Approximating the Crossing Number of Dense Graphs

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Abstract

We present a deterministic $n^{2+o(1)}$-time algorithm that approximates the crossing number of any graph G of order n up to an additive error of $o(n^4)$, as well as a randomized polynomial-time algorithm that constructs a drawing of G with $\text{cr}(G) + o(n^4)$ crossings. These results imply a $(1 + o(1))$-approximation algorithm for the crossing number of dense graphs. Our work builds on the machinery used by Fox, Pach and Súk [10], who obtained similar results for the rectilinear crossing number.

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1 Preliminaries and previous work

Let G be a finite simple graph. The *crossing number* of G, denoted by $\text{cr}(G)$, is the minimum number of crossing points between edges, where the minimum is taken over all drawings of G on the plane. A *straight-line drawing* of G is a drawing such that each edge is represented by a segment joining the corresponding endpoints. The *rectilinear crossing number* of G, $\overline{\text{cr}}(G)$, is the least number of crossings amongst all straight-line drawings of G. Clearly, $\text{cr}(G) \leq \overline{\text{cr}}(G)$, and it is known that there are graphs for which the inequality is strict [5]. The crossing number and the rectilinear crossing number have been studied extensively, and we refer the reader to the comprehensive monograph of Schaefer [17] for a review of the existing literature and several interesting questions.

Computing the crossing number is known to be NP-complete [11], while determining the rectilinear crossing numbers is complete for the existential theory of reals [4], and hence NP-hard. For any fixed k, there is a linear time algorithm that decides whether $\text{cr}(G) \leq k$. In contrast, it is NP-hard to determine if $\overline{\text{cr}}(G) \leq k$ holds. A considerable amount of work has been put into developing approximation algorithms for both $\text{cr}(G)$ and $\overline{\text{cr}}(G)$. A graph drawing technique of Bhatt and Leighton [3] and the approximation algorithm for optimal balanced cuts of Arora et al. [2] can be used to find, in polynomial time, a straight-line drawing of any bounded degree n-vertex graph G with no more than $O(\log^4 n(n + \text{cr}(G))$ crossings. It wasn't until several years later that Chuzhoy [6], using the edge planarization method from [8], found a polynomial-time $O(n^{9/10})$-approximation algorithm for $\text{cr}(G)$ for bounded degree graphs (by this, we mean a multiplicative approximation). Building on this method further, Kawarabayashi and Sidiropoulos [12, 13] improved the approximation ratio to $O(n^{1/2})$, and then Mahabadi and Tan [7] found a randomized $O(n^{1/2-\delta})$-approximation algorithm, where $\delta > 0$ is a constant.

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It is well known that if an n-vertex graph G is dense (i.e., it has $\Omega(n^2)$ edges) then both $\mathrm{cr}(G)$ and $\overline{\mathrm{cr}}(G)$ are of order $\Omega(n^4)$ (this follows from the celebrated crossing lemma of Ajtai at al. [1] and Leighton [14]). Fox, Pach and Suk [10] presented an algorithm that constructs a straight-line drawing of G with $\overline{\mathrm{cr}}(G) + o(n^4)$ crossings. If G is dense then this algorithm produces a drawing with $(1 + o(1))\overline{\mathrm{cr}}(G)$ crossings.

2 Our results

We have obtained a close analog of the result from [10] for the crossing number.

▶ **Theorem 1.** *There exists a deterministic $n^{2+o(1)}$-time algorithm that for any given n-vertex graph G approximates $\mathrm{cr}(G)$ up to an additive error of $O(n^4/(\log\log n)^{\delta})$. Furthermore, there is a randomized polynomial-time algorithm that, with probability $1 - o(1)$, computes a drawing of G with $\mathrm{cr}(G) + O(n^4/(\log\log n)^{\delta})$ crossings. Here, δ denotes an absolute positive constant.*

The approximation part of the algorithm follows the same strategy as the one for rectilinear crossing numbers:

1. We are given an n-vertex graph $G = (V, E)$ as input.
2. Set $\varepsilon = (\log\log n)^{-\frac{1}{2c}}$ for some suitable absolute constant c and find an equitable Frieze-Kannan ε-regular partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of G using the algorithm in [9], where $k \le O(2^{\sqrt{\log\log n}})$. This takes $n^{2+o(1)}$ time.
3. Construct the edge weighted graph $G/\mathcal{P}$ which has a vertex for each V_i and where the edge between V_i and V_j has weight equal to the number of edges between these two sets. Then, compute the crossing number of $G/\mathcal{P}$ (a crossing between edges of weights w_1 and w_2 has weight $w_1 w_2$) by brute force and output this quantity. This can be done in $n^{o(1)}$ time.

3 Overview of the proof of correctness

The main novel ingredient which makes it possible to prove the correctness of the above algorithm is the following bound on the difference between the crossing numbers of two graphs on the same vertex set in terms of their distance in the *(labeled) cut metric.* The definition of this metric can be found, for example, in [15].

▶ **Theorem 2.** *Let G_1 and G_2 be graphs with the same vertex set V. If $d_\square(G_1, G_2) \ge n^{-4}$, then $|\mathrm{cr}(G_1) - \mathrm{cr}(G_2)| \le C d_\square(G_1, G_2)^{1/4} n^4$, where C is an absolute constant.*

Proof sketch. Start with a drawing $\mathcal{D}$ of G_1 which attains $\mathrm{cr}(G_1)$; we will use $\mathcal{D}$ as a blueprint to construct a drawing of G_2 with few crossings. After adding a node at each crossing of $\mathcal{D}$, we arrive at a planar map. By carefully and repeatedly applying a planar cycle separator theorem due to Miller [16], along with some packing and covering arguments, for any $t \in (0, 1)$ it is possible to subdivide the plane into $r = O(1/t^2)$ connected regions which contain no more than $\lceil t^2 n \rceil$ vertices of G and satisfy the following key property: Any vertex of G inside the region and any point on its boundary can be connected by a curve that has no more than tn^2 intersection points with the edges of the drawing that have no endpoint in that same region.

Let $P_1, P_2, \ldots, P_r$ denote the sets of vertices within each of the r regions of the subdivision. As long as $d_\square(G_1, G_2)$ is small, the number of edges between P_i and P_j will be similar in G_1 and G_2. For each edge e in G_2 between P_i and P_j ($i \ne j$), choose an edge e' of G_1 between the same sets uniformly at random and route e along e'. Since e and e' might have different endpoints, we need to do some adjustments near the endpoints of e. Because of the aforementioned property that the regions of the subdivision posses, this adjustments can be carried out without incurring in too many additional crossings. The edges of G_2 that have both endpoints in P_i can be added to the drawing at the end without many complications. One con show that the expected number of crossings in such a drawing of G_2 is no more than $\mathrm{cr}(G_1) + Cd_{\square}(G_1, G_2)^{1/4} n^4$. ◀

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