Minimizing Switches in Cased Graph Drawings

Robert Ganian
Algorithms and Complexity Group, TU Wien, Austria

Martin Nöllenburg
Algorithms and Complexity Group, TU Wien, Austria

Sebastian Röder
TU Wien, Austria

Abstract

In cased drawings of graphs, edges are drawn in front of others in order to decrease the negative impact of crossings on readability. In this context, a switch on an edge is defined as two consecutive crossings, where the edge is drawn in the front at one crossing and behind another edge at the next crossing. We investigate the problem of minimizing the maximum number of switches on any edge – both in a fixed drawing as well as for non-embedded graphs. We resolve an open question by Eppstein, van Kreveld, Mumford, and Speckmann (2009) by establishing the NP-hardness of minimizing the number of switches in a fixed drawing, provide a fixed-parameter algorithm for this problem, and obtain a full characterization of the problem for non-embedded graphs.

2012 ACM Subject Classification Mathematics of computing → Graph theory; Human-centered computing → Graph drawings

Keywords and phrases beyond planarity, complexity theory, non-planar drawings, crossings

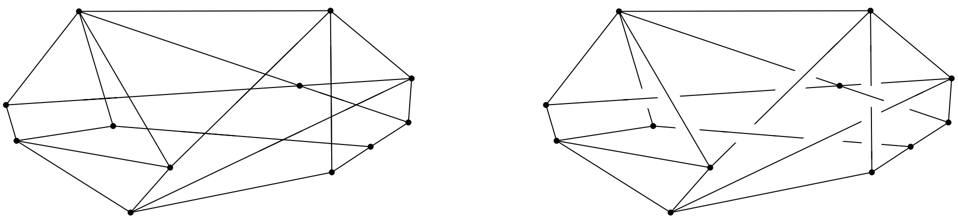
Digital Object Identifier 10.4230/LIPIcs.GD.2024.43

Category Poster Abstract

Funding The authors acknowledge support from the Vienna Science and Technology Fund (WWTF) [10.47379/ICT22029].
Robert Ganian: Austrian Science Fund (FWF) [10.55776/Y1329].

1 Introduction

Edge casing is a frequently used visual rendering method to improve the readability of crossings in non-planar graph drawings. In a cased drawing – introduced by Eppstein et al. [2] – two crossing edges are locally ordered into an upper and a lower edge and the curve representing the lower edge (called the *tunnel*) is locally interrupted to let the upper edge (called the *bridge*) pass through the created gap. This can be particularly important for graph drawings with regions of high feature density and many edge crossings, which, in non-cased drawings, are hard to discern from the small disk symbols typically representing the vertices; see Figure 1.



(a) **(b)**

Figure 1 **(a)** A drawing of a graph with crossings; **(b)** The same drawing with edge casing.

32nd International Symposium on Graph Drawing and Network Visualization (GD 2024).
Editors: Stefan Felsner and Karsten Klein; Article No. 43; pp. 43:1–43:3
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

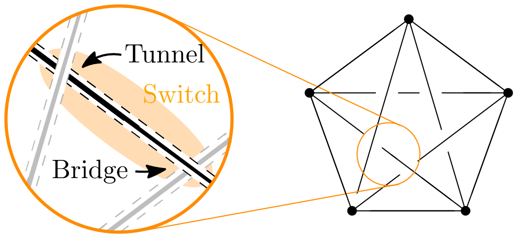


Figure 2 Cased drawing of K_5; the close-up shows a tunnel, bridge, and switch of the black edge.

Eppstein et al. defined several optimization problems, either concerning the number, length, or distance of the tunnels per edge, or concerning the number of *switches*, which are pairs of consecutive crossings of an edge e, one of which is a tunnel for e and the other one a bridge (see Figure 2). We revisit cased drawings and focus on the problem of minimizing the number of switches.

2 Results

We resolve an open question of Eppstein et al. [2] on the complexity of the MinMaxSwitches problem of minimizing the maximum number of switches per edge for a given graph drawing. We show that this problem is NP-complete even when the target number of switches per edge is 1, i.e., when we need to decide whether a given drawing δ_G can be embedded with at most 1 switch per edge. On the other hand, it is known that deciding whether δ_G can be embedded with 0 switches per edge is polynomial-time solvable [2]. We complement our hardness proof with a fixed-parameter algorithm. The structure of the input graph can be assumed to be trivial as every instance can be transformed into an equivalent matching. Therefore, we use the vertex cover number of the cell adjacency graph of the input drawing as a parameter. All results obtained also directly carry over to the straight-line setting.

▶ **Theorem 1.** MinMaxSwitches *is* NP*-complete.*

Proof Sketch. Inclusion in NP is immediate; we show hardness by reducing from NAE 3-SAT, i.e., the NP-complete variant of 3-SAT where clauses are required to contain at least one satisfied and at least one unsatisfied literal [4]. ◀

The structural parameter we use to achieve our tractability result is the *vertex cover number* ℓ of χ_G. The *cell adjacency graph* $\chi_G = (\mathcal{F}, E')$ is the graph, whose vertices are precisely the *cells* $\mathcal{F}$ of δ_G (i.e., the connected regions of $\mathbb{R}^2 \setminus \delta_G$) and where two cells are adjacent if and only they touch, i.e., share a edge segment or crossing on their boundary.

▶ **Theorem 2.** MinMaxSwitches *is fixed-parameter tractable when parameterized by the vertex cover number* ℓ *of* χ_G*.*

Proof Sketch. Two reduction rules are applied. First, remove any edge e with at most 2 crossings. Secondly, split the drawing into *bridgeless* subproblems. A drawing δ_G is *bridgeless* if there is no edge segment between two crossing points whose removal would disconnect δ_G. We claim that – after exhaustively applying the aforementioned reduction rules – the size of the obtained kernel is upper-bounded by $O(\ell^4)$. ◀

Moreover, following the definition of k-gap-planar graphs by Bae et al. [1], we define a graph to be k*-switch-planar* if it admits a cased drawing with at most k switches per edge. We give a full characterization of this notion. Recall that a graph G is biplanar if it has thickness at most 2, i.e., its edge set can be partitioned into two planar subgraphs.

▶ **Theorem 3.** *A graph G is* 0*-switch-planar if and only if it is biplanar.*

Theorem 3 implies that determining whether a graph admits a cased drawing without switches is NP-complete, as this is the case for testing a graph for biplanarity [3].

▶ **Theorem 4.** *Every graph G is 1-switch-planar.*

References

1 Sang Won Bae, Jean-Francois Baffier, Jinhee Chun, Peter Eades, Kord Eickmeyer, Luca Grilli, Seok-Hee Hong, Matias Korman, Fabrizio Montecchiani, Ignaz Rutter, and Csaba D. Tóth. Gap-planar graphs. *Theoretical Computer Science*, 745:36–52, October 2018. `doi:10.1016/j.tcs.2018.05.029`.

2 David Eppstein, Marc Van Kreveld, Elena Mumford, and Bettina Speckmann. Edges and switches, tunnels and bridges. *Computational Geometry*, 42(8):790–802, October 2009. `doi:10.1016/j.comgeo.2008.05.005`.

3 Anthony Mansfield. Determining the thickness of graphs is NP-hard. *Mathematical Proceedings of the Cambridge Philosophical Society*, 93(1):9–23, 1983. `doi:10.1017/S030500410006028X`.

4 Thomas J. Schaefer. The complexity of satisfiability problems. In *Theory of Computing (STOC'78)*, pages 216–226. ACM Press, 1978. `doi:10.1145/800133.804350`.