The Density Formula: One Lemma to Bound Them All

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Abstract

We introduce the Density Formula for (topological) drawings of graphs in the plane or on the sphere, which relates the number of edges, vertices, crossings, and sizes of cells in the drawing. We demonstrate its capability by providing several applications: we prove tight upper bounds on the edge density of various beyond-planar graph classes, including so-called k-planar graphs with $k = 1, 2$, fan-crossing / fan-planar graphs, k-bend RAC-graphs with $k = 0, 1, 2$, quasiplanar graphs, and k^+-real face graphs. In some cases (1-bend and 2-bend RAC-graphs and fan-crossing / fan-planar graphs), we thereby obtain the first tight upper bounds on the edge density of the respective graph classes. In other cases, we give new streamlined and significantly shorter proofs for bounds that were already known in the literature. Thanks to the Density Formula, all of our proofs are mostly elementary counting and mostly circumvent the typical intricate case analysis found in earlier proofs. Further, in some cases (simple and non-homotopic quasiplanar graphs), our alternative proofs using the Density Formula lead to the first tight lower bound examples.

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1 Introduction

Topological Graph Theory is concerned with the analysis of graphs drawn in the plane $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$ such that the drawing has a certain property often related to forbidden crossing configurations. The most prominent example is the class of planar graphs, which admit drawings without any crossings. Other well-studied examples include k-planar graphs where every edge can have up to k crossings, RAC-graphs where edges are straight line segments and every crossing happens at a right angle, or quasiplanar graphs where no three edges are allowed to pairwise cross each other. As all these include planar graphs as a special case, they are commonly known as *beyond-planar* graph classes. See [14] for a recent survey.

When studying a beyond-planar graph class $\mathcal{G}$, one of the most natural and important questions is to determine how many edges a graph in $\mathcal{G}$ can have. The *edge density* of $\mathcal{G}$ is the function giving the maximum number of edges over all n-vertex graphs in $\mathcal{G}$. For example, planar graphs with at least three vertices have edge density $3n - 6$. All the beyond-planar graph classes mentioned above have linear edge density, i.e., their edge density is in $\mathcal{O}(n)$. Proofs of precise linear upper bounds for the edge density of a specific class $\mathcal{G}$ are often times involved and very tailored to the specific drawing style that defines $\mathcal{G}$. In particular, getting a *tight* bound (even only up to an additive constant) was achieved only in a couple of cases. A particularly simple case is the class of planar graphs, whose edge density of $3n - 6$ can be easily derived from Euler's Formula. However, a comparable formula for general drawings (with crossings) that can be used to easily derive tight upper bounds for the edge density of beyond-planar graph classes was not known – until now.

Our Contribution. In this paper, we introduce a new tool, called the **Density Formula** (Lemma 3.1), which can be used to derive upper bounds on edge densities for many beyond-planar graph classes. It is an equation that relates the number of edges, vertices, crossings, and sizes[1] of cells[1] in a connected drawing of a graph and is parameterized by a real-valued parameter t. Intuitively, the Density Formula allows us to obtain density bounds by counting the cells of small size in a drawing, which is often times quite an elementary task. The parameter t is chosen in accordance with the desired density bound, e.g., when aiming for a bound of roughly $5n$, where n is the number of vertices, we might set $t = 5$, in which case the Density Formula states that the number of edges in the drawing is $5n - 10 - \sum_{c \in \mathcal{C}}(\|c\| - 5) - x$, where x is the number of crossings, $\mathcal{C}$ is the set of cells, and $\|c\|$ denotes the size of a cell c. Thus, any upper bound on $-(\sum_{c \in \mathcal{C}}(\|c\| - 5) + x)$ yields an upper bound on the number of edges. Since the quantity $(\|c\| - 5)$ is non-negative for cells of size at least 5, such a bound can indeed be obtained by counting the cells of small sizes (here, at most 4) and cross-charging them with the crossings.

We give the precise, more general, statement of the Density Formula in Section 3, where we also develop some general tools that help with the required counting / charging arguments. Before that, in Section 2, we formally define some basic notions, such as (connected) drawings, cells, cell sizes, etc., and discuss some further preliminaries. We demonstrate the capabilities of the Density Formula by providing several applications, which are discussed next.

Applications

***k*-Bend RAC-Drawings.** For an integer $k \geq 0$, a drawing Γ in the plane $\mathbb{R}^2$ of some graph G is *k-bend RAC*, which stands for right-angle crossing, if every edge of Γ is a polyline with at most k bends and every crossing in Γ happens at a right angle, and in this case G is called a *k-bend RAC-graph*. The k-bend RAC-graphs were introduced by Didimo, Eades, and Liotta [13], who prove that n-vertex 0-bend RAC-graphs have at most $4n - 10$ edges (and this is tight), while every graph is a 3-bend RAC-graph. The best known upper bound

[1] Loosely speaking, a cell of a drawing is a connected region of the plane (or sphere) after removing the drawing; its size is the number of vertex and edge segment occurrences along its boundary, see Figure 3 for examples.

Table 1 Overview of edge density bounds, i.e., the maximum number of edges in connected n-vertex (n large enough) graphs in that graph class. In particular, the third column lists previous work on upper bounds and the fourth column lists the upper bounds we obtain using the Density Formula. Previously unknown bounds are highlighted with boxes. Results from the literature that are written in light red rely on incomplete proofs as they use an incorrect statement from [16, 17], as we discuss in more detail in Section 5.1.

beyond-planar graph class	variant	upper bound (previous work)	upper bound (Density Formula)	lower bound
0-bend RAC	no constraint	$4n - 10$ [13]	$4n - 8$ full version [15]	$4n - 10$ [13]
1-bend RAC	non-homotopic	$5.4n - 10.8$ [5]	$5n - 10$ Theorem 4.2	$5n - 10$ [5]
2-bend RAC	non-homotopic	$20n - 24$ [21]	$10n - 19$ Theorem 4.2	$10n - 54$ Theorem 4.3
fan-crossing / fan-planar	simple	$5n - 10$ [11, 12, 16, 17]	$5n - 10$ Theorem 5.2	$5n - 10$ [16, 17]
fan-cr. / fan-pl. + bipartite	simple	$4n - 12$ [7, 12]	$4n - 10$ full version [15]	$4n - 16$ [7]
quasiplanar	simple	$6.5n - 20$ [4]	$6.5n - 20$ full version [15]	$6.5n - 20$ Theorem 6.2
	non-homotopic	$8n - 20$ [4]	$8n - 20$ full version [15]	$8n - 20$ Theorem 6.1
1^+-real face	non-homotopic	$5n - 10$ [10]	$5n - 10$ full version [15]	$5n - 10$ [10]
2^+-real face	non-homotopic	$4n - 8$ [10]	$4n - 8$ full version [15]	$4n - 8$ [10]
k^+-real face $k \geq 3$	no constraint	$\frac{k}{k-2}(n-2)$ [10]	$\frac{k}{k-2}(n-2)$ full version [15]	$\frac{k}{k-2}(n-2)$ [10]
1-planar	non-homotopic	$4n - 8$ [20]	$4n - 8$ full version [15]	$4n - 8$ [20]
2-planar	non-homotopic	$5n - 10$ [20]	$5n - 10$ full version [15]	$5n - 10$ [20]

for simple[2] 1-bend RAC-drawings is $5.4n - 10.8$ [5], while the lower bound is $5n - 10$ [5]. By means of the Density Formula, we give an improved upper bound of $5n - 10$ for the connected case, which is best-possible. Very recently, Tóth [21] established an upper bound of $20n - 24$ for simple graphs admitting 2-bend RAC-drawings, thereby improving the long

[2] Loosely speaking, in a simple drawing, every pair of edges intersects in at most one point, thereby forbidding digons formed by segments of two edges. In non-homotopic drawings, such digons are allowed as long as both regions bounded by a digon contain at least one vertex or crossing.

standing previous best upper bound of $74.2n$ [8]. Using the Density Formula, we derive a significantly improved upper bound of $10n - 19$ for simple drawings. We also show that this bound is tight up to an additive constant by constructing an infinite family of simple 2-bend RAC-drawings with $10n - 54$ edges. (A similiar construction of 2-bend RAC-drawings with $10n - 46$ edges was presented by Angelini et al. [6], but their drawings are not simple.) Both of our upper bound results in fact apply even to the non-homotopic[2] case. We prove these results in Section 4. For completeness, in the full version [15], we also apply the Density Formula to reprove the known upper bound for 0-bend RAC-graphs.

Fan-Crossing Drawings. A drawing Γ on the sphere $\mathbb{S}^2$ of some graph G is *fan-crossing* if for every edge e of G, the edges crossing e in Γ form a star in G, and in this case G is called a *fan-crossing graph*. A simple drawing is fan-crossing if and only if there is no configuration I and no triangle-crossing, as shown in Figure 1. Fan-crossing drawings generalize fan-planar drawings; but the story about fan-planar graphs is problematic and tricky. In a preprint from 2014, Kaufmann and Ueckerdt [16] introduced fan-planar drawings as the simple drawings in $\mathbb{R}^2$ without configuration I and II, as shown in Figure 1. These are today known as *weakly fan-planar* and they show that n-vertex weakly fan-planar graphs have at most $5n - 10$ edges [16]. However, recently, a first flaw in this proof was discovered [18]. It was fixed in the journal version [17] of [16] from 2022 by additionally forbidding configuration III, as shown in Figure 1. These, more restricted, graphs are today known as *strongly fan-planar* graphs, and it is known that this indeed is a different graph class [12]. However, for each n-vertex weakly fan-planar graph, there is a strongly fan-planar graph on the same number of vertices and edges [12]. So any density result could be lifted. As every triangle-crossing contains configuration II, weakly fan-planar graphs are also fan-crossing, while again these are indeed different graph classes [11]. However again, for each n-vertex fan-crossing graph, there is a weakly fan-planar graph on the same number of vertices and edges [11], and thus the density can be lifted to fan-crossing graphs.

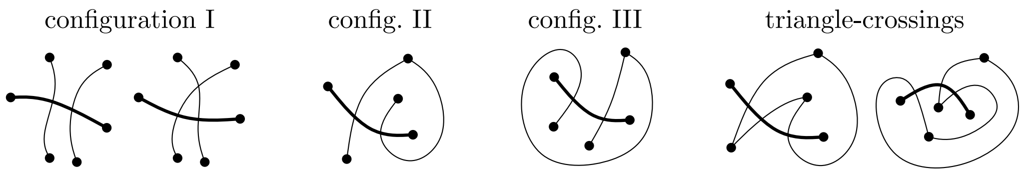

Figure 1 Simple fan-planar drawings have neither configuration I, nor II, nor III. Simple fan-crossing drawings have no configuration I and no triangle-crossings.

In Section 5, we prove an upper bound of $5n - 10$ for simple n-vertex connected fan-crossing drawings by applying the Density Formula. We also briefly describe in Section 5.1 another issue in the (updated) proof from [17] by providing a counterexample to one of their crucial statements. As all previous density results rest on [17], our result on fan-crossing drawings is the first complete proof for fan-crossing, weakly fan-planar, and strongly fan-planar drawings. Moreover, our proof is significantly simpler than the strategy used in [17]. In the full version [15], we discuss the special case of fan-crossing drawings of bipartite graphs, for which we obtain similar results.

We remark that, in a very recent preprint, Ackerman and Keszegh [3] also (independently of us) propose a new alternative proof for the $5n - 10$ upper bound for fan-crossing graphs. Moreover, Brandenburg [11] also considers *adjacency-crossing* graphs by just forbidding configuration I, but allowing triangle-crossings. He shows however that this class coincides with fan-crossing graphs, and hence our $5n - 10$ upper bound applies.

Quasiplanar Drawings. A drawing Γ on the sphere $\mathbb{S}^2$ of some graph G is *quasiplanar* if no three edges of G pairwise cross in Γ and in this case G is called a *quasiplanar graph*. Quasiplanar graphs were introduced by Pach [19]. It is known that simple n-vertex quasiplanar drawings have at most $6.5n - 20$ edges [4] and non-homotopic connected n-vertex quasiplanar drawings have at most $8n - 20$ edges [4]. However, the best known lower bounds [4] are just $6.5n - 29$ and $7n - 29$, respectively. In the full version [15], we reprove the known upper bounds using the Density Formula. In Section 6, inspired by insights gained in our upper bound proofs, we provide families of drawings showing that the previous upper bounds are actually best-possible.

Further applications. In the full version [15], we also use the Density Formula to reprove (and slightly generalize) the known upper bounds for so-called k^+-real face graphs and 1-planar and 2-planar graphs. All results are summarized in Table 1.

Some previously known density proofs already contain ideas that are similar to (parts of) our strategy, and we discuss this further in Section 7. But with the Density Formula, whose proof is merely a straight-forward application of Euler's Formula, we have a unified and simple approach that somewhat unveiled the essential tasks in this field of research. We believe it will serve as a useful tool for proving density bounds in the future. For example, very recently, the Density Formula was already applied by Bekos et al. [9] to give bounds on the density of k-planar graphs without short cycles. Moreover, given that the Density Formula behaves symmetrically when it comes to the number of edges and the number of crossings, it seems plausible that it can also be used to derive bounds on crossing numbers of beyond-planar graph classes.

2 Terminology, Conventions, and Notation

All graphs in this paper are finite and have no loops, but possibly parallel edges. We consider classic node-link drawings of graphs. More precisely, in a *drawing* Γ of a graph $G = (V, E)$ (in the plane $\mathbb{R}^2$ or on the sphere $\mathbb{S}^2$) the vertices are pairwise distinct points and each edge is a simple[3] Jordan curve connecting the two vertices. In particular, no edge crosses itself. In order to avoid special treatment of the unbounded region, we mostly consider drawings on the sphere $\mathbb{S}^2$. In one case (RAC-drawings), however, we consider drawings in the plane $\mathbb{R}^2$, as the drawing style involves straight lines and angles. In any case, we require throughout the usual assumptions of no edge passing through a vertex, having only proper crossings and no touchings, only finitely many crossings, and no three edges crossing in the same point.

So-called *simple* drawings are a particularly important and well-studied type of drawing. In such a drawing, any two edges have at most one point in common. In particular, simple drawings contain no two edges crossing more than once, no crossing adjacent edges, and no parallel edges. However, there is increasing interest in generalizations of simple drawings that allow these types of configurations, as long as the involved edge pairs are not just drawn in basically the same way within a narrow corridor. This notion is formalized as follows. A *lens* in a drawing Γ is a region whose boundary is described by a simple[3] closed Jordan curve γ such that γ is comprised of exactly two contiguous parts, each being formed by (a part of) one edge. So the curve γ consists of either two non-crossing parallel edges, or parts of two crossing adjacent edges, or parts of two edges crossing more than once; see Figure 2 for

[3] with no self-intersection

illustrations. Be aware that for drawings in $\mathbb{R}^2$, a lens might be an unbounded region. Now let us call a drawing Γ *non-homotopic*[4] if every lens contains a vertex or a crossing in its interior. This is indeed a generalization of simple drawings, as these cannot contain any lens.

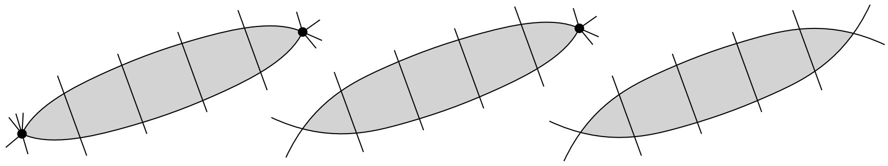

Figure 2 Lenses with no vertex and no crossing in their interior. Such configurations are forbidden in non-homotopic drawings.

Beyond-planar graph classes are implicitly defined as all graphs G that admit a drawing Γ with specific properties, such as all edges of G having at most one crossing in Γ. These for example are called 1-*planar drawings*[5] and the corresponding graphs are called 1-*planar graphs*. We extend this policy to the properties "simple" and "non-homotopic" in the same way, e.g., a non-homotopic 1-planar graph is a graph that admits a non-homotopic 1-planar drawing. Observe that this aligns with a simple graph being a graph with no loops (which we rule out entirely) and no parallel edges.

Fix a drawing Γ of some graph $G = (V, E)$. Setting up some notation, let $E_x \subseteq E$ be the set of all *crossed* edges of G, i.e., with at least one crossing in Γ, and $E_p = E \setminus E_x$ be the set of all *planar* edges (without crossings). Further, let $\mathcal{X}$ denote the set of all crossings in Γ. Each edge e is split into one or more *edge-segments* by the crossings along e. That is, an edge with exactly k crossings, $k \geq 0$, is split into exactly $k+1$ edge-segments. An *outer* edge-segment of Γ is incident to some vertex, while an *inner* edge-segment is not. The set of all edge-segments of Γ is denoted by $\mathcal{S}$ and the set of all inner edge-segments by $\mathcal{S}_{\text{in}}$.

▶ **Observation 2.1.** *Let Γ be any drawing of some graph $G = (V, E)$. Then*

$$|\mathcal{S}| = 2|\mathcal{X}| + |E| \quad \text{and} \quad |\mathcal{S}_{\text{in}}| = |\mathcal{S}| - 2|E_x| - |E_p| = 2|\mathcal{X}| - |E_x|.$$

The *planarization* Λ of the drawing Γ is the planar drawing obtained from Γ by replacing each crossing by a new vertex and replacing each edge by its edge-segments. We call the drawing Γ *connected* if the graph underlying its planarization Λ is connected. Let us remark that most density results in this paper assume for brevity the considered graphs to be connected, while our proofs actually only require the respective drawings to be connected.

The connected components of $\mathbb{S}^2$ or $\mathbb{R}^2$ after removing all edges and vertices in Γ are called the *cells* of Γ. The set of all cells is denoted by $\mathcal{C}$. The *boundary* ∂c of each cell c consists of a cyclic sequence alternating between $V \cup \mathcal{X}$ and $\mathcal{S}$, i.e., vertices/crossings and edge-segments of Γ. If Γ is not connected, ∂c might consist of multiple such sequences. Be aware that an edge-segment might appear twice on ∂c, a crossing might appear up to four times on ∂c, and a vertex v may appear up to $\deg(v)$ times on ∂c. Each appearance of an edge-segment / vertex / crossing on ∂c is called an edge-segment-incidence / vertex-incidence / crossing-incidence of c. The total number of edge-segment-incidences and vertex-incidences

[4] Usually, non-homotopic drawings require a vertex in each lens, but we only need our weaker requirement.
[5] In literature, planar drawings are also referred to as *plane drawings*, and a planar graph with a fixed plane drawing is called a *plane graph*. And there is a similar distinction for each beyond-planar graph class (e.g., 1-planar vs. 1-plane graphs). But for simplicity, we treat *planar* and *plane* as equivalent here.

of c is called the *size* of c and denoted by $\|c\|$. Note that $\|c\|$ does not take the number of crossings on ∂c into account, while edge-segments and vertices are counted with multiplicities; see Figure 3 for several examples. For an integer i, let $\mathcal{C}_i$ and $\mathcal{C}_{\geq i}$ denote the set of all cells of size exactly i and the set of all cells of size at least i, respectively.

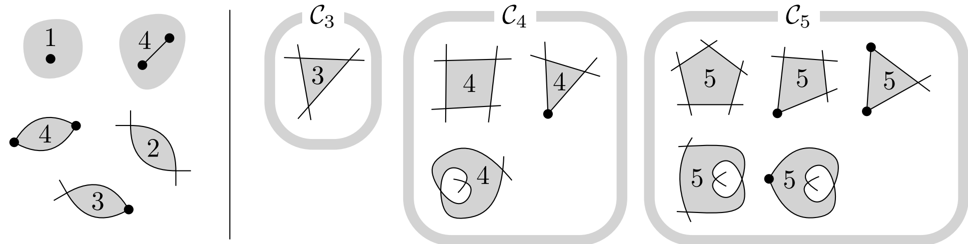

Figure 3 Left: Cells (and their sizes) that do not appear in simple or non-homotopic drawings on at least three vertices. Right: All types of cells c of size $\|c\| \leq 5$ in a non-homotopic connected drawing on at least three vertices (cf. Observation 2.2). The bottom row shows degenerate -cells, -cells, and -cells.

Figure 3(right) shows all possible types of cells of size at most 5 that can occur in connected non-homotopic drawings on $\mathbb{S}^2$ with at least three vertices; the bottom row shows the *degenerate* cells, i.e., those cells c with a crossing or vertex appearing repeated in ∂c. When proving edge density bounds by means of the Density Formula, our main task will be to count these "small cells"; we denote the different types for convenience with little pictograms, such as -cells, -cells, and -cells. More precisely, each pictogram describes a type of cell c in terms of the sequence of types of incidences found along its connected boundary ∂c. E.g., the boundary of a -cell consists of a vertex-incidence, an edge-segment-incidence, a crossing-incidence, an edge-segment-incidence, a crossing-incidence, and an edge-segment-incidence, in this order.

▶ **Observation 2.2.** *Let Γ be any non-homotopic connected drawing of some graph G with at least three vertices. Then*

- *$\mathcal{C}_3$ is the set of all -cells,*
- *$\mathcal{C}_4$ is the set of all -cells and -cells, and*
- *$\mathcal{C}_5$ is the set of all -cells, -cells, and -cells.* ◿

3 The Density Formula

In this section, we first state and prove the Density Formula, then derive some immediate consequences, and finally develop some general tools that are useful for its application.

▶ **Lemma 3.1** (Density Formula). *Let t be a real number. Let Γ be a connected drawing of a graph $G = (V, E)$ with at least one edge. Then*

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} \|c\| - t \right) - |\mathcal{X}|.$$

Proof. First recall that, by Observation 2.1,

$$|\mathcal{S}| = |E| + 2|\mathcal{X}|. \tag{1}$$

Considering the total sum of $\|c\|$ over all cells $c \in \mathcal{C}$, we count every vertex $v \in V$ exactly $\deg(v)$ times and every edge-segment exactly twice. (Here we use that $|E| \geq 1$ and, thus, $\deg(v) \geq 1$ for each $v \in V$ as Γ is connected.) Thus,

$$\sum_{c\in\mathcal{C}} \frac{1}{4}\|c\| = \frac{1}{4}\left(\sum_{v\in V} \deg(v) + 2|\mathcal{S}|\right) = \frac{1}{4}(2|E| + 2|\mathcal{S}|) \overset{(1)}{=} \frac{1}{4}(4|E| + 4|\mathcal{X}|) = |E| + |\mathcal{X}|. \quad (2)$$

Let $\Lambda = (V_\Lambda, E_\Lambda)$ be the planarization of G. It has exactly $|V_\Lambda| = |V| + |\mathcal{X}|$ vertices, $|E_\Lambda| = |\mathcal{S}|$ edges, and $|\mathcal{C}|$ faces. As Λ is connected, we can apply Euler's Formula $(*)$:

$$|E| + 2|\mathcal{X}| \overset{(1)}{=} |\mathcal{S}| = |E_\Lambda| \overset{(*)}{=} |V_\Lambda| + |\mathcal{C}| - 2 = |V| - 2 + |\mathcal{C}| + |\mathcal{X}|,$$

which gives the following two equations:

$$|E| = \quad |V| - 2 + |\mathcal{C}| - |\mathcal{X}| \quad = (|V| - 2) - \sum_{c\in\mathcal{C}} (-1) \quad - |\mathcal{X}| \quad (3)$$

$$0 = \quad |V| - 2 - (|E| + |\mathcal{X}|) + |\mathcal{C}| \quad \overset{(2)}{=} (|V| - 2) - \sum_{c\in\mathcal{C}} \left(\frac{1}{4}\|c\| - 1\right) \quad (4)$$

Adding (3) and $(t-1)$ times (4) gives the result. ◀

The Density Formula can be used to find upper bounds on edge densities by counting cells of small size. To see how this works, let us plug in two specific values for t ($t = 4$ and $t = 5$, which we use quite often throughout the paper) resulting in the following statements:

▶ **Corollary 3.2.** *For any connected drawing Γ of a graph $G = (V, E)$ with $|E| \geq 1$ we have*

$$|E| = 4|V| - 8 - \sum_{c\in\mathcal{C}} (\frac{3}{4}\|c\| - 4) - |\mathcal{X}| \leq 4|V| - 8 + \frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - |\mathcal{X}|.$$

▶ **Corollary 3.3.** *For any connected drawing Γ of a graph $G = (V, E)$ with $|E| \geq 1$ we have*

$$|E| = 5|V| - 10 - \sum_{c\in\mathcal{C}} (\|c\| - 5) - |\mathcal{X}| = 5|V| - 10 + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| - \sum_{c\in\mathcal{C}_{\geq 5}} (\|c\| - 5).$$

So indeed, Corollary 3.2 allows us to derive upper bounds on $|E|$ by proving upper bounds on $\frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - |\mathcal{X}|$, which can be done by counting cells of sizes 3, 4, and 5 and cross-charging them with the crossings. Similarly, noting that $\sum_{c\in\mathcal{C}_{\geq 5}}(\|c\| - 5)$ is non-negative, Corollary 3.3 allows us to derive upper bounds on $|E|$ by proving upper bounds on $2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|$. In fact, by taking into account the cells of larger sizes, one can sometimes obtain more precise bounds. Thus, in the remainder of the section, we will devise some general tools that help with the required counting / charging arguments. Moreover, we give a first concrete example of such an argument by proving Lemma 3.4, which is a simple but very general statement – in fact, it immediately gives two bounds of $4n - 8$ in Table 1.

▶ **Lemma 3.4.** *Let Γ be a non-homotopic connected drawing of a graph $G = (V, E)$ with $|V| \geq 3$ and with no 3-cells, no 4-cells, no 5-cells, no 4-cells, and no 5-cells. Then $|E| \leq 4|V| - 8$.*

Proof. By assumption and Observation 2.2, we have $|\mathcal{C}_3| = 0$ and $|\mathcal{C}_4| = 0$ and $|\mathcal{C}_5| = \#$5-cells. Clearly, every crossing is incident to at most four 5-cells and every 5-cell has one incident crossing. In particular, it follows that $\#$5-cells $\leq 4|\mathcal{X}|$. Therefore, the Density Formula with $t = 4$ (Corollary 3.2) immediately gives

$$|E| \leq 4|V| - 8 + \frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - |\mathcal{X}| = 4|V| - 8 + \frac{1}{4}\#\text{5-cells} - |\mathcal{X}| \leq 4|V| - 8. \quad ◀$$

▶ **Lemma 3.5.** *Let Γ be any non-homotopic drawing. Then* $\#\triangle_4$*-cells* $\leq |\mathcal{X}|$*. Moreover, one can assign each* $\triangle_4$*-cell c a crossing in* ∂c *such that each crossing is assigned at most once.*

Proof. At every crossing incident to a $\triangle_4$-cell there is one inner edge-segment and one outer edge-segment. As Γ is non-homotopic, every inner edge-segment is incident to at most one $\triangle_4$-cell. This implies that every crossing is incident to at most two $\triangle_4$-cells, while every $\triangle_4$-cell has two distinct incident crossings, which implies the claim. ◀

▶ **Lemma 3.6.** *Let Γ be a connected non-homotopic drawing of some graph G with at least three vertices. Then*

$$|\mathcal{S}_{\text{in}}| \geq \#\triangle_4\text{-cells} + 2\cdot\#\square_4\text{-cells} + 3\cdot\#\triangledown_3\text{-cells} \quad \text{and} \quad |\mathcal{S}_{\text{in}}| + \#\triangle_4\text{-cells} \geq 2|\mathcal{C}_4| + 3|\mathcal{C}_3|.$$

Proof. The second inequality follows by combining the first inequality with Observation 2.2. To prove the first inequality, let us call an inner edge-segment *bad* if it is incident to a $\triangle_4$-cell or $\triangledown_3$-cell in Γ. As Γ is non-homotopic, every bad edge-segment is incident to only one $\triangle_4$-cell or $\triangledown_3$-cell. Hence, for the set $\mathcal{B} \subseteq \mathcal{S}_{\text{in}}$ of all bad edge-segments we have $|\mathcal{B}| = \#\triangle_4\text{-cells} + 3\cdot\#\triangledown_3\text{-cells}$. Define an auxiliary graph $J = (V_J, E_J)$ with vertex set $V_J = \mathcal{S}_{\text{in}}$ and with two edge-segments being adjacent in J if and only if they are an opposite pair of edge-segments for some $\square_4$-cell. Note that this and the following is true whether the $\square_4$-cells are degenerate or not. Then $|V_J| = |\mathcal{S}_{\text{in}}|$ and $|E_J| = 2\cdot\#\square_4\text{-cells}$, and the maximum degree in J is at most two. Observe that J contains no cycle, as such a cycle would correspond to a cyclic arrangement of $\square_4$-cells and therefore two edges in G with no endpoints. Hence, J is a disjoint union of paths (possibly of length 0) and every bad edge-segment is an endpoint of one such path. Further, no path in J on two or more vertices can have two bad endpoints, as such a path would correspond to a lens in Γ containing no vertex and no crossing (as illustrated in Figure 2), contradicting the fact that Γ is non-homotopic. Note that this implies $|V_J| \geq |E_J| + |\mathcal{B}|$. Recalling that $|\mathcal{B}| = \#\triangle_4\text{-cells} + 3\cdot\#\triangledown_3\text{-cells}$, $|V_J| = |\mathcal{S}_{\text{in}}|$ and $|E_J| = 2\cdot\#\square_4\text{-cells}$, we obtain the first inequality of the lemma. ◀

4 *k*-Bend RAC-Graphs

In this section, we present our results for 1-bend and 2-bend RAC-graphs. We begin with the upper bounds, for which we only require the following lemma.

▶ **Lemma 4.1.** *Let* $k \in \{1, 2\}$ *and Γ be a non-homotopic drawing of a connected graph* $G = (V, E)$ *such that every crossed edge* $e \in E_x$ *is a polyline with at most k bends, and every crossing is a right-angle crossing. Then* $2|\mathcal{C}_3| + |\mathcal{C}_4| \leq |\mathcal{X}| + \frac{k-1}{2}(|E_x| + 1)$.

Proof. Lemma 3.6 gives

$$|\mathcal{S}_{\text{in}}| \geq \#\triangle_4\text{-cells} + 2\cdot\#\square_4\text{-cells} + 3\cdot\#\triangledown_3\text{-cells}. \tag{5}$$

Now, each $\triangle_4$-cell and each $\triangledown_3$-cell c is a polygon, and as all crossings have right angles, c has at least one convex corner that is a bend, except when c is the unbounded cell. As every bend is a convex corner for only one cell, we have

$$k|E_x| \geq \#\triangle_4\text{-cells} + \#\triangledown_3\text{-cells} - 1. \tag{6}$$

Together this gives the desired

$$4|\mathcal{C}_3| + 2|\mathcal{C}_4| \overset{(5),(6)}{\leq} |\mathcal{S}_{\text{in}}| + k|E_x| + 1 = 2|\mathcal{X}| + (k-1)|E_x| + 1,$$

where the last equality uses $|\mathcal{S}_{\text{in}}| = 2|\mathcal{X}| - |E_x|$ from Observation 2.1. Dividing by 2 and realizing that $2|\mathcal{C}_3| + |\mathcal{C}_4|$ is an integer, concludes the proof. ◀

▶ **Theorem 4.2.** *For every $k \in \{1,2\}$ and every $n \geq 3$, every connected non-homotopic n-vertex k-bend RAC-graph G has at most $k(5n-10)+(k-1)$ edges.*

Proof. Let Γ be a non-homotopic k-bend RAC-drawing of $G=(V,E)$. As G is connected, so is Γ. The Density Formula with $t=5$ (Corollary 3.3) and Lemma 4.1 immediately give

$$|E| \leq 5|V| - 10 + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| \leq 5|V| - 10 + \frac{k-1}{2}(|E_x|+1),$$

which implies the desired $|E| \leq |E| + (k-1)|E_p| \leq k(5|V|-10)+(k-1)$. ◀

The lower bound construction in [6, Theorem 6] gives 2-bend RAC-graphs with n vertices and $10n-46$ edges, but the provided drawings are not simple (not even non-homotopic). We modify it giving simple 2-bend RAC-graphs with n vertices and $10n-54$ edges.

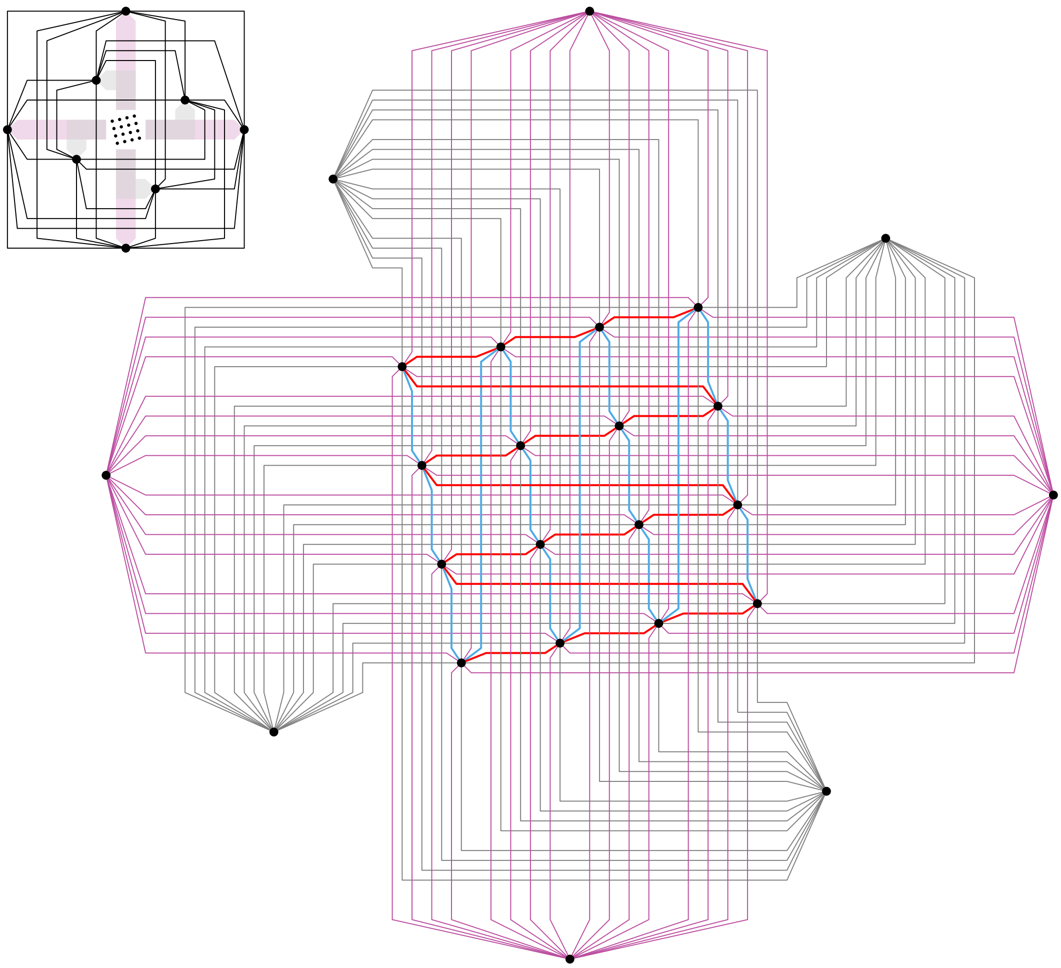

Figure 4 (Illustration of) a simple 2-bend RAC-drawing of G_4 from Theorem 4.3.

▶ **Theorem 4.3.** *For every integer $k \geq 1$ there exists a simple connected 2-bend RAC-graph G_k with $n = k^2+8$ vertices and $10n-54$ edges.*

Proof. For $k \geq 1$, a simple 2-bend RAC-drawing of the graph G_k (Figure 4) consists of

- a set Q of $k^2 = n - 8$ vertices in a regular but slightly rotated $k \times k$ grid,
- an x-monotone 2-bend edge between any two vertices of Q with consecutive y-coordinates (red), ($n - 9$ edges)
- a y-monotone 2-bend edge between any two vertices of Q with consecutive x-coordinates (blue), ($n - 9$ edges)
- a set P of eight vertices around Q, each connected to all vertices of Q with either all (weakly) x-monotone 2-bend edges or all (weakly) y-monotone 2-bend edges (gray and purple), ($8(n - 8)$ edges)
- a 2-bend edge between any two vertices of P (black). (28 edges)

The routing of the edges is illustrated in Figure 4. ◀

5 Fan-Crossing Graphs

Here, we present our upper bound for fan-crossing graphs, starting with the key lemma.

▶ **Lemma 5.1.** *Let Γ be a simple connected fan-crossing drawing of a graph with at least three vertices. Then $|\mathcal{C}_4| \leq |\mathcal{X}|$.*

Proof. First, observe that there are no degenerate $\boxed{4}$-cells since Γ is simple. We shall map each cell $c \in \mathcal{C}_4$ onto one of its incident crossings $\phi(c)$ in such a way that no crossing is used more than once, i.e., the mapping $\phi\colon \mathcal{C}_4 \to \mathcal{X}$ is injective.

As an auxiliary structure, we orient edge-segments incident to $\boxed{4}$-cells as follows. Let c be a $\boxed{4}$-cell and s, s' be a pair of *opposite* edge-segments in ∂c (that do not share a crossing). As Γ is simple, the corresponding edges e, e' are distinct. Now orient s and s', each towards the (unique) common endpoint of e and e', which exists as Γ is fan-crossing. Doing this for every $\boxed{4}$-cell and every pair of opposite edge-segments, we obtain a well-defined orientation:

▷ Claim. An edge-segment s shared by two $\boxed{4}$-cells c_1, c_2 has the same orientation in both.

Proof. Observe that the six crossings incident to c_1 and c_2 are pairwise distinct since Γ is a simple drawing. Let $e = uv$ be the edge containing s and e_1, e_2 be the two (distinct) edges crossing e at the endpoints of s (which are crossings in Γ). Further, let f_1, f_2 be the two edges containing the edge-segment opposite to s in c_1, c_2, respectively. In particular, e, f_1, f_2 all cross e_1 and all cross e_2. As Γ is fan-crossing[6], e, f_1, f_2 have a common endpoint, say u. But then s is oriented consistently towards u according to both incident $\boxed{4}$-cells c_1, c_2. ◁

▷ Claim. For each $\boxed{4}$-cell c, there is at least one crossing x incident to c such that both edge-segments incident to c and x are oriented outgoing from x.

Proof. Assuming otherwise, the edge-segments would be oriented cyclically around ∂c. Consider two crossings x_1, x_2 that are *opposite* along c (do not belong to the same edge segment of c). The edges of the two (distinct) edge-segments of c that are outgoing from x_1, x_2 have a common endpoint u, as Γ is fan-crossing; see Figure 5. The edges of the two edge-segments of c that are outgoing from the remaining two opposite crossings y_1, y_2 behave symmetrically and share an endpoint v, which is distinct from u, as Γ is simple. The four parts of the mentioned edges that join the vertices u, v with the crossings x_1, x_2, y_1, y_2 are pairwise crossing-free since Γ is simple. Hence, using these edge parts, we can obtain a planar drawing of the bipartite

[6] Here it is crucial that e, f_1 and f_2 do not form a triangle-crossing.

graph $K_{3,3} - e$ (obtained from $K_{3,3}$ by removing an edge) so that the bipartition classes are $\{x_1, x_2, v\}$ and $\{y_1, y_2, u\}$ and where the four degree-3 vertices form a face. However, the unique[7] planar embedding of $K_{3,3} - e$ has no such face; see again Figure 5. ◁

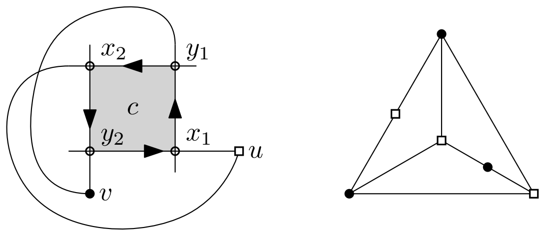

Figure 5 A cyclic orientation of a $\square_4$-cell leading to a double-crossing (left) or the unique[7] planar embedding of $K_{3,3} - e$ (right).

Now for every $\square_4$-cell c, we set $\phi(c)$ to be a crossing x in ∂c whose two edge-segments in ∂c are oriented outgoing from x. Moreover, by Lemma 3.5 for every $\triangle_4$-cell c, we can set $\phi(c)$ to be a crossing in ∂c such that $\phi(c) \neq \phi(c')$ for any distinct $\triangle_4$-cells c, c'.

▷ Claim. The mapping $\phi\colon \mathcal{C}_4 \to \mathcal{X}$ is injective.

Proof. For a $\square_4$-cell c and a $\triangle_4$-cell or $\square_4$-cell c' with $\phi(c) = x \in \partial c'$, we shall show $\phi(c') \neq x$.

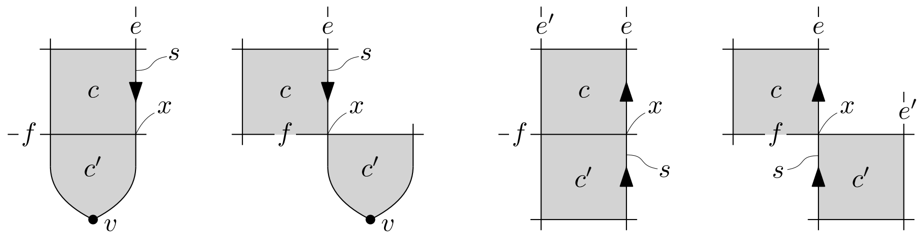

Figure 6 The four cases of a $\square_4$-cell c sharing a crossing x with a $\triangle_4$-cell or $\square_4$-cell c'.

If c' is a $\triangle_4$-cell, let e be the edge that is incident to the vertex $v \in \partial c'$ and contains x. Further, let f be the other edge at x (containing the inner edge-segment of c') and let s be the edge-segment of e in ∂c; see Figure 6. Evidently, v is the common endpoint of all edges crossing f. In particular, s is oriented inwards at x, which is a contradiction to $x = \phi(c)$.

If c' is a $\square_4$-cell, let s be an edge-segment that ends at x and belongs to $\partial c'$, but not to ∂c. Let e be the edge containing s, let f be the other edge at x, and let e' be the edge containing the edge-segment opposite of s in $\partial c'$; see Figure 6. As $\phi(c) = x$, the edge-segment of e in ∂c is oriented outwards at x and towards the common endpoint of all edges crossing f. As e and e' cross f, edge-segment s is oriented inwards at x and thus $\phi(c') \neq x$. ◁

Clearly, the last claim implies the desired $|\mathcal{C}_4| \leq |\mathcal{X}|$. ◀

Let us prove the edge density of $5n - 10$ for connected simple fan-crossing graphs in a slightly stronger form.

[7] All planar embeddings of $K_{3,3} - e$ are combinatorially isomorphic since it is a subdivision of the 3-connected complete graph K_4.

▶ **Theorem 5.2.** *Let Γ be a simple connected fan-crossing drawing of some graph $G = (V, E)$ with $|V| \geq 3$. Then*

$$|E| \leq 5|V| - 10 - \sum_{c \in \mathcal{C}_{\geq 5}} (\|c\| - 5).$$

Proof. As every edge is crossed only by adjacent edges and adjacent edges do not cross (Γ is simple), there are no $\triangledown$-cells in Γ and, hence, $|\mathcal{C}_3| = 0$. Therefore Corollary 3.3 (i.e., the Density Formula with $t = 5$) immediately gives

$$|E| = 5|V| - 10 + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| - \sum_{c \in \mathcal{C}_{\geq 5}} (\|c\| - 5) \leq 5|V| - 10 - \sum_{c \in \mathcal{C}_{\geq 5}} (\|c\| - 5),$$

where the last inequality uses Lemma 5.1. ◀

5.1 Flaws in the Original Proofs from Related Work

Recall that fan-planar graphs are a special case of fan-crossing graphs, defined by admitting drawings in $\mathbb{R}^2$ without configuration I and II (original definition [16]), respectively without configurations I, II, and III (revised definition [17]); cf. Figure 1. The proofs in [16, 17] involve a number of statements, each carefully analysing the possible routing of edges in a fan-planar drawing. In the past decade, many papers on (generalizations of) fan-planar graphs appeared and many rely (implicitly or explicitly) on said statements. As mentioned above, a flaw in one of the statements from [16] was discovered [18]. In this section, we will describe additional issues existing in both [16] and [17], thereby outlining why the previous proofs of the density bounds for fan-crossing, weakly fan-planar, and strongly fan-planar graphs are indeed incomplete.

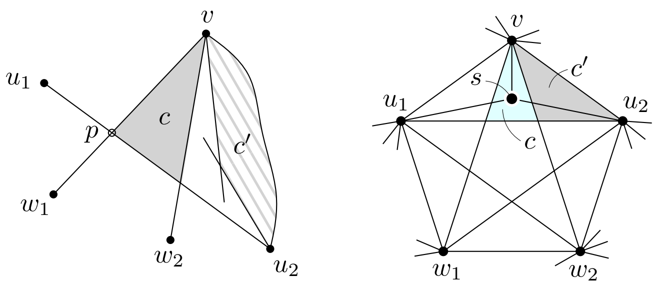

Figure 7 Left: Illustration of [17, Corollary 5] taken from the paper. Right: A counterexample.

- The authors try to guarantee [17, Corollary 5] that no cell of size 4 of any *subdrawing* of a fan-planar drawing Γ contains vertices of G. In fact, if c is a $\triangle$-cell with incident vertex v and inner edge-segment of an edge u_1u_2, and some set S of vertices lies inside c, it is suggested to move the drawing of $G[S]$ to a cell c' incident to an uncrossed edge vu_1 or vu_2 as illustrated in Figure 7(left). However, in the particular situation of Figure 7(right) with S just being a single vertex s, moving s into c' would cause edge u_1s to cross edge vw_2, which is already crossed by the independent edge u_2w_1; thus loosing fan-planarity.
- In a later proof [17, Lemma 11], induction is applied to the induced subdrawing of an induced subgraph G' of G. However throughout, the drawing Γ was chosen to satisfy (i) having the maximum number of planar edges, and (ii) being inclusionwise edge-maximal with that property [17, Section 3]. It is not shown or clear why the subdrawing for the induction still satisfies (i) and (ii).

6 Quasiplanar Graphs

The lower bounds for simple and non-homotopic quasiplanar graphs presented in this section are based on properties that tight examples must have that arise from a thorough reading of our upper bound proof, as provided in the full version [15]. For instance, the removal of any vertex leaves a cell of size 2 in the non-homotopic case, while in the simple case, the uncrossed edges must form a matching.

▶ **Theorem 6.1.** *For every $n \geq 4$, there exists a non-homotopic n-vertex connected quasiplanar graph with $8n - 20$ edges.*

Proof. For $n = 4$, let us simply refer to the construction illustrated in Figure 8(top-left).

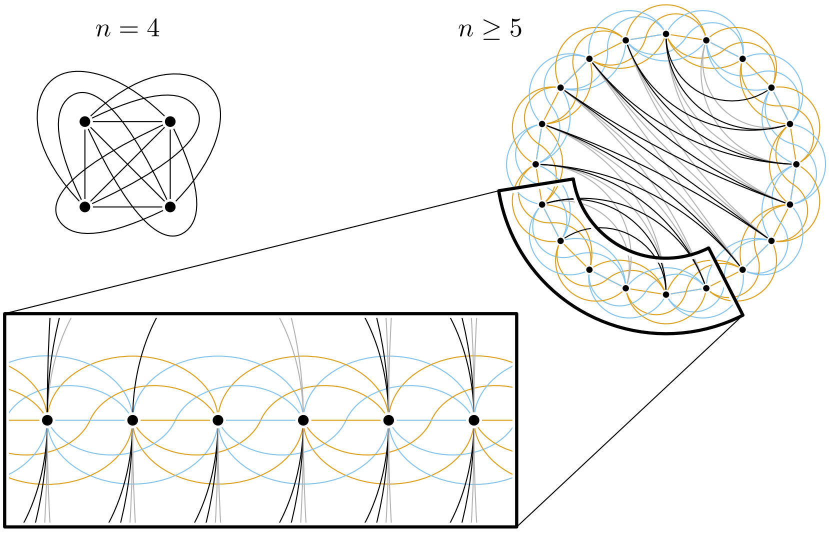

Figure 8 Illustrations of non-homotopic quasiplanar drawings with n vertices and $8n - 20$ edges. For better readability, the two zig-zag paths outside the cycle are omitted. The edge-coloring (works only for even n) just indicates four crossing-free sub-drawings, which helps to verify quasiplanarity.

For $n \geq 5$, the desired graph G_n consists of (for illustrations refer to Figure 8(right))

- an n-vertex cycle C drawn in a non-crossing way, (n edges)
- an edge between any two vertices at distance 2 on C drawn inside C, (n edges)
- an edge between any two vertices at distance 2 on C drawn outside C, (n edges)
- an edge between any two vertices at distance 3 on C, starting inside C, crossing C at distance 1.5, and ending outside C, (n edges)
- a zig-zag path of edges drawn inside C where the endpoints of each edge have distance at least 3 on C, ($n - 5$ edges)
- another (different) zig-zag path of edges drawn inside C where the endpoints of each edge have distance at least 3 on C, ($n - 5$ edges)
- a zig-zag path of edges drawn outside C where the endpoints of each edge have distance at least 3 on C, ($n - 5$ edges)
- another (different) zig-zag path of edges drawn outside C where the endpoints of each edge have distance at least 3 on C, ($n - 5$ edges)

Thereby, all edges are drawn without unnecessary crossings. For example, two edges drawn inside C cross only if the respective endpoints appear in alternating order around C.

Evidently, G_n has n vertices and $8n - 20$ edges, and it is straightforward to check that the described drawing of G_n is non-homotopic and quasiplanar. ◀

▶ **Theorem 6.2.** *For every even $n \geq 8$, there exists a simple n-vertex quasiplanar graph with $6.5n - 20$ edges.*

Proof. Our construction is a subgraph of the corresponding graph in the proof of Theorem 6.1; see Figure 9 for an illustration. For every even $n \geq 8$, the desired simple quasiplanar graph G_n is missing all the orange edges depicted in Figure 8 except the ones at distance 1, i.e.

- the edges between any two black vertices at distance 2 on C drawn inside C, ($n/2$ edges)
- the edges between any two white vertices at distance 2 on C drawn outside C, ($n/2$ edges)
- the edges from each black vertex to its white vertex clockwise at distance 3 on C, starting inside C, crossing C at distance 1.5, and ending outside C, ($n/2$ edges)

As $n \geq 8$, the four zig-zag paths can be chosen without introducing parallel edges. Again, all edges are drawn without unnecessary crossings. For example, two edges drawn inside C cross only if the respective endpoints appear in alternating order around C.

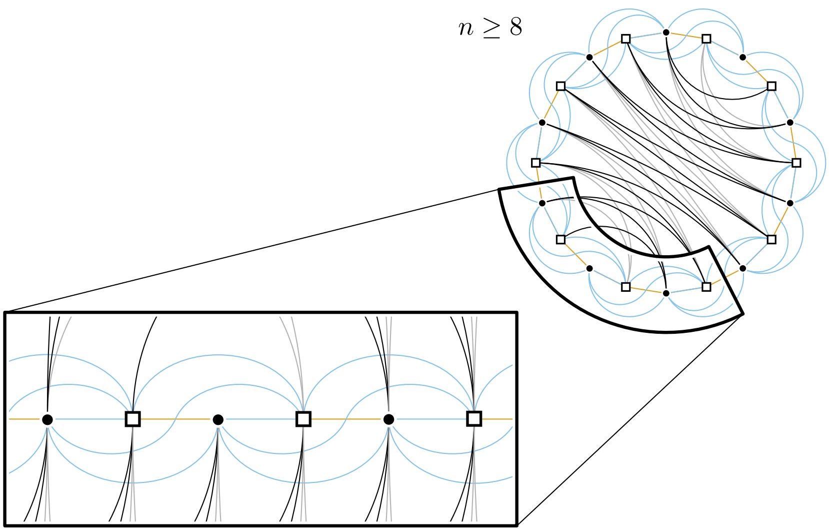

Figure 9 Illustration of simple quasiplanar drawings with n vertices and $6.5n - 20$ edges, for even $n \geq 8$. For better readability, the two zig-zag paths outside the cycle are omitted. The edge-coloring just indicates four crossing-free sub-drawings, which helps to verify quasiplanarity.

Evidently, G_n has n vertices and $6.5n - 20$ edges and it is straightforward to check that the described drawing of G_n is simple and quasiplanar. ◀

7 Concluding Remarks

Some previously known proofs already contain ideas that are similar to (parts of) our approach. Often times, this is phrased in terms of a discharging argument, instead of a direct counting. For example, some discharging steps in [4], [1], [2], and [3] (dealing with k-planar, so-called k-quasiplanar graphs, and fan-crossing graphs) directly correspond to our proof of Lemma 3.6. In these four cases, but also in [10], the total sum of all charges is $\sum_{c \in \mathcal{C}}(\|c\| - 4)$ (although stated a bit differently). In [5], which concerns 1-bend RAC-graphs, there is a charging involving the convex bends. Further, the concept of the size of a cell and the quantity $\sum_{c \in \mathcal{C}}(\|c\| - 5)$ already appear in the papers [16, 17] on fan-planar graphs. But with the Density Formula, we have a unified approach that somewhat unveiled the essential tasks in this field of research. A valuable asset of our approach are very streamlined and clean combinatorial arguments, as well as substantially shorter proofs, as certified by the number of beyond-planar graph classes that we can treat in about 20 pages (referring to the full version [15]).

Additionally, it is straightforward to derive from the particular application of the Density Formula properties that must be fulfilled by all tight examples. For example, from our proof for k^+-real face graphs (in the full version [15]), we immediately see that all tight examples of k^+-real face graphs with $k \geq 3$ are planar. Similarly, from our proof for 2-planar graphs (in the full version [15]), we see that no tight example of a 2-planar graph has a 3-cell or 4-cell. And (together with a short calculation) our proof for quasiplanar graphs (in the full version [15]) implies that in all tight examples of simple quasiplanar graphs the planar edges form a perfect matching. Specifically, this approach of analyzing the situation in which the proof with the Density Formula is tight, allowed us to find the first tight examples for simple and non-homotopic quasiplanar graphs (cf. Theorems 6.1 and 6.2).

The only cases presented here in which upper and lower bounds still differ by an absolute constant are k-bend RAC-graphs; cf. Table 1. This is due to the fact that these are drawings in the plane $\mathbb{R}^2$ and the unbounded cell behaves crucially different from all other cells. It is possible to reduce our upper bounds by an absolute constant by a separate analysis of the unbounded cell, but we did not pursue this here. On the other hand, it may well be that our bounds are already optimal for k-bends RAC-drawings on the sphere $\mathbb{S}^2$ (for the natural definition of this concept) – they definitely are for $k = 0$.

Finally for open problems, there is a number of beyond-planar graph classes for which the exact asymptotics of their edge density is not known yet. This includes for example non-homotopic fan-crossing graphs, k-quasiplanar graphs for $k \geq 4$, and k-planar graphs for $k \geq 4$. Let us refer again to the survey [14] from 2019 for more such cases and more beyond-planar graph classes in general. Additionally, each class could be considered in a "bipartite variant" (as we do for fan-crossing graphs in the full version [15]) and/or an "outer variant" where one additionally requires that there is one cell that is incident to every vertex; see for example [7].

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