Harborth's Conjecture for 4-Regular Planar Graphs

Daniel J. Chang
Department of Computer Science, San Francisco State University, CA, USA

Timothy Sun
Department of Computer Science, San Francisco State University, CA, USA

Abstract

We show that every 4-regular planar graph has a straight-line embedding in the plane where all edges have integer length. The construction extends earlier ideas for finding such embeddings for 4-regular planar graphs with diamond subgraphs or small edge cuts.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Planar graph, straight-line embedding, Diophantine equation

Digital Object Identifier 10.4230/LIPIcs.GD.2024.38

1 Introduction

In this paper, graphs are *simple*, i.e., they do not have self-loops or parallel edges. A *Fáry embedding* of a graph is an embedding in the plane where each edge is drawn as a straight line segment. It is a well-known fact that such drawings exist for all planar graphs [6, 12, 15], and a longstanding conjecture of Harborth (see [8]) posits that every planar graph has a Fáry embedding where every edge has integer (or equivalently, rational) length. Geelen, Guo, and McKinnon [7], using a result of Berry [2], showed that all planar graphs of maximum degree 3 satisfy Harborth's conjecture. Since then, there have been simpler proofs of the cubic case [3, 13] and extensions of their application of Berry's theorem to graphs of higher maximum degree [1, 3, 5].

Sun [14] showed how these ideas can be used to find rational Fáry embeddings of two families of 4-regular graphs. We correct an error in one of those constructions and show that those techniques can be strengthened to cover all 4-regular planar graphs:

▶ **Theorem 1.** *Every 4-regular planar graph has a rational Fáry embedding.*

2 Background

Given a planar graph $G = (V, E)$, we treat Fáry embeddings of G as injective mappings $\phi\colon V \to \mathbb{R}^2$. We say that a Fáry embedding is *rational* if all its edge lengths are rational and *fully rational* if it is rational and the vertices are at rational coordinates. Two Fáry embeddings ϕ and ϕ' of G are said to be *ε-close* if, for every vertex $v \in V$, the Euclidean distance $d(\phi(v), \phi'(v))$ between the images of v is less than ε. A planar graph G is said to be *fully approximable* if, for every Fáry embedding ϕ of G and every $\varepsilon > 0$, there exists a fully rational Fáry embedding ϕ' that is ε-close to ϕ. Geelen, Guo, and McKinnon [7] were the first to consider such a property because of the following result on Diophantine equations:

▶ **Theorem 2** (Berry [2]). *Let x, y, and z be non-collinear points in the plane such that $d(x, y)$, $d(y, z)^2$, and $d(z, x)^2$ are rational. Then, the set of points at rational distance from all three points is dense in the plane.*

32nd International Symposium on Graph Drawing and Network Visualization (GD 2024).
Editors: Stefan Felsner and Karsten Klein; Article No. 38; pp. 38:1–38:9
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

We note that the squares of the distances are automatically rational if the points have rational coordinates. Using Theorem 2, Geelen et al. constructed fully rational Fáry embeddings of planar graphs of maximum degree 3, but Biedl [3] was the first to observe that their method extends to a more general family of graphs. Let G be a graph on n vertices. An ordering of its vertices $v_1, \dots, v_n$ is said to be a *3-elimination order* if:

- G has only one vertex, i.e., $n = 1$, or
- v_1 has degree at most 2 and $v_2, \dots, v_n$ is a 3-elimination order for $G - v_1$, or
- v_1 has degree 3 and there are two neighbors u and u' of v_1 such that $v_2, \dots, v_n$ is a 3-elimination order for $(G - v_1) \cup \{uu'\}$.

Given a 3-elimination order, one can recursively construct a fully rational Fáry embedding of $G - v_1$ or $(G - v_1) \cup \{uu'\}$, and then add v_1 using Theorem 2:

▶ **Theorem 3** (Geelen et al. [7], Biedl [3]). *Every planar graph with a 3-elimination order is fully approximable.*

A graph is called $(2,1)$-*sparse* if every nonempty induced subgraph H satisfies $|E(H)| \leq 2|V(H)| - 1$. Biedl [3] identified the $(2,1)$-sparse graphs as a rich family of graphs with 3-elimination orders, but unfortunately, the original proof has an error. For our purposes, we do not explicitly need $(2,1)$-sparseness (a discussion and a corrected proof are deferred to Appendix A). Instead, we rely on just a special case: a graph is said to be *subquartic* if it has maximum degree 4, but is not 4-regular. Biedl [3] showed that connected subquartic graphs are $(2,1)$-sparse while Benediktovich [1] directly proved that such graphs have 3-elimination orders:

▶ **Theorem 4** (Benediktovich [1], Biedl [3]). *Every connected subquartic graph has a 3-elimination order. Hence, planar connected subquartic graphs are fully approximable.*

4-regular planar graphs cannot have 3-elimination orders because there is no possible first vertex v_1. To circumvent this, Sun [14] considered two approaches. If the graph has low edge connectivity, then a rational Fáry embedding can be pieced together from its subquartic blocks. The other idea is to apply a theorem of Kemnitz and Harborth [8] to remove a vertex of a diamond subgraph. In Theorems 5 and 14, we improve upon each approach.

3 The low connectivity case

Using rigidity theory, Sun [14] proved that 4-regular planar graphs with edge connectivity 2 have rational Fáry embeddings. We upgrade this result to *vertex* cuts of size 2:

▶ **Theorem 5.** *Every connected 4-regular planar graph that is not 3-connected has a rational Fáry embedding.*

Proof. By the handshaking lemma, minimal edge cuts in 4-regular graphs must have even size. If the graph has an edge cut of size 2 (which includes when the graph has a cutvertex), then the result of Sun [14] applies. Now suppose the graph G is 4-edge-connected and has a vertex cut $\{u, v\}$. Deleting the cut disconnects the graph into exactly two connected components C_1 and C_2. Furthermore, since G does not have an edge cut of size 2, u and v both have exactly two neighbors in both C_1 and C_2.

For $i = 1, 2$, define H_i to be the induced subgraph $G[V(C_i) \cup \{u, v\}]$, with the edge uv added. Since C_{3-i} is connected, there is a path in C_{3-i} from u to v, so H_i is planar. Choose Fáry embeddings of H_1 and H_2 where uv is on the boundary of the embeddings' convex hulls. By applying Theorem 4, we obtain two nearby rational Fáry embeddings of H_1 and

H_2. As shown in Figure 1, after applying a rational scaling to one of the embeddings so that the length of the edge uv is the same in both, they can be glued together at this edge to obtain a rational Fáry embedding of G. ◀

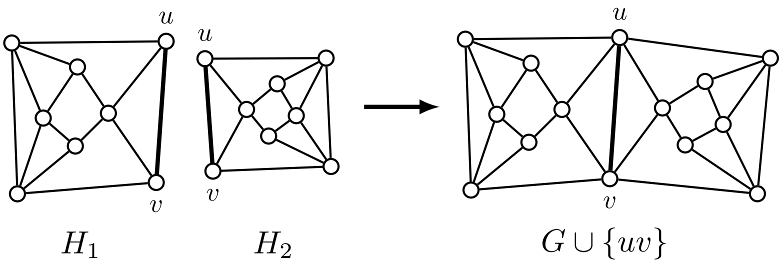

Figure 1 Pasting together two rational Fáry embeddings at two vertices.

4 The 3-connected case

4.1 The geometric part

Let $vpqr$ be a quadrilateral where vq is an internal diagonal and $d(v,p)$, $d(v,q)$, $d(v,r)$, $d(p,q)^2$, and $d(q,r)^2$ are rational. Kemnitz and Harborth [8] applied the theory of Diophantine equations to find a point u on the line $\overline{vq}$ at rational distance from v, p, q, r. If v, p, q, r are vertices of a graph, then the lengths required to be rational form a complete bipartite graph $K_{1,3}$, where v is the vertex of degree 3. We say that a Fáry embedding of $K_{1,3}$ is *permissible* if the rational constraints above are satisfied and Kemnitz and Harborth's solution for u lies on the interior of the line segment vq.

Sun [14] gave an explicit example of a permissible quadrilateral: we call a Fáry embedding of $K_{1,3}$ a *good kite* if $vpqr$ forms a convex quadrilateral, $d(p,q) = d(p,v) = 3$, $d(r,q) = d(r,v) = 4$, and $d(q,v) = 5$. A good kite and the location of the new vertex u are shown in Figure 2(a) and (b), respectively.

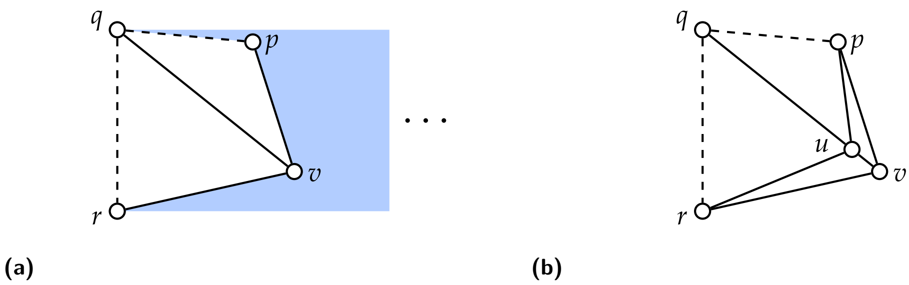

Figure 2 A special "one-sided" polygon (a) enables a useful solution to a certain system of Diophantine equations (b).

▶ **Proposition 6** (Sun [14]). *Let ϕ be a permissible embedding of $K_{1,3}$. Then, there exists $\varepsilon_0 > 0$ such that any fully rational Fáry embedding ϕ' that is ε_0-close to ϕ has a point on the interior of the line segment qv at rational distance to each of v, p, q, r.*

However, when trying to use Proposition 6, Sun [14] erroneously claimed that any drawing of an internal face as a convex polygon can be extended to a Fáry embedding of the entire graph. A simple counterexample is the wheel graph on 5 vertices whose quadrangular face is drawn as a square: the only way to add the remaining vertex is inside of the square.

Fortunately, the convex quadrilateral found in a good kite can be extended. Using the terminology in Mchedlidze, Nöllenburg, and Rutter [10], given a maximal plane graph with a specified unbounded face, we say that a cycle is *outerchordless* if all of its chords are in the interior of the cycle. A convex polygon is said to be *one-sided* if there is a point in the exterior of the polygon which is visible from every vertex of the polygon.

▶ **Theorem 7** (Mchedlidze et al. [10], Theorem 2)**.** *Let G be a maximal plane graph, and let C be an outerchordless cycle in G. Then any drawing of C as a one-sided polygon can be extended to a Fáry embedding ϕ of G, possibly with a different unbounded face, where a face is internal to C in ϕ if and only if it is internal to C in G.*

The original statement of the above result in [10] does not mention preserving internal faces, but it is implicit in their proof. We note that this is not their main result (Theorem 4 in [10]), which applies to arbitrary convex polygons and is able to preserve the unbounded face, though it has additional requirements on so-called "petals" of the cycle.

▶ **Lemma 8.** *Let G be a plane graph, where the rotation at some vertex v is of the form $(\dots p, q, r \dots)$, and p and r are not adjacent. Then, there is a Fáry embedding of G where the restriction to the $K_{1,3}$ subgraph formed by the edges vp, vq, and vr forms a good kite, and vq is the only edge that intersects the interior of the convex hull of the $K_{1,3}$ subgraph.*

Proof. Triangulate the graph to obtain a maximal plane graph G' so that there are faces $[v, q, p]$ and $[v, r, q]$ (which may require changing the locations of the edges pq and qr, if they already exist) and the edge pr is still missing, introducing additional vertices if necessary. Draw the $K_{1,3}$ subgraph as a good kite and consider the cycle $vpqr$. Since the edge vq is in the interior of the cycle and there is no edge pr, the cycle is outerchordless. In Figure 2(a), there is a point in the shaded region that extends perpendicularly from qr where all four vertices are visible from that point, so the cycle has been drawn as a one-sided quadrilateral. Thus, we may apply Theorem 7 to $vpqr$ to obtain a Fáry embedding of all of G'. Since G' is 3-connected, Whitney's theorem ensures that the new drawing has the same set of rotations, and hence the same set of faces. Consequently, no other edge besides vq intersects the interior of the convex quadrilateral $vpqr$. ◀

Adding a vertex using Proposition 6 creates what we call a *diamond*, two triangular faces meeting at an edge. We call that intersecting edge the *central* edge and its endpoints *central vertices.* As seen in Figure 2, undoing the vertex addition is combinatorially equivalent to contracting the edge uv. We can subsequently summarize the aforementioned geometric results in a graph-theoretic manner:

▶ **Lemma 9.** *Let G be a plane graph with a diamond where one of its central vertices has degree 4, and the two non-central vertices are not adjacent. Let G' be the graph after contracting the diamond's central edge. If G' has a 3-elimination order, then G has a rational Fáry embedding.*

Proof. Reusing earlier notation, let the rotation at u be (v, p, q, r) and let the diamond's faces be $[u, p, v]$ and $[u, v, r]$. Contract the edge uv and call the new vertex v. Apply Lemma 8 to obtain a Fáry embedding ϕ of G'. There exists $\varepsilon_1 > 0$ such that, in any Fáry embedding

ε_1-close to ϕ, the quadrilateral $vpqr$ is convex, and no other edge intersects the interior of $vpqr$ (i.e., the same guarantees as in Lemma 8). In particular, adding a new vertex anywhere in the interior of $vpqr$ and connecting it to v, p, q, r would not create any crossings. Since G' has a 3-elimination order, use Theorem 3 to find a fully rational Fáry embedding $\min(\varepsilon_0, \varepsilon_1)$-close to ϕ. Finally, use Proposition 6 to find a point in the interior of vq to add back vertex u and its incident edges so that those edges have rational length. ◀

Guaranteeing non-adjacency for applications of Lemma 9 is aided by the following facts:

▶ **Proposition 10** (Sun [14]). *In a 4-edge-connected 4-regular plane graph, every 3-cycle is facial.*

▶ **Corollary 11.** *In a 4-edge-connected 4-regular plane graph, if the rotation at a vertex is of the form (a, b, c, d), then a and c are not adjacent.*

We note that, for 4-regular graphs, 3-connectivity implies 4-edge-connectivity by the same parity argument mentioned in the proof of Theorem 5.

4.2 The combinatorial part

Given a 3-connected 4-regular plane graph, we say that a vertex is of *type* (t_1, t_2, t_3, t_4), where $t_1 \leq t_2 \leq t_3 \leq t_4$, if the lengths of the faces $f_1 \leq f_2 \leq f_3 \leq f_4$ incident with the vertex satisfy $f_i \leq t_i$, for $i = 1, \ldots, 4$.

▶ **Theorem 12** (Lebesgue [9]). *Every 3-connected 4-regular plane graph has a vertex of type $(3, 3, 3, \infty)$, $(3, 3, 4, 11)$, $(3, 3, 5, 7)$, or $(3, 4, 4, 5)$.*

We extract a few configurations of faces from Lebesgue's criterion. A *bowtie* consists of two triangular faces intersecting at a vertex, and a *house* consists of a triangular face and a quadrangular face intersecting at an edge.

▶ **Corollary 13.** *Every 3-connected 4-regular plane graph contains a diamond, a bowtie, or a house.*

Proof. If the vertex in Theorem 12 is incident with at least two triangular faces, then there is a diamond or a bowtie. Otherwise, the vertex is of type $(3, 4, 4, 5)$. At least one of the quadrangular faces intersects the triangular face at an edge, forming a house. ◀

Note that the latter two configurations are necessary: of the three configurations above, the medial graphs of the dodecahedron and cuboctahedron graphs have only bowties and houses, respectively.

▶ **Theorem 14.** *Every planar 3-connected 4-regular graph has a rational Fáry embedding.*

Proof. For each such graph G, we will create a diamond (if one does not already exist) by adding an edge near a triangular face, and then verify that this graph satisfies the conditions in Lemma 9. In each case, the diamond's non-central vertices will not be adjacent by Corollary 11, so it remains to show that contracting the diamond's central edge yields a graph G' with a 3-elimination order. If G already has a diamond, then G' would be connected and subquartic, so assume otherwise.

We note that in the remaining cases, the inclusion of another edge causes G' to have degree sequence $3, 4, \ldots, 4, 5$, which implies that it has too many edges to be $(2, 1)$-sparse. Instead, we will have to specify the first few vertices of the 3-elimination order until we are able to invoke Theorem 4 to generate the rest of the ordering.

By Corollary 13, G must have a bowtie or a house. If there is a bowtie, let its two triangular faces be $[x, a, b]$ and $[x, c, d]$, as seen on the left of Figure 3. In the subsequent drawings, blue triangles and red pentagons denote vertices of degree 3 and 5, respectively. Suppose that the rotation at vertex d is (c, x, y, z). By Corollary 11, x and z are not adjacent. Adding the edge cz creates a diamond where the edge cd is shared between two triangular faces, and vertex d has degree 4. After contracting cd, x and c are the vertices of degree 3 and 5, respectively. Let x be the first vertex in the 3-elimination order for G', and choose a and b to be its neighbors. The subsequent graph $(G' - x) \cup \{ab\} = G' - x$ is subquartic and connected, since it is a spanning supergraph of $G - \{x, d\}$, and G is 3-connected.

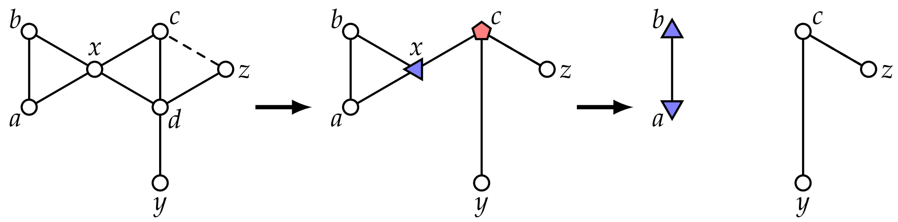

Figure 3 Reducing a bowtie by specifying the first vertex in the 3-elimination order.

If there is a house, label the vertices so that its two faces are $[w_1, w_\ell, b, c]$ and $[w_1, c, d]$, the rotation at vertex d is (c, w_1, y, z), and $[w_1, w_2, \ldots, w_\ell]$ is the face sharing the edge $w_1 w_\ell$ with the quadrangular face. Since the graph is simple, there are four distinct neighbors of w_1. If, say, $c = w_i$, for some $i \in \{2, \ldots, \ell - 1\}$, then deleting w_1 and c would disconnect w_2 from w_ℓ, contradicting the assumption that G is 3-connected. Similarly, $d \neq w_i$ as well. Thus, the vertices $c, d, w_1, \ldots, w_\ell$ are all distinct.

Like in the bowtie case, add the edge cz and contract the edge cd. As depicted in Figure 4, the first vertices in the 3-elimination order are $w_1, w_2, \ldots, w_\ell$, where the neighbors u, u' of w_i, for each $i = 1, \ldots, \ell - 1$, are chosen to be the two that are not w_{i+1}. Each vertex w_i has degree 3 by the time it is deleted, since they each started with degree 4. For w_ℓ, we choose b and c as its neighbors. The edges bc and bw_ℓ are in the original graph, but cw_ℓ was only added when deleting vertex w_1.

The resulting graph is now subquartic, so it remains to check that it is connected. It is a spanning supergraph of $(G/cd) - \{w_1, \ldots, w_\ell\}$, but since $c, d \neq w_i$, we obtain the same graph if we reverse the order of these two operations. Face boundaries in 3-connected plane graphs are non-separating cycles (see, e.g., Proposition 2.4.7 of Diestel [4]). Thus, deleting $w_1, \ldots, w_\ell$ does not disconnect the graph, and neither would contracting the edge cd afterwards. ◀

5 Future Directions

We showed that the solutions to certain Diophantine equations [2, 8] can be used to construct rational Fáry embeddings for all 4-regular planar graphs. However, the proof of Theorem 2 is quite complicated and does not give an explicit method for calculating the locations of such points. Is there a simpler construction, perhaps exploiting the additional condition that vertices are placed at rational coordinates?

Because the proof of Theorem 5 possibly applies a rotation to a Fáry embedding, the vertices are not guaranteed to be at rational coordinates. Is it possible to use the proof technique for 3-connected graphs (or other methods) to find fully rational Fáry embeddings in the low connectivity case?

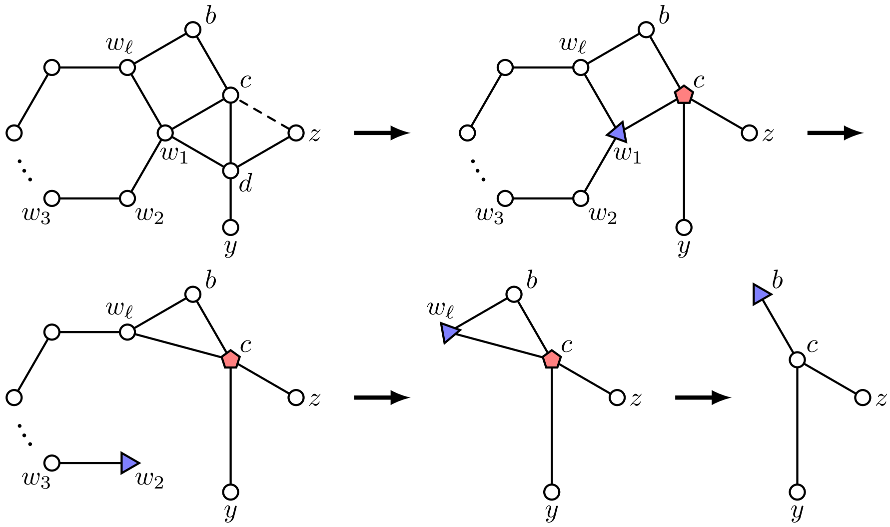

Figure 4 Reducing a house by transforming the quadrangular face into a triangular one.

Finally, Harborth's conjecture is still wide open. Are there any methods for finding a point at rational distance to families of five-point sets? What other interesting families of graphs, especially those with $3n - O(1)$ edges, have 3-elimination orders?

References

1 Vladimir I. Benediktovich. On rational approximation of a geometric graph. *Discrete Mathematics*, 313(20):2061–2064, 2013. doi:10.1016/J.DISC.2013.06.018.

2 T. G. Berry. Points at rational distance from the vertices of a triangle. *Acta Arithmetica*, 62:391–398, 1992.

3 Therese Biedl. Drawing some planar graphs with integer edge-lengths. In *Canadian Conference on Computational Geometry*, pages 291–296, 2011.

4 Reinhard Diestel. *Graph Theory*, volume 173. Springer, 2017.

5 Artūras Dubickas. On some rational triangles. *Mediterranean Journal of Mathematics*, 9(1):95–103, 2012.

6 István Fáry. On straight-line representation of planar graphs. *Acta Sci. Math. (Szeged)*, 11:229–233, 1948.

7 Jim Geelen, Anjie Guo, and David McKinnon. Straight line embeddings of cubic planar graphs with integer edge lengths. *Journal of Graph Theory*, 58(3):270–274, 2008. doi:10.1002/JGT.20304.

8 Arnfried Kemnitz and Heiko Harborth. Plane integral drawings of planar graphs. *Discrete Mathematics*, 236:191–195, 2001. doi:10.1016/S0012-365X(00)00442-8.

9 Henri Lebesgue. Quelques conséquences simples de la formule d'Euler. *Journal de Mathématiques Pures et Appliquées*, 19(1-4):27–43, 1940.

10 Tamara Mchedlidze, Martin Nöllenburg, and Ignaz Rutter. Drawing planar graphs with a prescribed inner face. In *International Symposium on Graph Drawing*, pages 316–327. Springer, 2013. doi:10.1007/978-3-319-03841-4_28.

11 Anthony Nixon and John Owen. An inductive construction of (2, 1)-tight graphs. *arXiv preprint arXiv:1103.2967*, 2011.

12 Sherman K. Stein. Convex maps. *Proceedings of the American Mathematical Society*, 2(3):464–466, 1951.

13 Timothy Sun. Rigidity-theoretic constructions of integral Fary embeddings. In *Canadian Conference on Computational Geometry*, pages 287–290, 2011.

14 Timothy Sun. Drawing some 4-regular planar graphs with integer edge lengths. In *Canadian Conference on Computational Geometry*, pages 193–198, 2013.

15 Klaus Wagner. Bemerkungen zum Vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.

A $(2, 1)$-sparse graphs have 3-elimination orders

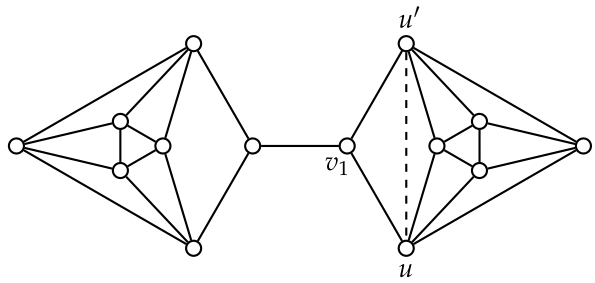

Figure 5 A potential dead end in a graph with a 3-elimination order.

The original proof in [3] attempts to show that $(2, 1)$-sparse graphs have 3-elimination orders via induction. In particular, the inductive step claimed that when deleting a vertex v_1 of degree 3, adding *any* edge between its neighbors still results in a $(2, 1)$-sparse graph. Figure 5 illustrates a case where the choice of neighbors is important: deleting vertex v_1 and adding an edge between its neighbors u and u' create a 4-regular component. Consequently, this new graph is not $(2, 1)$-sparse and does not have a 3-elimination order, even though the original graph is connected and subquartic.

Given a subset of vertices $X \subseteq V$, let $e(X)$ denote the number of edges in the subgraph induced by X. In this notation, a graph is $(2, 1)$-sparse if for all nonempty $X \subseteq V$, $e(X) \leq 2|X| - 1$. The key argument in the following proof is due to Nixon and Owen [11], who used it to characterize the $(2, 1)$-sparse graphs G with exactly $|E(G)| = 2|V(G)| - 1$ edges.

▶ **Lemma 15** (Biedl [3]). *Every* $(2, 1)$*-sparse graph has a 3-elimination order.*

Proof. We induct on the number of vertices. Given a $(2, 1)$-sparse graph G, the result is true when G has one vertex. Since G has average degree strictly less than 4, there is a vertex v of degree at most 3. If v has degree at most 2, or has degree 3 and two of its neighbors are adjacent, then $G - v$ is also $(2, 1)$-sparse by definition. The remaining case is when v has degree 3, and none of its neighbors w_1, w_2, w_3 are adjacent.

We show that there is at least one choice of neighbors w_i and w_j such that $(G - v) \cup \{w_i w_j\}$ is still $(2, 1)$-sparse. Assume that no such choice exists, so that for each pair of neighbors w_i and w_j, there is a subset of vertices X_{ij} such that $v \notin X_{ij}$, $w_i, w_j \in X_{ij}$, and $e(X_{ij}) = 2|X_{ij}| - 1$ (i.e., a subset that cannot afford having another edge). Then, consider the subgraph induced by $X' = X_{12} \cup X_{23} \cup \{v\}$. This graph has

$$\begin{aligned}
e(X') &= e(X_{12} \cup X_{23}) + 3 \\
&= e(X_{12}) + e(X_{23}) - e(X_{12} \cap X_{23}) + 3 \\
&= (2|X_{12}| - 1) + (2|X_{23}| - 1) - e(X_{12} \cap X_{23}) + 3 \\
&\geq (2|X_{12}| - 1) + (2|X_{23}| - 1) - (2|X_{12} \cap X_{23}| - 1) + 3 \\
&= 2|X_{12} \cup X_{23}| + 2 \\
&= 2|X'|
\end{aligned}$$

edges, violating $(2,1)$-sparseness. The inequality relies on the fact that $X_{12} \cap X_{23}$ is nonempty (since w_2 is in it), which allows us to apply the definition of $(2,1)$-sparseness to it. ◀