Holes in Convex and Simple Drawings

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Abstract

Gons and holes in point sets have been extensively studied in the literature. For simple drawings of the complete graph a generalization of the Erdős–Szekeres theorem is known and empty triangles have been investigated. We introduce a notion of k-holes for simple drawings and study their existence with respect to the convexity hierarchy. We present a family of simple drawings without 4-holes and prove a generalization of Gerken's empty hexagon theorem for convex drawings. A crucial intermediate step will be the structural investigation of pseudolinear subdrawings in convex drawings.

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1 Introduction

A classic theorem from combinatorial geometry is the Erdős–Szekeres theorem [15]. It states that for every $k \in \mathbb{N}$ every sufficiently large point set in general position (i.e., no three points on a line) contains a subset of k points that are the vertices of a convex polygon, a so called *k-gon*. In this article we will focus on a prominent variant of the Erdős–Szekeres theorem suggested by Erdős himself [14], which asks for the existence of *empty* k-gons, also known as k-holes. A *k-hole* H in a point set P is a k-gon with the property that there are no points of P in the interior of the convex hull of H. It is known that every sufficiently large point set contains a 6-hole [18, 23] and that there are arbitrarily large point sets without 7-holes [21].

Point sets in general position are in correspondence with *geometric drawings* of the complete graph where vertices are mapped to points and edges are drawn as straight-line segments between the vertices. In this article we generalize the notion of holes to simple drawings of the complete graph K_n. In a *simple drawing*, vertices are mapped to distinct points in the plane (or on the sphere) and edges are mapped to simple curves connecting the two corresponding vertices such that two edges have at most one point in common, which is either a common vertex or a proper crossing. In the course of this article, we will see that many important properties do not depend on the full drawing but only on the underlying combinatorics, more specifically on the isomorphism class of a drawing. We call two simple drawings of the same graph *isomorphic*[1] if there is a bijection between their vertex sets such that the corresponding pairs of edges cross. Note that this isomorphism is independent of the choice of the outer cell and thus only encodes the simple drawing on the sphere.

To study k-holes, we first extend the notion of k-gons to simple drawings of K_n. A *k-gon* $\mathcal{C}_k$ is a subdrawing isomorphic to the geometric drawing on k points in convex position; see Figure 1(a). In terms of crossings, a k-gon $\mathcal{C}_k$ is a (sub)drawing with vertices $v_1, \dots, v_k$ such that $\{v_i, v_\ell\}$ crosses $\{v_j, v_m\}$ exactly if $i < j < \ell < m$. In contrast to the geometric setting where every sufficiently large geometric drawing contains a k-gon, simple drawings of complete graphs do not necessarily contain k-gons [19]. For example, the twisted drawing $\mathcal{T}_n$ depicted in Figure 1(b) does not contain any 5-gon. In terms of crossings, $\mathcal{T}_n$ can be characterized as a drawing of K_n with vertices $v_1, \dots, v_n$ such that $\{v_i, v_m\}$ crosses $\{v_j, v_\ell\}$ exactly if $i < j < \ell < m$. A theorem by Pach, Solymosi and Tóth [24] states that, for every k, every sufficiently large simple drawing of K_n contains $\mathcal{C}_k$ or $\mathcal{T}_k$. The currently best known estimate is due to Suk and Zeng [28] who showed that every simple drawing of K_n with $n > 2^{9 \cdot \log_2(a) \log_2(b) a^2 b^2}$ contains $\mathcal{C}_a$ or $\mathcal{T}_b$. Convex drawings, which we define in the next paragraph, are a class of drawings nested between geometric drawings and simple drawings. In particular, convex drawings do not contain $\mathcal{T}_5$ as a subdrawing. Hence every convex drawing of K_n contains a k-gon $\mathcal{C}_k$ with $k = (\log n)^{1/2 - o(1)}$.

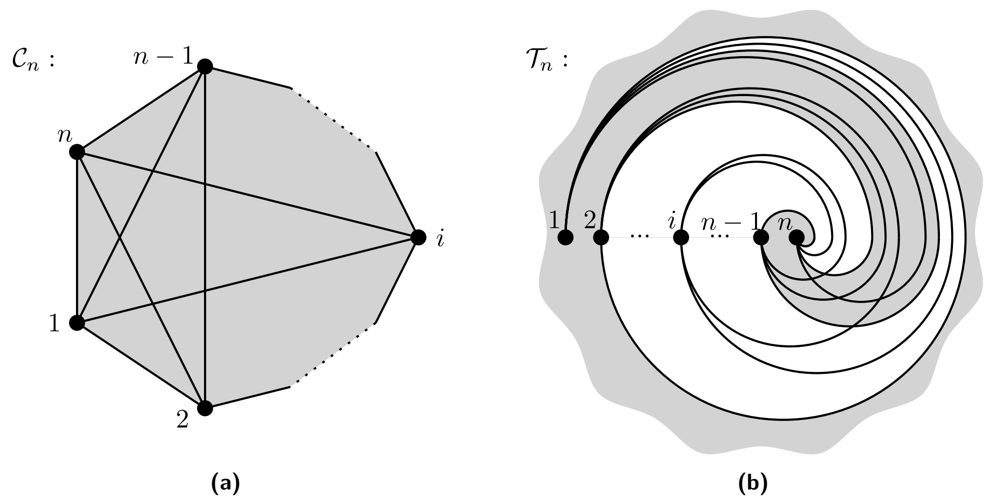

Figure 1 A drawing of (a) an n-gon $\mathcal{C}_n$ and (b) a twisted $\mathcal{T}_n$ for $n \geq 4$.

In the last decades, holes were intensively studied for the setting of point sets. Our focus will be on determining the existence of holes in various layers of the convexity hierarchy introduced by Arroyo et al. [5], which give a more fine-grained layering between geometric drawings and simple drawings. The basis to define convexity are *triangles*, which are

[1] This isomorphism is often referred to as "weak isomorphism" since there also exist stronger notions.

subdrawings induced by three vertices. Since in a simple drawing incident edges do not cross, a triangle separates the plane (resp. the sphere) into two connected components. The closure of each of the components is called a *side* of the triangle. A side S is *convex* if, for every pair of vertices in S, the connecting edge is fully contained in S. A simple drawing $\mathcal{D}$ of K_n is

- *convex* if every triangle in $\mathcal{D}$ has a convex side;
- *h-convex* (hereditarily convex) if there is a choice of a convex side S_T for every triangle T such that, for every triangle T' contained in S_T, it holds $S_{T'} \subseteq S_T$;
- *f-convex* (face convex) if there is a marking face F in the plane such that for all triangles the side not containing F is convex.

The class of f-convex drawings is related to pseudolinear drawings. A *pseudolinear drawing* is a simple drawing in the plane such that the edges can be extended to an arrangement of pseudolines. A *pseudoline* is a simple curve partitioning the plane into two unbounded components and in an *arrangement* each pair of pseudolines has exactly one point in common, which is a proper crossing. As shown by Arroyo et al. [4], a simple drawing of K_n is pseudolinear if and only if it is f-convex and the marking face F is the unbounded face. For more information about the convexity hierarchy we refer the reader to [4, 5, 6, 10].

Before we define k-holes, consider the case of 3-holes, also known as empty triangles. A triangle is *empty* if one of its two sides does not contain any vertex in its interior. Harborth [19] proved that every simple drawing of K_n contains at least two empty triangles and conjectured that the minimum among all simple drawings of K_n is $2n-4$. While $2n-4$ is obtained by $\mathcal{T}_n$ and all generalized twisted drawings [17], the best known lower bound is n [3].

In the geometric setting, the number of empty triangles behaves differently: every point set has $\Omega(n^2)$ empty triangles, and this bound is asymptotically optimal [8]. Note that the notion of empty triangles in point sets slightly differs from the one in simple drawings since the complement of the convex hull of a point set can be an empty triangle. The class of convex drawings behaves similarly to the geometric setting: the minimum number of empty triangles is asymptotically quadratic [4, Theorem 5].

In the drawing $\mathcal{C}_k$ with $k \geq 4$, all triangles have exactly one empty side, which is the unique convex side. The *convex side* of $\mathcal{C}_k$ is the union of convex sides of its triangles; see the grey shaded regions in Figure 1. Given a k-gon $\mathcal{C}_k$ in a simple drawing of K_n, we call vertices in the interior of the convex side of $\mathcal{C}_k$ *interior vertices*. A *k-hole* in a simple drawing of K_n is a k-gon that has no interior vertices. For example, the vertices $1, 2, n-1, n$ form a 4-hole in $\mathcal{T}_n$; marked grey in Figure 1(b). In convex drawings, as in the geometric setting, edges from an interior vertex to a vertex of $\mathcal{C}_k$ and edges between two interior vertices are contained in the convex side of $\mathcal{C}_k$ [5, Lemma 3.5]. For more details see the full version [11].

In this paper, using the notion of k-holes in simple drawings defined above, we resolve the questions of existence of 4-, 5- and 6-holes in simple and convex drawings of K_n. In particular, we show the existence of 6-holes in sufficiently large convex drawings (Theorem 2.1), generalizing Gerken's empty hexagon theorem [18]. The key ingredient of the proof is that any subdrawing induced by a minimal k-gon together with its interior vertices is f-convex (Lemma 2.2). This allows to transfer various existential results from the geometric, pseudolinear, and f-convex settings to convex drawings. Besides the existence of 6-holes, we also show the existence of monochromatic generalized 4-holes in two-colored convex drawings (Corollary 3.1), generalizing a result by Aichholzer et al. [2]. For this we discuss two variants of generalized holes (Section 3) in the setting of simple drawings of K_n and show the existence of plane cycles of length 4 such that one side does not contain other vertices (Theorem 3.3).

2 Holes in convex drawings

In this section, we show that convex drawings behave similarly to geometric point sets when it comes to the existence of holes. We show that every sufficiently large convex drawing contains a 6-hole and hence a 5-hole and a 4-hole. This is tight, as the construction by Horton [21] gives arbitrarily large point sets, that is geometric drawings, without 7-holes.

▶ **Theorem 2.1** (Empty Hexagon theorem for convex drawings). *For every sufficiently large n, every convex drawing of K_n contains a 6-hole.*

For the proof we use the existence of k-gons in sufficiently large convex drawings [24, 28]. Our key lemma is that the subdrawing induced by a minimal k-gon together with its interior vertices is f-convex, a fact that had been known only for h-convex drawings [5, Lemma 4.7]. A k-gon is *minimal* if its convex side does not contain the convex side of another k-gon.

▶ **Lemma 2.2.** *Let $\mathcal{C}_k$ be a minimal k-gon in a convex drawing $\mathcal{D}$ of K_n with $n \geq k \geq 5$. Then the subdrawing $\mathcal{D}'$ induced by the vertices in the convex side of $\mathcal{C}_k$ is f-convex.*

Proof. Let $v_1, \ldots, v_k$ be the vertices of the minimal k-gon $\mathcal{C}_k$ in $\mathcal{D}$ and F be a face contained in the non-convex side of $\mathcal{C}_k$. We show that for every triangle spanned by three vertices of the convex side of $\mathcal{C}_k$, the side not containing F is convex and hence $\mathcal{D}'$ is f-convex. Suppose towards a contradiction that there exists a triangle spanned by vertices t_1, t_2, t_3 from the convex side of $\mathcal{C}_k$, such that the side not containing F is not convex. The non-convex side S_N is the side contained in the convex side of $\mathcal{C}_k$. Since $\mathcal{D}$ is convex, the other side containing F and all vertices $v_1, \ldots, v_k$ is convex and is denoted by S_C. If we additionally assume that S_N is not contained in (the closure of) a single cell of the subdrawing induced by $\mathcal{C}_k$, then some edge $\{v_i, v_j\}$ has a crossing with one of the edges $\{t_\ell, t_m\}$. This shows that S_C is not convex; a contradiction. Hence, S_N lies in (the closure of) a cell of $\mathcal{C}_k$.

Since $\mathcal{C}_k$ is minimal, there are no interior vertices in the convex side of a triangle $\{v_i, v_{i+1}, v_{i+2}\}$. For details see the full version [11].

Since all cells in the convex side of $\mathcal{C}_k$ incident to the vertex v_{i+1} are inside this triangle, the vertex v_{i+1} is not part of the triangle spanned by t_1, t_2, t_3. This holds for every $i = 1, \ldots, k$ and hence the vertices t_1, t_2, t_3 are interior vertices of $\mathcal{C}_k$ and S_N lies in a cell of the convex side of $\mathcal{C}_k$ that is not covered by the convex side of any triangle $\{v_i, v_{i+1}, v_{i+2}\}$. Since S_N is not convex, there exists a vertex z in the interior of S_N such that the subdrawing induced by $\{t_1, t_2, t_3, z\}$ has a crossing [5, Corollary 2.5]. We assume without loss of generality that the edge $\{t_1, z\}$ crosses $\{t_2, t_3\}$. Moreover, exactly one of the following two conditions holds: Either the triangle $\{t_1, t_3, z\}$ separates t_2 and F or the triangle $\{t_1, t_2, z\}$ separates t_3 and F. We assume that the former holds as otherwise we exchange the roles of t_2 and t_3. Figure 2 gives an illustration.

Now we consider all edges from t_2 to the vertices $v_1, \ldots, v_k$ of $\mathcal{C}_k$. Since S_C is convex and contains $v_1, \ldots, v_k$, the edges $\{t_2, v_i\}$ are contained in S_C. This shows that none of the edges $\{t_2, v_i\}$ crosses any of the triangle edges and, in particular, they do not cross $\{t_1, t_3\}$.

The edges $\{t_2, v_1\}, \ldots, \{t_2, v_k\}$ partition the convex side of $\mathcal{C}_k$ into triangles t_2, v_i, v_{i+1}. Hence there is an index i such that the three vertices t_1, t_3, z lie in the convex side of the triangle $\{t_2, v_i, v_{i+1}\}$. However, the edge $\{t_1, z\}$ is not fully contained in this side; a contradiction to convexity. This completes the proof of Lemma 2.2. ◀

Recently, Heule and Scheucher [20] used SAT to show that every set of 30 points has a 6-hole. Since their result is about the more general case of pseudoconfigurations of points, it holds for pseudolinear drawings. To prove Theorem 2.1, we combine this fact with Lemma 2.2.

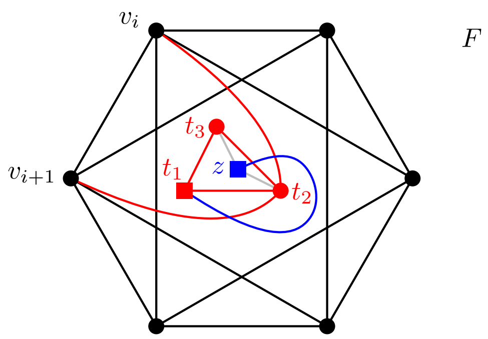

Figure 2 Illustration of the proof of Lemma 2.2.

Proof of Theorem 2.1. Let $\mathcal{D}$ be a convex drawing of K_n with $n > 2^{225 \log_2(5) \cdot 30^2 \log_2(30)}$. Since convex drawings do not contain the twisted drawing $\mathcal{T}_5$, it follows from [28] that $\mathcal{D}$ contains a 30-gon. To find a 6-hole in $\mathcal{D}$, we choose a minimal 30-gon G. By Lemma 2.2, the subdrawing $\mathcal{D}'$ induced by G and its interior vertices is f-convex. Since the existence of holes is invariant under the choice of the outer cell, we can assume without loss of generality that $\mathcal{D}'$ is pseudolinear as we may otherwise choose the face F as the unbounded face. According to [7], $\mathcal{D}'$ corresponds to a pseudoconfiguration of points, and hence there exists a 6-hole H in $\mathcal{D}'$ [20]. Hence the convex side of H does not contain any vertex of $\mathcal{D}'$. Moreover, every vertex of $\mathcal{D}$ in the convex side of H would be an interior vertex of G and therefore belong to $\mathcal{D}'$. This shows that H is a 6-hole in $\mathcal{D}$. ◀

The existence of 6-holes further implies the existence of 4- and 5-holes. However, it remains a challenging task to determine the smallest integer $n(k)$ such that every convex drawing of K_n with $n \geq n(k)$ contains a k-hole for $k = 4, 5, 6$.

For 6-holes, one can slightly improve the estimate from Theorem 2.1 by utilizing the fact that every 9-gon in a point set yields a 6-hole [18]. As shown in [27] this result transfers to pseudolinear drawings. It follows from Lemma 2.2 that every convex drawing of K_n with $n > 2^{225 \log_2(5) \cdot 9^2 \log_2(9)}$ contains a 6-hole.

A similar improvement is possible for 5-holes: as the textbook proof for the existence of 5-holes in every 6-gon of a point set (see e.g. Section 3.2 in [22]) applies to pseudolinear drawings, every convex drawing with more than $2^{225 \log_2(5) \cdot 6^2 \log_2(6)}$ vertices contains a 5-hole.

For 4-holes, we can combine the proof of Bárány and Füredi [8, Theorem 3.3] for the quadratic number of empty 4-holes in point sets and the proof of Arroyo et al. [4, Theorem 5] for the quadratic number of empty triangles in convex drawings to obtain:

▶ **Lemma 2.3.** *Every crossed edge in a convex drawing of K_n is a chord of a 4-hole, that is, it is one of the crossing edges of a 4-hole.*

Since the number of uncrossed edges in drawings of K_n is at most $2n - 2$ [26], Lemma 2.3 implies that there are $\Omega(n^2)$ empty 4-holes in every convex drawing of K_n. A detailed proof is provided in the full version [11]. Since every drawing of K_5 contains a crossing, Lemma 2.3 also implies that every convex drawing of K_n with $n \geq 5$ contains a 4-hole. In contrast to the convex setting, 4-holes can be avoided in simple drawings as we show in the next section.

3 Generalized Holes

Devillers et al. [13] showed that sufficiently large two-colored point sets in general position contain a monochromatic 3-hole and constructed arbitrarily large two-colored sets without monochromatic 5-holes. The existence of monochromatic 4-holes, however, remains a longstanding open problem [12, Problem 8.2.7]. A weaker version was shown by Aichholzer et al. [2]. They showed that every two-colored point set $P = A \dot{\cup} B$ contains a monochromatic generalized 4-hole. A *generalized k-hole* is a simple polygon (not necessarily convex) which is spanned by k points of P and does not contain any point of P in its interior. Apparently, their proof transfers to the pseudolinear setting, which allows us to generalize this result to convex drawings in the same way as the Empty Hexagon theorem 2.1 using Lemma 2.2.

▶ **Corollary 3.1.** *Every sufficiently large convex drawing on vertices $V = A \dot{\cup} B$ has an empty 4-triangulation induced only by vertices from A or only by vertices from B.*

To define generalized k-holes in simple drawings we consider plane cycles. A plane cycle divides the plane into two components whose closures we call *sides*. An *empty k-cycle* in a simple drawing is a plane cycle of length k such that one of its sides is empty. For $k = 3$ this definition coincides with empty triangles. Since polygons in point sets can be triangulated, we say that an empty k-cycle is an *empty k-triangulation* if its empty side is the disjoint union of empty triangles. As the following construction (Figure 3) shows, there are simple drawings of K_n without empty 4-triangulations. For the construction, we start with the twisted drawing $\mathcal{T}_n$ and reroute some edges such that the drawing is still crossing maximal. The resulting drawing $\mathcal{T}'_n$ does not contain 4-holes. A precise description and proof of Proposition 3.2 is given in the full version [11].

▶ **Proposition 3.2.** *For $n \geq 6$ the simple drawing $\mathcal{T}'_n$ contains no empty 4-triangulation.*

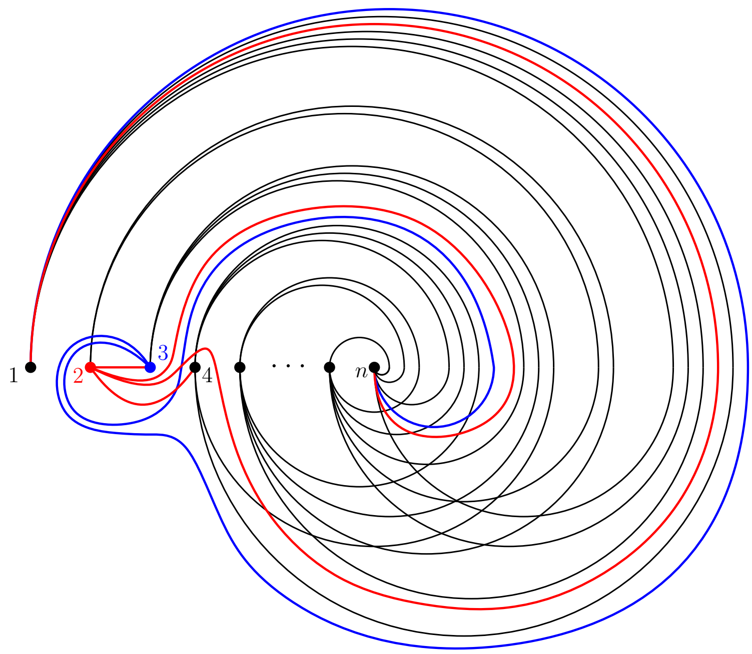

Figure 3 The drawing $\mathcal{T}'_n$ without empty 4-triangulations for $n \geq 6$.

If instead of empty 4-triangulations we only ask for empty 4-cycles, then we can actually guarantee their existence in all simple drawings of K_n.

▶ **Theorem 3.3.** *Let $\mathcal{D}$ be a simple drawing of K_n with $n \geq 4$ and let v be a vertex of $\mathcal{D}$. Then $\mathcal{D}$ contains an empty 4-cycle passing through v.*

This resolves one case of a recent conjecture by Bergold et al. [9]. They showed that every convex drawing contains an empty k-cycle for all $3 \leq k \leq n$ and conjectured that this holds for simple drawings.

▶ **Conjecture 3.4** ([9]). *Every simple drawing of K_n contains an empty k-cycle for each $3 \leq k \leq n$.*

While the case $k = 3$ follows by Harborth's result [19], the $k = n$ case coincides with Rafla's conjecture concerning the existence of plane Hamiltonian cycles in all simple drawings of K_n [25]. For the proof of the case $k = 4$ of Conjecture 3.4 (Theorem 3.3), we use results on plane subdrawings by García, Pilz, and Tejel [16].

Proof of Theorem 3.3. For a fixed vertex v, we consider the spanning star S_v centered at v. By [16, Corollary 3.4], there is a plane subdrawing $\mathcal{D}'$ of $\mathcal{D}$ that consists of the star S_v and some spanning tree T on the other $n-1$ vertices. Note that $\mathcal{D}'$ has exactly $2n-3$ edges and $n-1$ faces. Every face F of $\mathcal{D}'$ contains v on its boundary because the tree T is cycle-free and since $\mathcal{D}'$ is 2-connected [16, Theorem 3.1], F is bounded by exactly two edges of S_v.

If there is a face of $\mathcal{D}'$ with exactly 4 boundary edges or if there are two adjacent triangular faces, we obtain an empty 4-cycle passing through v and the statement follows. Otherwise we count the number of edges $|E|$ in $\mathcal{D}'$: At most half of the $n-1$ faces are triangles so that none of them are adjacent. All other faces have at least 5 boundary edges. Since every edge is incident to exactly two faces, we have $|E| \geq \frac{1}{2}\left(5(n-1) - 2\left\lfloor\frac{n-1}{2}\right\rfloor\right) \geq 2n-2$. This is a contradiction to the fact that $\mathcal{D}'$ contains exactly $2n-3$ edges. ◀

The above theorem implies a linear lower bound on the number of empty 4-cycles. This is similar to the minimum number of empty triangles which is asymptotically linear as well [3].

▶ **Corollary 3.5.** *Every simple drawing of K_n with $n \geq 4$ contains at least $\frac{n}{4}$ empty 4-cycles.*

While the twisted drawing $\mathcal{T}_n$ is conjectured to minimize the number of empty triangles, it contains $\Theta(n^3)$ empty 4-cycles. This is certainly not minimal as there exist drawings with $\Theta(n^2)$ empty 4-cycles; see Figure 4 and the full version [11].

This seems to be in contrast to the geometric setting, where the number of empty k-cycles with $k \geq 4$ is conjectured to be super-quadratic [1].

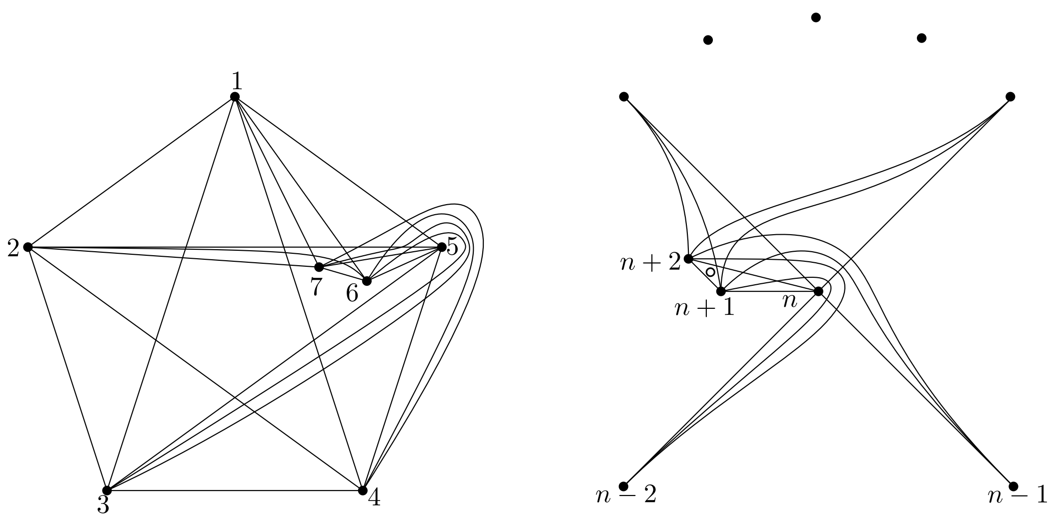

Figure 4 Constructing the drawing $\mathcal{D}_n$ of K_n, n odd, with few empty 4-cycles from K_5.

References

1 Oswin Aichholzer, Ruy Fabila-Monroy, Hernán González-Aguilar, Thomas Hackl, Marco A. Heredia, Clemens Huemer, Jorge Urrutia, Pavel Valtr, and Birgit Vogtenhuber. On k-gons and k-holes in point sets. *Computational Geometry*, 48(7):528–537, 2015. doi:10.1016/j.comgeo.2014.12.007.

2 Oswin Aichholzer, Thomas Hackl, Clemens Huemer, Ferran Hurtado, and Birgit Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. *SIAM Journal on Discrete Mathematics*, 23(4), 2010. doi:10.1137/090767947.

3 Oswin Aichholzer, Thomas Hackl, Alexander Pilz, Pedro Ramos, Vera Sacristán, and Birgit Vogtenhuber. Empty triangles in good drawings of the complete graph. *Graphs and Combinatorics*, 31:335–345, 2015. doi:10.1007/s00373-015-1550-5.

4 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Levi's Lemma, pseudolinear drawings of K_n, and empty triangles. *Journal of Graph Theory*, 87(4):443–459, 2018. doi:10.1002/jgt.22167.

5 Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Convex drawings of the complete graph: topology meets geometry. *Ars Mathematica Contemporanea*, 22(3), 2022. doi:10.26493/1855-3974.2134.ac9.

6 Alan Arroyo, R. Bruce Richter, and Matthew Sunohara. Extending drawings of complete graphs into arrangements of pseudocircles. *SIAM Jorunal on Discrete Mathematics*, 35(2):1050–1076, 2021. doi:10.1137/20M1313234.

7 Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing Numbers and Combinatorial Characterization of Monotone Drawings of K_n. *Discrete & Computational Geometry*, 53(1):107–143, 2015. doi:10.1007/s00454-014-9644-z.

8 Imre Bárány and Zoltán Füredi. Empty simplices in Euclidean space. *Canadian Mathematical Bulletin*, 30(4):436–445, 1987. doi:10.4153/cmb-1987-064-1.

9 Helena Bergold, Stefan Felsner, Meghana M. Reddy, Joachim Orthaber, and Manfred Scheucher. Plane Hamiltonian cycles in convex drawings. In *Proceedings of the 40th International Symposium on Computational Geometry (SoCG 2024)*, pages 18:1–18:16, 2024. doi:10.4230/LIPIcs.SoCG.2024.18.

10 Helena Bergold, Stefan Felsner, Manfred Scheucher, Felix Schröder, and Raphael Steiner. Topological Drawings meet Classical Theorems from Convex Geometry. *Discrete & Computational Geometry*, 2022. doi:10.1007/s00454-022-00408-6.

11 Helena Bergold, Joachim Orthaber, Manfred Scheucher, and Felix Schröder. Holes in convex and simple drawings. *arXiv*, 2024. URL: https://arxiv.org/abs/2409.01723.

12 Peter Brass, William O. J. Moser, and János Pach. *Research Problems in Discrete Geometry*. Springer, 2005. doi:10.1007/0-387-29929-7.

13 Olivier Devillers, Ferran Hurtado, Gyula Károlyi, and Carlos Seara. Chromatic variants of the Erdős–Szekeres theorem on points in convex position. *Computational Geometry*, 26(3):193–208, 2003. doi:10/bb9h93.

14 Paul Erdős. Some problems on elementary geometry. *Australian Mathematical Society*, 2:2–3, 1978. URL: https://users.renyi.hu/~p_erdos/1978-44.pdf.

15 Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935. URL: http://www.renyi.hu/~p_erdos/1935-01.pdf.

16 Alfredo García, Alexander Pilz, and Javier Tejel. On plane subgraphs of complete topological drawings. *Ars Mathematica Contemporanea*, 20(1):69–87, 2021. doi:10.26493/1855-3974.2226.e93.

17 Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Empty triangles in generalized twisted drawings of K_n. In *Graph Drawing and Network Visualization*, volume 13764 of *LNCS*, pages 40–48. Springer, 2022. doi:10.1007/978-3-031-22203-0_4.

18 Tobias Gerken. Empty Convex Hexagons in Planar Point Sets. *Discrete & Computational Geometry*, 39(1):239–272, 2008. doi:10.1007/s00454-007-9018-x.

19 Heiko Harborth. Empty triangles in drawings of the complete graph. *Discrete Mathematics*, 191(1–3):109–111, 1998. doi:10.1016/S0012-365X(98)00098-3.

20 Marijn J. H. Heule and Manfred Scheucher. Happy ending: An empty hexagon in every set of 30 points. In Bernd Finkbeiner and Laura Kovács, editors, *Proc. 30th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS'24)*, volume 14570 of *LNCS*, pages 61–80. Springer, 2024. doi:10.1007/978-3-031-57246-3_5.

21 Joseph D. Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26:482–484, 1983. doi:10.4153/CMB-1983-077-8.

22 Jiří Matoušek. *Lectures on Discrete Geometry*. Springer, 2002. doi:10.1007/978-1-4613-0039-7.

23 Carlos M. Nicolás. The Empty Hexagon Theorem. *Discrete & Computational Geometry*, 38(2):389–397, 2007. doi:10.1007/s00454-007-1343-6.

24 János Pach, József Solymosi, and Géza Tóth. Unavoidable configurations in complete topological graphs. *Discrete & Computational Geometry*, 30(2):311–320, 2003. doi:10.1007/s00454-003-0012-9.

25 Nabil H. Rafla. *The good drawings D_n of the complete graph K_n*. PhD thesis, McGill University, Montreal, 1988. URL: https://escholarship.mcgill.ca/concern/theses/x346d4920.

26 Gerhard Ringel. Extremal problems in the theory of graphs. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, volume 8590. Publ. House Czechoslovak Acad. Sci Prague, 1964.

27 Manfred Scheucher. A SAT Attack on Erdős–Szekeres Numbers in $\mathbb{R}^d$ and the Empty Hexagon Theorem. *Computing in Geometry and Topology*, 2(1):2:1–2:13, 2023. doi:10.57717/cgt.v2i1.12.

28 Andrew Suk and Ji Zeng. Unavoidable patterns in complete simple topological graphs. In *Graph Drawing and Network Visualization*, volume 13764 of *LNCS*, pages 3–15. Springer, 2022. doi:10.1007/978-3-031-22203-0_1.