From Planar via Outerplanar to Outerpath – Engineering NP-Hardness Constructions

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Abstract

A typical question in graph drawing is to determine, for a given graph drawing style, the boundary between polynomial-time solvability and NP-hardness. For two examples from the area of drawing graphs with few slopes, we sharpen this boundary. We suggest a framework for a certain type of NP-hardness constructions where graphs have some parts that can only be realized as rigid structures, whereas other parts allow a controllable degree of flexibility. Starting with an NP-complete problem involving planarity (here, we use planar monotone rectilinear 3-SAT), we consider first a reduction to a planar graph, which can be adjusted to an outerplanar graph, and finally to an outerpath. An *outerplanar graph* is a graph admitting an *outerplanar drawing*, that is, a crossing-free drawing where every vertex lies on the outer face, and an *outerpath* is a graph admitting an outerplanar drawing where the weak dual is a path. The *(weak) dual* of a graph drawing is the graph that has a node for every (inner) face and a link if two faces share an edge.

Specifically, we first show that, for every upward-planar directed outerpath G, it is NP-hard to decide whether G admits an upward-planar straight-line drawing where every edge has one of three distinct slopes, and we second show that, for every undirected outerpath G, it is NP-hard to decide whether G admits a proper level-planar straight-line drawing where every edge has one of two distinct slopes. For both problems, NP-hardness has been known before for outerplanar graphs.

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1 The Framework

Consider a problem where the input is a graph G with specific properties and the task is to draw G in a specific drawing style without crossings. When reducing from planar monotone rectilinear 3-SAT [2], we model variables by variable gadgets, and clauses by clause gadgets. The gadgets are subgraphs of G that can only be drawn in a specific way – the boundaries (/frames/skeletons) are rigid building blocks, while other parts can be drawn with a small degree of flexibility allowing a mapping of truth values to variable gadgets and a crossing-free drawing of a clause gadget only if the clause is satisfied. First, we construct G such that it is planar and connected by following the planar incidence graph of the 3-SAT instance.

Second, we try to make G an outerplanar graph G' while the reduction remains applicable. To this end, we add gaps to the rigid boundary such that every vertex in a drawing of G' lies on the outer face but G' stays connected. Note that not every NP-hardness construction in this flavor is directly suited here because the rigid and flexible structures should be “thin”.

Third, we try to make G' an outerpath G'' while the reduction remains applicable. To this end, we trace the boundary of the embedding of G'. Note that this boundary is a cycle that encounters every vertex of G'. Replace the boundary by a chain of the same rigid building

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blocks as before (just "smaller"). For the flexible parts, one needs to carefully re-design them as (multiple parts of) outerpaths such that the functionality in the reduction is preserved. We keep a gap in the boundary such that the weak dual is not a wheel graph but a path.

2 Upward-Planar 3-Slope Drawings

An upward-planar graph is a directed graph that admits a planar drawing where every directed edge uv is drawn as a y-monotone curve such that $\mathrm{y}(u) < \mathrm{y}(v)$. Determining for a directed graph whether it is upward planar is NP-hard in general [4], but polynomial-time solvable if an embedding is given [1] or if the input graph is outerplanar [13]. We consider the setting of straight-line drawings with a limited number of slopes. Determining the minimum number of slopes is $\exists\mathbb{R}$-hard for undirected and directed graphs [7, 14]. For three slopes (w.l.o.g., 45, 90, 135 degrees w.r.t. the x-axis), Klawitter and Zink [9, 10] observe that a specific (sub)graph can only be drawn as a square with a diagonal (we call them *rigid square* here). Using them as the rigid structure, and using as the flexible structure *sliders*, which are two parallel edges attached to rigid squares that can extend only in one dimension, they show NP-hardness for directed outerplanar graphs. We extend their result as follows.

▶ **Theorem 1.** *Given a directed outerpath G, which optionally can be equipped with an upward-planar outerpath embedding, it is* NP*-hard to decide whether G admits an upward-planar straight-line drawing where every edge has one of three distinct slopes.*

According to the framework, we arrange rigid squares and triangles along the boundary. We replace each slider by two sliders that are attached to two chains of rigid squares that are twisted into each other such that we cannot move the two sliders much and the replacement of the slider behaves like the original. For details see the theses by Geis [5] and Zink [15].

3 Proper Level-Planar 2-Slope Drawings

A *level-planar* graph drawing is crossing free and every vertex is placed at a specific level (levels are equidistant horizontal lines). A level-planar drawing is *proper* if, for every edge e, the endpoints of e lie on consecutive levels. The setting where a leveling of the vertices is given in addition to the graph is most common. This is testing upward planarity with prescribed y-coordinates, which is polynomial-time solvable [3, 6, 8]. We focus on the case of straight-line edges with a limited set of slopes. Brückner, Krisam, and Mchedlidze show for this case that deciding if a graph given with a leveling and an arbitrary number of slopes admits a level-planar drawing can be solved in polynomial time if the leveling is proper, but otherwise it is NP-hard even for just two slopes. Here, we study the case where no leveling (or edge directions) are given but the generated drawing shall be proper. The problem becomes NP-hard with two slopes even for outerplanar graphs as shown by Kraus [11]. Again, squares are used as rigid structure and the flexible structure are edges going either a level up or down, and edges having either the first or the second slope. We extend this result to outerpaths.

▶ **Theorem 2.** *Given an outerpath G, it is* NP*-hard to decide whether G admits a proper level-planar straight-line drawing where every edge has one of two distinct slopes.*

4 Conclusion and Open Questions

The framework seems to be suited for settings with a small number of edge slopes or edge lengths. The two NP-hardness constructions presented here are very similar to each other and to Nöllenburg's [12] NP-hardness proof for planar octilinear metro maps. However, it is not obvious how to change his construction to work for outerplanar graphs or even outerpaths. Are there more examples of graph drawing problems where NP-hardness for (outer)planar graphs is known, while the "simpler" trees and cactus graphs are polynomial-time solvable, and the known construction can be adjusted to outerplanar graphs or outerpaths?

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