1-Planar Unit Distance Graphs

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Abstract

A matchstick graph is a plane graph with edges drawn as unit distance line segments. This class of graphs was introduced by Harborth who conjectured that a matchstick graph on n vertices can have at most $\lfloor 3n - \sqrt{12n-3} \rfloor$ edges. Recently his conjecture was settled by Lavollée and Swanepoel. In this paper we consider 1-planar unit distance graphs. We say that a graph is a 1-planar unit distance graph if it can be drawn in the plane such that all edges are drawn as unit distance line segments while each of them are involved in at most one crossing. We show that such graphs on n vertices can have at most $3n - \sqrt[4]{n}/10$ edges.

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1 Introduction

A graph is called a matchstick graph if it can be drawn in the plane with no crossings such that all edges are drawn as unit segments. This graph class was introduced by Harborth in 1981 [7, 9]. He conjectured that the maximum number of edges of a matchstick graph with n vertices is $\lfloor 3n - \sqrt{12n-3} \rfloor$. He managed to prove it in a special case where the unit distance is also the smallest distance among the points [8]. Recently his conjecture was settled by Lavollée and Swanepoel [12].

Other interesting classes of graphs are the k-planar graphs. For any $k \geq 0$, a graph G is called k-planar if it can be drawn in the plane such that each edge is involved in at most k crossings. Let $e_k(n)$ denote the maximum number of edges of a k-planar graph on n vertices. Since 0-planar graphs are the well known planar graphs, $e_0(n) = 3n - 6$ for $n \geq 3$. We have $e_1(n) = 4n - 8$ for $n \geq 4$ [19], $e_2(n) \leq 5n - 10$, which is tight for infinitely many n [19], $e_3(n) \leq 5.5n - 11$, which is tight up to an additive constant [17] and $e_4(n) \leq 6n - 12$, which is also tight up to an additive constant [1]. For general k we have $e_k(n) \leq c\sqrt{k}n$ for some constant c, which is tight apart from the value of c [19, 1].

A k-planar unit distance graph is a graph that can be drawn in the plane such that each edge is a unit segment and involved in at most k crossings. Let $u_k(n)$ be the maximum number of edges of a k-planar unit distance graph. Since 0-planar unit distance graphs are exactly the matchstick graphs, by the result of Lavollée and Swanepoel we have $u_0(n) = \lfloor 3n - \sqrt{12n-3} \rfloor$. We do not have any better lower bound for $u_1(n)$ than the value of $u_0(n)$. That is, allowing to use one crossing on each edge does not seem to help, still a proper piece of the triangular grid is the best known construction. Somewhat surprisingly, we prove an almost matching upper bound.

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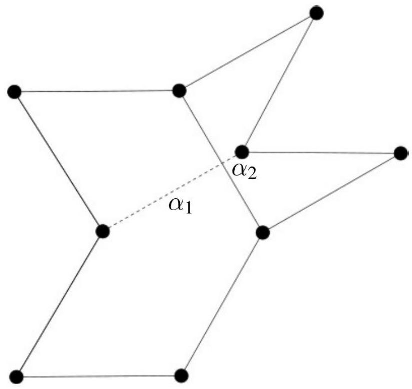

Figure 1 An edge in E_1 can be partitioned into two halfedges, α_1 and α_2.

▶ **Theorem 1.** *For the maximum number of edges of a* 1*-planar unit distance graph,* $u_1(n)$*, we have* $\lfloor 3n - \sqrt{12n-3} \rfloor \leq u_1(n) \leq 3n - \sqrt[4]{n}/10$.

For general k, the best known lower bound is due to Günter Rote (personal communication, 2023).

▶ **Theorem 2** (Rote)**.** *For the maximum number of edges of a* k*-planar unit distance graph,* $u_k(n)$*, we have* $u_k(n) \geq 2^{\Omega(\log k/\log\log k)} n$.

We have the following upper bound.

▶ **Theorem 3.** *For any* $n, k \geq 0$ *we have* $u_k(n) \leq c\sqrt[4]{k}n$ *for some constant* $c > 0$.

2 1-planar unit distance graphs

Proof of Theorem 1. The lower bound follows directly from Harborth's construction for matchstick graphs [8]. We prove the upper bound. Let G be a 1-planar unit distance graph with n vertices and consider a 1-plane unit distance drawing of G. Let E be the set of edges, $|E| = e$. Let G_0 be a plane subgraph of G with maximum number of edges, and among those one with the minimum number of triangular faces. Let $E_0 \subset E$ denote the set of edges of G_0 and $E_1 = E \setminus E_0$ denote the set of remaining edges, $|E_0| = e_0$, $|E_1| = e_1$. Let f be the number of faces of G_0, including the unbounded face and let $\Phi_1, \Phi_2 \dots \Phi_f$ be the faces of G_0. For any face Φ_i, $|\Phi_i|$ is the number of bounding edges of it, counted with multiplicity. That is, if an edge bounds Φ_i from both sides, then it is counted twice. Due to the maximality of G_0 and 1-planarity of G, every edge $\alpha \in E_1$ crosses an edge in E_0 and connects two vertices that belong to neighbouring faces of G_0. Therefore, we can partition every edge $\alpha \in E_1$ into two *halfedges*, α_1 and α_2 at the unique crossing point on α. See Figure 1. Each halfedge is contained in a face Φ, one of its endpoints is a vertex of Φ and the other endpoint is an interior point of a bounding edge.

▷ Claim 4. A triangular face of G_0 does not contain any halfedge.

Proof. Let $\Phi = uvw$ be a triangular face of G_0 that contains a halfedge α_1, which is part of the edge $\alpha = ux$. Then α crosses the edge vw. Replace the edge vw by α in G_0. See Figure 2. Since vw is the only edge of G that crosses α, we obtain another plane subgraph of G. It has the same number of edges.

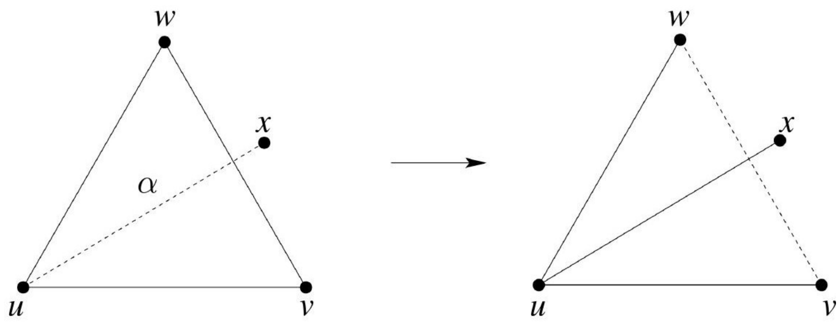

Figure 2 The number of triangles in G_0 can be reduced by edge flips.

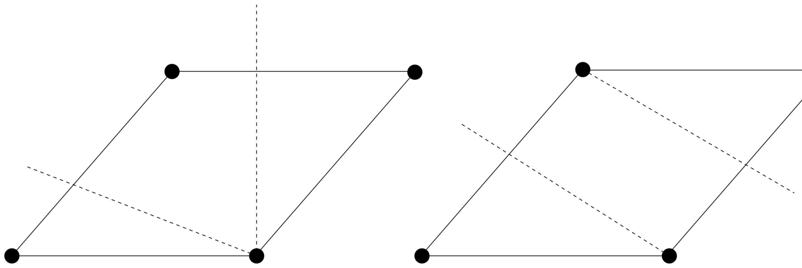

Figure 3 A quadrilateral can have at most two halfedges.

We claim that it has fewer triangular faces. The triangular face Φ disappeared. Suppose that we have created a new triangular face. Then α should be a side of it. But then either uv or uw is also a side, suppose without loss of generality that it is uv. But then uvx is also a unit equilateral triangle. If the two equilateral triangles uvw and uvx are on the same side of uv then $x = w$, if they are on opposite sides then vw and ux can not cross. ◁

Assign weight 1/2 to each halfedge. For any face Φ_i, let $s(\Phi_i)$ be the sum of the weights of its halfedges. Clearly, we have $\sum_{i=1}^{f} s(\Phi_i) = |E_1|$. For any face Φ of G_0, let $t(\Phi)$ denote the number of edges in a triangulation of Φ. A straightforward consequence of Euler's formula is the following statement. If the boundary of Φ_i has m connected components, then $t(\Phi_i) = |\Phi_i| + 3m - 6$.

▷ Claim 5. For any face Φ of G_0 we have (a) $s(\Phi) \leq t(\Phi)$, and if $|\Phi| \geq 5$ then (b) $s(\Phi) \leq t(\Phi) - |\Phi|/10$.

Proof. Suppose first that the boundary of Φ is not connected, that is, $m \geq 2$. Each of the $|\Phi_i|$ edges on the boundary of Φ_i is crossed by at most one halfedge, therefore, $s(\Phi) \leq |\Phi|/2$. On the other hand, $t(\Phi) \geq |\Phi|$. Therefore, $t(\Phi) \geq |\Phi| \geq |\Phi|/2 + |\Phi|/10 \geq s(\Phi) + |\Phi|/10$ and we are done in this case.

Suppose now that the boundary of Φ_i is connected, that is, $m = 1$. If $|\Phi| = 3$, then Φ is a triangle. Then $t(\Phi) = 0$ and by Claim 4, $s(\Phi) = 0$. If $|\Phi| = 4$, then Φ is a quadrilateral (actually, a rhombus). Then $t(\Phi) = 1$. Figure 3 shows all possible cases when Φ has two halfedges. On the other hand, it is shown in [19] by an easy case analysis that no more halfedges can be added. Therefore, $s(\Phi) \leq 1 = t(\Phi)$. This finishes part (a).

Suppose that $|\Phi_i| \geq 5$. We can assume that Φ_i has at least two halfedges, otherwise we are done. A halfedge α in Φ divides Φ into two parts. Let $a(\alpha)$ and $b(\alpha)$ be the number of vertices of Φ in the two parts. If a vertex appears on the boundary more than once, then

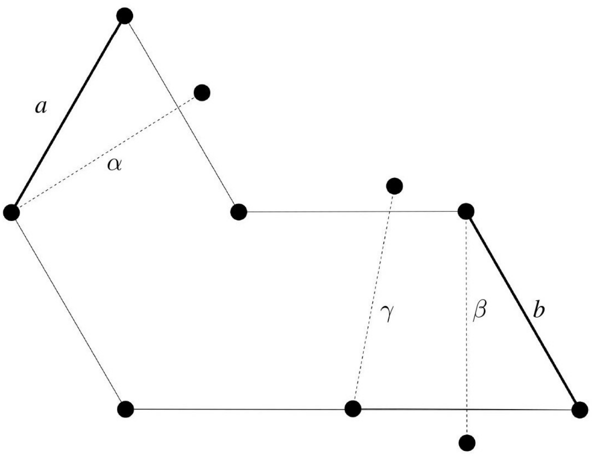

Figure 4 Halfedges α and β are minimal, edges a and b are uncrossed.

it is counted with multiplicity. Since the halfedges in Φ do not cross each other, all other halfedges are entirely in one of these two parts. If one part does not contain any halfedge, then α is called a *minimal halfedge*. Let α be a halfedge for which $M = \min\{a(\alpha), b(\alpha)\}$ is minimal. Then there are M vertices of Φ_i on one side of α. Clearly, this part cannot contain any halfedge, so α is minimal. Now for any other halfedge $\beta \neq \alpha$, let $c(\beta)$ be the number of vertices of Φ_i on the side of β not containing α. Take a halfedge β for which $c(\beta)$ is minimal. Then β is also a minimal halfedge. So, we can conclude that there are at least two minimal halfedges in Φ, say, α and β.

Then α and β together partition Φ into three parts, two parts contain no other halfedges but both contain an edge of Φ. So, at most $|\Phi| - 2$ edges of Φ are crossed by a halfedge, therefore, there are at most $|\Phi| - 2$ halfedges in Φ, consequently $s(\Phi) \leq (|\Phi| - 2)/2$. See Figure 4.

On the other hand, $t(\Phi) = |\Phi| - 3$. Since $|\Phi| \geq 5$, we have $t(\Phi) = |\Phi| - 3 \geq (|\Phi| - 2)/2 + |\Phi|/10 \geq s(\Phi) + |\Phi|/10$. This concludes the proof of the Claim. ◁

Return to the proof of Theorem 1. For $i \geq 3$, let f_i denote the number of faces Φ of G_0 with $|\Phi| = i$. By definition, $\sum_{i=3}^{\infty} f_i = f$ and $\sum_{i=3}^{\infty} i f_i = 2e_0$. Let $F_{\geq 5} = \sum_{i=5}^{\infty} i f_i$. By the maximality of G_0, every edge in E_1 crosses an edge in E_0, and by 1-planarity, every edge in E_0 is crossed by at most one edge in E_1. Consequently, $|E_0| = e_0 \geq |E_1| = e_1$.

If $e_0 \leq n$, then $e = e_0 + e_1 \leq 2e_0 \leq 2n < 3n - c\sqrt[4]{n}$, so we are done. Therefore, for the rest of the proof we can assume that $e_0 \geq n$. It follows that $3f_3 + 4f_4 + F_{\geq 5} = 2e_0 \geq 2n$.

▷ Claim 6. Suppose that $F_{\geq 5} \geq p$. Then $e = e_0 + e_1 \leq 3n - p/10$.

Proof. By the previous observations,

$$e = e_0 + e_1 = e_0 + \sum_{\substack{\alpha \text{ is a} \\ \text{halfedge}}} 1/2 = e_0 + \sum_{i=1}^{f} s(\Phi_i) = e_0 + \sum_{|\Phi|=3} s(\Phi) + \sum_{|\Phi|=4} s(\Phi) + \sum_{|\Phi|\geq 5} s(\Phi)$$

$$\leq e_0 + \sum_{|\Phi|=3} t(\Phi) + \sum_{|\Phi|=4} t(\Phi) + \sum_{|\Phi|\geq 5} (t(\Phi) - |\Phi|/10)$$

$$\leq e_0 + \sum_{|\Phi|} t(\Phi) - \sum_{|\Phi|\geq 5} |\Phi|/10 \leq 3n - 6 - F_{\geq 5}/10 \leq 3n - p/10.$$ ◁

▷ Claim 7. Suppose that $f_3 \geq p$. Then $e = e_0 + e_1 \leq 3n - \sqrt{p}/5$.

Proof. We can assume that Ψ, the unbounded face of G_0 has at least 5 edges. If not, the statement holds trivially. Since we have p equilateral triangles in G_0, the union of all bounded faces, R, has area at least $\sqrt{3}p/4$. The Isoperimetric inequality states that if a polygon has perimeter l and area A, then $l^2 \geq 4\pi A$ [5]. It implies that R has perimeter at least $\sqrt[4]{3}\sqrt{\pi p} > 2\sqrt{p}$. That is $|\Psi| \geq 2\sqrt{p}$. Therefore,

$$e = e_0 + e_1 = e_0 + \sum_{\substack{\alpha \text{ is a} \\ \text{halfedge}}} 1/2 = e_0 + \sum_{i=1}^{f} s(\Phi_i) = e_0 + \sum_{\Phi \neq \Psi} s(\Phi) + s(\Psi)$$

$$\leq e_0 + \sum_{\Phi \neq \Psi} t(\Phi) + t(\Psi) - |\Psi|/10 = 3n - 6 - |\Psi|/10 \leq 3n - 6 - \sqrt{p}/5. \qquad \triangleleft$$

We can assume that $n \geq 5$, otherwise Theorem 1 holds trivially. If $F_{\geq 5} \geq n/2$, then by Claim 6, $e \leq 3n - n/20 \leq 3n - \sqrt[4]{n}/10$ and we are done. If $f_3 \geq n/9$, then by Claim 7, $e \leq 3n - \sqrt{n}/15 \leq 3n - \sqrt[4]{n}/10$ and we are done again. So, we can assume that $F_{\geq 5} \leq n/2$, $f_3 \leq n/9$. Since $3f_3 + 4f_4 + F_{\geq 5} = 2e_0 \geq 2n$, it follows that $f_4 \geq n/4$.

Suppose without loss of generality that none of the edges of G are vertical. Otherwise apply a rotation. Define an auxiliary graph H as follows. The vertices represent the quadrilateral faces of G_0. Since all edges are of unit length, all these faces are rhombuses. Two vertices are connected by an edge if the corresponding rhombuses have a common edge. The edges of H correspond to the edges of G_0 with a rhombus face on both sides. For every edge of H define its weight as the slope of the corresponding edge of G_0. A path in H, such that all of its edges have the same weight w, is called a *w-chain*, or briefly a *chain*. A chain corresponds to a sequence of rhombuses such that the consecutive pairs share a side and all these sides are parallel. A chain, with at least two vertices (rhombuses) is called *maximal* if it cannot be extended. With one-vertex chains we have to be careful. Suppose that v is a vertex of H, R is the corresponding rhombus, and let w_1, w_2 be the slopes of its sides. The one-vertex chain v is *maximal* if it cannot be extended to a larger w_1-chain or a larger w_2-chain. Each vertex of H is in exactly two maximal chains.

▷ Claim 8. The intersection of two chains is empty or forms a chain.

Proof. If the intersection is just one vertex then the statement clearly holds. Suppose that A and B are chains with at least two common vertices, and their intersection is not a chain. Let $A = v_1, v_2, \dots v_a$. We can assume without loss of generality that $v_1, v_a \in B$ but no other vertex of A is in B. Otherwise we can delete some vertices of A to obtain this situation. Delete all vertices of B which are not between v_1 and v_a. Now $B = u_1, u_2, \dots u_b$ where $v_1 = u_1$, $v_a = u_b$ and these are the only common points of A and B. Let R be the rhombus that represents $v_1 = u_1$ in G_0. Its sides have slopes w_1 and w_2 such that A is a w_1-chain, B is a w_2-chain. Apply an affine transformation so that R is a unit square, w_1 is the horizontal, w_2 is the vertical direction. Suppose that Q is the rhombus that represents $v_a = u_b$. Then its sides also have slopes w_1 and w_2, so Q is also an axis-parallel unit square. Represent each vertex $v_1, v_2, \dots v_a, u_1, u_2, \dots u_b$ by the center of the corresponding rhombus. For simplicity we call these points also $v_1, v_2, \dots v_a, u_1, u_2, \dots u_b$, respectively. Assume without loss of generality that the point $v_a = u_b$ has larger x and y coordinates, than $v_1 = u_1$. Connect the consecutive points in both chains by straight line segments. Since A is a w_1-chain and w_1 is the horizontal direction, the polygonal chain $P_A = v_1, v_2, \dots v_a$ is y-monotone, and similarly,

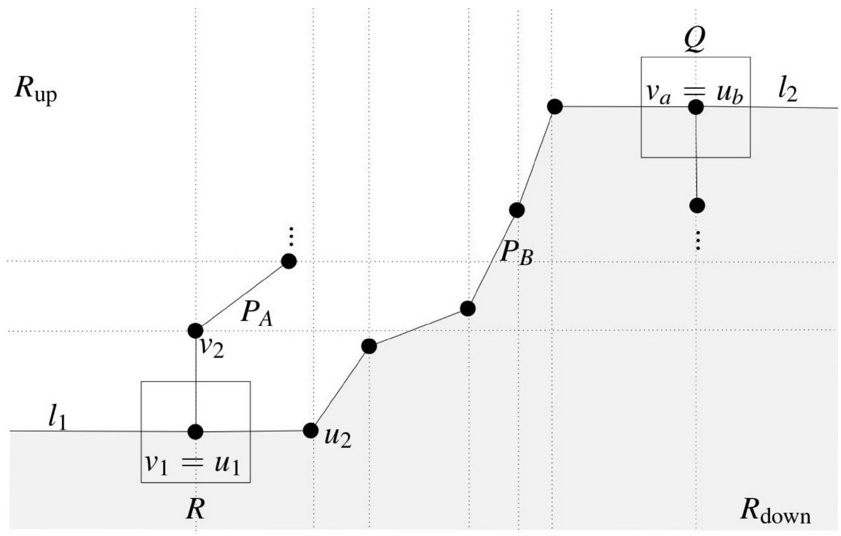

Figure 5 The intersection of two chains is empty or forms a chain.

the polygonal chain $P_B = u_1, v_2, \dots v_b$ is x-monotone. Let l_1 be the horizontal halfline from $v_1 = u_1$, pointing to the left and let l_2 be the horizontal halfline from $v_a = u_b$, pointing to the right. The bi-infinite curve $l_1 \cup P_B \cup l_2$ is simple, because P_B is x-monotone. It divides the plane into two regions, R_{down}, which is below it and its complement, R_{up}, see Figure 5.

Observe that the initial part of P_A, near $v_1 = u_1$ is in R_{up}, while the final part, near $v_a = u_b$ is in R_{down}. On the other hand, P_A does not intersect the boundary of R_{down} and R_{up}. Indeed, it does not intersect l_1 and l_2 since it is y-monotone, and does not intersect P_B by assumption. This is clearly a contradiction which proves the Claim. ◁

▷ Claim 9. There are at least $\sqrt{n}/\sqrt{2}$ disjoint maximal chains.

Proof. For any vertex of H (that is, for any rhombus face in G_0) there are exactly two maximal chains containing it. Therefore, the total length of all the maximal chains is $2f_4 \geq n/2$. If there are less than $\sqrt{n}/\sqrt{2}$ disjoint maximal chains, then one of them has length at least $\sqrt{n}/\sqrt{2}$. Through each of its vertices, there is another maximal chain and by Claim 8 all of these chains are different. ◁

By Claim 9, we have at least $\sqrt{n}/\sqrt{2}$ disjoint maximal chains. Each of them has two ending rhombuses with sides that bound a face of size different than 4. All of these bounding edges are different, therefore, $3f_3 + F_{\geq 5} \geq \sqrt{2}\sqrt{n}$, which implies that either $3f_3 \geq \sqrt{n}/\sqrt{2}$, or $F_{\geq 5} \geq \sqrt{n}/\sqrt{2}$.

In the first case, by Claim 7 we have $e \leq 3n - \sqrt[4]{n}/10$. In the second case, by Claim 6 we have $e \leq 3n - \sqrt{n}/10$. This concludes the proof of Theorem 1. ◀

3 *k*-planar unit distance graphs

Proof of Theorem 2. Suppose that $n, k > 100$. The following is a well known result in number theory (see [14], [16]). For any m, there is an $r < m$ such that r can be written as $a^2 + b^2$ in $2^{\Omega(\log m/\log\log m)}$ different ways where a and b are integers. For any fixed m let r be the product of the first l primes congruent to 1 mod 4, such that l is maximal with the property that $r < m$. This r satisfies the requirements.

Erdős [6] used it to construct a set of n points that determine $n2^{\Omega(\log n/\log\log n)}$ unit distances. Clearly, r is square-free, therefore, whenever $r = a^2 + b^2$, $(a, b) = 1$.

Apply the above result for $m = \sqrt{k}/5$. We obtain $r < \sqrt{k}/5$ that can be written as the sum of two integer squares, $r = a^2 + b^2$ in $2^{\Omega(\log m/\log\log m)} = 2^{\Omega(\log k/\log\log k)}$ different ways. Take a $\sqrt{n} \times \sqrt{n}$ unit square grid and connect two points by a straight line segment if they are at distance $\sqrt{r}$. Then each vertex has degree $2^{\Omega(\log k/\log\log k)}$, so our graph has $n2^{\Omega(\log k/\log\log k)}$ edges. Observe that no edge contains a vertex in its interior.

Let uv be an edge. Consider all vertices adjacent to an edge that crosses uv. All these vertices are at distance at most $\sqrt{r}$ from uv. This region has area $(2 + \pi)r$, so the number of vertices in this region is less than $6r$. Each of these vertices have degree at most $4r$, so uv is crossed by at most $24r^2 < k$ edges. Scale the picture by a factor of $1/\sqrt{r}$ and we obtain a k-planar unit distance graph of n vertices and $2^{\Omega(\log k/\log\log k)}$ edges. ◀

For the proof of Theorem 3 we need some introduction. Let $\text{CR}(G)$ denote the *crossing number* of graph G, that is, the minimum number of edge crossing over all drawings of G in the plane. According to the Crossing Lemma [3, 13], for every graph G with n vertices and $e \geq 4n$ edges, $\text{CR}(G) \geq \frac{1}{64}\frac{e^3}{n^2}$. It is asymptotically tight in general for simple graphs [19]. However, there are better bounds for graphs satisfying some monotone property [15], or for monotone drawing styles [10].

A drawing style $\mathcal{D}$ is a subset of all drawings of a graph G. so some drawings belong to $\mathcal{D}$, others do not. It is monotone if removing edges retains the drawing style. A vertex split is the following operation. (a) Replace a vertex v of G by two vertices, v_1 and v_2, both very close to v. Connect each edge of G incident to v either to v_1 or v_2 by locally modifying them such that no additional crossing is created. Or as an extreme or limiting case, (b) place both v_1 and v_2 to the same point where v was, connect each edge incident to v either to v_1 or v_2 without modifying them, such that the edges incident to v in G that are connected to v_1 (resp. v_2) after the split form an interval in the clockwise order from v. A drawing style $\mathcal{D}$ is split-compatible if performing vertex splits retains the drawing style.

The bisection width $b(G)$ of a graph G is the smallest number of edges whose removal splits G into two graphs, G_1 and G_2, such that $|V(G_1)|, |V(G_2)| \geq |V(G)|/5$. For a drawing style $\mathcal{D}$ the $\mathcal{D}$-bisection width $b_{\mathcal{D}}(G)$ of a graph G in drawing style $\mathcal{D}$ is the smallest number of edges whose removal splits G into two graphs, G_1 and G_2, both in drawing style $\mathcal{D}$ such that $|V(G_1)|, |V(G_2)| \geq |V(G)|/5$. Let $\Delta(G)$ denote the maximum degree in G. The following result is a generalization of the Crossing Lemma.

▶ **Theorem 10** (Kaufmann-Pach-Tóth-Ueckerdt [10]). *Suppose that $\mathcal{D}$ is a monotone and split-compatible drawing style, and there are constants k_1, k_2, $k_3 > 0$ and $b > 1$ such that each of the following holds for every n-vertex e-edge graph G in drawing style $\mathcal{D}$:*

1. *If $\text{CR}_{\mathcal{D}}(G) = 0$, then $e \leq k_1 \cdot n$.*
2. *The $\mathcal{D}$-bisection width satisfies $b_{\mathcal{D}}(G) \leq k_2\sqrt{\text{CR}_{\mathcal{D}}(G) + \Delta(G) \cdot e + n}$.*
3. *$e \leq k_3 \cdot n^b$.*

Then there exists a constant $\alpha > 0$ such that for any n-vertex e-edge graph G in drawing style $\mathcal{D}$ we have $\text{CR}_{\mathcal{D}}(G) \geq \alpha\frac{e^{1/(b-1)+2}}{n^{1/(b-1)+1}}$ provided $e > (k_1 + 1)n$.

In [10] only vertex split of type (a) was allowed, but the proof works also for type (b).

▶ **Theorem 11** (Spencer-Szemerédi-Trotter [20]). *Let G be a unit distance graph on n verices. The number of edges in G is at most $cn^{4/3}$ where $c > 0$ is a constant.*

Proof of Theorem 3. Consider now the following drawing style $\mathcal{D}$ for a graph G.

1. Vertices are represented by not necessarily distinct points.
2. Edges are represented by unit segments between the corresponding points.
3. The intersection of two edges is empty or a point, that is, they cannot overlap.
4. If a point p represents more than one vertex, say, $v_1, \ldots, v_m$, then the sets of edges incident to $v_1, \ldots, v_m$, respectively, form an interval in the clockwise order from point p.

Clearly, $\mathcal{D}$ satisfies the following properties.

1. The drawing style $\mathcal{D}$ is monotone and split-compatible.
2. If $\text{CR}(G) = 0$, then $e \leq 3n - 6$. In fact, by [12], $e \leq \lfloor 3n - \sqrt{12n - 3} \rfloor$.
3. For any graph G, we have $b(G) \leq 10\sqrt{\text{CR}(G) + \Delta(G) \cdot e + n}$ by the result of Pach, Shahrokhi and Szegedy [18]. But if G is drawn in drawing style $\mathcal{D}$, then all of its subgraphs are also drawn in drawing style $\mathcal{D}$. Therefore, $b_{\mathcal{D}}(G) \leq 10\sqrt{\text{CR}(G) + \Delta(G) \cdot e + n}$.
4. By [2], any n-vertex graph in drawing style $\mathcal{D}$ has less than $1.94n^{4/3}$ edges.

Summarizing, we can apply Theorem 10 with $k_1 = 3$, $k_2 = 10$, $k_3 = 1.94$, $b = 4/3$ and obtain the following. For any graph G in drawing style $\mathcal{D}$ with n vertices and $e > 4n$ edges we have $\text{CR}_{\mathcal{D}}(G) \geq \alpha \frac{e^{1/(b-1)+2}}{n^{1/(b-1)+1}} = \alpha \frac{e^5}{n^4}$ for some $\alpha > 0$.

Consider now a k-plane drawing of a unit distance graph G with n vertices and e edges. If $e \leq 4n$, we are done, suppose that $e \geq 4n$. Since each edge contains at most k crossings, the total number of crossings $c(G)$ satisfies $c(G) \leq ek/2$. On the other hand, we have $c(G) \geq \alpha \frac{e^5}{n^4}$. Therefore, $ek/2 \geq \alpha \frac{e^5}{n^4}$ so $e \leq \beta \sqrt[4]{k} n$ for some $\beta > 0$. ◀

4 Open questions

In this paper we proved that a 1-planar unit distance graph on n vertices can have at most $u_1(n) \leq 3n - \sqrt[4]{n}/10$ edges. However, the best known lower bound construction for $u_1(n)$ is the same as for $u_0(n)$.

▶ **Problem 12.** *Is it true that $u_0(n) = u_1(n)$?*

For $k = 2$ there is a slightly better construction by Dániel Simon (personal communication, 2023) of roughly $3n - \sqrt{\frac{192}{23} n}$ edges and for $k = 3$ there is an easy construction (a piece of a unit triangular grid and its shifted copy by a unit vector) with $3.5n - c\sqrt{n}$ edges.

For a larger k our lower and upper bounds for $u_k(n)$ are very far from each other.

▶ **Problem 13.** *Determine the maximum number of edges of a k-planar unit distance graph.*

There are r-regular matchstick graphs for $r \leq 4$ [9, 21] and there are no r-regular matchstick graphs for $r \geq 5$ [4, 11]. It follows from Theorem 1 that there are no r-regular 1-planar unit distance graphs for $r \geq 6$.

▶ **Problem 14.** *Are there 5-regular 1-planar unit distance graphs?*

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