String Graph with Cop Number 4

Stephane Durocher
Department of Computer Science, University of Manitoba, Winnipeg, Canada

Myroslav Kryven
Department of Computer Science, University of Manitoba, Winnipeg, Canada

Maarten Löffler
Department of Information and Computing Sciences, Utrecht University, The Netherlands

Abstract

Cops and Robbers is a well-studied pursuit-evasion game in which a set of cops seeks to catch a robber in a graph G, where cops and the robber move along edges of G. The *cop number* of G is the minimum number of cops that is sufficient to catch the robber. The game of Cops and Robbers has been well-studied on *beyond-planar graphs* (that is, graphs that can be drawn with only few crossings) [1, 4] as well as *intersection graphs* (that is, graphs where the vertices represent geometric objects, and an edge exists between two vertices if the corresponding objects intersect). We consider a well-known subclass of intersection graphs called *string graphs* where the objects are curves. So far no string graph with cop number larger than three was known. We construct the first string graph with cop number four, which improves the previous bound and answers an open question by Gavenčiak et al. [5].

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Category Poster Abstract

The game of cops and robbers

One of the most common and well-studied pursuit-evasion problems is the game of *Cops and Robbers* on graphs, which was formalized by Quilliot [8] and Nowakowski and Winkler [7] in the 1980s; see also the recent book by Bonato and Nowakowski [2]. The game is played in a graph by two players: the robber player and the cop player. The common assumption that we also adopt here is that each player has full information about the graph and the other player's moves. The game consists of rounds (or steps) on a given graph. In the initial round, the cop player selects starting vertices for a set of cops, and then the robber player selects a starting vertex for a robber. In the subsequent rounds, the players alternate turns; during the cops' turn, the cop player may move some of the cops to adjacent vertices. Similarly, the robber player may move the robber to an adjacent vertex during the robber's turn. The cop player wins if the robber and any of the cops are simultaneously on the same vertex; otherwise, when the game continues indefinitely, the robber player wins. If a single cop suffices to catch the robber in a graph G, even when the robber plays adversarially, then G is a *cop-win graph*; otherwise, G is a *robber-win graph*. The minimum number of cops necessary to catch the robber in G, denoted $c(G)$, is called the *cop number* of G.

String graphs

The class of *string graphs* is the class of intersection graphs of *strings* where each string is a bounded curve in the plane, i.e., a continuous image of the interval $[0, 1]$ into $\mathbb{R}^2$. One of the most notable subclasses of string graphs is the class of *k-string graphs*, which are the graphs admitting a string representation in which every two curves intersect in at most k points. It is known that k-string graphs is a strict subclass of $k+1$-string graphs for all $k \geq 1$ and this inclusions is strict [3, 6].

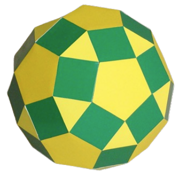


Figure 1 Rhombicosidodecahedron.

The *girth* of a graph is the length of a shortest cycle in the graph.

▶ **Lemma 1** ([1]). *A graph with min degree δ and girth at least five has cop number at least δ.*

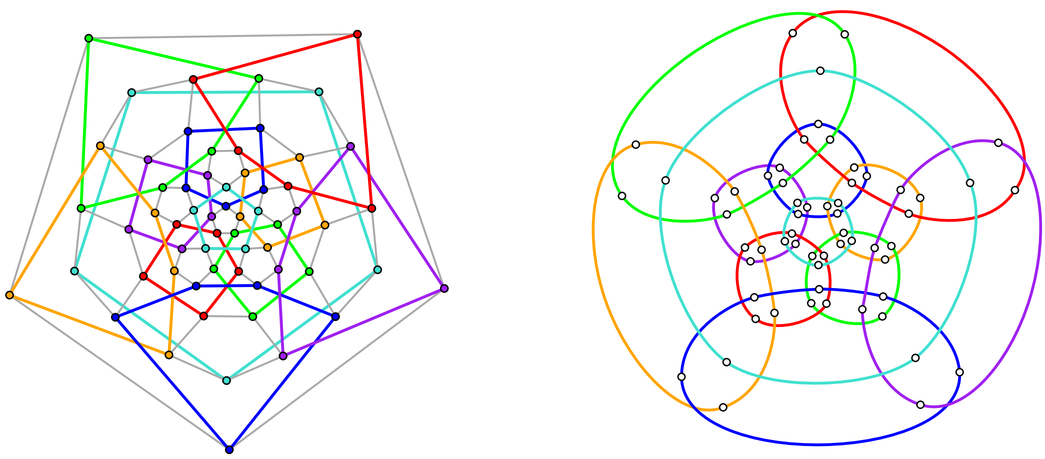
In the following we construct a 1-string graph with minimum degree four and girth five. And thus, as implied by Lemma 1, its cop number is at least four. This answers an open question by Gavenčiak et al. [5] as well as an open question on Math Stack Exchange [9].

1-string graph with cop number 4

▶ **Theorem 2.** *There is a 1-string graph G with minimum degree four and girth at least five.*

Proof. Our construction is based on an Archemidian solid Rhombicosidodecahedron R; see Figure 1. We consider a planar drawing of R and subdivide each edge incident to a triangular face with a *subdivision* vertex, let the resulting graph be R'; see Figure 2b. To distinguish the vertices of R in R' from the subdivision vertices (marked with dots in Figure 2b), we call the vertices of R the *original* vertices (the crossings in Figure 2b).

To construct our string graph G we cover the edges of R' with the strings so that the strings cross *internally* (that is, on each string s participating in the crossing, the crossing is between two other intersections on s) at the original vertices and the endpoints of the strings intersect at the subdivision vertices. More precisely, we partition the edges of R' into twelve *colored* cycles; see colored cycles in Figure 2b. Each such cycle consists of five strings. Each string is between two consecutive subdivision vertices of a cycle and two consecutive strings in the cycle intersect at their endpoints. Figure 2a shows a straight-line embedding of G.



(a) Straight-line drawing of the graph G.

(b) Planar drawing of the graph R' and the string representation of the graph G.

Figure 2 The string graph G and its string representation.

It is easy to check that every string s intersects with exactly four other strings: two strings intersect s at its endpoints and two strings cross s internally.

Now let us reassure that there are no cycles of length three or four in G. Notice that such a cycle, must have its edges from at most four different colored cycles in Figure 2b. By simply checking every set of at most four colored cycles, we can see that there is no cycle of length three or four. ◀

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