

Separable Drawings: Extendability and Crossing-Free Hamiltonian Cycles

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Abstract

Generalizing pseudospherical drawings, we introduce a new class of simple drawings, which we call *separable drawings*. In a separable drawing, every edge can be closed to a simple curve that intersects each other edge at most once. Curves of different edges might interact arbitrarily.

Most notably, we show that (1) every separable drawing of any graph on n vertices in the plane can be extended to a simple drawing of the complete graph K_n , (2) every separable drawing of K_n contains a crossing-free Hamiltonian cycle and is plane Hamiltonian connected, and (3) every generalized convex drawing and every 2-page book drawing is separable. Further, the class of separable drawings is a proper superclass of the union of generalized convex and 2-page book drawings. Hence, our results on plane Hamiltonicity extend recent work on generalized convex drawings by Bergold et al. (SoCG 2024).

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1 Introduction

A *simple drawing* of a graph G is a representation of G in the plane (or on the sphere) such that the vertices of G are mapped to distinct points and the edges of G are mapped to Jordan arcs connecting their respective end-vertices. Furthermore, every pair of edges is allowed to have at most one point in common, which is either a common end-vertex or a proper crossing. Simple drawings of graphs are widely studied combinatorial objects that have received considerable attention in different areas of graph drawing, for example, every crossing-minimizing drawing of a graph is simple.

Several classes of simple drawings have been considered, including *straight-line drawings* (where the edges are straight-line segments), *x -monotone drawings* (where the edges are x -monotone curves), *2-page book drawings*¹ (where all vertices lie on a straight line and the

¹ 2-page book drawings are also known as linear layouts.



edges are drawn as half-circles), *pseudolinear drawings* (for which there exists an arrangement of pseudolines such that every edge lies on one pseudoline), and *pseudocircular drawings* (for which there exists an arrangement of pseudocircles such that every edge lies on one pseudocircle). For details on and relations between these and several more classes (some of which are mentioned later) see for example [3].

A drawing class that was introduced by Arroyo, Richter, and Sunohara [7] and that is of special interest for this work is the class of *pseudospherical drawings*. These are pseudocircular drawings with the additional property that every edge of the drawing intersects every pseudocircle of the underlying arrangement at most once. Stated differently, a pseudospherical drawing \mathcal{D} of a graph G is a simple drawing in which every edge e is contained in a simple closed curve γ_e such that

1. the only two vertices of \mathcal{D} on γ_e are the end-vertices of e ,
2. for any two edges $e \neq f$ the curves γ_e and γ_f intersect in exactly two crossing points, and
3. γ_e intersects every edge $f \neq e$ of \mathcal{D} at most once, either in a crossing or in an end-vertex.

In this work we introduce a new class of simple drawings, which we call *separable drawings*. These are all simple drawings that fulfill Properties 1 and 3 of pseudospherical drawings (but not necessarily Property 2). Separable drawings can also be seen as “locally pseudospherical” because locally for every edge, they look like pseudospherical drawings, but the curves γ_e and γ_f of different edges e and f of \mathcal{D} may interact arbitrarily. This additional freedom gives the advantage that for recognizing separable drawings, it is sufficient to independently find a curve for each edge of the drawing. That is, we do not have to consider the set of potential such curves for all edges simultaneously, which can be relevant from a computational point of view. Moreover, we show that it is a real additional freedom in the sense that the class of separable drawings is strictly larger than the one of pseudospherical drawings.

Note that pseudocircular drawings are all simple drawings that fulfill Properties 1 and 2 of the definition of pseudospherical drawings. Hence the class of pseudospherical drawings is the intersection of the classes of separable and pseudocircular drawings.

Our motivation for studying separable drawings stems from the quest of solving two classic graph drawing questions for simple drawings, namely, the extendability to simple drawings of complete graphs and the existence of crossing-free Hamiltonian cycles in drawings of complete graphs. In this work, we answer both questions for the class of separable drawings and elucidate the relation of separable drawings to further classes of simple drawings.

Edge extension of simple drawings. It is easy to see that every straight-line drawing in the plane on n vertices in general position can be extended to a straight-line drawing of the complete graph K_n . As a consequence of Levi’s Enlargement Lemma [17], an analogous statement is true for pseudolinear drawings. For simple drawings the situation is very different. Kynčl showed that extendability to complete graphs is not always possible [14]. Further, there exist simple drawings of graphs with only a linear number of edges, which cannot be extended by any of the missing edges without violating simplicity [11]. The decision problem of whether a given drawing can be extended by some given edges is NP-complete [4], even for a single edge and if the drawing is pseudocircular [5]. To the positive, the edge extension problem is fixed-parameter tractable (FPT) when parameterized by the number of edges to insert and an upper bound on newly created crossings [9]. The complexity of deciding whether a simple drawing (of some class) can be extended to a simple drawing of the complete graph is still an open problem. Recently, Kynčl and Soukup [16] showed that every x -monotone drawing admits an extension to an x -monotone drawing of the complete graph.

Crossing-free Hamiltonian cycles and paths. It is well known that every straight-line drawing of K_n contains a crossing-free Hamiltonian cycle, and that this property does not hold for straight-line drawings of general graphs (it already breaks for K_n minus one edge). In 1988, Rafla [20] conjectured that the same is true for simple drawings of K_n .

► **Conjecture 1.** *Every simple drawing of K_n with $n \geq 3$ vertices contains a crossing-free Hamiltonian cycle.*

If Conjecture 1 is true, then every simple drawing of K_n also contains at least n crossing-free Hamiltonian paths and 2 crossing-free perfect matchings. Pach, Solymosi, and Tóth [18] made the study of crossing-free subdrawings popular. For simple drawings, a lot of effort went into the search for crossing-free matchings, with the current best lower bound for their size being $\Omega(\sqrt{n})$ [2]. With regard to special drawing classes, the existence of a crossing-free Hamiltonian cycle is an easy exercise for 2-page book drawings and x -monotone drawings. Further, Conjecture 1 was proven to hold for generalized twisted drawings on an odd number of vertices [2]. In a previous work, we also confirmed it for cylindrical drawings as well as strongly c -monotone drawings [3]. In that work, we further stated the following conjecture, which we showed to be a strengthening of Conjecture 1 in the sense of an affirmative answer for all simple drawings of K_n (but not necessarily for a restricted class of simple drawings). Further, we showed that the implication holds for cylindrical and strongly c -monotone drawings and confirmed both conjectures for them.

► **Conjecture 2.** *Every simple drawing \mathcal{D} of K_n on $n \geq 2$ vertices contains, for each pair of vertices $v \neq w$ in \mathcal{D} , a crossing-free Hamiltonian path with end-vertices v and w .*

Very recently, both conjectures have been verified for the large class of g -convex² (short for generalized convex) drawings [8], where the authors also coined the term *plane Hamiltonian connected* for drawings fulfilling Conjecture 2.

A simple drawing \mathcal{D} of K_n is called *g -convex* if every *triangle* in \mathcal{D} has a *convex side*. A *triangle* in \mathcal{D} is the simple closed curve formed by an induced subdrawing on three vertices in \mathcal{D} . Every triangle splits the plane (or sphere) into two connected components, their closures are the *sides* of the triangle. A side S of a triangle is called *convex* if the subdrawing of \mathcal{D} that is induced by all vertices in S is completely contained in S (that is, no edge between two such vertices crosses the triangle).

G -convex drawings have been introduced by Arroyo, McQuillan, Richter, and Salazar [6] as the largest class of a hierarchy of classes of simple drawings of K_n , all of which are combinatorial generalizations of straight-line drawings. Hence the results of [8] on plane Hamiltonicity are quite strong.

Our contribution. In Section 2 we introduce some more notation and show first properties of separable drawings, also explaining why we chose the name “separable”. We further observe that every 2-page book drawing is separable (Observation 6) and show that for simple drawings of K_n being separable is a property of the rotation system (Lemma 5).

In Section 3 we consider the extension problem. We prove that for every graph G on n vertices, every separable drawing of G can be completed to a simple drawing of K_n and that the same holds for crossing-minimizing drawings of G (Theorems 8 and 9). We further discuss that extension to simple drawings is the best we can hope for by presenting an example of a separable drawing that cannot be extended to any separable drawing of K_n (Figure 4).

² G -convex drawings are just called convex drawings in [6, 8]. However, we prefer the term generalized convex or g -convex to avoid confusion, since the term convex drawing classically refers to a straight-line drawing with vertices in convex position.

In Section 4, we turn our attention to the plane Hamiltonicity problem. We show that all separable drawings of K_n fulfill both Conjecture 1 (Theorem 11) and Conjecture 2 (Theorem 10). Further, we prove that separable drawings are a proper superclass of g-convex drawings (Theorem 14). Thus our results on plane Hamiltonicity constitute a strengthening of the according results on g-convex drawings in [8].

Finally, we consider the question of recognizing separable drawings in Section 5. We show that the recognition problem is solvable in polynomial time for simple drawings of K_n (Theorem 15) and NP-complete for simple drawings of general graphs (Theorem 16).

We conclude with some open problems in Section 6.

2 Preliminaries

Before we get to first properties of separable drawings, we introduce some more notation to facilitate argumentation. We call an edge e of a simple drawing \mathcal{D} a *separator edge* if there exists a simple closed curve γ_e containing e such that the only vertices of \mathcal{D} on γ_e are the end-vertices of e and such that, for each edge $f \neq e$ of \mathcal{D} , γ_e has at most one point in common with f . We call the curve γ_e a *witness* for e . With this definition, a simple drawing \mathcal{D} is separable if and only if every edge of \mathcal{D} is a separator edge.

Note that a simple closed curve γ partitions the plane into two connected components. We call the closures of these components the *sides* of γ . To ease reasoning, we sometimes refer to the bounded side of γ in the plane as the *inside* and the other side as the *outside*.

The following lemma motivates why we call separable drawings “separable”.

► **Lemma 3.** *Let γ_e be a witness of a separator edge e in a simple drawing \mathcal{D} . Then every edge f of \mathcal{D} that connects two vertices on the same side of γ_e is fully contained in that side.*

Proof. The statement is clear for e itself. Further, by the definition of a separator edge, each edge $f \neq e$ of \mathcal{D} has at most one point in common with γ_e . Every edge f incident to e already has an end-vertex in common with γ_e and, therefore, is contained in one side of γ_e . Finally, every edge f with both end-vertices on the same side of γ_e and not incident to e crosses γ_e an even number of times. Since f crosses γ_e at most once, it does not cross γ_e at all, which implies that f is contained in one side of γ_e . ◀

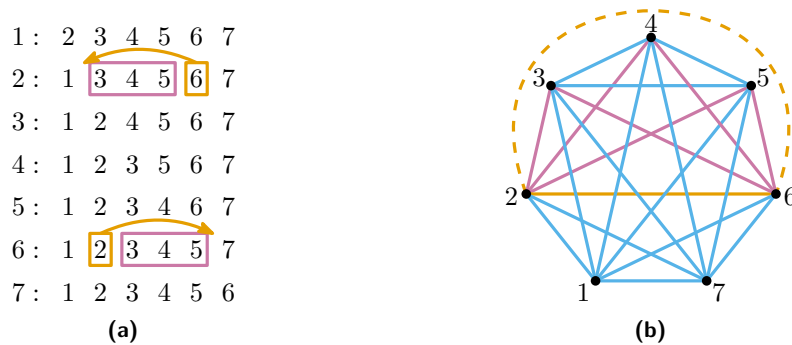
Lemma 3 tells us that, for every edge e in a separable drawing \mathcal{D} , each witness γ_e of e separates \mathcal{D} into two induced subdrawings that together cover all vertices of \mathcal{D} , and that do not interact with each other except for sharing the common edge e . In Lemma 5 (Item 2) we show that, for simple drawings of K_n , the existence of two such induced subdrawings is an equivalent characterization of separability. This implies that, for complete graphs, we do not need to check edges between the two sides of γ_e for multiple intersections with γ_e .

A special case of a separator edge is an uncrossed edge e . Indeed, we can close e to a simple curve γ_e in a small neighborhood of e itself. Then γ_e has one point in common with every edge incident to e and no point in common with any other edge. With respect to the separation into two subdrawings, this means that one of them only consists of the edge e .

► **Observation 4.** *Every uncrossed edge is a separator edge.*

A classic combinatorial abstraction of a simple drawing \mathcal{D} of K_n is its rotation system. The *rotation* of a vertex in a simple drawing is the (clockwise) cyclic order of its incident edges, which is classically given by an accordingly sorted list of its adjacent vertices. The *rotation system* of a simple drawing is the collection of the rotations of all of its vertices. Gioan [10] and Kynčl [13] independently showed that two simple drawings of K_n have the

same set of crossings if and only if they have the same rotation system. An *abstract rotation system* of K_n gives, for each vertex, a (potentially arbitrary) cyclic order of its incident edges. An abstract rotation system of K_n is called *realizable* if there exists a simple drawing of K_n with this rotation system. As shown by Kynčl [15] in combination with computational results from [1], an abstract rotation system is realizable if and only if all its subrotation systems on five vertices are. This implies that deciding whether an abstract rotation system of K_n is realizable can be done in $O(n^5)$ time.



■ **Figure 1** (a) A rotation system corresponding to a convex straight-line drawing of K_7 . The only possible flip of the edge $e = \{2, 6\}$ is marked. (b) As we implicitly show in the proof of Lemma 5, Item 3 \Rightarrow Item 1, a flip of e in the rotation system corresponds to redrawing e (the dashed version is after the flip, the solid version before) in any simple drawing realizing the rotation system.

We next discuss that for a simple drawing \mathcal{D} of K_n , separability only depends on the rotation system of \mathcal{D} . To this end, we first introduce local changes in rotation systems, which we call flips. A *flip* in the rotation system of a simple drawing \mathcal{D} of K_n , see Figure 1 for an example illustration, is the operation of removing an edge $e = \{v, w\}$ in the rotations of its two incident vertices and adding it again in a different position such that

1. in the counter-clockwise rotation of v and the clockwise rotation of w , the sets of vertices between the position of e before and after the operation coincide and are non-empty, and
2. the resulting (abstract) rotation system is realizable.

To relate separator edges to flips in rotation systems (Item 3 of Lemma 5) we will make use of a result by Schaefer [21]. It states that every pair of drawings of K_n minus a non-perfect matching having the same set of crossings can be transformed into each other via triangle mutations (the operation of moving an edge over the crossing between two other edges; see also Figure 2), plus a homeomorphism of the plane.

► **Lemma 5.** *Let \mathcal{D} be a simple drawing of K_n and let $e = \{v, w\}$ be an edge of \mathcal{D} . Then the following are equivalent:*

1. *The edge e is a separator edge.*
2. *The edge e can be closed to a simple curve γ'_e such that every edge f of \mathcal{D} that connects two vertices on the same side of γ'_e is fully contained in that side.*
3. *The edge e is either uncrossed or it can be flipped to an edge e' such that e and e' cross disjoint sets of edges.*

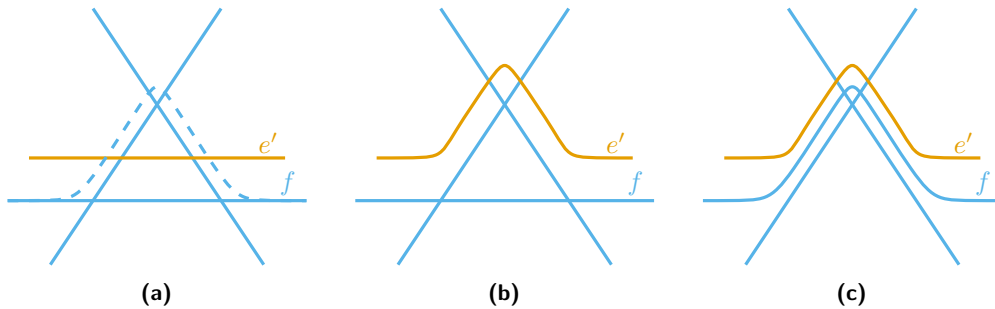
Proof. In the following we show that Item 1 is equivalent to Item 2 and that Item 1 is equivalent to Item 3. The implication Item 1 \Rightarrow Item 2 is given by Lemma 3. Hence, there are three implications left to show.

Item 2 \Rightarrow Item 1 Let \mathcal{D}_1 be the subdrawing of \mathcal{D} induced by all vertices of the inside of γ'_e and similarly \mathcal{D}_2 for all vertices of the outside of γ'_e . Since γ'_e separates \mathcal{D}_1 and \mathcal{D}_2 , any edge in \mathcal{D}_1 or \mathcal{D}_2 has at most one point in common with γ'_e . It remains to consider edges $f = \{v_1, v_2\}$ with $v_1 \in \mathcal{D}_1$ and $v_2 \in \mathcal{D}_2$. If f crosses e , which it can cross at most once, then f lies in the inside of the crossing K_4 on the vertices $\{v, v_1, w, v_2\}$, or, in other words, it is separated from $\gamma'_e \setminus e$ by the 4-cycle v, v_1, w, v_2 . Hence it cannot cross γ'_e a second time. The remaining case is that f crosses $\gamma'_e \setminus e$. Let \mathcal{B} be the boundary of the unbounded cell of \mathcal{D}_1 . We show that f crosses \mathcal{B} exactly once.

Assume for a contradiction that f crosses \mathcal{B} more than once. Let x_1 and x_2 be two consecutive such crossings along f such that the part f' between x_1 and x_2 lies inside \mathcal{B} . Then f' crosses \mathcal{D}_1 and separates the inside of \mathcal{B} into two connected components F_1 and F_2 . Let F_1 be the component that contains e . If no vertex of \mathcal{D}_1 lies in F_2 , then every edge in \mathcal{D}_1 that is crossed by f' would have to be crossed at least twice, a contradiction to \mathcal{D} being simple. Hence, there is some vertex z of \mathcal{D}_1 in F_2 . If v_1 lies in F_1 , then $\{v_1, z\}$ crosses its incident edge f . If v_1 lies in F_2 , then $\{v_1, v\}$ and $\{v_1, w\}$ cross their incident edge f . Since we have a contradiction in both cases, f crosses \mathcal{B} exactly once.

We reroute $\gamma'_e \setminus e$ arbitrarily close to \mathcal{B} along the outside of \mathcal{B} . This does not change any crossings with \mathcal{D}_1 or \mathcal{D}_2 and, by the arguments above, every edge f between the two subdrawings \mathcal{D}_1 or \mathcal{D}_2 is crossed exactly once by the adapted curve γ'_e . Consequently, e is a separator edge.

Item 1 \Rightarrow Item 3 Recall that γ_e has at most one point in common with every edge $f \neq e$ in \mathcal{D} . Hence, replacing e by $e' = \gamma_e \setminus e$ gives a simple drawing \mathcal{D}' . If \mathcal{D} and \mathcal{D}' have the same crossings, then both e and e' are uncrossed. Otherwise the rotation system must have changed and, by Lemma 3, the change is exactly as defined for a flip.



■ **Figure 2** (a) If after a triangle mutation the redrawn edge f (dashed) would cross e' twice, then (b) we first move e' over the respective crossing and then (c) redraw f as planned.

Item 3 \Rightarrow Item 1 If e is uncrossed, then e is a separator edge by Observation 4. So assume that e can be flipped to e' such that no edge is crossed by both e and e' , and let \mathcal{D}' be a simple drawing realizing the rotation system with e' instead of e .

We first show that $\mathcal{D} - e$ and $\mathcal{D}' - e'$ have the same crossing edge pairs. Recall that for a rotation system of K_n , $n \geq 4$, and any pair f, g of independent edges, the subrotation system on the 4-tuple of the four end-vertices of f and g determines whether or not f and g cross. When flipping e to e' , the subrotation system of any 4-tuple that contains at most one end-vertex of e remains unchanged. Hence it remains to consider 4-tuples v, w, v_1, v_2 that involve both end-vertices of e . If in \mathcal{D} , such a 4-tuple forms a crossing that does not involve e , then the order of v_1 and v_2 in the counter-clockwise rotation around v and the clockwise rotation around w (when starting with $\{v, w\}$) is inverse, implying that the subrotation system after the flip is the same as the one before the flip and hence that

the same edge pair also crosses in \mathcal{D}' . If in \mathcal{D} , such a 4-tuple is non-crossing or contains the crossing between e and $\{v_1, v_2\}$, then the order of v_1 and v_2 in the counter-clockwise rotation around v and the clockwise rotation around w (when starting with $\{v, w\}$) is the same. Hence, after the flip, the subrotation system again is either non-crossing or forms the crossing between e' and $\{v_1, v_2\}$. Altogether, this implies that all crossing edge pairs that exist in exactly one of \mathcal{D} and \mathcal{D}' involve e or e' .

Since $\mathcal{D} - e$ and $\mathcal{D}' - e'$ have the same crossing edge pairs, they at most differ in the order of crossings along edges. Hence we can apply Schaefer's generalization of Gioan's theorem [21] to transform $\mathcal{D}' - e'$ to $\mathcal{D} - e$ via triangle mutations, potentially also transforming e' . More exactly, whenever we want to move an edge f over a crossing and e' lies between the edge and the crossing, we first move e' over that crossing and then make the originally planned move with f ; see Figure 2 for an illustration. This process may change the order of crossings along e' , but changes neither the crossing edge pairs nor the rotation at any vertex. Hence, once we have transformed $\mathcal{D}' - e'$ to $\mathcal{D} - e$, we have obtained a transformed edge e' such that $\mathcal{D} - e + e'$ is a realization of the flipped rotation system. Since no edge of $\mathcal{D} - e$ is crossed by both e and e' and no edge incident to e crosses any of e and e' , the union of e and e' fulfills the properties of a witness γ_e , which shows that e is a separator edge. ◀

Note that in the proof of Item 2 \Rightarrow Item 1 the subdrawings \mathcal{D}_1 and \mathcal{D}_2 are interchangeable. That is, we could also reroute γ_e close to the boundary of \mathcal{D}_2 . Furthermore, in the proof of Item 3 \Rightarrow Item 1 the drawing \mathcal{D} can be an arbitrary realization of the respective rotation system. Therefore, every simple drawing that corresponds to a rotation system that has at least one realization as a separable drawing is itself a separable drawing. This makes being separable a property of the rotation system for simple drawings of K_n .

Before we come to the main results of this paper, let us mention 2-page book drawings \mathcal{D} . There the vertices lie on a common line and all edges e are drawn as half-circles. Hence, by closing e to a circle, we get a simple closed curve that has at most one point in common with any edge $f \neq e$ of \mathcal{D} .

► **Observation 6.** *Every 2-page book drawing is separable.*

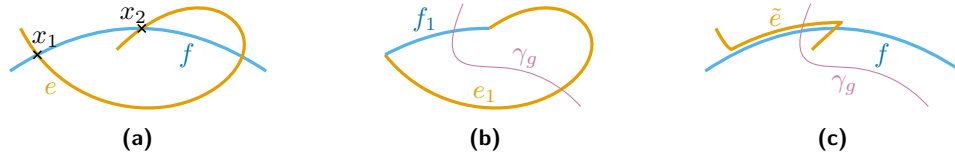
3 Extendability

In the following we prove that every separable drawing \mathcal{D} of a graph G on n vertices can be completed to a simple drawing of K_n . As a first step we show how to add one edge to \mathcal{D} . To do so, we impose a minimality condition regarding the witnesses of all edges in \mathcal{D} . In particular, we call a collection \mathcal{D}° of witnesses, one for every edge in \mathcal{D} , a *witness set* for \mathcal{D} . Further, for an edge $\{u, v\}$ not in G , we call a continuous curve that connects the drawn end-vertices $\mathcal{D}(u)$ and $\mathcal{D}(v)$ in \mathcal{D} a *realization* of $\{u, v\}$ in \mathcal{D} .

► **Lemma 7.** *Let \mathcal{D} be a separable drawing of a non-complete graph G and let \mathcal{D}° be a witness set for \mathcal{D} . For a fixed edge $\{u, v\}$ not in G , let e be a realization of $\{u, v\}$ in \mathcal{D} that, over all possible realizations, minimizes the number of crossings with the witness set \mathcal{D}° . Then the drawing $\mathcal{D}' = \mathcal{D} + e$ is simple.*

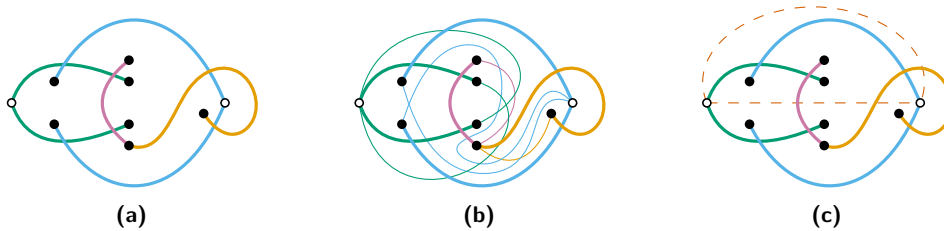
Proof. Let e be as described and assume, to the contrary, that \mathcal{D}' is not simple. The minimality condition implies that e is self-avoiding, hence, the assumption implies that e has more than one point in common with an edge f of \mathcal{D} ; see Figure 3 for an example illustration. Let x_1 and x_2 be two of those common points that are consecutive along f . Then the parts

e_1 and f_1 of e and f , respectively, between x_1 and x_2 each, form a simple closed curve. Since every witness γ_g in \mathcal{D}° for an edge g in \mathcal{D} has at most one point in common with f it follows that, if γ_g crosses f_1 , then γ_g also has to cross e_1 . Therefore, rerouting e along f between x_1 and x_2 reduces the number of crossings of e with \mathcal{D}° by at least one; a contradiction to the minimality condition on e . ◀



■ **Figure 3** (a) The edges e and f have more than one point in common, with x_1 and x_2 being consecutive common points on f . (b) Every witness γ_g that crosses f_1 also has to cross e_1 . (c) The result of rerouting e along f between x_1 and x_2 .

A natural way to get to a simple drawing of K_n would be to iterate the argument of Lemma 7. However, we would need the drawing in each step to be separable, which might not be the case. In particular, Figure 4(a) shows an example of a separable drawing \mathcal{D} on 9 vertices that cannot be completed to a *separable* drawing of K_n . Figure 4(b) shows a witness set for \mathcal{D} , and Figure 4(c) indicates that, with respect to crossings, there are only two different ways to add the edge e between the leftmost and rightmost vertex in \mathcal{D} . Hence the witness of e would have to be the union of these two options. However, both cross the rightmost edge in \mathcal{D} (orange), which is not allowed for a witness.



■ **Figure 4** A separable drawing of a non-complete graph that cannot be extended to any *separable* drawing of a complete graph.

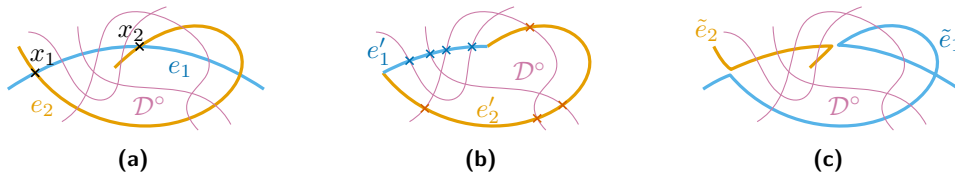
By imposing a second minimality condition, however, we can still extend to a *simple* drawing of K_n .

► **Theorem 8.** *Let \mathcal{D} be a separable drawing of a non-complete graph on n vertices. Then \mathcal{D} can be extended to a simple drawing of K_n .*

Proof. Let \mathcal{D}° be a witness set for \mathcal{D} . We extend \mathcal{D} to a drawing \mathcal{D}' of K_n such that (1) each added edge e creates a minimum number of additional crossings when being added to \mathcal{D}° and such that under this condition (2) \mathcal{D}' has the least total number of crossings. Then, by Lemma 7, $\mathcal{D} + e$ is simple for each of those added edges.

Hence, an obstruction to simplicity can only occur between two added edges e_1 and e_2 in \mathcal{D}' ; see Figure 5 for an example illustration. Let x_1 and x_2 be two consecutive common points on e_1 , and let e'_1 and e'_2 be the respective parts between x_1 and x_2 . By the first minimality condition, e'_1 and e'_2 must have the same number of crossings with \mathcal{D}° , otherwise

we could reroute one of e'_1 or e'_2 along (a part of) the other to get fewer crossings. But then exchanging e'_1 and e'_2 produces a drawing \mathcal{D}' fulfilling the first minimality condition but with fewer crossings than \mathcal{D}' ; a contradiction to the second minimality condition on \mathcal{D}' . ◀



■ **Figure 5** (a) The edges e_1 and e_2 have more than one point in common, with x_1 and x_2 being consecutive common points on e_1 . (b) The parts e'_1 and e'_2 must have the same number of common points (4 each in this example) with the witness set \mathcal{D}° . (c) The result of exchanging e'_1 and e'_2 .

Inspired by the question whether every crossing-minimizing drawing of K_n is pseudospherical, we also investigate the extendability of crossing-minimizing drawings of non-complete graphs. Interestingly, as we detail in the full version of this paper, the proof works rather similar to that for separable drawings, we only need to replace the arguments regarding the witness set with arguments using that the initial drawing is crossing-minimizing.

► **Theorem 9.** *Let \mathcal{D} be a crossing-minimizing drawing of a non-complete graph on n vertices. Then \mathcal{D} can be extended to a simple drawing of K_n .*

4 Crossing-free Hamiltonian cycles and paths

This section is about separable drawings of the complete graph K_n . We first show that they are plane Hamiltonian connected, that is, there exists a crossing-free Hamiltonian path between each pair of vertices, which proves Conjecture 2 for this class.

► **Theorem 10.** *Every separable drawing \mathcal{D} of K_n contains, for each pair of vertices $v \neq w$ in \mathcal{D} , a crossing-free Hamiltonian path with end-vertices v and w .*

Proof. The proof is by induction on n . For $n \leq 2$ the statement is trivially true. For the induction step, let $n \geq 3$, let $v \neq w$ be two arbitrary vertices in \mathcal{D} , and consider some edge $e = \{v, v'\}$ with $v' \neq w$ and witness γ_e . Further, let \mathcal{D}_1 be the subdrawing of \mathcal{D} induced by the set of vertices on the side of γ_e not containing w and let \mathcal{D}_2 be the subdrawing of \mathcal{D} induced by the set of vertices on the other side of γ_e but without vertex v .

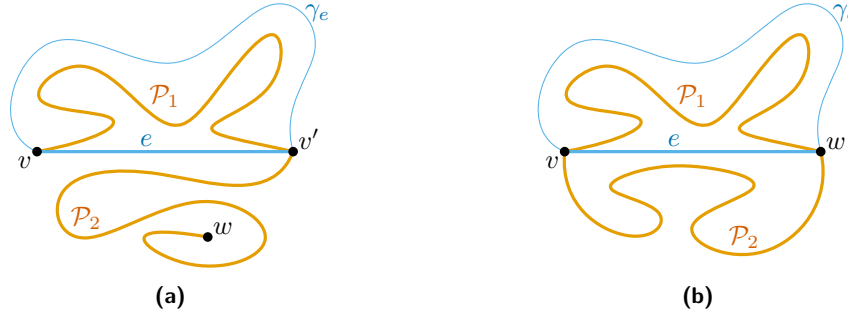
Then \mathcal{D}_1 and \mathcal{D}_2 are both proper subdrawings of \mathcal{D} . Hence, by the induction hypothesis, there exists a crossing-free Hamiltonian path \mathcal{P}_1 in \mathcal{D}_1 with end-vertices v and v' and there exists a crossing-free Hamiltonian path \mathcal{P}_2 in \mathcal{D}_2 with end-vertices v' and w . By Lemma 3, no edge of the path \mathcal{P}_1 crosses any edge of the path \mathcal{P}_2 . Consequently, the union of \mathcal{P}_1 and \mathcal{P}_2 forms a crossing-free Hamiltonian path in \mathcal{D} with end-vertices v and w . ◀

Figure 6(a) gives an illustration of the proof. With a similar approach we obtain that separable drawings of K_n also contain a crossing-free Hamiltonian cycle, by this proving Conjecture 1 for them. Figure 6(b) shows an illustration of how to get the Hamiltonian cycle.

► **Theorem 11.** *Every separable drawing \mathcal{D} of K_n with $n \geq 3$ vertices contains a crossing-free Hamiltonian cycle.*

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Proof. Let $e = \{v, w\}$ be an arbitrary edge in \mathcal{D} with witness γ_e and let \mathcal{D}_1 and \mathcal{D}_2 be the subdrawings of \mathcal{D} induced by the vertices on the two sides of γ_e , respectively. By Theorem 10, there exists a crossing-free Hamiltonian path \mathcal{P}_i in \mathcal{D}_i with end-vertices v and w , for $i \in \{1, 2\}$. By Lemma 3, no edge of \mathcal{P}_1 crosses any edge of \mathcal{P}_2 . Hence, the union of \mathcal{P}_1 and \mathcal{P}_2 forms a crossing-free Hamiltonian cycle in \mathcal{D} . ◀



■ **Figure 6** (a) Finding a crossing-free Hamiltonian path between two given vertices and (b) finding a crossing-free Hamiltonian cycle in a separable drawing of K_n .

For the proofs of Theorems 10 and 11 it is actually sufficient that for every pair of vertices v and w , one of them is incident to a separator edge that is not $\{v, w\}$. In particular, this is the case when every vertex is incident to at least 2 separator edges. We call this property *degree-2-separable*. In the proof we further rely on induction. Therefore, we call a class \mathcal{S} of simple drawings *subset-closed* if every subdrawing of a drawing in \mathcal{S} is itself in \mathcal{S} . With this we get the following observation, which might be helpful to show Conjecture 1 for even larger classes of simple drawings.

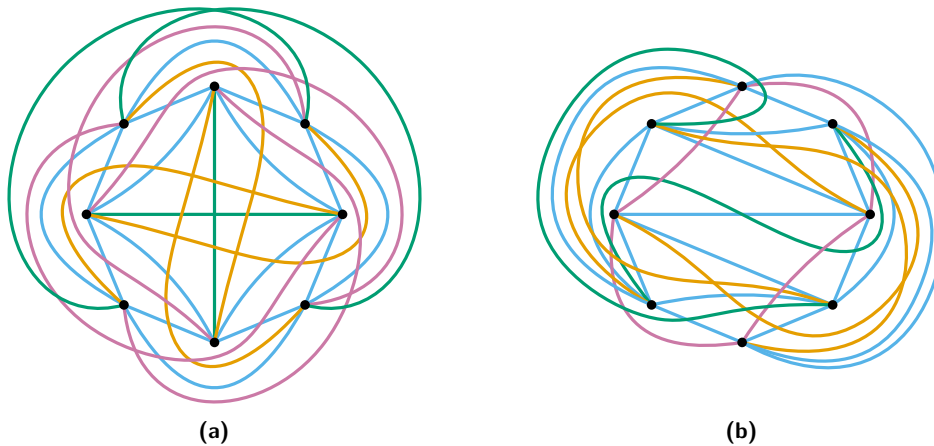
► **Observation 12.** *Let \mathcal{S} be a subset-closed class of simple drawings of complete graphs such that every drawing in \mathcal{S} is degree-2-separable. Then every drawing in \mathcal{S} contains a crossing-free Hamiltonian cycle.*

Let us further mention that a single separator edge is enough to find a crossing-free matching of linear size; let us call this property *1-separable* for a subset-closed class of simple drawings. Indeed, we can add the separator edge e to the matching and then recurse on the subdrawings in the two sides of the witness γ_e . In the worst case, for each edge that we add, we get two subdrawings with only one vertex each that cannot be matched anymore.

► **Observation 13.** *Let \mathcal{S} be a subset-closed class of simple drawings of complete graphs such that every drawing in \mathcal{S} is 1-separable. Then every drawing in \mathcal{S} contains a crossing-free matching of linear size in n .*

Unfortunately there exist simple drawings of K_n without a single separator edge. Figure 7 shows the (up to weak isomorphism³) only two simple drawings of K_8 with this property; the different edge colors are just for better visibility. This result has been obtained by applying the algorithm of Theorem 15 to all different rotation systems of K_8 as given in [1]. Note that Harborth and Mengersen [12] proved that simple drawings of K_n for $n \leq 7$ always have uncrossed edges, and therefore, they have separator edges by Observation 4. Hence, the drawings depicted in Figure 7 are the smallest examples without any separator edge.

³ Two simple drawings of a graph are *weakly isomorphic* if they have the same crossing edge pairs. For simple drawings of K_n , weak isomorphism is equivalent to having the same rotation system [10, 13].



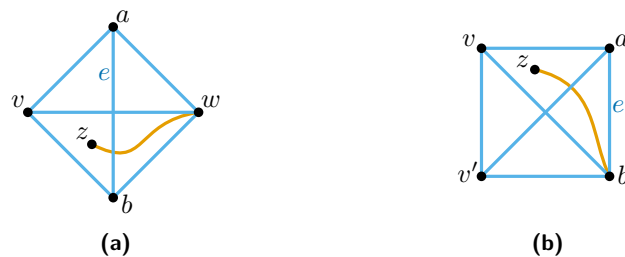
■ **Figure 7** The two simple drawings of K_8 that do not have a single separator edge.

We conclude this section by proving that all g -convex drawings are separable, therefore showing that our results on plane Hamiltonicity improve upon the work of Bergold, Felsner, M. Reddy, Orthaber, and Scheucher [8]. Our proof is inspired by the proof of Arroyo, Richter, and Sunohara [7] that all so-called hereditarily convex drawings (of K_n) are pseudospherical.

► **Theorem 14.** *Every g -convex drawing (of K_n) is separable.*

Proof. We show that every edge $e = \{a, b\}$ in a g -convex drawing \mathcal{D} is a separator edge. If e is uncrossed, then it is a separator edge by Observation 4. Hence, we can assume that e is crossed by at least one edge. In the following we find a simple closed curve γ_e fulfilling Item 2 of Lemma 5, thereby showing that e is a separator edge. In particular, we find vertex sets V_L and V_R that will correspond to the vertices on the two sides of γ_e , respectively.

We fix an orientation of e and say that a vertex v of \mathcal{D} lies on *the left* or on *the right* of e if the convex side of the triangle spanned by e and v lies to the left or right of the oriented edge e , respectively. Recall that both sides of such a triangle can be convex and that the convex side is unique if and only if it is part of a crossing $K_4 = \{a, b, v, w\}$. In this K_4 , e can either be a diagonal or a boundary edge. In the first case v and w lie on different sides of e and in the second case they lie on the same side of e ; see Figure 8 for an illustration.



■ **Figure 8** In a crossing K_4 , e is either (a) a diagonal edge or (b) a boundary edge.

We start with $V_L = V_R = \{a, b\}$. In a first step, we consider crossing K_4 's where e is a diagonal and we add the respective vertices v that are to the left of e to V_L . Since \mathcal{D} is g -convex, the respective vertices w that are to the right of e will never be added to V_L . In a second step, we successively add vertices v to V_L if there exists a crossing $K_4 = \{a, b, v', v\}$

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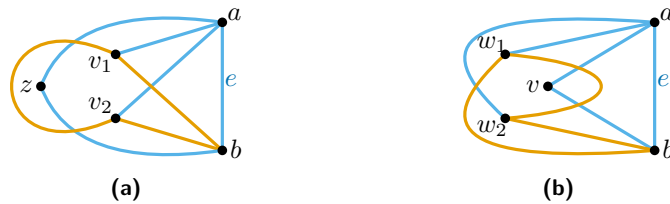
such that e is a boundary edge and v' was already added to V_L before. Once we cannot add anymore vertices to V_L in this manner, we add all remaining vertices to V_R . Note that for all vertices v in V_L the unique convex side of the triangle spanned by v and e is to the left of e .

Let \mathcal{D}_L and \mathcal{D}_R be the subdrawings of \mathcal{D} induced by the vertices V_L and V_R , respectively. Note that e is uncrossed in both those subdrawings. Consider the cell F_∞ in \mathcal{D}_L that is incident to e and to its right. We show that all vertices of V_R lie in F_∞ .

Assume first that a vertex z of V_R lies in the convex side of a triangle spanned by e and a vertex v added to V_L in the first step, that is, there is an edge $\{v, w\}$ that crosses e . Then, by convexity, the edge $\{z, w\}$ must lie in the crossing side of the K_4 spanned by $\{v, a, w, b\}$ and hence $\{z, w\}$ also crosses e ; see Figure 8(a) for an example. But this implies that z lies to the left of e and was added to V_L in the first step; a contradiction to $z \in V_R$.

Assume next that z lies in the unique convex side of a triangle added to V_L in the second step but not in one added in the first step. Then either $\{z, a\}$ or $\{z, b\}$ has to cross a triangle spanned by e and a vertex v' in V_L ; see Figure 8(b) for an example. Therefore, z is added to V_L in the second step; again a contradiction.

Assume last that z lies neither in F_∞ nor in any of the triangles spanned by e and V_L . Then the edges $\{z, a\}$ and $\{z, b\}$ cannot cross any of those triangle edges as argued before. Further, no vertex of V_L can lie in the triangle spanned by z and e to the right of e because we would not have added it to V_L then. Since z does not lie in F_∞ , some part of \mathcal{D}_L separates it from there. Hence, there is an edge $f = \{v_1, v_2\}$ in \mathcal{D}_L crossing $\{z, a\}$ or $\{z, b\}$. Since v_1 and v_2 are on the same side of the triangle $\{z, a, b\}$ and f does not cross e , f crosses both $\{z, a\}$ and $\{z, b\}$; see Figure 9(a) for an illustration. Consequently, the triangles $\{v_1, v_2, a\}$ and $\{v_1, v_2, b\}$ have no convex side; a contradiction to \mathcal{D} being g-convex.



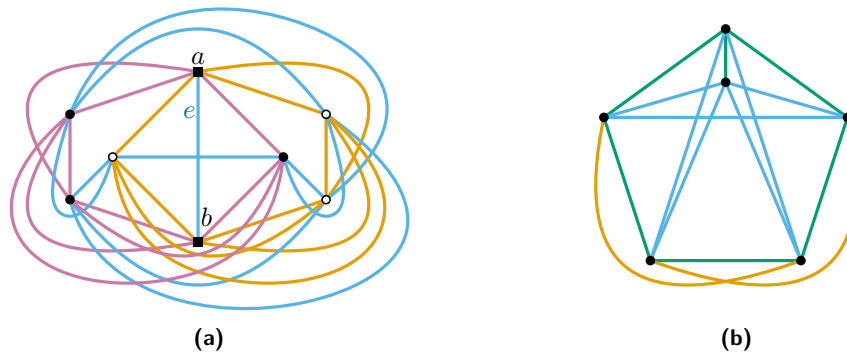
■ **Figure 9** Two situations leading to a triangle (marked orange) not having a convex side.

It remains to show that no edge $f = \{w_1, w_2\}$ of \mathcal{D}_R can cross any edge of \mathcal{D}_L . We first show that f cannot cross any edge incident to e . If f itself is incident to e , this cannot happen similar to before. Otherwise, since both w_1 and w_2 are in F_∞ , f has to pass through a triangle spanned by e and a vertex v of V_L ; see Figure 9(b) for an illustration. This results in the triangles $\{w_1, w_2, a\}$ and $\{w_1, w_2, b\}$ not having a convex side. Consequently, f can only cross some edge in \mathcal{D}_L that is independent to e . But then f has to cross that edge more than once. This is a contradiction in both cases.

Hence, we can close the edge e in F_∞ close to the boundary of \mathcal{D}_L to a simple curve γ_e that fulfills all properties of a witness, that is, e is a separator edge. ◀

Note that we could not just add *all* vertices to the left or right of e to V_L or V_R , respectively. Figure 10(a) shows an example where this would not result in two separated subdrawings.

Further, to see that separable drawings are not only the union of g-convex and 2-page book drawings, for example, consider a straight-line drawing with at least 5 vertices $\{v_1, \dots, v_k\}$ on the convex hull and reroute the edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ outside of the convex hull. This is always separable, not g-convex, and in most cases also not weakly isomorphic to any 2-page book drawing. Figure 10(b) shows the smallest such example.



■ **Figure 10** (a) A g-convex drawing where a complete left-right splitting via convex sides is not possible. (b) A separable drawing that is neither g-convex nor a 2-page book drawing.

5 Recognition

We have shown in Section 2 that for simple drawings \mathcal{D} of the complete graph K_n being separable is a property of the rotation system (Item 3 of Lemma 5). From this we obtain a polynomial time recognition algorithm for separable drawings (and rotation systems) of K_n .

► **Theorem 15.** *It can be decided in $\mathcal{O}(n^6)$ time whether a given simple drawing \mathcal{D} of K_n is separable.*

Proof. We check, for each edge $e = \{v, w\}$ in \mathcal{D} , whether it is a separator edge. If e is uncrossed, then it is a separator edge by Observation 4 and we are done. Otherwise we use the relation between separator edges and flips in rotation systems given by Item 3 of Lemma 5 to determine all possible flips of e in the rotation system. Recall that by Lemma 5, separability of a simple drawing \mathcal{D} of K_n is a property of the rotation system of \mathcal{D} . Given the drawing \mathcal{D} , its rotation system can be computed in $\mathcal{O}(n^2)$ time in a straight forward way. In the following, we use this rotation system to check separability.

By the definition of a flip of e in the rotation system, the subsets in the counter-clockwise rotation of v and the clockwise rotation of w between the position of e before and after the flip must coincide. We get all such possibilities of potential flips for e in $\mathcal{O}(n)$ time, by going through the rotations of v and w in parallel (starting with $\{v, w\}$) and keeping a parity list of all vertices how often they appeared in the subsets. A counter is used to see how many of the vertices appeared an odd number of times, that is, showed up in only one of the two subsets so far. Every time this counter is zero we have a potential flip.

Checking whether the new rotation system after a potential flip is realizable takes $\mathcal{O}(n^3)$ time because we only need to test all 5-tuples that contain the flipped edge [15]. Further, checking whether the flipped edge has all different crossings from the original edge takes $\mathcal{O}(n^2)$ time by testing for all $\mathcal{O}(n^2)$ new crossings whether they also existed before.

In total there are $\mathcal{O}(n^2)$ many edges e , each of them has $\mathcal{O}(n)$ potential flips, and testing whether such a flip yields a witness for e takes $\mathcal{O}(n^3)$ time as argued. Hence, we can decide in $\mathcal{O}(n^6)$ time whether a simple drawing of K_n is separable. ◀

Unfortunately, the situation is very different for simple drawings of arbitrary graphs. In particular, we construct simple drawings of matchings in the following where it is NP-hard to decide whether they are separable. For this we use a reduction from linked planar 3-SAT with negated edges on one side, which was shown to be NP-hard by Pilz [19, Theorem 10].

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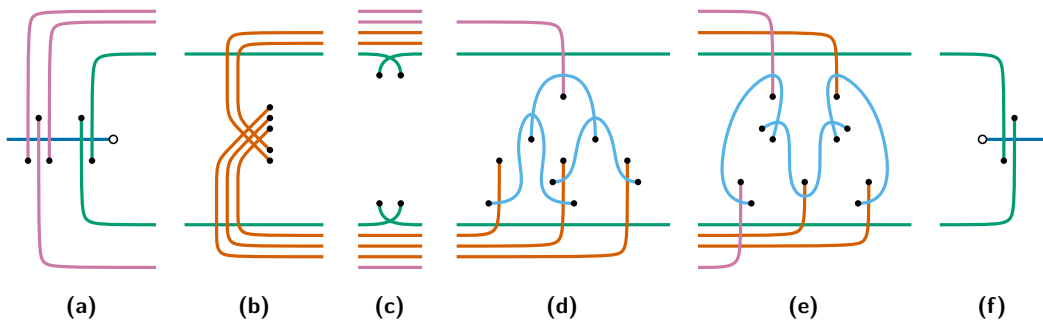
The incidence graph G_ϕ of a 3-SAT formula ϕ has one vertex for each variable and each clause in ϕ and an edge between a variable vertex and a clause vertex if the variable occurs in the clause (as a positive or negative literal). If G_ϕ is a planar graph, then ϕ is a *planar* 3-SAT instance. For *linked* planar 3-SAT there is a Hamiltonian cycle \mathcal{C} that first visits all variable vertices and then all clause vertices such that the union of G_ϕ and \mathcal{C} is still a planar graph. Further, in the restriction “with negated edges on one side”, there exists an embedding of $G_\phi \cup \mathcal{C}$ such that all edges in G_ϕ corresponding to positive literals are drawn inside of \mathcal{C} and all edges corresponding to negative literals are drawn outside of \mathcal{C} .

► **Theorem 16.** *It is NP-complete to decide whether a given simple drawing of an arbitrary graph is separable.*

Proof. Given a 3-SAT formula ϕ that is an instance of linked planar 3-SAT with negated edges on one side, we construct a simple drawing \mathcal{D} containing a special edge e such that e is a separator edge if and only if ϕ is satisfiable. Therefore, it is NP-hard to decide whether e is a separator edge. Moreover, we show that all other edges in \mathcal{D} are definitely separator edges. Consequently, it is NP-hard to decide whether \mathcal{D} is separable. In Figure 11 we illustrate the individual gadgets of the following construction and in Figure 12 we show an example of the complete drawing \mathcal{D} corresponding to a small 3-SAT formula.

Given an embedding of the union of the incidence graph G_ϕ and the Hamiltonian cycle \mathcal{C} , we let e be the part of \mathcal{C} between the clause vertices and the variable vertices. We then add four *boundary* edges, close to \mathcal{C} and on both sides next to the variable and the clause part each, crossing e and crossing each other in the middle; see Figures 11(a), 11(c), and 11(f). Thereby we restrict the potential witness γ_e of e to be drawn within a strip close to \mathcal{C} .

For each edge of G_ϕ we also add an edge to \mathcal{D} . We call these edges *literal* edges. Instead of the variable vertices of G_ϕ we let the incident literal edges in \mathcal{D} cross in a grid such that edges for positive literals are drawn in one direction and those for negative literals in the other direction; see Figure 11(b). This is possible because \mathcal{C} splits those edges into inside and outside, respectively. In that way we force γ_e to either cross all positive or all negative literal edges of the corresponding variable. In the following, crossing the positive side encodes the variable being set to FALSE and vice versa.



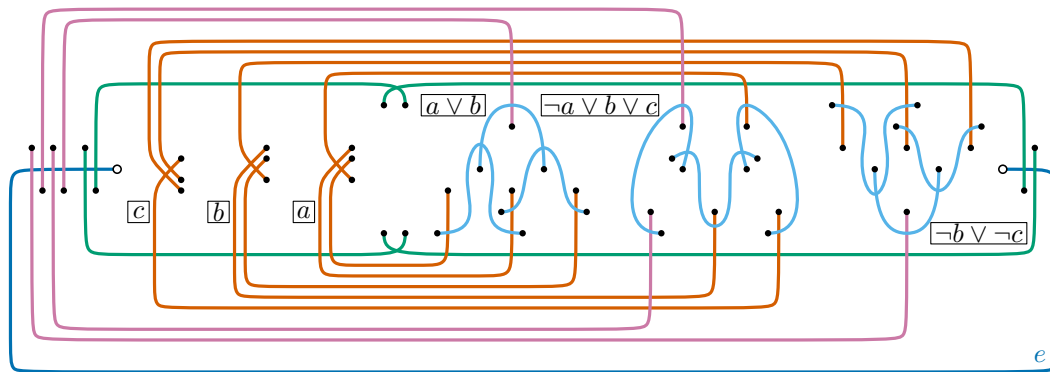
■ **Figure 11** The variable gadget (b) and the two clause gadgets (d) and (e). Boundary edges are drawn green, literal edges darkorange, auxiliary edges purple, and local edges lightblue.

For the clause variables of G_ϕ we construct special clause gadgets depending on how many positive/negative literals are in the clause. We can assume, for simplicity and without loss of generality, that all clauses contain exactly three literals (duplicating one literal if necessary). Hence we have two cases, either all literals are of the same type (negated or not) or two are of one type and one of the other. See Figures 11(d) and 11(e) for the two

constructions of clause gadgets. In addition to the literal edges, we need some *auxiliary* edges that cross e and a boundary edge, and some *local* edges in the gadgets. Since auxiliary edges cross e , they cannot be crossed by γ_e again. Further, literal edges can only be crossed if they were not yet crossed in the variable gadget, that is, if they have the value TRUE. Finally, γ_e can pass through a clause gadget without crossing any local edge twice if and only if it can cross at least one literal edge.

So far we have shown that e is a separator edge if and only if ϕ is satisfiable. It remains to show that all other edges in the construction are separator edges in any case. The local edges can be closed locally within the gadget. The boundary edges can be closed next to the boundary edge on the other side of the strip. Further, we let the auxiliary edges in the construction cross e in reverse order (nested) to how they enter the strip within the boundary edges. Therefore they pairwise do not cross and can be closed outside of the other side of the strip. For the literal edges we go back next to the boundary on the other side of the strip. Since G_ϕ is planar, we can cross all other literal edges except for those corresponding to the same variable. In each clause gadget we can cross the auxiliary edge or at least one of the three literal edges. Between the clause and variable gadgets (Figure 11(c)) we change sides and then cross all literal edges on the same side.

This finishes the proof for NP-hardness. For NP-completeness observe that a witness set for \mathcal{D} can be encoded and checked in polynomial space and time. ◀



■ **Figure 12** The simple drawing \mathcal{D} corresponding to the 3-SAT formula $(a \vee b) \wedge (\neg a \vee b \vee c) \wedge (\neg b \vee \neg c)$ as an instance to decide whether the edge e (darkblue) is a separator edge. The first and third clause use the same gadget, just upside down, and the literals a and $\neg c$, respectively, are duplicated to have exactly 3 literals in all clauses.

6 Future Work

Extending the question whether all crossing-minimizing drawings of K_n are g-convex [6] we ask the following.

► **Question 17.** *Is every crossing-minimizing drawing of an arbitrary graph separable?*

Further, we showed NP-hardness for recognizing separable drawings. The corresponding question for pseudospherical drawings [7] is still open to the best of our knowledge.

► **Question 18.** *Is it NP-hard to decide whether a simple drawing of an arbitrary graph is pseudospherical?*

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