Rectilinear Crossing Number of Graphs Excluding a Single-Crossing Graph as a Minor

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Abstract

The rectilinear crossing number of G is the minimum number of crossings in a straight-line drawing of G. A single-crossing graph is a graph whose crossing number is at most one. We prove that every n-vertex graph G that excludes a single-crossing graph as a minor has rectilinear crossing number $O(\Delta n)$, where Δ is the maximum degree of G. This dependence on n and Δ is best possible. The result applies, for example, to K_5-minor-free graphs, and bounded treewidth graphs. Prior to our work, the only bounded degree minor-closed families known to have linear rectilinear crossing number were bounded degree graphs of bounded treewidth as well as bounded degree $K_{3,3}$-minor-free graphs. In the case of bounded treewidth graphs, our $O(\Delta n)$ result is again tight and it improves on the previous best known bound of $O(\Delta^2 n)$ by Wood and Telle, 2007.

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1 Introduction

In this article graphs are undirected, simple, and finite, unless stated otherwise. For a graph G, with vertex set $V(G)$ and edge set $E(G)$, let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. [1]

A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A drawing is *rectilinear* if each edge is a line segment. [2] A *crossing* is a point of intersection between two edges (other than a common endpoint). A drawing with no crossings is *crossing-free*. A graph is *planar* if it has a crossing-free drawing. The *crossing number* of a graph G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in any drawing of G. The *rectilinear crossing number* of a graph G, denoted by $\overline{\mathsf{cr}}(G)$, is the minimum number of crossings in any rectilinear drawing of G.

Crossing number is a fundamental and extensively studied graph parameter with wide ranging applications and rich history (see the survey by Schaefer [37] for over 700 references on the crossing number and its variants). Computationally the problem of determining the

[1] For each vertex v of G, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ be the neighbourhood of v in G. The *degree* of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. Let $\Delta(G)$ be the maximum degree of G. When the graph is clear from the context, we will sometimes write $\deg(v)$ instead of $\deg_G(v)$ and Δ instead of $\Delta(G)$.

[2] Rectilinear drawings are also known as *straight-line drawings* in the literature.

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crossing number of a given graph, is notoriously difficult. Computing the crossing number is NP-hard by Garey and Johnson [19], even for planar graph plus an edge [8]. It is hard to approximate even for cubic graphs [7] and until recently there were no approximation algorithms with sub-polynomial in n approximation factor even for bounded degree graphs [10]. On the positive side, Kawarabayashi and Reed [23] give an $O(f(k)\cdot|G|)$ algorithm for deciding whether a given graph G has crossing number at most k.

Since computing the exact, or even an asymptotic, crossing number of a graph is hard, a great deal of past research has been focused on deriving asymptotic bounds. Regardless of the applications, be it visualization or circuit design [4, 26, 27], having as few crossings as possible is a desirable property in a drawing of a graph. This naturally leads to a study on upper bounds and lower bounds on the crossing number of various graph families.

Trivially, the (rectilinear) crossing number of every graph G is at most $O(\|G\|^2)$. For some graphs this bound is asymptotically tight, including for example, the complete graph and a random cubic graph. For others, it is far from optimal. We would like to understand what classes of graphs have rectilinear crossing number better than $O(\|G\|^2)$ and by how much. To understand that, we consider next the known lower bounds on the crossing number, which clearly apply to rectilinear crossing number.

Every planar triangulation G is known to have $3|G|-6$ edges (if $|G|\geqslant 3$). Consequently, for every $n\geqslant 3$, there are n-vertex graphs with $3n-6$ edges that can be drawn with zero crossings. The following result, known as the *crossing lemma*, tells us that as soon as a graph has a little more than $3|G|$ edges, it must have vastly more than zero crossings in every drawing. Specifically, the crossing lemma, proposed by Erdős and Guy [18], proved by Leighton [28] and Ajtai et al. [3]; and, subsequently improved [2, 29, 31] states the following.

► **Theorem 1.** *For any $\epsilon>0$, there exists c_ϵ such that, every graph G with $\|G\|>(3+\epsilon)\cdot|G|$ edges,* $\mathsf{cr}(G)\geqslant c_\epsilon\frac{\|G\|^3}{|G|^2}$

An immediate consequence of this theorem is that all graphs G that have at least $(3+\epsilon)\cdot|G|$ edges have crossing number at least $\Omega(|G|)$. We say that a family of graphs has a *linear crossing number* if there exists a constant c such that every graph G in the family has $\mathsf{cr}(G)\leqslant c\cdot|G|$. The crossing lemma tells us that for most graph families the best one can hope of is that they have linear crossing number. By the crossing lemma, any family of graphs that has members with the number of edges superlinear in the number of vertices cannot have linear crossing number. Thus the only candidates for linear crossing number are families of graphs whose members all have a linear number of edges. One example of such a family is a family of graphs whose members have the maximum degree bounded by a constant. Another example is a family of graphs that exclude some fixed graph H as a minor. [3] Kostochka [24] and Thomason [39] proved independently that H-minor free graphs G have $O(|H|\sqrt{\log|H|}\cdot|G|)$, and thus a linear, number of edges.

It turns out that neither of these two families of graphs have linear crossing number. Consider, for example, the graph $K_{3,n}$. It has a linear number of edges and it is K_5-minor-free, yet it is known to have crossing number $\Omega(n^2)$ [30, 32]. Similarly, consider the family of all cubic graphs. They have a linear number of edges and yet, for every large enough n, there is a cubic n-vertex graph whose crossing number is $\Omega(n^2)$ [25, 13, 14, 28].

[3] Let vw be an edge of a graph G. Let G' be the graph obtained by identifying the vertices v and w, deleting loops, and replacing parallel edges by a single edge. Then G' is obtained from G by *contracting* vw. A graph X is a *minor* of a graph G if X can be obtained from a subgraph of G by contracting edges. A graph G is *X-minor-free* if X is not a minor of G. A family of graphs $\mathcal{F}$ is *minor-closed* if $G\in\mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. $\mathcal{F}$ is *proper* if it is not the family of all graphs.

Thus to admit a linear crossing number, it is not enough for a family of graphs to have a bounded degree only or to only exclude a fixed graph as a minor. Having both of these properties however is enough. Böröczky et al. [22] first showed such a result for bounded Euler genus graphs (see Theorem 2). Note that by the above mentioned result by Kostochka [24] and Thomason [39], for every graph G from a proper minor closed family of graphs $||G|| \in O(|G|)$. That fact will be used throughout this article, starting with the next theorem.

▶ **Theorem 2** ([1, 22]). *For every integer $\gamma \geqslant 0$, there is a function f such that every graph G with Euler genus γ has crossing number $\mathsf{cr}(G) \leqslant f(\gamma)\cdot\sum_{v\in V(G)} \deg(v)^2 \leqslant 2\cdot f(\gamma)\cdot\Delta(G)\cdot||G|| \in O(f(\gamma)\cdot\Delta(G)\cdot|G|)$*

An improvement on the dependence on γ in Theorem 2 for orientable surfaces was shown by Djidjev and Vrťo [15], with $\mathsf{cr}(G) \leqslant c\cdot\gamma\cdot\Delta(G)\cdot|G|$ for some constant c. Wood and Telle [41] were the first to show that excluding a minor and bounding the maximum degree were sufficient to ensure a linear crossing number, as stated in the next theorem.

▶ **Theorem 3** ([41]). *For every graph H, there is a constant $c := c(H)$ such that every H-minor-free graph G has crossing number $\mathsf{cr}(G) \leqslant c\cdot\Delta(G)^2\cdot|G|$*

Theorem 3 was improved by Dujmović et al. [16] by reducing $\Delta(G)^2$ to $\Delta(G)$.

▶ **Theorem 4** ([16]). *For every graph H, there is a constant $c := c(H)$ such that every H-minor-free graph G has crossing number $\mathsf{cr}(G) \leqslant c\cdot\Delta(G)\cdot|G|$*

In addition, the result in Theorem 4 was shown to have the best possible dependence of $\Delta(G)$ and $|G|$. These results show that we know very strong, in fact best possible, bounds on the crossing number of all proper minor-closed families of graphs of bounded degree.

Much less is known about the rectilinear crossing number. Fáry [21] and Wagner [40] proved independently that every planar graph has a rectilinear drawing with no crossings. Hence, every planar graph G has the rectilinear crossing number 0, and thus for planar graphs G, $\overline{\mathsf{cr}}(G) = \mathsf{cr}(G)$. One may be tempted to conjecture that the rectilinear crossing number and crossing number are tied. However, that is not the case. In particular, Bienstock and Dean [5] proved that for every m and every $k \geqslant 4$, there exists a graph G with $\mathsf{cr}(G) = k$, but $\overline{\mathsf{cr}}(G) \geqslant m$. Therefore, Theorem 3 and Theorem 4 do not imply that bounded degree proper minor-closed families of graphs have linear rectilinear crossing number.

In fact, in addition to planar graphs, we are only aware of the following two meinor-closed families of bounded degree admitting linear rectilinear crossing number. The first is the result on $K_{3,3}$-minor-free graphs by Dujmović et al. [16].

▶ **Theorem 5** ([16]). *Every $K_{3,3}$-minor-free graph G has rectilinear crossing number $\overline{\mathsf{cr}}(G) \leqslant \sum_{v\in V(G)} \deg(v)^2 \leqslant 2\cdot\Delta(G)\cdot||G|| \in O(\Delta(G)\cdot|G|)$*

The second is a result on the convex crossing number of bounded treewidth graphs by Wood and Telle [41]. Rectilinear drawings where vertices are required to be in convex positions are called *convex drawings*. For a graph G, the minimum number of crossings over all convex drawings of G is called *convex crossing number* of G and is denoted by $\mathrm{cr}^*(G)$. Clearly, for every G, $\mathsf{cr}(G) \leqslant \overline{\mathsf{cr}}(G) \leqslant \mathrm{cr}^*(G)$.

▶ **Theorem 6** ([41]). *Every graph G of treewidth k has convex crossing number $\mathrm{cr}^*(G) \in O(k^2\cdot\Delta(G)^2\cdot||G||) \in O(k^3\cdot\Delta(G)^2\cdot|G|)$*

In the case of the rectilinear crossing number a stronger bound is known but still with a quadratic dependence on $\Delta(G)$ in the worst case.

► **Theorem 7** ([16]). *Every graph G of treewidth k has rectilinear crossing number $\overline{\mathsf{cr}}(G) \in O(k \cdot \Delta(G) \cdot \sum_{v \in V(G)} \deg(v)^2) \in O(k \cdot \Delta(G)^2 \cdot |G|)$*

Our goal in this article is to extend this result to much wider minor-closed families of graphs of bounded degree, in particular, a family of graphs that exclude a single-crossing graph as a minor. *A single-crossing graph* is a graph whose crossing number is at most one. Single-crossing minor-free graphs have been studied by the algorithms community [17, 12, 9] where at times these results were precursors to algorithms and techniques applicable to more general minor-closed classes [12, 11]. $K_{3,3}$, K_5 and every planar graph are examples of single-crossing graphs. Note however that a minor of a single-crossing graph is not necessarily a single-crossing graph itself (see [12] for easy examples). Note finally that a graph excluding a single-crossing graph as a minor may have arbitrarily large crossing number. For example, any n-vertex graph G, composed of disjoint union of $\lfloor \frac{n}{6} \rfloor$ copies of $K_{3,3}$, excludes K_5 as a minor (K_5 is a single-crossing graph) and yet the crossing number of G is $\Theta(n)$.

The following theorem is our main result.

► **Theorem 8.** *Let X be a single-crossing graph. There exists a constant $c := c(X)$, such that every X-minor-free graph G has the rectilinear crossing number of at most $c \cdot \Delta(G) \cdot |G|$.*

The dependence on Δ and $|G|$ in the above theorem is best possible. A standard lower bound constructions implies it (see for example [22, 41]). Specifically, consider a graph comprised of the disjoint union of $\Omega(n/\Delta)$ copies of $K_{3,3}$ where each $K_{3,3}$ is transformed into a maximum degree Δ graph by adding $\Omega(\Delta)$ paths of length two between every pair of adjacent vertices in each copy of $K_{3,3}$. This graph has maximum degree Δ and is still K_5-minor-free (and thus single-crossing minor-free) and yet has crossing number $\Omega(\Delta \cdot n)$. It also has treewidth at most 5. Thus the following two corollaries of Theorem 8 are both tight. Since K_5 is a single-crossing graph, the following is an immediate corollary of Theorem 8.

► **Corollary 9.** *There exists a constant c such that every K_5-minor-free graph G has a rectilinear crossing number of at most $c \cdot \Delta(G) \cdot |G|$.*

It is known that the family of graphs of treewidth at most k excludes a planar grid of size k^c as a minor (for some constant c) [35]. Since every planar graph is a single-crossing graph, Theorem 8 implies the following result.

► **Corollary 10.** *For every integer $k > 0$, there exists a constant c_k such that every graph G of treewidth at most k has a rectilinear crossing number at most $c_k \cdot \Delta(G) \cdot |G|$.*

In the process of proving the main result, Theorem 8, we establish a more precise upper bound for treewidth-k graphs: $k \cdot (k+2) \cdot \Delta(G) \cdot ||G||$. See Theorem 19.

This corollary improves the previous best known bound on rectilinear crossing number of bounded treewidth graphs from $O(\Delta(G)^2 \cdot |G|)$ (see Theorems 6 and 7 above) to the optimal $O(\Delta(G) \cdot |G|)$ bound. It should be noted however that Theorem 6 by Wood and Telle [41] gives an $O(\Delta(G)^2 \cdot |G|)$ bound for the convex crossing number of bounded treewidth graphs, and that bounds still stands as the best known for convex drawings.

In the next section, Section 2, we introduce key concepts that will be helpful in proving Theorem 8. In Section 3, we present the proof of Theorem 8. Finally, we conclude in Section 4.

2 Preliminaries

Recall that the definition of drawing of a graph has the following condition: "no three edges intersect at a common point (except at a common endpoint)". In relation to this, we define a set of points P to be in *general position* if no three points of P lie on one line and if no three line segments between pairs of points in P intersect in one point unless all three share a common endpoint. For the ease of presentation we will add to the definition of rectilinear drawings the condition that all endpoints of G are in general position.

2.1 Multigraphs

Our proof of the main result, Theorem 8, will require the use of multigraphs. Recall that a multigraph is a graph that may have parallel edges but no loops. For the remainder of this paper, we will use the term "multigraph" when parallel edges are allowed, and "graph" when they are not, i.e., when the graph is simple. The degree of a vertex v in a multigraph Q, denoted by $\deg_Q(v)$, is the number of edges of Q incident to v. However, unlike in simple graphs, $\deg_Q(v)$ is not necessarily equal to $|N_Q(v)|$.

A *rectilinear drawing* of a multigraph Q represents vertices, $V(Q)$, by a set of $|V(Q)|$ points in the plane in general position and represents each edge by a line segment between its endpoints. The general position assumption implies that the only vertices an edge intersects are its own endpoints, and no point in the drawing is in three *distinct* line segments (unless all three share a common endpoint). It should be noted that parallel edges between the same pair of vertices in such a drawing overlap, as they are represented by the same line segment. A *crossing-pair* is a pair of edges in a rectilinear drawing that do not have a common endpoint and whose line segments intersect at a common point. The number of crossings in a rectilinear drawing of a multigraph is the number of crossing-pairs in the drawing. The *rectilinear crossing number* of a multigraph Q, denoted by $\overline{\mathsf{cr}}(Q)$, is the minimum number of crossings over all rectilinear drawings of Q.

Note that by these definitions, a pair of overlapping edges in a rectilinear drawing of a multigraph is not considered a crossing-pair. This is because, in our main proof, we start with a rectilinear drawing of a certain multigraph Q. We eventually replace parallel edges (and their overlapping line segments) with incident edges (and thus line segments that have one endpoint in common) and such edges can never cross. Notice also that if one is allowed to replace line segments by arcs in a rectilinear drawing of a multigraph Q, then it is trivial to redraw Q such that the resulting "arc" drawing of Q has no overlapping edges and has the same number of crossings as the starting rectilinear drawing of Q. Finally, if Q is a simple graph, these definitions of rectilinear drawing and rectilinear crossing number are equivalent to the earlier ones for simple graphs.

2.2 Decompositions and Treewidth

For graphs G and H, an *H-decomposition* of G is a collection $(B_x \subseteq V(G) : x \in V(H))$ of sets of vertices in G (called *bags*) indexed by the vertices of H, such that

1. for every edge vw of G, some bag B_x contains both v and w, and
2. for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H.

The *width* of a decomposition is the size of the largest bag minus 1. The *adhesion* of a decomposition is the size of the largest intersection between two bags that share an edge in H. If H is a tree, then an H-decomposition is called a *tree decomposition*. The *treewidth* of a graph G is the minimum width of any tree decomposition of G. Tree decomposition and treewidth are key concepts in graph minor structure theory and they have been extensively studied ever since their introduction by Halin [20] and by Robertson and Seymour [34].

2.3 Rectilinear Drawings

In the process of proving our main result, Theorem 8 in Section 3.1, we will construct drawings of graphs where at one stage we will replace a point (representing some vertices) with disks that fulfill certain criteria. The following lemma will be helpful for that stage.

For any positive integer h, let $[h]$ denote the sequence of numbers $[1, \cdots, h]$. When it is clear from the context, we will make no distinction between a vertex v of a graph and the point that represents it in a drawing. Specifically, we will refer to both as v when no confusion can arise. The same will be true for an edge e and the line segment representing it in a drawing.

▶ **Lemma 11.** *Let D be a rectilinear drawing of any graph G. Then for each vertex $w \in V(G)$, there exists a disk C_w of positive radius centered at w such that the following is true. Let $v_1, \ldots, v_d$ be the neighbours of w in G. Let P_w be* any *set of at most d points in C_w such that $V(G) \cup P_w$ is in general position. For each $i \in [d]$, replace the line segment $\overline{wv_i}$ of D by a line segment between v_i and any point in P_w. Denote that point by p_i. The resulting drawing D' (of the resulting graph G') has the following properties:*

1. *Any two edges in G, neither of which is incident to w, cross in D' if and only if they cross in D.*
2. *For each $i \in [d]$, the edge wv_i and any edge xy where $\{x, y\} \subseteq V(G) - \{w, v_i\}$ cross in D if and only if v_ip_i and xy cross in D'.*
3. *All the remaining crossings in D' are crossings between pairs of segments with distinct endpoints in P_w.*

It should be noted that if $|P_w| = 1$, that is if P_w has exactly one point, then D' is a rectilinear drawing of G where a pair of edges of G cross in D' if and only if they cross in D.

Proof. Start with the drawing D of G and a disk C centered at w such that the only parts of D that intersect C are w and the edges incident to w. Then, for each $i \in [d]$, let S_i be the union of all possible line segments from v_i to any point in C. Let S denote the union of all S_i, $i \in [d]$. By reducing the radius of C to some positive radius r and then redefining S accordingly, the following becomes true for D and C.

- No vertex of G is in S other than $w, v_1, \ldots, v_d$.
- For each $i \in [d]$, the only vertices of G that are in S_i are v_i and w.
- For each $i \in [d]$, the only crossing points of D in S_i are crossings between wv_i and the edges not incident to w in G.
- No segment between two crossings in D is fully contained in S, unless it lies on one of the edges wv_i, $i \in [d]$.

Such a positive radius r exists by continuity and the resulting disk meets the conditions imposed on C_w. ◀

3 Main Result

In order to prove our main result, Theorem 8, we will use, as one of the tools, Robertson and Seymour's structure theorem for graphs that exclude a single-crossing graph as a minor [33]. This structure theorem uses the notion of clique-sum, that we define next. Let G_1 and G_2 be two disjoint graphs. Let $C_1 = \{v_1, v_2, \cdots, v_k\}$ be a clique in G_1 and $C_2 = \{w_1, w_2, \cdots, w_k\}$ be a clique in G_2, each of size k, for some integer $k \geqslant 1$. Let G be a graph obtained from G_1 and G_2 by identifying v_i and w_i for each $i \in [k]$ and possibly deleting some of the edges $u_i u_j$ in the resulting clique $C = \{u_1, u_2, \cdots, u_k\}$ of G. Then we say that G is *obtained* by *k-clique-sums* of graphs G_1 and G_2 (at C_1 and C_2). A *($\leqslant k$)-clique-sum* is an l-clique-sum for any $l \leqslant k$. The following theorem by Robertson and Seymour [33] describes a structure of graphs that exclude a single-crossing graph as a minor. [4]

▶ **Theorem 12** ([33]). *For every single-crossing graph X, there exists a positive integer $t := t(|X|)$ such that if G is an X-minor-free graph, then G can be obtained by $(\leqslant 3)$-clique-sums of graphs $G_1, \ldots, G_h$ such that for each $i \in [h]$, G_i is a planar graph (with no separating triangles) or the treewidth of G_i is at most t.*

The graphs $G_1, \cdots, G_h$ in Theorem 12 are called the *pieces* of the decomposition.

Theorem 12 is equivalent to stating that every X-minor-free graph G has a tree decomposition of adhesion at most 3 such that the vertices in each bag of the decomposition induce in G either a planar graph (with no separating triangles) or a graph of treewidth at most t. Armed with these notions, we are now ready to state a more precise version of our main result.

▶ **Theorem 13.** *Let X be a single-crossing graph. Let G be an X-minor-free graph and let $t := t(|X|)$ be the integer from Theorem 12. Then $\overline{\mathsf{cr}}(G) \leqslant 3 \cdot (t^2 + 2t + 2) \cdot \Delta(G) \cdot ||G||$.*

Theorem 13 is a strengthened version of Theorem 8 by Theorem 12 and the fact that $\|G\| \in O(|X|\sqrt{\log |X|} \cdot |G|)$ [24, 39] (as discussed earlier). Hence, the remainder of this section will be dedicated to proving Theorem 13. To do so, one has to be able to produce rectilinear drawings of the pieces, $G_1, \ldots, G_h$, of the decomposition (from Theorem 12) with the claimed number of crossings and then combine these drawings by conducting clique-sums. The following is a sketch of the two main steps our proof will take.

Step 1. Foremost, Theorem 13 has to be true for the pieces G_i of the decomposition, namely the planar graphs and bounded treewidth graphs. By the Fáry-Wagner theorem [21, 40], we know that Theorem 13 is true for all planar graphs. In fact, it is true with bound zero for the rectilinear crossing number. On the contrary, if G_i is a bounded treewidth graph, the required $O(\Delta(G_i) \cdot |G_i|)$ bound on its rectilinear crossing number was not known prior to our work. Thus one of the goals of this paper is to prove that bound for bounded treewidth graphs as one of the necessary steps in the proof of Theorem 13.

Step 2. Suppose now that for each piece, G_i of the decomposition, we have already established the $O(\Delta(G_i) \cdot |G_i|)$ bound for the rectilinear crossing of G_i. The main goal then becomes demonstrating that the rectilinear drawings of $G_1, G_2, \ldots, G_h$ can be joined by performing clique-sums without increasing the number of crossings in the final drawing of G by too much. Prior to this work it was not known how to conduct clique-sums on rectilinear drawings while achieving that goal. In particular, we need to join rectilinear drawings of $G_1, G_2, \ldots, G_h$ in such a way that the resulting number of crossings in the rectilinear drawing of G is $O(\Delta(G) \cdot |G|)$.

[4] Note that the original statement of Theorem 12 by Robertson and Seymour [36] does not mention separating triangles. The reason such a statement can be made is that any planar graph G containing a separating triangle can itself be obtained by 3-clique-sums of two strictly smaller planar graphs, G_1 and G_2, where the clique-sum is performed on that separating triangle.

The main challenge for proving Theorem 13 is Step 2 above. To overcome that challenge, we introduce the notion of simplicial blowups of graphs. The use of these simplicial blowups however impacts Step 1. In particular, it is not enough anymore to prove that the pieces of the decomposition have $O(\Delta(G_i) \cdot |G_i|)$ rectilinear crossing number. We must prove a stronger condition, namely that the simplicial blowups of the pieces have such a rectilinear crossing number.

In Section 3.1 we introduce simplicial blowups and demonstrate how to achieve Step 2. In Section 3.2 we introduce graph partitions and present a helpful lemma for producing rectilinear drawings. In Section 3.3 and 3.4, we then prove that Step 1 above can be accomplished, or more precisely that simplicial blowups of planar graphs and bounded treewidth graphs have the desired rectilinear crossing number. Once those two steps have been achieved, we will conclude the proof of Theorem 13 in Section 3.5.

3.1 Bound for Rectilinear Crossing Number Using Clique-Sums

A multigraph Q is called a $(\leqslant k)$-*simplicial blowup* of a graph G if Q can be obtained from G by adding an independent set of vertices S to G, and performing the following steps for each vertex u in S:

1. Make u adjacent to all the vertices of some clique of size at most k of G
2. Add zero or more parallel edges between u and its neighbours in G.

and finally, once Steps 1 and 2 are conducted on all vertices of S, delete zero or more edges from each clique of G involved in Step 1.

Theorem 14 is the key technical tool of this paper. It shows how rectilinear drawings (of simplicial blowups) of the pieces of a decomposition can be combined into a rectilinear drawing of a graph obtained by clique-sums of the pieces, all while not increasing the final number of crossings by too much. The previous result on the crossing number of minor-closed families (Theorem 4) by Dujmović et al. [16], also had to deal with performing clique-sums on drawings while controlling the crossing number. Our proof of Theorem 14 is inspired by their proof. However the drawings produced by their theorem have many bends per edge and are thus far from rectilinear drawings.

The following theorem is stated in a form that is more general than we will require. Specifically, the theorem does not require the pieces G_i of the decomposition to be planar or of bounded treewidth. As such, Theorem 14 may be useful in future work on rectilinear crossing numbers of X-minor-free graphs where X is not necessarily a single-crossing graph and thus the pieces of the decomposition are the almost embeddable graphs from the Robertson and Seymour graph minor theory.

We say that a graph R is (k, c)-*agreeable* if for every induced subgraph R' of R and every $(\leqslant k)$-simplicial blowup R^* of R', $\overline{\text{cr}}(R^*) \leqslant c \cdot \Delta(R^*) \cdot ||R^*||$.

▶ **Theorem 14.** *Let c be a positive number, k a positive integer, and $G_1, \cdots, G_h$ a collection of graphs such that every G_i is (k, c)-agreeable. Then every graph G that can be obtained by $(\leqslant k)$-clique-sums of graphs $G_1, \cdots, G_h$ has rectilinear crossing number $\overline{\text{cr}}(G) \leqslant k \cdot (c+2) \cdot \Delta(G) \cdot ||G||$.*

Proof. Since clique-sums identify vertices, to avoid confusion, we will assume that the vertices of the final graph G have names and that each vertex in each piece G_i, $i \in [h]$ inherits its name from G. Thus vertices that are identified by clique-sums have the same name in the pieces involved. Consequently, there may be multiple vertices with the same name in the disjoint union of $G_1, G_2, \ldots G_h$.

We may assume that the indices $1,\ldots,h$ are such that for all $j \geqslant 2$, there exists a minimum i such that $i < j$ where G_i and G_j are joined at some clique C of G_i when constructing G. We define G_i to be the *parent* of G_j, with $P_j = V(C)$ being the *parent clique* of G_j. The parent clique of G_1 is the empty set. Note that, by the introductory paragraph of this proof, it makes sense to talk about the clique P_j as existing in both in G_i and in G_j.

Let T be a rooted tree with vertex set $\{1,\ldots,h\}$, where ij is an edge of T if and only if G_j is a child of G_i. Let T_i denote the subtree of T rooted at i and U_i be the set of the children of i in T.

For each $i \in [h]$, let $G_i^- = G_i - P_i$. Note that for each $v \in V(G)$, there is exactly one $i \in [h]$ such that v is in $V(G_i^-)$. Thus $V(G_1^-),\cdots,V(G_h^-)$ is a partition of $V(G)$. We say that a vertex v of G *belongs* to vertex i of T if $v \in G_i^-$. For each $i \in [h]$, let $G[T_i]$ denote the graph induced in G by the vertices of G that belong to the vertices of T_i, that is the graph induced in G by $\bigcup\{V(G_j^-) : j \in T_i\}$.

Defining the $(\leqslant k)$-simplicial blowups of pieces. To prove the theorem, we now define, for each G_i^-, $i \in [h]$, a specific $(\leqslant k)$-simplicial blowup, denoted by Q_i^-. To define Q_i^-, start with G_i^-. For each child G_j of G_i, add a new vertex c_j to G_i^-. We call c_j a *dummy* vertex and say that c_j *represents* G_j in Q_i^-. Note that for all $j \in [2,\cdots,h]$, there is exactly one $i < j$ such that Q_i^- has a vertex that represents G_j (namely, the vertex c_j). For the clarity of the next statement, note first that $V(G_i^-) \cap P_j$ is not empty as otherwise P_j would also exist in some G_f where $f < i$ and G_j would not be a child of G_i. For each edge $vw \in E(G)$, where $v \in V(G_i^-) \cap P_j$ and $w \in G_\ell^-$, where $i < \ell$ and $\ell \in V(T_j)$, connect v to c_j by an edge. Label that edge with the triple $(v, w, \mathcal{P}_{vw})$, where $\mathcal{P}_{vw}$ is the path in T from i to ℓ. We call the edge labelled $(v, w, \mathcal{P}_{vw})$ in Q_i^- an *isthmus* edge. It represents the edge vw in the final drawing of G. We consequently refer to the edge vw of G as isthmus edge as well. We say that two isthmus edges are *siblings* if they are adjacent to the same dummy vertex. For a vertex u in Q_i^- such that u is in P_j for some child G_j of G_i, we say that u is *involved* in a clique-sum in Q_i^-. Thus each isthmus edge of Q_i^- has an endpoint in G_i^- that is involved in some clique-sum in Q_i^-. We finally remove from Q_i^- the edges in P_j that are not in G. We set the resulting multigraph to be Q_i^-.

Notice that Q_i^- is a $(\leqslant k)$-simplicial blowup of G_i^-. Since G_i is (k, c)-agreeable (by the assumption) and since G_i^- is an induced subgraph of G_i, it follows that $\overline{\mathsf{cr}}(Q_i^-) \leqslant c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$.

For $i \in [h]$, consider a rectilinear drawing of Q_i^- with at most $c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$ crossings. We will construct the desired rectilinear drawing of G by joining these rectilinear drawings of Q_i^-. Consider for a moment solely the disjoint union of these rectilinear drawings. The resulting rectilinear drawing of the disjoint union has at most $\sum_{i\in[h]} c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$ crossings.

Notice that there is one-to-one mapping between the edges of G and the edges in the union of all $Q_1^-, Q_2^-, \ldots, Q_h^-$, that is in $\bigcup_{i\in[h]} E(Q_i^-)$ (where the isthmus edges of G map to the isthmus edges in the union and where the non-isthmus edges of G map to the non-isthmus edges of the union). Thus $||G|| = \sum_{i\in[h]} ||Q_i^-||$. Hence, if for all $i \in [h]$, $\Delta(Q_i^-) \leqslant k \cdot \Delta(G)$, the above sum would be upper bounded by $c \cdot k \cdot \Delta(G) \cdot \sum_{i\in[h]} ||Q_i^-|| = c \cdot k \cdot \Delta(G) \cdot ||G||$. This is akin to the upper bound that we want on the rectilinear crossing number of G. Thus we want to first bound the degree of each vertex in Q_i^- by $k \cdot \Delta(G)$. This is not completely obvious due to the addition of the dummy vertices in the construction of Q_i^- and also due to the fact that clique-sums allow for edge deletions from the cliques.

▷ Claim 1. For every $i \in [h]$ and every $v \in Q_i^-$, $\deg_{Q_i^-}(v) \leqslant k \cdot \Delta(G)$.

Proof. There are three cases to consider.

Case 1. *v is a dummy vertex of Q_i^-.*
By construction, for some $j \in U_i$, v represents some G_j and is adjacent to at most k vertices of the parent clique P_j in G_i. Each edge between a vertex $u \in P_j$ and v corresponds to an (isthmus) edge in G adjacent to u. Since $\deg_G(u) \leqslant \Delta(G)$, v is incident to at most $k \cdot \Delta(G)$ edges, giving $\deg_{Q_i^-}(v) \leqslant k \cdot \Delta(G)$.

Case 2: *v is in G_i^- (and thus not a dummy vertex) and v is not involved in any clique-sums.*
Then, it follows that $\deg_{Q_i^-}(v) \leqslant \deg_G(v)$.

Case 3: *v is in G_i^- (and thus not a dummy vertex) and is involved in at least one clique-sum.*
Consider every $j \in U_i$ such that $v \in P_j$. Then v has a least one neighbour in $G[T_j]$ and thus at least one edge connecting it to c_j, otherwise the clique-sum could have omitted v. Additionally, there exists a one-to-one mapping between the set of edges in G between v and its neighbours in $G[T_j]$ and the set of (parallel isthmus) edges between v and c_j in Q_i^-. In other words, there is a one-to-one mapping between the isthmus edges incident to v in Q_i^- and the isthmus edges incident to v in G. Finally, consider the non-isthmus edges incident to v in G_i^-. Each edge of G_i that has been removed in the construction of Q_i^- (namely the edges removed from P_j) was also removed in G, thus $\deg_{Q_i^-}(v) \leqslant \deg_G(v)$. ◁

With degrees of the vertices of Q_i^-sorted out, we are ready to describe how to construct a rectilinear drawing of G from the rectilinear drawings of $Q_1^-, Q_2^-, \dots, Q_h^-$.

Constructing the rectilinear drawing of G from the rectilinear drawings of $Q_1^-, Q_2^-, \dots, Q_h^-$. Since for each $i \in [h]$, Q_i^- is (k,c)-agreeable, $\overline{\mathsf{cr}}(Q_i^-) \leqslant c \cdot \Delta(Q_i^-) \cdot ||Q_i^-||$. By Claim 1, $\overline{\mathsf{cr}}(Q_i^-) \leqslant c \cdot k \cdot \Delta(G) \cdot ||Q_i^-||$. Let $D(Q_i^-)$ denote a rectilinear drawing of Q_i^- with at most $c \cdot k \cdot \Delta(G) \cdot ||Q_i^-||$ crossings. For the remainder of the proof, we will show how to construct a rectilinear drawing, $D(G)$, of G by combining the rectilinear drawings $D(Q_i^-)$ of Q_i^-, $i \in [h]$, such that the resulting number of crossings in $D(G)$ is as claimed in the theorem.

Note that removing dummy vertices (and their incident isthmus edges) from $D(Q_i^-)$ gives a rectilinear drawing of G_i^-. Denote these rectilinear drawings by $D(G_i^-)$. In the final drawing, $D(G)$, the drawing of each G_i^- will be identical to $D(G_i^-)$, possibly scaled and/or rotated. In other words, in $D(G)$, the implied rectilinear drawing of the disjoint union of $G_1^-, G_2^-, \dots, G_h^-$ will be the disjoint union of $D(Q_1^-), D(Q_2^-), \dots, D(Q_h^-)$ without the isthmus edges. The isthmus edges will be redrawn in this construction.

We will join the rectilinear drawings $D(Q_i^-)$, $i \in [h]$ in the order of their indices. For $\ell \in [h]$, D_ℓ denotes the rectilinear drawing obtained by joining $D(Q_1^-), D(Q_2^-), \dots, D(Q_\ell^-)$ (joining is detailed below). The rectilinear drawing D_h will thus be the desired rectilinear drawing $D(G)$ of G. While joining these drawings, we will maintain the invariant that for each $j > \ell$, such that the parent of G_j is some G_i with $i \in [\ell]$, the rectilinear drawing D_ℓ contains the representative dummy vertex (c_j) of each G_j. Furthermore we maintain that D_ℓ minus the dummy vertices (that is $D_\ell - \cup_{j>\ell} c_j$) is isomorphic to $G - \cup_{j>\ell} V(G[T_j])$.

We start by defining $D_1 = D(Q_1^-)$. D_1 satisfies the above invariant. For $j \in [2, \cdots, h]$ we construct D_ℓ from $D_{\ell-1}$ and $D(Q_\ell^-)$ as follows. By the invariant, $D_{\ell-1}$ has a dummy vertex c_ℓ representing G_ℓ. Let C_ℓ be a disk centered in the point c_ℓ in $D_{\ell-1}$ that meets the conditions of Lemma 11. Let $v_1, v_2, \dots, v_d$ be the neighbours of c_ℓ in $D_{\ell-1}$. Construct D_ℓ by the following steps.

1. Remove c_ℓ and its incident (isthmus) edges.
2. Scale down $D(Q_\ell^-)$. Place it inside C_ℓ and rotate it such that all the vertices of D_ℓ are in general position.
3. For each isthmus edge labelled with (x, y, P_{xy}) that was incident to c_ℓ, (re)draw it as the line segment from x to y if y in Q_ℓ^-. Otherwise, by construction, $D(Q_\ell^-)$ has a point c_j, $j > \ell$ and $y \in G[T_j]$. In that case, draw a line segment between x and c_j. By Lemma 11, the only new crossings (pairs) that this introduces are crossings between (a) a pair of (re)drawn sibling isthmus edges (that were both incident to c_ℓ) or (b) one such isthmus edge (incident to c_ℓ) and edges strictly inside the disk C_ℓ (that is, edges in G_ℓ^-).

The resulting drawing D_ℓ satisfies the invariant. Note that at the end of this process, when $\ell = h$, there are no more dummy vertices and each edge labelled (x, y, P_{xy}) in D_h is an actual line segment connecting vertex x and y in G and thus actually represents the isthmus edge xy of G. The final drawing D_h is a rectilinear drawing $D(G)$ of G. It remains to prove that $D(G)$ has the claimed number of crossings.

Before joining the drawings $D(Q_i^-)$, $i \in [h]$, the total number of crossings in the disjoint union of all drawings was at most $c \cdot k \cdot \Delta(G) \cdot ||G||$, as argued earlier. We name this quantity the *initial sum*. We now prove that joining these drawings into a drawing of G does not increase the initial sum by much. Specifically, we will show that all new crossings can be charged to the edges of G such that each edge is charged at most $2 \cdot k \cdot \Delta(G)$ new crossings, which will complete the proof.

By the construction, the new crossings must involve at least one isthmus edge. Consider such an isthmus edge e labelled (v, w, P_{vw}), where $v \in Q_i^-$ and $w \in Q_p^-$, $i < p$ and $w \in G[T_j]$ where $j \in U_i$ (and thus $p \in T_j$). There are four cases to consider.

Case 1. Consider first a crossing in $D(G)$ between e and a non-isthmus edge e' in Q_i^-. That crossing is already accounted for in the initial sum by the crossing in $D(Q_i^-)$ between e' and edge vc_j labelled (v, w, P_{vw}).

Case 2. Consider next a crossing between e and an isthmus edge e' labelled (x, y, P_{xy}), where $x \in Q_i^-$ and $y \in G[T_r]$, with $r \in U_i$. 2a) If $r \neq j$ (so e and e' are not sibling isthmus edges), then crossing between e and e' was accounted for as well in the initial sum by the crossing in $D(Q_i^-)$ between the edge vc_j labelled (v, w, P_{vw}) and the edge xc_r labelled (x, y, P_{xy}). 2b) If $r = j$, it must be that $v \neq x$ as otherwise e and e' cannot cross. By construction both w and y are in the disk C_r. In the construction of G, $G[T_j]$ is added via a ($\leqslant k$)-clique-sum to G_i (with parent clique P_j). Thus at most $k \cdot \Delta(G)$ (isthmus) edges cross the cycle bounding C_r. Thus e can be crossed by at most $k\Delta(G)$ such edges e'. We charge these at most $k \cdot \Delta(G)$ crossings to e.

Case 3. Consider next a crossing between e and any edge e' where both endpoints of e' are in $G[T_j]$. The endpoints of e' are thus in Q_a^- and Q_b^- where $j \leqslant a \leqslant b$. We charge the crossing to e'. (Think of that crossing being charged to e' in Q_a^-). As argued above, at most $k \cdot \Delta(G)$ (isthmus) edges cross the cycle C_a that replaced the dummy vertex c_a thus each such edge e' is charged at most $k \cdot \Delta(G)$ new crossings.

Case 4. Finally consider a crossing between e and an isthmus edge e' labelled (x, y, P_{xy}), where $x \in Q_f^-$ with $f < i$. Then there exists $g \in U_f$ such that $i \in T_g$. In that case both endpoints of e are in $G[T_g]$ and we are in Case 3 with the roles of e and e' reversed. Thus at most $k \cdot \Delta(G)$ crossings are charged to e.

By the arguments above, each edge of G is charged at most $2 \cdot k \cdot \Delta(G)$ new crossings (at most $k \cdot \Delta(G)$ in Case 2b and at most $k \cdot \Delta(G)$ in Case 4). Together with the initial sum that results in at most $(c + 2) \cdot k \cdot \Delta(G) \cdot ||G||$ crossings. ◀

3.2 Rectilinear Drawings of Multigraphs via Graph Partitions

As mentioned previously, in order to prove our main result, (Theorem 13), we will use as a main tool the theorem that we have just proved, Theorem 14. Theorem 12 tells us that in order to use Theorem 14, we need to show that planar graphs and bounded treewidth graphs are $(3, c)$-agreeable for some constant c. In this section, we define graph partitions and prove a lemma that will be helpful in proving that planar graphs are $(3, c)$-agreeable in Section 3.3 and that bounded treewidth graphs are (k, c)-agreeable for any $k \geqslant 1$ in Section 3.4.

An H-*partition* of a (multi)graph G is comprised of a graph H and a partition of vertices of G such that

- each vertex of H is a non-empty set of vertices of G (called a *bag*),
- every vertex of G is in exactly one bag of H, and
- if an edge of G has one endpoint in A and the other endpoint in B and A and B are distinct, then AB is an edge of H.

The *width* of a partition is the maximum number of vertices in a bag. The *density* of a bag of an H-partition is the number of edges of G with at least one endpoint in that bag. The *density* of an H-partition is the maximum density over all bags of H. A bag is said to be *solitary* if it contains exactly one vertex of G.

The proof of the following lemma is a slight modification of a similar result by Wood and Telle [41].

▶ **Lemma 15.** *Let K be a multigraph and H a simple graph such that K has an H-partition of width w and density d. Let X be the set of all vertices of K that are not in solitary bags of H. Then we have the following.*

1. $\overline{\mathsf{cr}}(K) \leqslant \overline{\mathsf{cr}}(H) \cdot w^2 \cdot \Delta(K)^2 + (w-1) \cdot \sum_{v \in X} \deg_K(v)^2$
2. *if H is planar, then*
   - **(a)** *there exists a rectilinear drawing of K with a most $2 \cdot d$ crossings per edge.*
   - **(b)** *if in addition, the non-solitary bags of H form an independent set in H, then there is a rectilinear drawing of K with at most d crossings per edge.*

Proof. We start with a rectilinear drawing $D(H)$ of H with $\overline{\mathsf{cr}}(H)$ crossings. Consider any vertex (bag) B of H. Let $C_\epsilon(B)$ be a disk of radius $\epsilon > 0$ centered at B in $D(H)$. For each edge AB of H, let $C_\epsilon(AB)$ be the region defined by the union of all the line segments with one endpoint in $C_\epsilon(A)$ and the other in $C_\epsilon(B)$. Note that there exists an ϵ small enough such that all of the following conditions are met:

- $C_\epsilon(A) \cap C_\epsilon(B) = \emptyset$ for all distinct bags A and B of H;
- $C_\epsilon(AB) \cap C_\epsilon(PQ) = \emptyset$ for every pair of edges AB and PQ of H that have no endpoints in common and do not cross in $D(H)$;
- $C_\epsilon(AB) \cap C_\epsilon(Q) = \emptyset$ for every triple of distinct bags A, B, Q of H where AB is an edge of H;
- For each crossing-pair of edges AB and PQ in $D(H)$, $C_\epsilon(AB) \cap C_\epsilon(PQ)$ is non-empty. We call that region, $C_\epsilon(AB) \cap C_\epsilon(PQ)$, of the plane *busy region* of pair AB and PQ. Finally, the busy regions of all distinct pair of edges are pairwise disjoint.

For each vertex v of K such that v is in a bag B of H, draw v as a point in $C_\epsilon(B)$ such that the final set of points representing $V(K)$ is in general position. Draw every edge of K straight. This defines a rectilinear drawing $D(K)$ of K, since no edge in $D(K)$ contains a vertex other than its own endpoints and no three edges of $D(K)$ cross at one point.

We first prove that the number of crossings in $D(K)$ is at most $\overline{\text{cr}}(H) \cdot w^2 \cdot \Delta(K)^2 + (w - 1) \cdot \sum_{v \in X} \deg_K(v)^2$ which will prove the first part of the theorem. Consider two crossing edges e and f in $D(K)$. There are two cases to consider (based on two types of crossings that can occur in $D(K)$).

- Case 1: there is bag B of H that has at least one endpoint of e and at least one endpoint of f. Order all the vertices of $B = \{v_1, v_2, \cdots, v_\ell\}$, $l \leqslant w$ such that $\deg_K(v_1) \leqslant \cdots \leqslant \deg_K(v_\ell)$. Let v_i be an endpoint of e and v_j and endpoint of f, $i < j$. We charge the crossing between e and f to v_j.
  Thus the number of crossings charged to v_j is at most
  $$\sum_{i<j} \deg_K(v_i) \cdot \deg_K(v_j) \leqslant \sum_{i<j} \deg_K(v_j)^2 \leqslant (\ell - 1) \cdot \deg_K(v_j)^2 \leqslant (w - 1) \cdot \deg_K(v_j)^2$$
  The vertices in the solitary bags of H are charged 0 crossings, rendering the total number of crossings in Case 1 is at most $(w - 1) \sum_{v \in X} \deg_K(v)^2$.
- Case 2: there is no bag of H that has both an endpoint of e and an endpoint of f. This implies that four endpoints of e and f are in four distinct bags, A, B, P, Q of H. Let $e \in C_\epsilon(AB)$ and $f \in C_\epsilon(PQ)$. Since e and f cross, their crossing point must be the busy region of AB and PQ. Denote that region by R. There are at most $\Delta(K) \cdot w$ edges of K drawn inside $C_\epsilon(AB)$ that intersect R and at most $\Delta(K) \cdot w$ edges of K drawn inside $C_\epsilon(PQ)$ that intersect R. We charge the crossings between these pairs of edges to the busy region R. Thus the number of crossings charged to R is at most $w\Delta(K) \cdot w\Delta(K) = w^2 \cdot \Delta(K)^2$. Since $D(H)$ has $\overline{\text{cr}}(H)$ crossings, there are exactly $\overline{\text{cr}}(H)$ busy regions determined by crossing edges in $D(H)$. Thus the total number of crossings in Case 2 is at most $\overline{\text{cr}}(H) \cdot w^2 \cdot \Delta(K)^2$.

Thus $\overline{\text{cr}}(K) \leqslant \overline{\text{cr}}(H) \cdot w^2 \cdot \Delta(K)^2 + (w - 1) \cdot \sum_{v \in X} \deg_K(v)^2$ as stated in part 1.

We now prove the second part of the theorem. In this case, H is planar. By the Fáry-Wagner theorem [40, 21], there is a rectilinear drawing $D(H)$ of H with no crossings. Starting with such crossing-free drawing $D(H)$, we produce a rectilinear drawing $D(K)$ of K using the algorithm described above. Let e be an edge of K with an endpoint in some bag A of H. We now prove that the number of crossings on e in $D(K)$ is at most $2d$ as claimed in part 2a. There are two cases to consider:

- Case 1: both endpoints of e are in A. Then, e is only crossed by the edges that have at least one endpoint in A. As there are at most d such edges, there is at most d crossings on e in $D(K)$.
- Case 2: the other endpoint of e is in a bag B of K distinct from A. Then, since $D(H)$ is crossing-free, e can only be crossed by the edges that have at least one endpoint in A or in B. There is at most $2d$ such edges, thus there is at most $2d$ crossings on e in $D(K)$.

In either case, e is crossed by at most $2d$ edges in $D(K)$ as required by part 2a.

Finally, consider the case when the non-solitary bags of H form an independent set in H. Let e be an edge of K. If two endpoints of e are in two distinct solitary bags of H then no edge of K crosses e since $D(H)$ is crossing-free. Therefore, in that case, trivially, there are at most d crossings on e in $D(K)$. Thus we may assume that at least one endpoint of e is in a non-solitary bag of H. Let A denote that bag. If the other endpoint of e is also in A, the result follows from Case 1 above. Therefore, we may assume that the other endpoint, v, of e is in a bag B of H distinct from A. B is then a solitary bag (by the independent set assumption). Since the edges incident to the same vertex (v in this case) cannot cross, the only edges that can cross e are those with an endpoint in A. There is at most d edges with endpoints in A and thus there are at most d crossings on e in $D(K)$. ◀

3.3 Rectilinear Crossing Number of Simplicial Blowups of Planar Graphs

Theorem 12 tells us that in order to use Theorem 14, it is enough to consider ($\leqslant 3$)-simplicial blowups of planar graphs with no separating triangles. In other words, it is enough to prove that planar graphs with no no separating triangles are $(3, c)$-agreeable for some constant c. The next lemma achieves that.

▶ **Lemma 16.** *Every planar graph G that has no separating triangles is $(3, 3)$-agreeable.*

Proof. Since every induced subgraph of G is also planar and with no separating triangles, it is enough to show that every ($\leqslant 3$)-simplicial blowup Q of G has rectilinear crossing number $\overline{\mathsf{cr}}(Q) \leqslant 3 \cdot \Delta(Q) \cdot ||Q||$.

Let $S = V(Q) - V(G)$. Since adding a 1-simplicial or 2-simplicial vertex to a planar graph results in a planar graph, we may assume that each vertex in S has exactly 3 neighbours in G. We now define an H-partition of Q. To start, we make H isomorphic to G and put each $v \in V(G)$ in the bag B_v in H. Currently, all the bags in H are solitary bags. Since G, and therefore the current H, has no separating triangles and since S is an independent set in Q, we have that for each $v \in S$, $N_Q(v)$ induces a face in an embedding of G and thus it is a face in the equivalent embedding of H. For each vertex set $\{x, y, z\}$ in H that forms such a face, we add a bag B_{xyz} adjacent to x, y and z in H. The resulting graph H is simple and planar. For each vertex $v \in S$ adjacent to x, y and z in Q, add v to the corresponding bag B_{xyz} in H. Thus the defined graph H and the assignment of the vertices of Q to its bags defines an H-partition of Q.

As every vertex of Q in bag B_{xyz} is adjacent to all vertices in $\{x, y, z\}$, the maximum number of edges of Q with an endpoint in a non-solitary bag B_{xyz} is at most $\deg_Q(x) + \deg_Q(y) + \deg_Q(z) \leqslant 3 \cdot \Delta(Q)$. The maximum number of edges of Q with an endpoint in a solitary bag of H is clearly $\Delta(Q)$. Thus the density of the H-partition is at most $3 \cdot \Delta(Q)$. Additionally, the non-solitary bags of H form an independent set in H which, by Lemma 15 (2b), implies that Q has a rectilinear drawing with at most $3 \cdot \Delta(Q)$ crossings per edge, giving the desired result, $\overline{\mathsf{cr}}(Q) \leqslant 3 \cdot \Delta(Q) \cdot ||Q||$. ◀

3.4 Rectilinear Crossing Number of Simplicial Blowups of Treewidth-*k* Graphs

In this section, we prove that bounded treewidth graphs are (k, c)-agreeable for some constants k and c. We start with the following trivial bound applicable to all graphs.

▶ **Lemma 17.** *Every graph G is $(|G|,\ |G| - 1)$-agreeable.*

Proof. If $|G| = 1$, the statement is trivial since every ($\leqslant 1$)-simplicial blowup of G is a star thus the crossing number of every such blowup is zero. Assume now that $|G| \geqslant 2$. Since every induced subgraph of G is also in the class of all graphs, it is enough to show that every ($\leqslant |G|$)-simplicial blowup Q of G has rectilinear crossing number $\overline{\mathsf{cr}}(Q) \leqslant (|G|-1) \cdot \Delta(Q) \cdot ||Q||$.

Let $S = V(Q) - V(G)$. We build an H-partition of Q as follows. Start with $H := K_2$ with $V(H) = \{v, w\}$. Place one vertex of G in B_v and all the remaining vertices of G in B_w. Add an independent set of $|S|$ of vertices to H and make each connected to v and w. It is simple to verify that H is a simple planar graph. Place each vertex of S in a new vertex (bag) of H. That defines an H-partition of Q where H is a simple planar graph and where all bags of H are solitary except for one bag, that is B_w. Trivially, that one non-solitary bag forms an independent set in H. Since H is planar and since the density of H is at most $(|G| - 1) \cdot \Delta(Q)$, we obtain the desired result, $\overline{\mathsf{cr}}(Q) \leqslant (|G| - 1) \cdot \Delta(Q) \cdot ||Q||$ by Lemma 15 (2b). ◀

The following result, obtained by setting $|G| = t$, is an immediate corollary of Lemma 17.

▶ **Corollary 18.** *The complete graph, K_t, is $(t,\ t-1)$-agreeable.*

We are now ready to prove that every bounded treewidth graph G has $\overline{\mathsf{cr}}(G) \in O(\Delta(G) \cdot |G|)$.

▶ **Theorem 19.** *For $k \geqslant 1$, let $\mathcal{G}$ denote a family of graphs of treewidth at most k. For every graph $G \in \mathcal{G}$, $\overline{\mathsf{cr}}(G) \leqslant k \cdot (k+2) \cdot \Delta(G) \cdot ||G||$.*

Proof. It is well known (see [6] for example) that G can be obtained by ($\leqslant k$)-clique-sums on graphs $G_1, G_2 \cdots$, where each G_i, is the complete graph on at most $k{+}1$ vertices. Corollary 18 implies that, for each i, $i \in [h]$, G_i is $(k+1,\ k)$-agreeable and thus $(k,\ k)$-agreeable. This fulfills the sole condition of Theorem 14. Thus $\overline{\mathsf{cr}}(G) \leqslant k \cdot (k+2) \cdot \Delta(G) \cdot ||G||$. ◀

Theorem 19 gives an $O(\Delta(G) \cdot |G|)$ bound for the rectilinear crossing number of bounded treewidth graphs G. As discussed in the introduction, the bound is optimal and it improves on the previously known bounds (see Theorems 6 and 7).

Since every k-simplicial blowup of any graph of treewidth at most k itself has treewidth at most k, we get the following immediate corollary of Theorem 19.

▶ **Lemma 20.** *For every positive integer k, every graph of treewidth at most k is $(k,\ k{\cdot}(k{+}2))$-agreeable.*

3.5 Proof of Theorem 13

Recall that Theorem 13 of Robertson and Seymour states that each piece in the decomposition is either a graph of treewidth at most t or it is a planar graph with no separating triangle. Lemma 16 then implies that every planar piece G_i of the decomposition is $(3, 3)$-agreeable. Consider the non-planar pieces of the decomposition. By Theorem 12, they have treewidth at most t, where $t \geqslant 3$, as graphs of treewidth at most 2 are planar [6]. Lemma 20 states that every treewidth at most t graph is $(t,\ t \cdot (t+2))$-agreeable. Since every non-planar piece of the decomposition has treewidth at most t with $t \geqslant 3$, these pieces are $(3,\ t \cdot (t+2))$-agreeable. Since $t \geqslant 1$ for all pieces of the decomposition, if we choose $c := t \cdot (t+2)$ all the pieces of the decomposition are $(3, c)$-agreeable. Theorem 14 (and Theorem 12 by Robertson and Seymour) then implies that G has rectilinear crossing number at most $3 \cdot (t \cdot (t+2) + 2) \cdot \Delta(G) \cdot ||G|| = 3 \cdot (t^2 + 2t + 2) \cdot \Delta(G) \cdot ||G||$, as claimed.

4 Conclusion and Open Problems

In this article, we proved that n-vertex bounded degree single-crossing minor-free graphs have $O(n)$ rectilinear crossing number. More strongly we proved that for any single-crossing graph X, every n-vertex X-minor-free graph G has rectilinear crossing number at most $O(\Delta(G) \cdot n)$ and the bound is best possible. The result represents a strong improvement over the previous state of the art on the rectilinear crossing numbers of minor-closed families of graphs, as argued in the introduction.

The ultimate goal for future work would be to obtain the above result for any fixed graph X. For such families an $O(f(\Delta) \cdot n)$ bound is not known for any function f. In fact, the best known bound on the rectilinear crossing number of bounded degree proper minor-closed families is $O(n \log n)$ [38].

In order to attempt to prove an $O(f(\Delta) \cdot n)$ bound, that is, a linear rectilinear crossing number for all proper minor-closed families of graphs of bounded degree, Robertson and Seymour's graph minor theory tells us that one should provide two ingredients. The first ingredient is to prove the result for k-almost embeddable graphs. The second is to be able to handle clique-sums of those. Proving the result for almost embeddable graphs entails proving it for bounded Euler genus graphs, that is, proving a result akin to Theorem 2 by Pach and Tóth [1] but with the crossing number replaced by the rectilinear crossing number. However, such a result is not even known for all bounded degree toroidal graphs.

The second ingredient however, handling the clique-sums of rectilinear drawings, can be achieved by our Theorem 14. In particular, one can change the definition of (k, c)-agreeable to $(k, f(\Delta))$-agreeable so as to allow for any function $f(\Delta)$ and not just the linear function, $c \cdot \Delta$, and then recall that the proof of Theorem 14 in fact shows that the rectilinear drawings of $(k, f(\Delta))$-agreeable graphs can be joined by $(\leqslant k)$-clique sums into a rectilinear drawing of the resulting graph G while only increasing the total number of crossings by $2 \cdot k \cdot \Delta(G) \cdot ||G||$. Suppose, in the future, one could provide the first ingredient above, that is, show that almost embeddable n-vertex graphs G have linear rectilinear crossing number, that is $\overline{\mathsf{cr}}(G) \leqslant g(\Delta) \cdot n$ for some function g. In that case the following lemma, Lemma 21, would imply that simplicial blowups of almost embeddable graphs are $(k, f(\Delta))$–agreeable with $f(\Delta) \in O(\Delta^4) \cdot g(\Delta)$. That and Theorem 14, as discussed in this paragraph, would imply that all proper minor-closed families of graphs of bounded degree have linear rectilinear crossing number.

▶ **Lemma 21.** *For every graph G and every $(\leqslant k)$-simplicial blowup Q of G, $\overline{\mathsf{cr}}(Q) \leqslant (\Delta(Q) + 1)^2 \cdot \overline{\mathsf{cr}}(G) + \Delta(Q)^4 \cdot ||Q||$.*

Proof. Let $S = V(Q) - V(G)$. We now define an H-partition of Q. To start, we make H isomorphic to G and put each $v \in V(G)$ in the bag B_v in H. For each vertex $u \in S$, u is adjacent to all the vertices of some clique C in G. Place u in a bag B_v where $v \in C$. This does not change H since v is adjacent to all the neighbours of u in G. This defines an H-partition of multigraph Q.

For each $v \in H$, each vertex of S in B_v is adjacent to v in Q thus the width of H is at most $\Delta(Q) + 1$. Thus by Lemma 15, $\overline{\mathsf{cr}}(Q) \leqslant (\Delta(Q) + 1)^4 \cdot \overline{\mathsf{cr}}(H) + \Delta(Q)^2 \cdot ||Q||$ which is equal to $(\Delta(Q) + 1)^4 \cdot \overline{\mathsf{cr}}(G) + \Delta(Q)^2 \cdot ||Q||$ since H is isomorphic to G. ◀

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