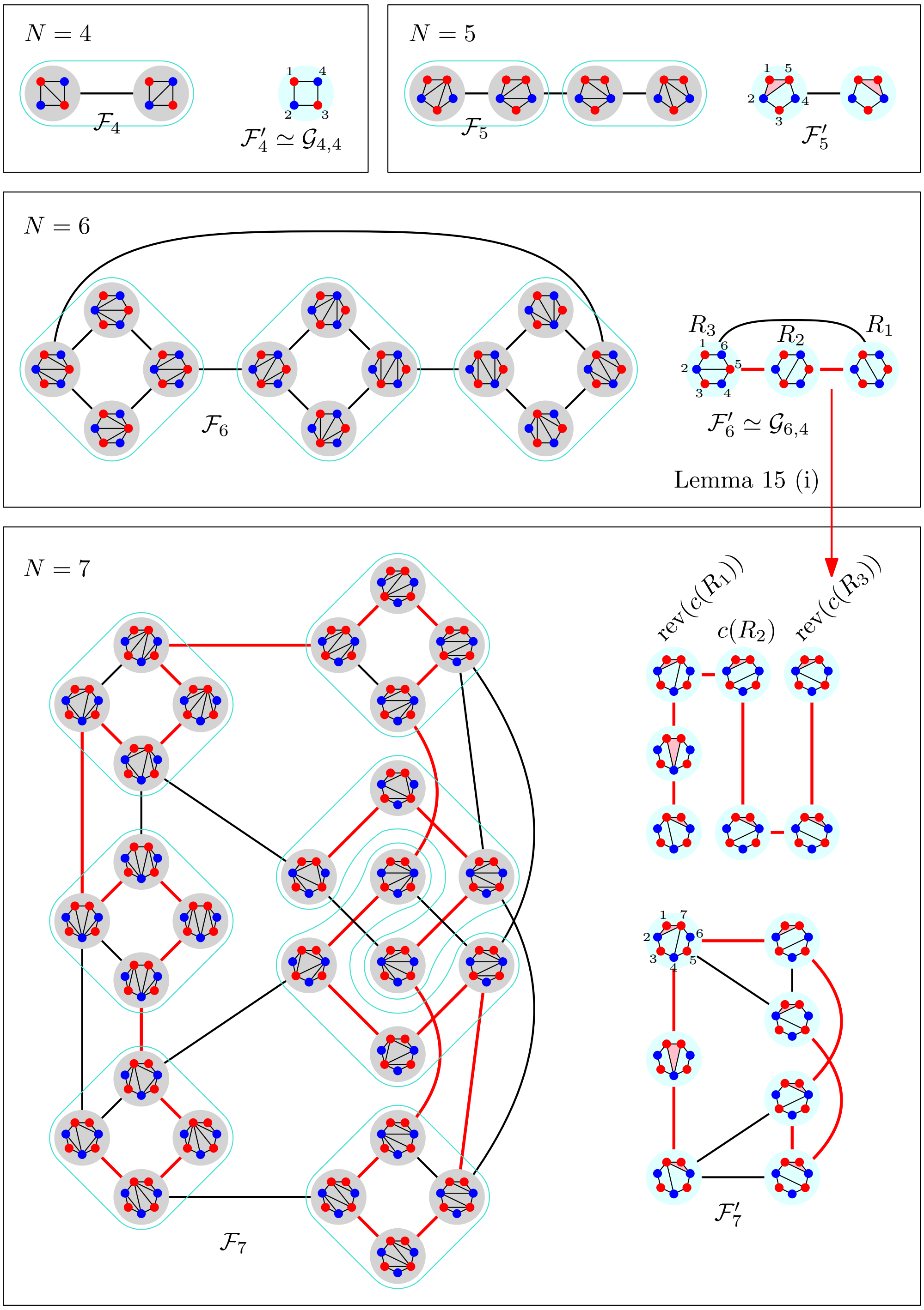

Figure 12 Flip graphs of colorful triangulations and reduced graphs for $N = 4, 5, 6, 7$.

T has $N - 2 = n$ triangles, $r(T)$ has $q := (N - 2)/2 = n/2$ quadrangles. Furthermore, there are 2^q many colorful triangulations that yield the same quadrangulation $r(T)$ by removing monochromatic edges. They are obtained from $r(T)$ by placing a diagonal in each of the q quadrangles in one of the two ways. Note that the subgraph of $\mathcal{F}_N$ induced by

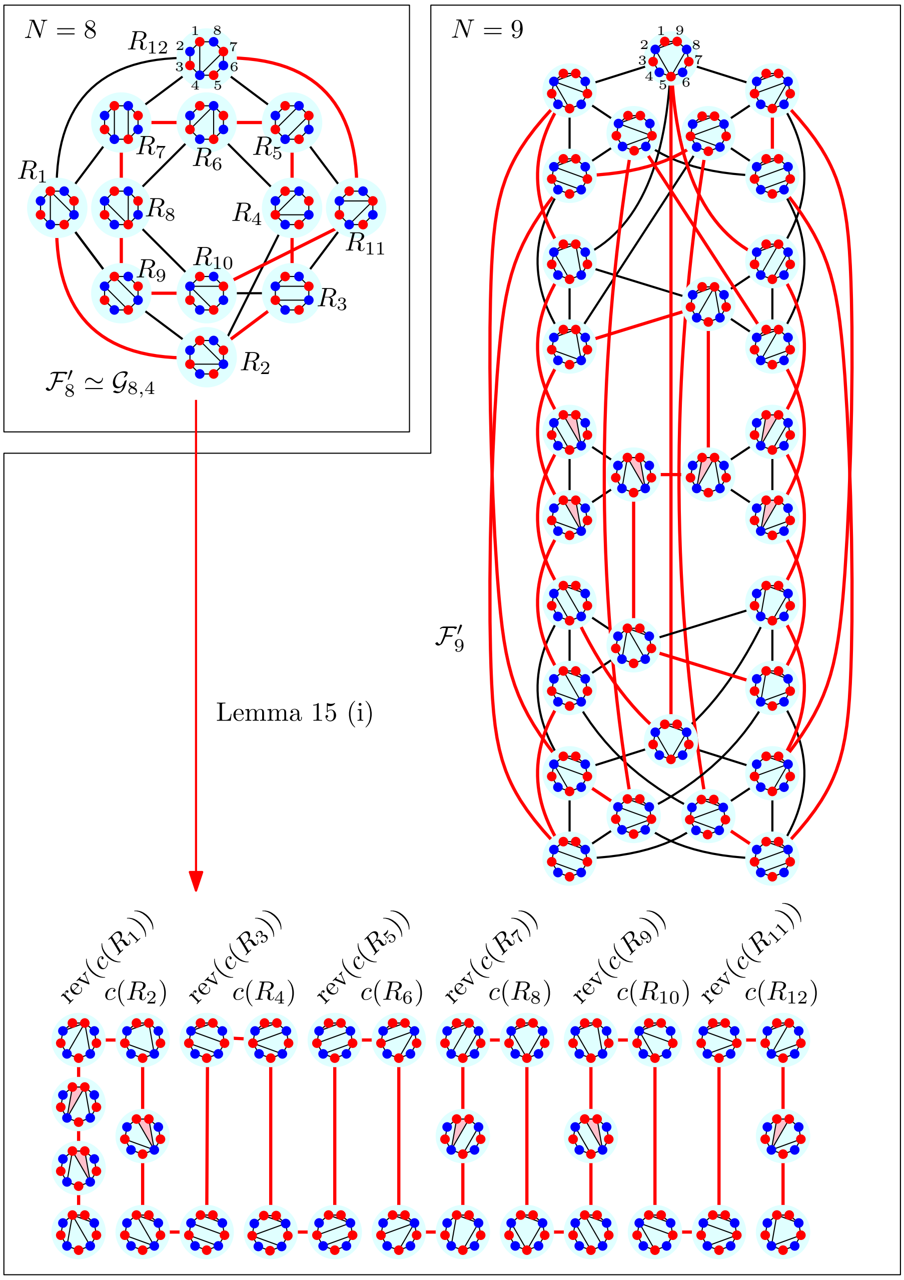

Figure 13 Reduced graphs of colorful triangulations for $N = 8, 9$.

those 2^q triangulations is isomorphic to the q-dimensional hypercube Q_q, as each of the q monochromatic edges in T can be flipped independently from the others. We thus obtain a partition of $\mathcal{F}_N$ into hypercubes Q_q, plus edges between them. These copies of hypercubes are highlighted by blue bubbles in Figure 12.

We also note that every quadrangulation R on N points equals $r(T)$ for some colorful triangulation $T \in \mathcal{C}_N$. Indeed, given R, then coloring the N points red and blue alternatingly will make all edges colorful. We define a reduced graph $\mathcal{F}'_N$, that has as nodes all quadrangulations on N points, and for any two colorful triangulations T and T' that differ in a flip of a colorful edge, we add an edge between $r(T)$ and $r(T')$ in $\mathcal{F}'_N$; see Figures 12 and 13. We observe that $\mathcal{F}'_N$ is isomorphic to the flip graph of quadrangulations $\mathcal{G}_{N,4}$, i.e., we have $\mathcal{F}'_N \simeq \mathcal{G}_{N,4}$. These arguments yield a direct combinatorial proof for the first equality in Theorem 10.

▶ **Lemma 13.** *Let $q \geq 3$ be an integer and $N := 2q+2$, and let $\mathcal{S}$ be a spanning tree of $\mathcal{F}'_N$.*
(i) *If $q = 3$ and $\Delta(\mathcal{S}) \leq 2$ (i.e., $\mathcal{S}$ is a Hamilton path), then $\mathcal{F}_N$ has a Hamilton cycle.*
(ii) *If $q \geq 4$ and $\Delta(\mathcal{S}) \leq 3$, then $\mathcal{F}_N$ has a Hamilton cycle.*

The conclusions of the lemma do not hold when $q = 2$: Indeed, while $\mathcal{F}'_6$ has a Hamilton path (the graph is a triangle), there is no Hamilton path or cycle in $\mathcal{F}_6$; see Figure 12.

Proof. The idea is to "uncompress" the spanning tree $\mathcal{S}$ in $\mathcal{F}'_N$ to a Hamilton cycle in $\mathcal{F}_N$. Specifically, for any quadrangulation R we consider the 2^q colorful triangulations $C(R) := \{T \in \mathcal{C}_N \mid r(T) = R\}$, and we let $Q(R)$ denote the subgraph of $\mathcal{F}_N$ spanned by the triangulations in $C(R)$. Recall that $Q(R) \simeq Q_q$, i.e., $Q(R)$ is isomorphic to the q-dimensional hypercube. In the first step of the uncompression, we replace each edge (R, R') of $\mathcal{S}$ by two edges (T_1, T'_1), (T_2, T'_2) with $T_1, T_2 \in C(R)$ and $T'_1, T'_2 \in C(R')$. We refer to the edges (T_1, T'_1) and (T_2, T'_2) as *connectors*, and to their end nodes T_1, T_2, T'_1, T'_2 as *terminals*. In the second step, each quadrangulation R with degree d in $\mathcal{S}$ is replaced by d paths that together visit all nodes in $Q(R)$ and which join the connectors at their terminals to a single Hamilton cycle.

We now describe both steps in detail. For a given quadrangulation R, we associate each of the colorful triangulations $T \in C(R)$ by a bitstring $b(T) \in \{0,1\}^q$ as follows: We label the q quadrangles of R arbitrarily by $j = 1, \ldots, q$, and we define $b(T)_j := 0$ if the monochromatic edge of T that sits inside the jth quadrangle of R connects the two red points, and otherwise (if it connects two blue points) $b(T)_j := 1$.

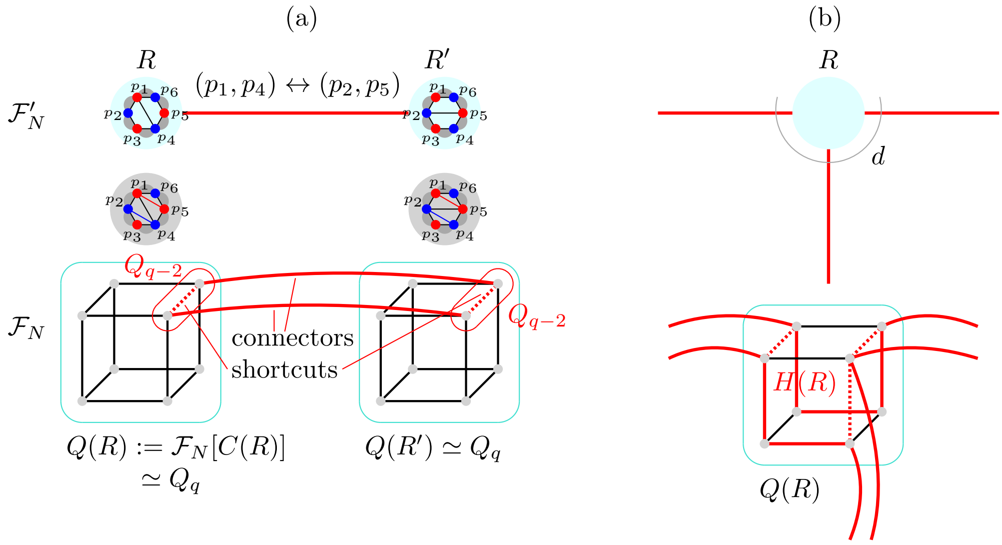

Figure 14 Illustration of the proof of Lemma 13.

Now consider an edge (R, R') of $\mathcal{S}$, which we aim to replace by two connectors (T_1, T'_1), (T_2, T'_2) with $T_1, T_2 \in C(R)$ and $T'_1, T'_2 \in C(R')$. We denote the edge flipped in R by (p_1, p_4), and we label the points of the adjacent 4-gons in circular order by p_1, p_2, p_3, p_4 and p_4, p_5, p_6, p_1, respectively, such that the edge (p_1, p_4) is replaced by (p_2, p_5); see Figure 14 (a). It follows that T_1, T'_1, T_2, T'_2 must be triangulations that contain the two monochromatic edges (p_2, p_4)

and (p_5, p_1) (but neither (p_1, p_3) nor (p_4, p_6)). Consequently, the two corresponding bits of $b(T_1), b(T_1'), b(T_2), b(T_2')$ must have a prescribed value, and therefore T_1, T_2 and T_1', T_2' can be chosen from a $(q-2)$-dimensional subcube of $Q(R)$ and $Q(R')$, respectively. Also, we will choose the connectors so that the pairs of terminals (T_1, T_2) and (T_1', T_2') differ only in a single flip, i.e., we select the two pairs of terminals as edges in their respective cubes, and we call these edges in $Q(R)$ and $Q(R')$ *shortcuts*. Note that the two connectors with the two shortcuts form the 4-cycle (T_1, T_1', T_2', T_2). By the assumption $q \geq 3$ we have $q - 2 \geq 1$, i.e., there is at least one choice for each prescribed edge.

If the node R has degree d in $\mathcal{S}$, then we have to choose d distinct shortcut edges in the hypercube $Q(R)$, each selected from a distinct (but not necessarily disjoint) $(q-2)$-dimensional subcube, and to find a Hamilton cycle $H(R)$ in $Q(R)$ that contains all of these edges; see Figure 14 (b). By Lemma 11, it is enough to ensure that the shortcut edges together form paths in $Q(R)$. If $d \leq 2$ (case (i) of the lemma), then this is clear, as one or two edges always form one or two paths. If $d = 3$ (case (ii) of the lemma), one has to avoid that all three shortcut edges are incident to the same node, which is easily possible under the stronger assumption $q \geq 4$.

Then the Hamilton cycle in $\mathcal{F}_N$ is obtained by taking the symmetric difference of the edge sets of the cycles $H(R) \subseteq Q(R)$ for all quadrangulations R on N points with the 4-cycles formed by the connectors and shortcuts (i.e., the shortcuts are removed, and the connectors are added instead). This completes the proof. ◀

3.2 General coloring patterns

We now consider an arbitrary coloring sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and the corresponding coloring pattern defined in (1). Recall that $\mathcal{C}_\alpha$ denotes the set of all colorful triangulations with this coloring pattern, and that the corresponding flip graph is denoted by $\mathcal{F}_\alpha$. The graph $\mathcal{F}_\alpha$ is an induced subgraph of the associahedron $\mathcal{G}_N$, where $N = \sum_{i=1}^{\ell} \alpha_i$. As in the previous section, a colorful triangulation $T \in \mathcal{C}_\alpha$ has two types of edges, namely monochromatic and colorful edges. We write E_α for the set of boundary edges that are monochromatic in T, i.e., these are the pairs of points $(i, i+1)$ for $i = 1, \ldots, N$ (modulo N) where both endpoints receive the same color. Generalizing the discussion from the previous section, we observe the following:

(i) Every triangle of T has exactly one monochromatic edge.
(ii) Except the edges in E_α, every monochromatic edge of T is an inner edge.

Consequently, if we remove from T all monochromatic inner edges (the edges in E_α are boundary edges and hence not removed), keeping only the colorful ones, then the resulting dissection $r(T)$ has $t := N - \ell$ triangles that contain the edges in E_α and $q := (\ell - 2)/2$ quadrangles. Furthermore, there are 2^q many colorful triangulations that yield the same dissection $r(T)$ by removing monochromatic edges. They are obtained from $r(T)$ by placing a diagonal in each of the q quadrangles in one of the two ways. Note that the subgraph of $\mathcal{F}_\alpha$ induced by those 2^q triangulations is isomorphic to the q-dimensional hypercube Q_q. We thus obtain a partition of $\mathcal{F}_\alpha$ into hypercubes Q_q, plus edges between them.

We refer to a dissection of a convex N-gon into q quadrangles and t triangles that contain all the edges of E_α as an *α-angulation*, and we write $\mathcal{D}_\alpha$ for the set of all such dissections. We also note that every α-angulation R on N points equals $r(T)$ for some colorful triangulation $T \in \mathcal{C}_\alpha$. Indeed, given R, then coloring the N points according to the pattern in (1) will make all edges except the ones in E_α colorful. We define a reduced graph $\mathcal{F}_\alpha'$ that has as nodes all α-angulations on N points, and for any two colorful triangulations T and T' that differ in a flip of a colorful edge, we add an edge between $r(T)$ and $r(T')$ in $\mathcal{F}_\alpha'$; see Figures 12, 13 and 15.

The proof of Lemma 13 presented in the previous section generalizes straightforwardly, yielding the following statement. Note that the variable ℓ in Lemma 14 below plays the role of $N = 2q + 2$ in Lemma 13, and so the assumptions $\ell = 8$ and $\ell \geq 10$ translate to $q = (\ell - 2)/2 = 3$ and $q \geq 4$ used in the proof of Lemma 13, respectively.

▶ **Lemma 14.** *Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a coloring sequence of (even) length $\ell \geq 8$, and let $\mathcal{S}$ be a spanning tree of $\mathcal{F}'_\alpha$.*

(i) *If $\ell = 8$ and $\Delta(\mathcal{S}) \leq 2$ (i.e., $\mathcal{S}$ is a Hamilton path), then $\mathcal{F}_\alpha$ has a Hamilton cycle.*

(ii) *If $\ell \geq 10$ and $\Delta(\mathcal{S}) \leq 3$, then $\mathcal{F}_\alpha$ has a Hamilton cycle.*

The next lemma allows us to duplicate the occurrence of a color that appears only once (i.e., we change $\alpha_i = 1$ to some larger number $\alpha_i > 1$), while inductively maintaining spanning trees with small degrees in the corresponding reduced flip graphs.

▶ **Lemma 15.** *Let $\beta = (\beta_1, \ldots, \beta_\ell)$ and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be coloring sequences of (even) length $\ell \geq 4$ that agree in all but the ith entry such that $\beta_i = 1$ and $\alpha_i > 1$.*

(i) *If $\mathcal{F}'_\beta$ has a Hamilton path and $\alpha_i = 2$, then $\mathcal{F}'_\alpha$ has a Hamilton path.*

(ii) *If $\mathcal{F}'_\beta$ has a spanning tree $\mathcal{T}$ with $\Delta(\mathcal{T}) \leq 3$, then $\mathcal{F}'_\alpha$ has a spanning tree $\mathcal{S}$ with $\Delta(\mathcal{S}) \leq 3$.*

In Figure 12, part (i) of this lemma is applied to construct a Hamilton path in $\mathcal{F}'_7$ from one in $\mathcal{F}'_6$. Similarly, in Figure 13, a Hamilton path in $\mathcal{F}'_9$ is constructed from one in $\mathcal{F}'_8$.

$\alpha = (\alpha_1, \ldots, \alpha_6) = (2, 1, 2, 2, 1, 1)$
$\ell = 6, N = 9$
$t = N - \ell = 3$
$q = (\ell - 2)/2 = 2$

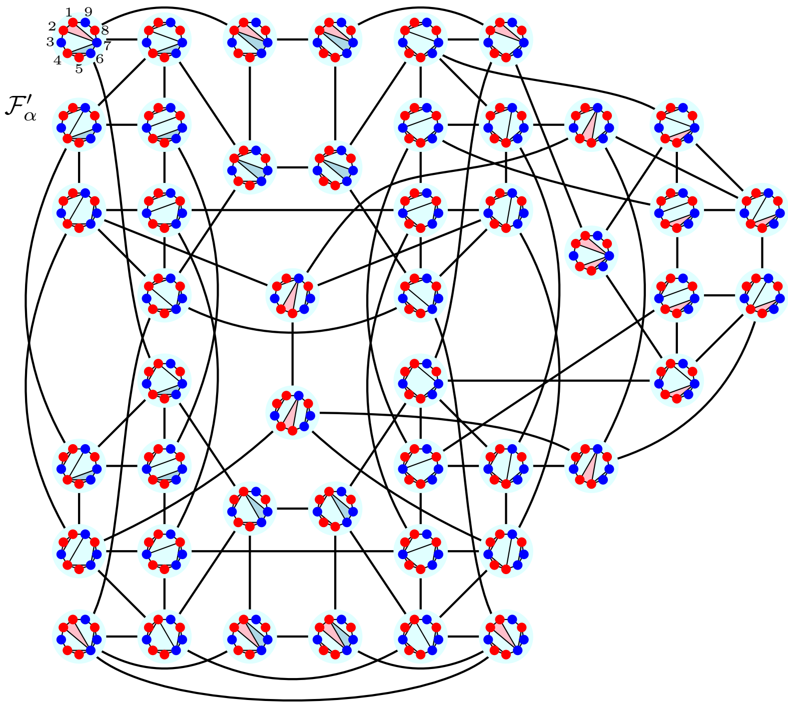

Figure 15 Reduced graph of colorful triangulations for the coloring sequence $\alpha = (2, 1, 2, 2, 1, 1)$ (rrbrrbbrb).

The idea for the proof of part (i) is the same as the one used by Hurtado and Noy [16].

Proof. We consider the point $p := \sum_{j=1}^{i} \beta_j$ on the boundary, which is neighbored by two points $p-1$ and $p+1$ (modulo $N = \sum_{j=1}^{\ell} \beta_j$) of the opposite color. We also define $a := \alpha_i - 1$, i.e., we want to add a points of the same color as p next to p. Let $R \in \mathcal{D}_\beta$ be a β-angulation, and let $(p,q_1),(p,q_2),\ldots,(p,q_d)$ be the edges incident with the point p in R in counterclockwise order (all these edges are colorful), such that $q_1 = p+1$ and $q_d = p-1$; see Figure 16. If $a=1$, then for $j=1,\ldots,d$ we let R^j be the α-angulation obtained from R by inflating the edge (p,q_j) to a triangle (p,p',q_j). Specifically, the single point p is split into two consecutive points p and p' on the boundary joined by an edge, and $q_1,\ldots,q_j$ remain connected to p', whereas $q_j,q_{j+1},\ldots,q_d$ remain connected to p. More generally, we define $J(R) := \{(j_1,\ldots,j_a) \mid 1 \leq j_1 \leq j_2 \leq \cdots \leq j_a \leq d\}$, $\check{\jmath}(R) := 1^a$ and $\hat{\jmath}(R) := d^a$, and for any $(j_1,j_2,\ldots,j_a) \in J(R)$ we let $R^{(j_1,\ldots,j_a)}$ be the β-angulation obtained from R by inflating each of the edges $(p,q_{j_1}),\ldots,(p,q_{j_a})$ to a triangle. Note that the same edge may be inflated multiple times; see the bottom rows with labels $a=2$ and $a=3$ in Figure 16. Specifically, if some value j_b, $b \in \{1,\ldots,a\}$, appears c times in the list $j_1,\ldots,j_a$, then the edge (p,q_{j_b}) is inflated to c many triangles. Furthermore, observe that $R^{(j_1,\ldots,j_a)}$ differs from $R^{(j'_1,\ldots,j'_a)}$ in a flip if and only if $(j_1,\ldots,j_a)$ and $(j'_1,\ldots,j'_a)$ differ in a single entry by ± 1, i.e., the subgraph of $\mathcal{F}'_\alpha$ induced by the α-angulations $R^{(j_1,\ldots,j_a)}$, $(j_1,\ldots,j_a) \in J(R)$, is isomorphic

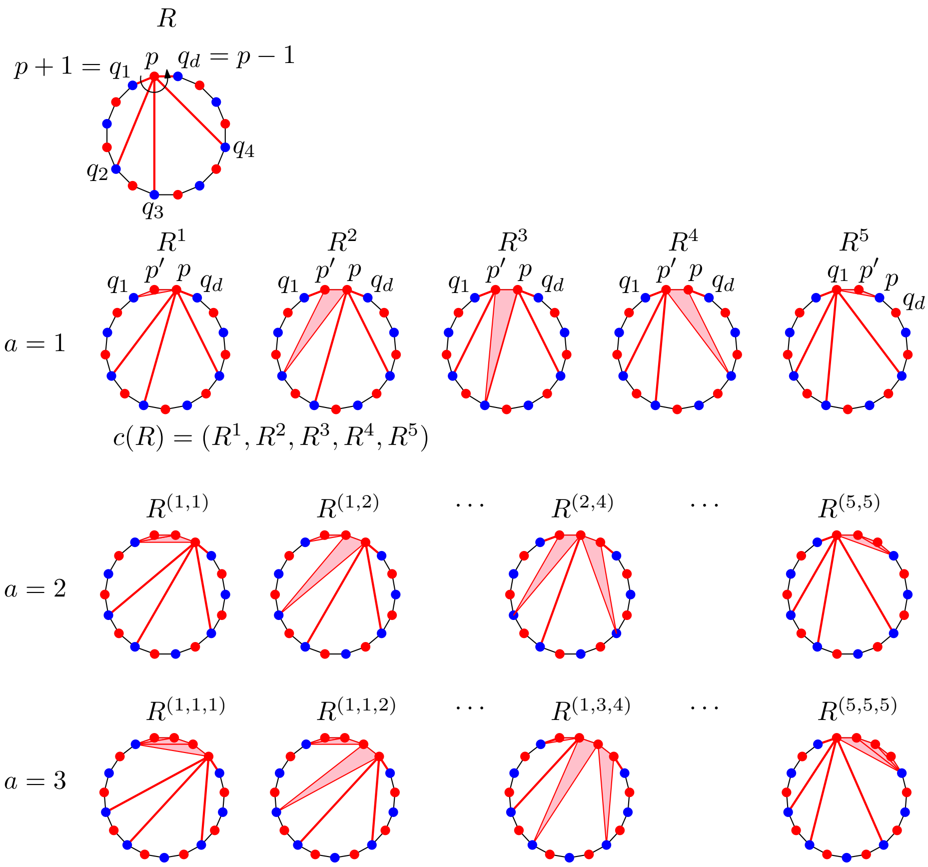

Figure 16 Illustration of the proof of Lemma 15. Edges of the β-angulation R that are not incident to the point p are not shown for clarity.

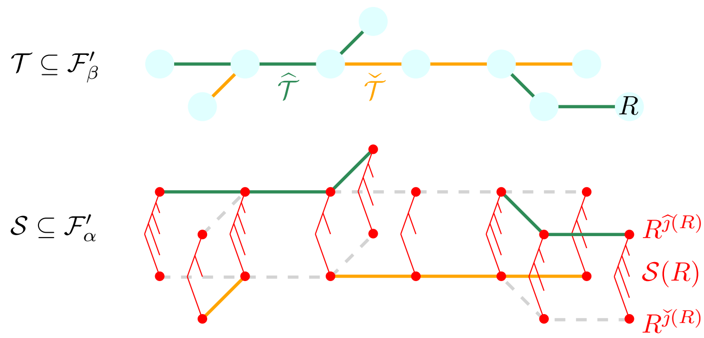

Figure 17 Illustration of the proof of part (ii) of Lemma 15. The fork-like structures are the spanning trees obtained from Lemma 12 (cf. Figure 11).

to the graph $G(a, d)$ defined in Section 2.4. By Lemma 12, it admits a spanning tree $\mathcal{S}(R)$ with $\Delta(\mathcal{S}(R)) \leq 3$, in which the nodes $R^{\check{\jmath}(R)}$ and $R^{\hat{\jmath}(R)}$ have degree 1. If $a = 1$, then this subgraph and spanning tree is simply a path, and we refer to it as children sequence $c(R) := (R^1, R^2, \ldots, R^d)$. Also note that if (R, Q) is an edge in $\mathcal{F}'_\beta$, then $(R^{\check{\jmath}(R)}, Q^{\check{\jmath}(Q)})$ and $(R^{\hat{\jmath}(R)}, Q^{\hat{\jmath}(Q)})$ are both edges in $\mathcal{F}'_\alpha$.

We now prove (i), using the assumption $\alpha_i = 2$, i.e., $a = 1$. Let $P = (R_1, \ldots, R_L)$ be a Hamilton path in $\mathcal{F}'_\beta$. Then a Hamilton path in $\mathcal{F}'_\alpha$ is given by $P' := (\mathrm{rev}(c(R_1)), c(R_2), \mathrm{rev}(c(R_3)), c(R_4), \ldots)$; see Figures 12 and 13.

For proving (ii), let $\mathcal{T}$ be a spanning tree in $\mathcal{F}'_\beta$ with $\Delta(\mathcal{T}) \leq 3$. We partition its edges into two disjoint forests of paths $\check{\mathcal{T}}$ and $\widehat{\mathcal{T}}$, i.e., we have $\Delta(\check{\mathcal{T}}) \leq 2$ and $\Delta(\widehat{\mathcal{T}}) \leq 2$; see Figure 17. We then define the spanning tree $\mathcal{S}$ as the union of the trees $\mathcal{S}(R)$ for all β-angulations R plus the edges $\{(R^{\check{\jmath}(R)}, Q^{\check{\jmath}(Q)}) \mid (R, Q) \in \check{\mathcal{T}}\}$ and $\{(R^{\hat{\jmath}(R)}, Q^{\hat{\jmath}(Q)}) \mid (R, Q) \in \widehat{\mathcal{T}}\}$. It is easy to check that $\mathcal{S}$ is indeed a spanning tree of $\mathcal{F}'_\alpha$ with $\Delta(\mathcal{S}) \leq 3$. ◀

3.3 Proofs of Theorems 1 and 2

Proof of Theorem 2. For the given coloring sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of length $\ell \geq 10$, we consider the alternating coloring sequence $\beta = 1^\ell$ of length ℓ, i.e., all repetitions of colors are reduced to a single occurrence, and β corresponds to coloring ℓ points alternatingly red and blue. The corresponding reduced flip graph $\mathcal{F}'_\ell$ is isomorphic to the rotation graph of ternary trees, i.e., we have $\mathcal{F}'_\beta = \mathcal{F}'_\ell \simeq \mathcal{G}_{\ell,4}$. Theorem 7 yields a Hamilton path in the graph $\mathcal{F}'_\beta = \mathcal{F}'_\ell$. Applying Lemma 15 (ii) once for each α_i with $\alpha_i > 1$, we obtain that $\mathcal{F}'_\alpha$ has a spanning tree $\mathcal{S}$ with $\Delta(\mathcal{S}) \leq 3$. Lastly, applying Lemma 14 (ii) yields that $\mathcal{F}_\alpha$ has a Hamilton cycle. ◀

Proof of Theorem 1. For $N \geq 10$ the result is a special case of Theorem 2, so it remains to cover the cases $N = 8$ and $N = 9$. A Hamilton path P in $\mathcal{F}'_8$ is guaranteed by Theorem 7; see Figure 13. Applying Lemma 13 (i) to P yields that $\mathcal{F}_8$ has a Hamilton cycle. Applying Lemma 15 (i) to P proves that $\mathcal{F}'_9$ has a Hamilton path P'. Applying Lemma 13 (i) to P' shows that $\mathcal{F}_9$ has a Hamilton cycle. ◀

4 Open questions

Tables 1 and 2 show all coloring sequences α on up to $N \leq 11$ points for which the graph $\mathcal{F}_\alpha$ has no Hamilton path or cycle. The sequences are shown up to rotational symmetry, reversal, and exchange of the two colors. In several cases, Theorems 3, 4 and 5 provide an explanation

Table 1 Coloring sequences α for $N \leq 11$ for which $\mathcal{F}_\alpha$ has no Hamilton cycle but a Hamilton path.

N	α	Reason
6	$(4,2)$	Thm. 3
	$(2,1,1,2)$	
7	$(5,2)$	Thm. 3
	$(2,1,1,1,1,1)$	Thm. 5
8	$(6,2)$	Thm. 3
	$(4,4)$	Thm. 3
9	$(7,2)$	Thm. 3
	$(4,1,1,1,1,1)$	Thm. 5
10	$(8,2)$	Thm. 3
	$(6,4)$	Thm. 3
	$(5,1,2,2)$	
	$(5,1,1,1,1,1)$	Thm. 5
	$(4,1,3,2)$	
	$(4,1,2,3)$	
11	$(9,2)$	Thm. 3
	$(6,1,1,1,1,1)$	Thm. 5
	$(5,1,2,3)$	
	$(5,1,1,4)$	

Table 2 Coloring sequences α for $N \leq 11$ for which $\mathcal{F}_\alpha$ has no Hamilton path.

N	α	Reason
6	$(3,1,1,1)$	Thm. 4
	$(3,3)$	Thm. 3
	$(1,1,1,1,1,1)$	Thm. 5
7	$(4,3)$	Thm. 3
	$(3,1,2,1)$	Thm. 4
	$(3,1,1,2)$	
8	$(5,1,1,1)$	Thm. 4
	$(5,3)$	Thm. 3
	$(4,1,2,1)$	Thm. 4
	$(4,1,1,2)$	
	$(3,1,3,1)$	Thm. 4
	$(3,1,1,1,1,1)$	Thm. 5
	$(3,1,1,3)$	
	$(3,2,1,2)$	
9	$(6,1,1,1)$	Thm. 4
	$(6,3)$	Thm. 3
	$(5,1,2,1)$	Thm. 4
	$(5,1,1,2)$	
	$(5,4)$	Thm. 3
	$(4,1,3,1)$	Thm. 4
	$(4,1,1,3)$	
	$(3,1,3,2)$	
	$(3,1,2,3)$	
10	$(7,1,1,1)$	Thm. 4
	$(7,3)$	Thm. 3
	$(6,1,2,1)$	Thm. 4
	$(6,1,1,2)$	
	$(5,1,3,1)$	Thm. 4
	$(5,1,1,3)$	
	$(5,5)$	Thm. 3
	$(4,1,4,1)$	Thm. 4
	$(4,1,1,4)$	
	$(3,1,3,3)$	
11	$(8,1,1,1)$	Thm. 4
	$(8,3)$	Thm. 3
	$(7,1,2,1)$	Thm. 4
	$(7,1,1,2)$	
	$(7,4)$	Thm. 3
	$(6,1,3,1)$	Thm. 4
	$(6,1,1,3)$	
	$(6,5)$	Thm. 3
	$(5,1,4,1)$	Thm. 4
	$(5,1,3,2)$	
	$(4,1,3,3)$	

for the non-Hamiltonicity; see the third column in the tables. In the other cases, we are still missing such an explanation. Based on this data, we feel that Theorem 2 can be strengthened and the requirement $\ell \geq 10$ relaxed to $\ell \geq 8$. Furthermore, it seems that for $\ell = 6$ there is always a Hamilton path in $\mathcal{F}_\alpha$ unless $\alpha \in \{(1,1,1,1,1,1),(3,1,1,1,1,1)\}$.

Also, Theorem 9 on twists in 3-colored triangulations invites deeper investigation. We conjecture that the graph $\mathcal{H}_N$ has a Hamilton cycle for all $N \geq 9$ that are divisible by 3. Furthermore, it seems that if $N = 2 \pmod 3$ the graph $\mathcal{H}_N$ is not connected. What are the properties of the flip graphs for general coloring patterns with three or more colors?

Another interesting question concerns bijections between k-ary trees and classes of permutations. For $k = 2$ (binary trees), there is a natural bijection to 231-avoiding permutations. Are there similar correspondences between k-ary trees and pattern-avoiding permutations for $k \geq 3$? In particular, do tree rotations translate to nice operations on the permutations, specifically to so-called jumps heavily used in [12, 13, 20, 5]?

Going back to the uncolored setting and the associahedron $\mathcal{G}_N$, Theorem 2 shows that $\mathcal{G}_N$ admits cycles of many different lengths. What is the *cycle spectrum* of $\mathcal{G}_N$, i.e., the set $S(\mathcal{G}_N)$ of all possible lengths of cycles in $\mathcal{G}_N$? We conjecture that almost all lengths are possible.

▶ **Conjecture 16.** *We have* $|S(\mathcal{G}_N)|/|\mathcal{D}_{N,3}| = 1 - o(1)$ *as* $N \to \infty$*.*

Baur, Bergerova, Voon and Xu [2] recently introduced another family of flip graphs on triangulations in which the triangles are colored, not the vertices. The resulting graphs are disconnected in general, and their structure is still not very well understood (in [11] these graphs are related to the famous Four Color Theorem).

References

1. R. Acharya, T. Mütze, and F. Verciani. Flips in colorful triangulations. *arXiv:2406.03783*, June 2024. `doi:10.48550/arXiv.2406.03783`.
2. K. Baur, D. Bergerova, J. Voon, and L. Xu. Flip graphs of coloured triangulations of convex polygons. *arXiv:2402.06546*, February 2024.
3. L. A. Berry, B. Reed, A. Scott, and D. R. Wood. A logarithmic bound for the chromatic number of the associahedron. *arXiv:1811.08972*, November 2018.
4. M. Buck and D. Wiedemann. Gray codes with restricted density. *Discrete Math.*, 48(2–3):163–171, 1984. `doi:10.1016/0012-365X(84)90179-1`.
5. J. Cardinal, A. Merino, and T. Mütze. Combinatorial generation via permutation languages. IV. Elimination trees. To appear in *ACM Transactions on Algorithms*; preprint available on *arXiv:2106.16204*, 2024.
6. C. Ceballos, F. Santos, and G. M. Ziegler. Many non-equivalent realizations of the associahedron. *Combinatorica*, 35(5):513–551, 2015. `doi:10.1007/s00493-014-2959-9`.
7. The Combinatorial Object Server: Generate k-ary trees and dissections. `http://www.combos.org/kary`.
8. T. Dvořák. Hamiltonian cycles with prescribed edges in hypercubes. *SIAM J. Discrete Math.*, 19(1):135–144, 2005. `doi:10.1137/S0895480103432805`.
9. P. Eades, M. Hickey, and R. C. Read. Some Hamilton paths and a minimal change algorithm. *J. Assoc. Comput. Mach.*, 31(1):19–29, 1984. `doi:10.1145/2422.322413`.
10. R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia, and D. R. Wood. On the chromatic number of some flip graphs. *Discrete Math. Theor. Comput. Sci.*, 11(2):47–56, 2009. `doi:10.46298/DMTCS.460`.
11. S. Gravier and C. Payan. Flips signés et triangulations d'un polygone. *European J. Combin.*, 23(7):817–821, 2002. `doi:10.1006/eujc.2002.0601`.

12 E. Hartung, H. P. Hoang, T. Mütze, and A. Williams. Combinatorial generation via permutation languages. I. Fundamentals. *Trans. Amer. Math. Soc.*, 375(4):2255–2291, 2022. doi:10.1090/tran/8199.

13 H. P. Hoang and T. Mütze. Combinatorial generation via permutation languages. II. Lattice congruences. *Israel J. Math.*, 244(1):359–417, 2021. doi:10.1007/s11856-021-2186-1.

14 C. Hohlweg and C. E. M. C. Lange. Realizations of the associahedron and cyclohedron. *Discrete Comput. Geom.*, 37(4):517–543, 2007. doi:10.1007/s00454-007-1319-6.

15 C. Huemer, F. Hurtado, and J. Pfeifle. The rotation graph of k-ary trees is Hamiltonian. *Inform. Process. Lett.*, 109(2):124–129, 2008. doi:10.1016/j.ipl.2008.09.013.

16 F. Hurtado and M. Noy. Graph of triangulations of a convex polygon and tree of triangulations. *Comput. Geom.*, 13(3):179–188, 1999. doi:10.1016/S0925-7721(99)00016-4.

17 J.-L. Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004. doi:10.1007/s00013-004-1026-y.

18 J. M. Lucas. The rotation graph of binary trees is Hamiltonian. *J. Algorithms*, 8(4):503–535, 1987. doi:10.1016/0196-6774(87)90048-4.

19 J. M. Lucas, D. Roelants van Baronaigien, and F. Ruskey. On rotations and the generation of binary trees. *J. Algorithms*, 15(3):343–366, 1993. doi:10.1006/jagm.1993.1045.

20 A. Merino and T. Mütze. Combinatorial generation via permutation languages. III. Rectangulations. *Discrete Comput. Geom.*, 70(1):51–122, 2023. doi:10.1007/s00454-022-00393-w.

21 V. Pilaud, F. Santos, and G. M. Ziegler. Celebrating Loday's associahedron. *Arch. Math. (Basel)*, 121(5-6):559–601, 2023. doi:10.1007/s00013-023-01895-6.

22 L. Pournin. The diameter of associahedra. *Adv. Math.*, 259:13–42, 2014. doi:10.1016/j.aim.2014.02.035.

23 B. E. Sagan. Proper partitions of a polygon and k-Catalan numbers. *Ars Combin.*, 88:109–124, 2008.

24 D. D. Sleator, R. E. Tarjan, and W. P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. *J. Amer. Math. Soc.*, 1(3):647–681, 1988. doi:10.2307/1990951.