


Solving Co-Path/Cycle Packing and Co-Path Packing Faster Than 3^k

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Abstract

The CO-PATH/CYCLE PACKING problem (resp. The CO-PATH PACKING problem) asks whether we can delete at most k vertices from the input graph such that the remaining graph is a collection of induced paths and cycles (resp. induced paths). These two problems are fundamental graph problems that have important applications in bioinformatics. Although these two problems have been extensively studied in parameterized algorithms, it seems hard to break the running time bound 3^k . In 2015, Feng et al. provided an $O^*(3^k)$ -time randomized algorithms for both of them. Recently, Tsur showed that they can be solved in $O^*(3^k)$ time deterministically. In this paper, by combining several techniques such as path decomposition, dynamic programming, cut & count, and branch-and-search methods, we show that CO-PATH/CYCLE PACKING can be solved in $O^*(2.8192^k)$ time deterministically and CO-PATH PACKING can be solved in $O^*(2.9241^k)$ time with failure probability $\leq 1/3$. As a by-product, we also show that the CO-PATH PACKING problem can be solved in $O^*(5^p)$ time with probability at least $2/3$ if a path decomposition of width p is given.

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1 Introduction

In the classic VERTEX COVER problem, the input is a graph G and an integer k , and the problem asks whether it is possible to delete at most k vertices such that the maximum degree of the remaining graph is at most 0. A natural generalization of VERTEX COVER is that: can we delete at most k vertices such that the maximum degree of the remaining graph is at most d ? Formally, for every integer $d \geq 0$, we consider the following d -BOUNDED-DEGREE VERTEX DELETION problem.

d -BOUNDED-DEGREE VERTEX DELETION

Instance: A graph $G = (V, E)$ and two integers d and k .

Question: Is there a set of at most k vertices whose removal from G results in a graph with maximum degree at most d ?



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The d -BOUNDED-DEGREE VERTEX DELETION problem finds applications in computational biology [7] and social network analysis [16]. In this paper, we focus on the case that $d = 2$, which is referred to as the CO-PATH/CYCLE PACKING problem. The CO-PATH/CYCLE PACKING problem also has many applications in computational biology [4]. Formally, the CO-PATH/CYCLE PACKING problem is defined as follows.

CO-PATH/CYCLE PACKING

Instance: A graph $G = (V, E)$ and an integer k .

Question: Is there a vertex subset $S \subseteq V$ of size at most k whose deletion makes the graph a collection of induced paths and cycles?

We also focus on a similar problem called CO-PATH PACKING, which allows only paths in the remaining graph, defined as follows.

CO-PATH PACKING

Instance: A graph $G = (V, E)$ and an integer k .

Question: Is there a vertex subset $S \subseteq V$ of size at most k whose deletion makes the graph a collection of induced paths?

Related Work. In this paper, we mainly consider parameterized algorithms. When d is an input, the general d -BOUNDED-DEGREE VERTEX DELETION problem is W[2]-hard with the parameter k [7]. Xiao [22] gave a deterministic algorithm that solves d -BOUNDED-DEGREE VERTEX DELETION in $O^*((d+1)^k)$ time for every $d \geq 3$, which implies that the problem is FPT with parameter $k+d$. In term of treewidth (tw), van Rooij [21] gave an $O^*((d+2)^{tw})$ -time algorithm to solve d -BOUNDED-DEGREE VERTEX DELETION for every $d \geq 1$. Lampis and Vasilakis [14] showed that no algorithm can solve d -BOUNDED-DEGREE VERTEX DELETION in time $(d+2-\epsilon)^{tw}n^{O(1)}$, for any $\epsilon > 0$ and for any fixed $d \geq 1$ unless the SETH is false. The upper and lower bounds have matched. This problem has also been extensively studied in kernelization. Fellows et al. [7] and Xiao [23] gave a generated form of the NT-theorem that study polynomial kernels for the problem with each fixed d .

For each fixed small d , d -BOUNDED-DEGREE VERTEX DELETION has also been paid certain attention. The 0-BOUNDED-DEGREE VERTEX DELETION problem is referred to as VERTEX COVER, which is one of the most fundamental problems in parameterized algorithms. For a long period of time, the algorithm of Chen et al. [3] held the best-known running time of $O^*(1.2738^k)$, and recently this result was improved by Harris and Narayanaswamy [11] to $O^*(1.25284^k)$. The 1-BOUNDED-DEGREE VERTEX DELETION problem is referred to as P_3 VERTEX COVER, where Tu [20] achieved a running time of $O^*(2^k)$ by using iterative compression. This result was later improved by Katrenič [12] to $O^*(1.8127^k)$. Then, Chang et. al. [2] gave an $O^*(1.7964^k)$ -time polynomial-space algorithm and an $O^*(1.7485^k)$ -time exponential-space algorithm. Xiao and Kou [24] gave an $O^*(1.7485^k)$ -time polynomial-space algorithm. This result was improved by Tsur [18] to $O^*(1.713^k)$ through a branch-and-search approach and finally by Červený and Suchý [1] to $O^*(1.708^k)$ through using an automated framework for generating parameterized branching algorithms.

CO-PATH/CYCLE PACKING is the special case of d -BOUNDED-DEGREE VERTEX DELETION with $d = 2$. A closely related problem is CO-PATH PACKING, where even cycles are not allowed in the remaining graph. Chen et al. [4] initially showed that CO-PATH/CYCLE PACKING and CO-PATH PACKING can be solved in $O^*(3.24^k)$ time, and a finding subsequently

■ **Table 1** Algorithms for CO-PATH/CYCLE PACKING and CO-PATH PACKING.

Years	References	CO-PATH/CYCLE	Deterministic	CO-PATH	Deterministic
2010	Chen et al. [4]	$O^*(3.24^k)$	Yes	$O^*(3.24^k)$	Yes
2015	Feng et al. [8]	$O^*(3^k)$	No	$O^*(3^k)$	No
2016	Xiao [22]	$O^*(3.07^k)$	Yes	–	–
2022	Tsur [19]	$O^*(3^k)$	Yes	$O^*(3^k)$	Yes
2024	This paper	$O^*(2.8192^k)$	Yes	$O^*(2.9241^k)$	No

refined to $O^*(3.07^k)$ for CO-PATH/CYCLE PACKING by Xiao [22]. Feng et al. [8] introduced a randomized $O^*(3^k)$ -time algorithm for the CO-PATH PACKING, which also works for CO-PATH/CYCLE PACKING. However, we do not know how to derandomize this algorithm. Recently, Tsur [19] provided $O^*(3^k)$ -time algorithms solving CO-PATH/CYCLE PACKING and CO-PATH PACKING deterministically. It seems that the bound 3^k is hard to break for the two problems. As shown in Tsur’s algorithms [19], many cases, including the case of handling all degree-4 vertices, lead to the same bottleneck. One of the main targets in this paper is to break those bottlenecks. Previous results and our results for CO-PATH/CYCLE PACKING and CO-PATH PACKING are summarized in Table 1.

Our Contributions. The main contributions of this paper are a deterministic algorithm for CO-PATH/CYCLE PACKING running in $O^*(2.8192^k)$ time and $O^*(2.5199^k)$ space and a randomized algorithm for CO-PATH PACKING running in $O^*(2.9241^k)$ time and space with failure probability $\leq 1/3$. To obtain this result, we need to combine path decomposition, dynamic programming, branch-and-search and some other techniques. In the previous $O^*(3^k)$ -time algorithms for both CO-PATH/CYCLE PACKING and CO-PATH PACKING, many cases, including dealing with degree-4 vertices and several different types of degree-3 vertices, lead to the same bottleneck. It seems very hard to avoid all the bottleneck cases by simply modifying the previous algorithms. The main idea of the algorithm in this paper is as follows. We first design some new reduction and branching rules to handle some good structures of the graph. After this, we prove that the remaining graph has a small pathwidth and then design an efficient dynamic programming algorithm based on a path decomposition. Specifically, our algorithm firstly runs the branch-and-search algorithm to handle the degree- ≥ 5 vertices and the degree-4 vertices adjacent to at least one degree- ≥ 3 vertex. In the branch-and-search phase, our algorithm runs in $O^*(2.8192^k)$ time for both CO-PATH/CYCLE PACKING and CO-PATH PACKING. When branching steps cannot be applied, we can construct a path decomposition of width at most $2k/3 + \epsilon k$ for any $\epsilon > 0$ and call our dynamic programming algorithm. The running time and space of the dynamic programming algorithm are bounded by $O^*(2.5199^k)$ for CO-PATH/CYCLE PACKING and bounded by $O^*(2.9241^k)$ with failure probability $\leq 1/3$ for CO-PATH PACKING. Therefore, the whole algorithm runs in $O^*(2.8192^k)$ time and $O^*(2.5199^k)$ space for CO-PATH/CYCLE PACKING and runs in $O^*(2.9241^k)$ time and space with failure probability $\leq 1/3$ for CO-PATH PACKING.

For the dynamic programming algorithms based on a path decomposition, the first algorithm is an $O^*((d+2)^p)$ -time algorithm to solve d -BOUNDED-DEGREE VERTEX DELETION for every $d \geq 1$, where p is the width of the given path decomposition. This was firstly found in [21]. We also present it in our way to make this paper self-contained. The second algorithm is designed for CO-PATH PACKING. In this algorithm, we use an algorithm framework called cut & count [6]. Given a path decomposition of width p , we show that CO-PATH PACKING can be solved in $O^*(5^p)$ time and space with failure probability $\leq 1/3$.

Reading Guide. Section 2 begins with a review of fundamental definitions and the establishment of notation. In Section 3 we solve CO-PATH/CYCLE PACKING and in section 4 we solve CO-PATH PACKING. In Section 3.1, we show that a proper graph has a small pathwidth and present a dynamic programming algorithm for CO-PATH/CYCLE PACKING. In Section 3.2, we give the branch-and-search algorithm for CO-PATH/CYCLE PACKING. Similarly, in Section 4.1, we present a randomized dynamic programming algorithm for CO-PATH PACKING. In Section 4.2, we give the branch-and-search algorithm for CO-PATH PACKING. Most of the content of this branching algorithm is the same as the branching algorithm for CO-PATH/CYCLE PACKING. In Section 5, we give a conclusion. Due to lack of space, the proof of theorems marked ♣ is omitted here and they can be found in the full version of this paper [15].

2 Preliminaries

In this paper, we only consider simple and undirected graphs. Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges. A vertex v is called a *neighbor* of a vertex u if there is an edge $uv \in E$. Let $N(v)$ denote the set of neighbors of v . For a subset of vertices X , let $N(X) = \bigcup_{v \in X} N(v) \setminus X$ and $N[X] = N(X) \cup X$. We use $d(v) = |N(v)|$ to denote the *degree* of a vertex v in G . A vertex of degree d is called a *degree- d vertex*. For a subset of vertices $X \subseteq V$, the subgraph induced by X is denoted by $G[X]$. The induced subgraph $G[V \setminus X]$ is also written as $G \setminus X$. A *path* P in G is a sequence of vertices v_1, v_2, \dots, v_t such that for any $1 \leq i < t$, $\{v_i v_{i+1}\} \in E$. Two vertices u and v are *reachable* to each other if there is a path v_1, v_2, \dots, v_t such that $v_1 = u$ and $v_t = v$. A *connected component* of a graph is a maximum subgraph such that any two vertices are reachable to each other. A vertex subset S is called a *cPCP-set* of graph G if the degree of any vertex in $G \setminus S$ is at most 2. A vertex subset S is called a *cPP-set* of graph G if $G \setminus S$ is a collection of disjoint paths. For a graph G , we will use $V(G)$ and $E(G)$ to denote the vertex set and edge set of it, respectively. A complete graph with 3 vertices is called a *triangle*. A singleton $\{v\}$ may be denoted as v .

Path decomposition. We will use the concepts of path decomposition and nice path decomposition of a graph.

► **Definition 1** ([5]). *A path decomposition of a graph G is a sequence $P = (X_1, X_2, \dots, X_r)$ of vertex subsets $X_i \subseteq V(G)$ ($i \in \{1, 2, \dots, r\}$) such that:*

(P1) $\bigcup_{i=1}^r X_i = V(G)$.

(P2) *For every $uv \in E(G)$, there exists $l \in \{1, 2, \dots, r\}$ such that X_l contains both u and v .*

(P3) *For every $u \in V(G)$, if $u \in X_i \cap X_k$ for some $i \leq k$, then $u \in X_j$ for all $i \leq j \leq k$.*

For a path decomposition (X_1, X_2, \dots, X_r) of a graph, each vertex subset X_i in it is called a *bag*. The *width* of the path decomposition is $\max_i \{|X_i| - 1\}$. The *pathwidth* of a graph G , denoted by $\text{pw}(G)$, is the minimum possible width of a path decomposition of G . A path decomposition (X_1, X_2, \dots, X_r) is *nice* if $X_1 = X_r = \emptyset$ and for every $i \in \{1, 2, \dots, r-1\}$ there is either a vertex $v \notin X_i$ such that $X_{i+1} = X_i \cup \{v\}$, or there is a vertex $w \in X_i$ such that $X_{i+1} = X_i \setminus \{w\}$. The following lemma shows that any path decomposition can be turned into a nice path decomposition without increasing the width.

► **Lemma 2** ([5]). *If a graph G admits a path decomposition of width at most p , then it also admits a nice path decomposition of width at most p . Moreover, given a path decomposition $P = (X_1, X_2, \dots, X_r)$ of G of width at most p , one can in time $O(p^2 \cdot \max(r, |V(G)|))$ compute a nice path decomposition of G of width at most p .*

There are also easy ways to reduce the length r of a path decomposition to a polynomial of the graph size. Next, we will also assume that r is bounded by a polynomial of the number of vertices. In terms of the pathwidth, there is a known bound.

► **Theorem 3** ([10]). *For any $\epsilon > 0$, there exists an integer n_ϵ such that for every graph G with $n > n_\epsilon$ vertices,*

$$pw(G) \leq \frac{1}{6}n_3 + \frac{1}{3}n_4 + n_{\geq 5} + \epsilon n,$$

where n_i $i \in \{3, 4\}$ is the number of vertices of degree i in G and $n_{\geq 5}$ is the number of vertices of degree at least 5. Moreover, a path decomposition of the corresponding width can be constructed in polynomial time.

Branch-and-Search Algorithm. For a branch-and-search algorithm, we use a parameter k of the instance to measure the running time of the algorithm. Let $T(k)$ denote the maximum size of the search tree generated by the algorithm when running on an instance with the parameter no greater than k . Assume that a branching operation generates l branches and the measure k in the i -th instance decreases by at least c_i . This operation generates a recurrence relation

$$T(k) \leq T(k - c_1) + T(k - c_2) + \dots + T(k - c_l) + 1.$$

The largest root of the function $f(x) = 1 - \sum_{i=1}^l x^{-c_i}$ is called the *branching factor* of the recurrence. Let γ denote the maximum branching factor among all branching factors in the search tree. The running time of the algorithm is bounded by $O^*(\gamma^k)$. For more details about analyzing branch-and-search algorithms, please refer to [13].

3 A Parameterized Algorithm for Co-Path/Cycle Packing

In this section, we propose a parameterized algorithm for CO-PATH/CYCLE PACKING. First, in Section 3.1, we show that CO-PATH/CYCLE PACKING on a special graph class, called *proper graph*, can be quickly solved by using the dynamic programming algorithm based on path decompositions in Theorem 6. The key point in this section is to bound the pathwidth of proper graphs by $2k/3 + \epsilon k$ for any $\epsilon > 0$. Second, in Section 3.2, we give a branch-and-search algorithm that will implement some branching steps on special local graph structures. When there is no good structure to apply our branching rules, we show that the graph must be a proper graph and then the algorithm in Section 3.1 can be called directly to solve the problem.

3.1 Proper Graphs with Small Pathwidth

A graph is called *proper* if it satisfies the following conditions:

1. The maximum degree of G is at most 4.
2. For any degree-4 vertex v , all neighbors are of degree at most 2.
3. For any degree-2 vertex v , at least one vertex in $N(v)$ is of degree at least 3.
4. Each connected component contains at least 6 vertices.

We are going to solve CO-PATH/CYCLE PACKING on proper graphs first. Next, we try to bound the pathwidth of proper graphs.

11:6 Solving Co-Path/Cycle Packing and Co-Path Packing Faster Than 3^k

► **Lemma 4 (♣).** *Let G be a proper graph. If G has a cPCP-set (resp. cPP-set) of size at most k , then it holds that*

$$|V(G)| \leq 100k \quad \text{and} \quad \frac{n_3}{6} + \frac{n_4}{3} \leq \frac{2k}{3}, \quad (1)$$

where n_3 and n_4 are the number of degree-3 and degree-4 vertices in G , respectively.

► **Lemma 5 (♣).** *Let G be a proper graph. For any $\epsilon > 0$, in polynomial time we can either decide that G has no cPCP-set (resp. cPP-set) of size at most k or compute a path decomposition of width at most $\frac{2k}{3} + \epsilon k$.*

The following theorem shows that there exists an algorithm for the general d -BOUNDED-DEGREE VERTEX DELETION problem based on a given path decomposition of the graph. The running time bound of the algorithm is $O^*((d+2)^p)$, where p is the width of the given path decomposition. Previously, an $O^*(3^p)$ -time algorithm for 1-BOUNDED-DEGREE VERTEX DELETION was known [2]. Recently, this result was extended for any $d \geq 1$ by van Rooij [21]. We also present it (in the full version of this paper) in our way to make this paper self-contained.

► **Theorem 6 (♣).** *Given a path decomposition of G with width p . For any $d \geq 1$, d -BOUNDED-DEGREE VERTEX DELETION can be solved in $O^*((d+2)^p)$ time and space.*

However, the algorithm given in Theorem 6 cannot be used to solve CO-PATH PACKING since there is a global connectivity constraint for CO-PATH PACKING. We will discuss it in Section 4. Based on Theorem 6, we have the following lemma.

► **Lemma 7.** *CO-PATH/CYCLE PACKING on proper graphs can be solved in $O^*(2.5199^k)$ time.*

Proof. We first call the algorithm in Lemma 5. If the algorithm decides that G has no cPCP-set of size at most k , we claim that (G, k) is a no-instance. Otherwise, we can obtain a nice path decomposition of width at most $\frac{2k}{3} + \epsilon k$. Then we call the algorithm in Theorem 6. This algorithm runs in $O^*(4^{2k/3 + \epsilon k}) = O^*(2.5199^k)$, where we choose $\epsilon < 10^{-6}$. This lemma holds. ◀

3.2 A Branch-and-Search Algorithm

In this subsection, we provide a branch-and-search algorithm for CO-PATH/CYCLE PACKING, which is denoted by $\text{cPCP}(G, k)$. Our algorithm contains several reduction and branching steps. After recursively executing these steps, we will get a proper graph and then call the dynamic programming algorithm in Section 3.1 to solve it.

3.2.1 Reduction and Branching Rules

Firstly we have a reduction rule to reduce small connected components.

► **Reduction-Rule 1.** *If there is a connected component C of the graph such that $|V(C)| \leq 6$, then run a brute force algorithm to find a minimum cPCP-set S in C , delete C and include S in the deletion set.*

► **Lemma 8.** *Let u and v be two adjacent vertices of degree at most 2 in G and G' be the graph after deleting edge uv from G . Then (G, k) is a yes-instance if and only if (G', k) is a yes-instance.*

This lemma holds because adding an edge between two vertices of degree at most 1 back to a graph will not make the two vertices of degree greater than 2. Based on this lemma, we have the following reduction rule.

► **Reduction-Rule 2.** *If there are two adjacent vertices u and v of degree at most 2, then return $\text{cPCP}(G' = (V(G), E(G) \setminus \{uv\}), k)$.*

► **Lemma 9.** *Let $\{u, v, w\}$ be a triangle such that $|N(\{u, v, w\})| = 1$. Let x be the vertex in $N(\{u, v, w\})$. There is a minimum cPCP-set containing x .*

This lemma holds because we must delete at least one vertex in $\{u, v, w, x\}$ and delete any vertex in $\{u, v, w\}$ cannot decrease the degree of vertices in $V(G) \setminus \{u, v, w, x\}$. Based on this lemma, we have the following reduction rule.

► **Reduction-Rule 3.** *If there is a triangle $\{u, v, w\}$ such that $|N(\{u, v, w\})| = 1$, then return $\text{cPCP}(G \setminus N[\{u, v, w\}], k - 1)$.*

After applying the three simple reduction rules, we will execute some branching steps. Although we have several branching steps, most of them are based on the following two branching rules.

For a vertex v of degree at least 3, either it is included in the deletion set or it remains in the graph. For the latter case, there are at most two vertices in $N(v)$ that can remain in the graph. So we have the following branching rule.

Branching-Rule (B1). *For a vertex v of degree at least 3, branch on it to generate $\binom{|N(v)|}{2} + 1$ branches by either (i) deleting v from the graph and including it in the deletion set, or (ii) for every pair of vertices u and w in $N(v)$, deleting $N(v) \setminus \{u, w\}$ from the graph and including $N(v) \setminus \{u, w\}$ in the deletion set.*

A vertex v dominates a vertex u if $N[u] \subseteq N[v]$. We have the following property for dominated vertices.

► **Lemma 10.** *If a vertex v dominates a vertex u , then there is a minimum cPCP-set either containing v or containing none of v and u .*

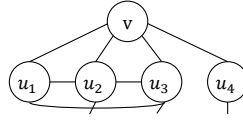
Proof. Let S be a cPCP-set. Assume to the contrary that $S \cap \{v, u\} = \{u\}$. Since v dominates u , we know that $S' = S \setminus \{u\} \cup \{v\}$ is still a feasible cPCP-set with $|S'| = |S|$. There is a minimum cPCP-set containing v . This lemma holds. ◀

Assume that v dominates u . By Lemma 10, we know that either v is included in the deletion set or both of v and u remain in the graph. For the latter case, there are at most two vertices in $N(v)$ that can remain in the graph. Thus, at least $|N(v)| - 2$ vertices in $N(v) \setminus \{u\}$ will be deleted. We get the following branching rule.

Branching-Rule (B2). *Assume that a vertex v of degree at least 3 dominates a vertex u . Branch on v to generate $1 + (|N(v)| - 1) = |N(v)|$ branches by either (i) deleting v from the graph and including it in the deletion set, or (ii) for each vertex $w \in N(v) \setminus \{u\}$, deleting $N(v) \setminus \{u, w\}$ from the graph and including $N(v) \setminus \{u, w\}$ in the deletion set.*

3.2.2 Steps

When we execute one step, we assume that all previous steps are not applicable in the current graph anymore. We will analyze each step after describing it.



■ **Figure 1** Degree-4 vertex v dominates a degree-3 vertex u_1 .

Step 1 (Vertices of degree at least 5). If there is a vertex v of $d(v) \geq 5$, then branch on v with Branching-Rule (B1) to generate $\binom{d(v)}{d(v)-2} + 1$ branches and return the best of

$$\text{cPCP}(G \setminus \{v\}, k - 1)$$

$$\text{and } \text{cPCP}(G \setminus (N(v) \setminus \{u, w\}), k - |N(v) \setminus \{u, w\}|) \text{ for each pair } \{u, w\} \subseteq N(v).$$

For this step, we get a recurrence

$$T(k) \leq T(k - 1) + \binom{d(v)}{d(v) - 2} \times T(k - (d(v) - 2)) + 1,$$

where $d(v) \geq 5$. For the worst case that $d(v) = 5$, the branching factor of it is 2.5445.

After Step 1, the graph contains only vertices with degree at most 4.

Step 2 (Degree-4 vertices dominating some vertex of degree at least 3). Assume that there is a degree-4 vertex v that dominates a vertex u_1 , where $d(u_1) \geq 3$. Without loss of generality, we assume that the other three neighbors of v are u_2, u_3 and u_4 , and u_1 is adjacent to u_2 and u_3 (u_1 is further adjacent to u_4 if $d(u_1) = 4$). See Figure 1 for an illustration. We branch on v with Branching-Rule (B2) and get $|N(v)|$ branches

$$\text{cPCP}(G \setminus \{v\}, k - 1), \text{cPCP}(G \setminus \{u_2, u_3\}, k - 2),$$

$$\text{cPCP}(G \setminus \{u_2, u_4\}, k - 2), \text{ and } \text{cPCP}(G \setminus \{u_3, u_4\}, k - 2).$$

The corresponding recurrence is

$$T(k) \leq T(k - 1) + 3 \times T(k - 2) + 1,$$

which has a branching factor of 2.3028.

A triangle $\{u, v, w\}$ is called *heavy* if it holds that $|N(\{u, v, w\})| \geq 4$.

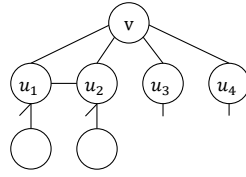
Step 3 (Degree-4 vertices in a heavy triangle). Assume that a degree-4 vertex v is in a heavy triangle. Let u_1, u_2, u_3 and u_4 be the four neighbors of v , where we assume without loss of generality that $\{v, u_1, u_2\}$ is a heavy triangle. See Figure 2 for an illustration. We branch on v with Branching-Rule (B1). In the branch of deleting $\{u_3, u_4\}$, we can simply assume that the three vertices v, u_1 and u_2 are not deleted, otherwise this branch can be covered by another branch and then it can be ignored. Since v, u_1 and u_2 form a triangle, we need to delete all the vertices in $N(\{v, u_1, u_2\})$ in this branch. Note that $\{v, u_1, u_2\}$ is a heavy triangle and we have that $|N(\{v, u_1, u_2\})| \geq 4$. We generate the following $\binom{4}{2} + 1$ branches

$$\text{cPCP}(G \setminus \{v\}, k - 1),$$

$$\text{cPCP}(G \setminus (\{u_1, u_i\}), k - |\{u_1, u_i\}|) \text{ for each } i = 2, 3, 4,$$

$$\text{cPCP}(G \setminus (\{u_2, u_i\}), k - |\{u_2, u_i\}|) \text{ for each } i = 3, 4,$$

$$\text{and } \text{cPCP}(G \setminus N(\{v, u_1, u_2\}), k - |N(\{v, u_1, u_2\})|).$$



■ **Figure 2** Degree-4 vertex v is in a heavy triangle $\{v, u_1, u_2\}$.

The corresponding recurrence is

$$T(k) \leq T(k-1) + \binom{4}{2} - 1 \times T(k-2) + T(k - |N(\{v, u_1, u_2\})|) + 1,$$

where $|N(\{v, u_1, u_2\})| \geq 4$. For the worst case that $|N(\{v, u_1, u_2\})| = 4$, the branching factor of it is 2.8186.

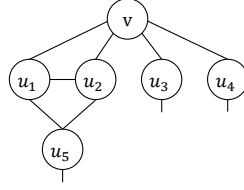
► **Lemma 11.** *If a vertex v has two degree-1 neighbors u_1 and u_2 , then there is a minimum cPCP-set containing none of u_1 and u_2 . Furthermore, If v is a vertex of degree at most 3, then there is a minimum cPCP-set containing none of v, u_1 , and u_2 .*

Proof. Let S be a minimum cPCP-set. If $v \in S$, then $S \setminus \{u_1, u_2\}$ is still a minimum cPCP-set. Next, we assume that $v \notin S$. If at least one of u_1 and u_2 is in S , then $S' = (S \setminus \{u_1, u_2\}) \cup \{v\}$ is a cPCP-set with $|S'| \leq |S|$. Thus, S' is a minimum cPCP-set not containing u_1 and u_2 .

If v is a degree-2 vertex, then the component containing v is a path of three vertices. None of the three vertices should be deleted. If v is a degree-3 vertex, we let u_3 be the third neighbor of v . Note that at least one vertex in $\{v, u_1, u_2, u_3\}$ should be deleted and then any solution S will contain at least one vertex in $\{v, u_1, u_2, u_3\}$. We can see that $S' = (S \setminus \{v, u_1, u_2, u_3\}) \cup \{u_3\}$ is still a cPCP-set with size $|S'| \leq |S|$. There is always a minimum cPCP-set containing none of v, u_1 , and u_2 . ◀

Step 4 (Degree-4 vertices in a triangle). Assume that there is still a degree-4 vertex v in a triangle $\{v, u_1, u_2\}$. We also let u_1, u_2, u_3 and u_4 be the four neighbors of v . Since triangle $\{v, u_1, u_2\}$ can not be a heavy triangle now, we know that $|N(\{v, u_1, u_2\})| \leq 3$. First, we show that it is impossible $|N(\{v, u_1, u_2\})| = 2$. Assume to the contrary that $|N(\{v, u_1, u_2\})| = 2$. Then u_1 and u_2 can only be adjacent to vertices in $N[v]$. If both of u_1 and u_2 are degree-2 vertices, then Reduction-Rule 1 should be applied. If one of u_1 and u_2 is a vertex of degree at least 3, then v would dominate this vertex, and then the condition of Step 2 would hold. Any case is impossible. Next, we assume that $|N(\{v, u_1, u_2\})| = 3$ and let u_3 denote the third vertex in $N(\{v, u_1, u_2\})$. We further consider several different cases.

Case 1: One of u_1 and u_2 , say u_1 is a degree-4 vertex. For this case, vertex u_1 is adjacent to u_5 , otherwise the degree-4 vertex v would dominate the degree-4 vertex u_1 . Since u_1 is of degree 4, we know that u_1 is also adjacent to one of u_3 and u_4 , say u_3 . Vertices u_2 and u_3 can only be adjacent to vertices in $N[v] \cup \{u_5\}$, otherwise $\{v, u_1, u_2\}$ or $\{v, u_1, u_3\}$ would form a heavy triangle and Step 3 should be applied. Thus, neither u_2 nor u_3 can be a degree-3 vertex, otherwise degree-4 vertex u_1 or v dominates a degree-3 vertex u_2 or u_3 and then Step 2 should be applied. Thus, we have that either $d(u_2) = d(u_3) = 2$ or $d(u_2) = d(u_3) = 4$. We further consider the following three cases:



■ **Figure 3** In the Case 2 of Step 4, degree-4 vertex v is in a triangle $\{v, u_1, u_2\}$ and both of u_1 and u_2 are degree-3 vertices.

Case 1.1: $d(u_2) = d(u_3) = 2$. For this case, we have that v dominates u_2 and u_3 . By Lemma 10, we have that there is a minimum cPCP-set either containing v or containing none of v , u_2 and u_3 . For the first branch, we delete v from the graph and include it in the deletion set. For the second branch, we delete u_1 and u_4 from the graph and include it in the deletion set. The corresponding recurrence is

$$T(k) \leq T(k-1) + T(k-2) + 1,$$

the branching factor of which is 1.6181.

Case 1.2: $d(u_2) = d(u_3) = 4$ and u_2 and u_3 are not adjacent. For this case, both of u_2 and u_3 are adjacent to u_4 and u_5 . Since v does not dominate u_4 , we know that u_4 is adjacent to at least one vertex out of $N[v]$. Since triangle $\{v, u_2, u_4\}$ is not a heavy triangle, we know that u_4 is not adjacent to any vertex other than $N[v] \cup \{u_5\}$. Thus, u_4 is also adjacent to u_5 . Since the maximum degree of the graph is 4 now, we know that this component only contains six degree-4 vertices $N[v] \cup \{u_5\}$, which should be eliminated by Reduction-Rule 2.

Case 1.3: $d(u_2) = d(u_3) = 4$ and u_2 and u_3 are adjacent. For this case, since v does not dominate u_3 , we know that u_3 is adjacent to at least one vertex out of $N[v]$. This vertex can only be u_5 . Thus, the four neighbors of u_3 are v, u_1, u_2 , and u_5 . Now u_1 is a degree-4 vertex dominating a degree-4 vertex u_3 . Step 2 should be applied. Thus, this case is impossible.

Case 2: Both of u_1 and u_2 are degree-3 vertices. It holds that $N(u_1) = \{v, u_2, u_5\}$ and $N(u_2) = \{v, u_1, u_5\}$, otherwise vertex v would dominate u_1 or u_2 . See Figure 3 for an illustration. For this case, we first branch on v with Branching-Rule (B1) to generate $\binom{d(v)}{d(v)-2} + 1$ branches. We can only get a recurrence relation

$$T(k) \leq T(k-1) + \binom{4}{2}T(k-2) + 1.$$

This recurrence is not good enough. Next, we look at the branch of deleting v and try to get some improvements on this subbranch. After deleting v , vertices u_1 and u_2 become degree-2 vertices in a triangle. We first apply Reduction-Rule 1 to delete edge u_1u_2 between u_1 and u_2 . Then, vertices u_1 and u_2 become degree-1 vertices adjacent to u_5 .

Case 2.1: u_5 is a degree-2 vertex. This case is impossible otherwise Reduction Rule 3 would be applied before this step.

Case 2.2: u_5 is a degree-3 vertex. By Lemma 11, we can delete $N[u_5]$ directly and include the third neighbor of u_5 in the deletion set. We generate the following $\binom{4}{2} + 1$ branches

$$\begin{aligned} & \text{cPCP}(G \setminus (\{v\} \cup N[u_5]), k-2) \\ & \text{and } \text{cPCP}(G \setminus (N(v) \setminus \{u, w\}), k - |N(v) \setminus \{u, w\}|) \text{ for each pair } \{u, w\} \subseteq N(v). \end{aligned}$$

The corresponding recurrence is

$$T(k) \leq T(k-2) + \binom{4}{2} T(k-2) + 1.$$

which has a branching factor of 2.6458.

Case 2.3: u_5 is a degree-4 vertex. By Lemma 11, we know that there is a minimum cPCP-set either containing u_5 or containing none of u_5, u_1 , and u_2 . For the first case, we delete u_5 and include it in the deletion set. For the second case, we delete $N[u_5]$ from the graph and include $N(u_5) \setminus \{u_5, u_1, u_2\}$ in the deletion set. Note that $|N(u_5) \setminus \{u_5, u_1, u_2\}| = 2$ since u_5 is a degree-4 vertex. Combining with the previous branching on v , we get the following $\binom{4}{2} + 2$ branches

$$\begin{aligned} & \text{cPCP}(G \setminus \{v, u_5\}, k-2), \\ & \text{cPCP}(G \setminus (\{v\} \cup N[u_5]), k-3), \end{aligned}$$

and $\text{cPCP}(G \setminus (N(v) \setminus \{u, w\}), k - |N(v) \setminus \{u, w\}|)$ for each pair $\{u, w\} \subseteq N(v)$.

The corresponding recurrence is

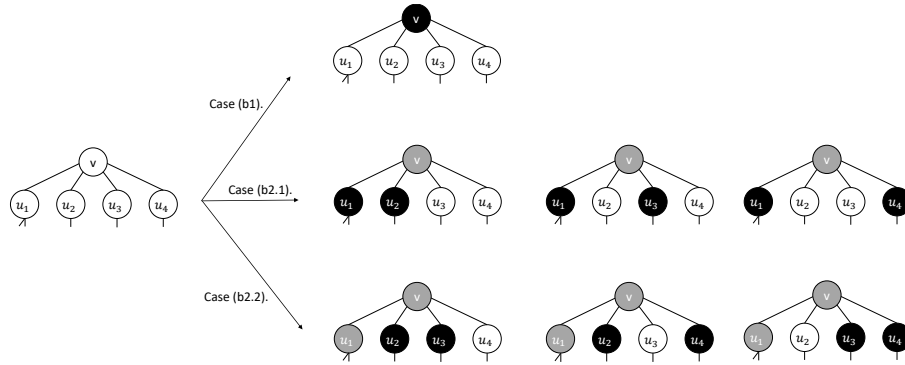
$$T(k) \leq T(k-1-1) + T(k-1-2) + \binom{4}{2} T(k-2) + 1,$$

which has a branching factor of 2.7145.

After Step 4, no degree-4 vertex is in a triangle.

Step 5 (Degree-4 vertices adjacent to some vertex of degree at least 3). Assume there is a degree-4 vertex v adjacent to at least one vertex of degree at least 3. Let u_1, u_2, u_3 and u_4 be the four neighbors of v , where we assume without loss of generality that $d(u_1) \geq 3$. First, we branch on v by either (b1) including it in the solution set or (b2) excluding it from the solution set. Next, we focus on the latter case (b2). In (b2), we further branch on u_1 by (b2.1) either including it in the solution set or (b2.2) excluding it from the solution set. For case (b2.1), there are at least $|N(v)| - 2 = 2$ vertices in $N(v)$ that should be deleted by the same argument for Branching-Rule (B1). Thus, we can generate three subbranches by deleting $\{u_1, u_2\}$, $\{u_1, u_3\}$, and $\{u_1, u_4\}$, respectively. For case (b2.2), there are still at least $|N(v)| - 2 = 2$ vertices in $N(v)$ that should be deleted, which can be one of the following three sets $\{u_2, u_3\}$, $\{u_2, u_4\}$, and $\{u_3, u_4\}$. Furthermore, there are also at least $|N(u_1)| - 2 \geq 1$ vertices in $N(u_1)$ that should be deleted. Note that $v \in N(u_1)$ is not allowed to be deleted now. Thus, at least $|N(u_1) \setminus \{v\}| - 1$ vertices in $N(u_1) \setminus \{v\}$ should be deleted. We will generate $\binom{|N(u_1) \setminus \{v\}|}{|N(u_1) \setminus \{v\}| - 1} = |N(u_1) \setminus \{v\}| = d(u_1) - 1$ branches by decreasing k by $|N(u_1) \setminus \{v\}| - 1 = d(u_1) - 2$. Since after Step 3, the degree-4 vertex v is not in any triangle, we know that $N(u_1) \setminus \{v\}$ is disjoint with $\{u_2, u_3, u_4\}$. For case (b2.2), we will generate $3 \times (d(u_1) - 1)$ subbranches by decreasing k by at least $2 + d(u_1) - 2 = d(u_1)$ in each, where $d(u_1) = 3$ or 4 . See Figure 4 for an illustration. In total, we will generate the following $1 + 3 + 3 \times (d(u_1) - 1)$ branches

$$\begin{aligned} & \text{cPCP}(G \setminus \{v\}, k-1), \\ & \text{cPCP}(G \setminus (\{u_1, u_i\}), k-2) \text{ for each } i = 2, 3, 4, \\ & \text{cPCP}(G \setminus (\{u_2, u_3\} \cup (N(u_1) \setminus \{v, w\})), k - d(u_1)), \text{ for each } w \in N(u_1) \setminus \{v\}, \\ & \text{cPCP}(G \setminus (\{u_2, u_4\} \cup (N(u_1) \setminus \{v, w\})), k - d(u_1)), \text{ for each } w \in N(u_1) \setminus \{v\}, \\ & \text{and } \text{cPCP}(G \setminus (\{u_3, u_4\} \cup (N(u_1) \setminus \{v, w\})), k - d(u_1)), \text{ for each } w \in N(u_1) \setminus \{v\}. \end{aligned}$$



■ **Figure 4** Vertices in the deletion set are denoted by black vertices, and vertices not allowed to be deleted are denoted by grey vertices.

■ **Table 2** The branching factors of each of the first five steps.

Steps	Step 1	Step 2	Step 3	Step 4	Step 5
Branching factors	2.5445	2.3028	2.8186	2.7145	2.8192

We get a recurrence

$$T(k) \leq T(k - 1) + 3 \times T(k - 2) + 3(d(u_1) - 1) \times T(k - d(u_1)) + 1,$$

where $d(u_1) = 3$ or 4 . For the case that $d(u_1) = 3$, the branching factor is 2.8192 , and for the case that $d(u_1) = 4$, the branching factor is 2.6328 .

The worst branching factors in the above five steps are listed in Table 2. After Step 5, any degree-4 vertex can be only adjacent to vertices of degree at most 2. It is easy to see that the remaining graph after Step 5 is a proper graph. We call the algorithm in Lemma 7 to solve the instance in $O^*(2.5199^k)$ time.

► **Theorem 12.** *CO-PATH/CYCLE PACKING can be solved in $O^*(2.8192^k)$ time.*

Fomin et al. [9] introduced the monotone local search technique for deriving exact exponential-time algorithms from parameterized algorithms. Under some conditions, a $c^k n^{O(1)}$ -time algorithm implies an algorithm with running time $(2 - 1/c)^{n+o(n)}$. By applying this technique on our $O^*(2.8192^k)$ -time algorithm in Theorem 12, we know that CO-PATH/CYCLE PACKING can be solved in $(2 - 1/2.8192)^{n+o(n)} = O(1.6453^n)$ time.

► **Corollary 13.** *CO-PATH/CYCLE PACKING can be solved in $O(1.6453^n)$ time.*

4 A Parameterized Algorithm for Co-Path Packing

In this section, we show a randomized $O^*(2.9241^k)$ time algorithm for CO-PATH PACKING. We also use the method for CO-PATH/CYCLE PACKING, which consists of two phases. The first one is the dynamic programming phase, and the second one is the branch-and-search phase.

In the dynamic programming phase, it is more difficult to solve CO-PATH PACKING based on a path decomposition in comparison to CO-PATH/CYCLE PACKING. The reason is that CO-PATH/CYCLE PACKING involves only local constraints, which means that the object's properties can be verified by checking each vertex's neighborhood independently. For this problem, a typical dynamic programming approach can be used to design a $c^{pw(G)} |V(G)|^{O(1)}$

time algorithm straightforwardly. In contrast, there is a global connectivity constraint for CO-PATH PACKING. The problem with global connectivity constraint is also called *connectivity-type problem*. For connectivity-type problems, the typical dynamic programming approach has to keep track of all possible ways the solution can traverse the corresponding separator of the path decomposition that is $\Omega(l^l)$, where l is the size of the separator and hence the pathwidth [6].

To obtain a single exponential algorithm parametrized by pathwidth for CO-PATH PACKING, we use the cut & count framework. Previously, cut & count significantly improved the known bounds for various well-studied connectivity-type problems, such as HAMILTONIAN PATH, STEINER TREE, and FEEDBACK VERTEX SET [6]. Additionally, for k -CO-PATH SET, cut & count has been used to obtain a fast parameterized algorithm [17]. In Section 4.1, we present a randomized fpt algorithm with complexity $O^*(5^{pw(G)})$ for CO-PATH PACKING.

In the branch-and-search phase, our algorithm for CO-PATH PACKING is similar to the branching algorithm for CO-PATH/CYCLE PACKING. Specifically, our branching algorithm for CO-PATH PACKING contains two reduction rules, two branching rules, and four steps, while the first reduction rule, both two branching rules, the first two steps, and the last step are the same as the branching algorithm for CO-PATH/CYCLE PACKING. The branch-and-search phase is shown in Section 4.2.

4.1 A DP Algorithm via Cut & Count for Co-Path Packing

In this section, we use the cut & count framework to design a $5^{pw}n^{O(1)}$ one-sided error Monte Carlo algorithm with a constant probability of a false negative for CO-PATH PACKING. Due to the limited space, we do not provide the proof of the following theorem here.

► **Theorem 14 (♣).** *Given a path decomposition of G with width p . CO-PATH PACKING can be solved in $O^*(5^p)$ time and space with failure probability $\leq 1/3$.*

4.2 The Whole Algorithm

In this section, we propose a parameterized algorithm for CO-PATH PACKING. First, in Section 4.2.1, we show that CO-PATH PACKING on a proper graph class can be quickly solved by using the dynamic programming algorithm based on path decompositions in Theorem 14. Similarly, we will use Lemma 5 to conclude this result. Second, in Section 4.2.2, we give a branch-and-search algorithm that will implement some branching steps on special local graph structures. Our branching algorithm for CO-PATH PACKING is similar to the branching algorithm for CO-PATH/CYCLE PACKING. Specifically, our branching algorithm for CO-PATH PACKING contains two reduction rules, two branching rules, and four steps, while the first reduction rule, both two branching rules, the first two steps and the last step are the same as the branching algorithm for CO-PATH/CYCLE PACKING. When all the steps cannot be applied, we show that the graph must be a proper graph and then the algorithm in Section 4.1 can be called directly to solve the problems.

4.2.1 Proper Graphs with Small Pathwidth

Recall that a graph is called proper if it satisfies the following conditions:

1. The maximum degree of G is at most 4.
2. For any degree-4 vertex v , all neighbors are of degree at most 2.
3. For any degree-2 vertex v , at least one vertex in $N(v)$ is of degree at least 3.
4. Each connected component contains at least 6 vertices.

Combine Lemma 5 for CO-PATH PACKING with Lemma 14, we have the following lemma.

► **Lemma 15 (♣).** *CO-PATH PACKING on proper graphs can be solved in $O^*(2.9241^k)$ time with probability at least $2/3$.*

4.2.2 A Branch-and-Search Algorithm

In this subsection, we provide a branch-and-search algorithm for CO-PATH PACKING, which is denoted by $\text{cPP}(G, k)$. Our algorithm contains several reduction and branching steps. After recursively executing these steps, we will get a proper graph and then call the dynamic programming algorithm in Lemma 15 to solve it.

4.2.2.1. Reduction and Branching Rules. Firstly, we present two reduction rules. The first reduction rule is Reduction Rule 1 for CO-PATH/CYCLE PACKING and Reduction Rule *2 is a new reduction rule for CO-PATH PACKING.

Reduction-Rule 1. *If there is a connected component C of the graph such that $|V(C)| \leq 6$, then run a brute force algorithm to find a minimum cPP-set S in C , delete C and include S in the deletion set.*

A path $v_0v_1 \dots v_{h-1}v_h$ is called a *degree-two-path* if the two vertices v_0 and v_h are of degree not 2 and the other vertices $v_1 \dots v_{h-1}$ are of degree 2, where we allow $v_0 = v_h$.

► **Lemma 16 (♣).** *For a degree-two-path $P = v_0v_1 \dots v_{h-1}v_h$ with $h \geq 4$ in $G = (V, E)$, let $G' = (V \setminus v_2, E \setminus \{v_1v_2, v_2v_3\} \cup \{v_1v_3\})$, we have that (G, k) is a yes-instance if and only if (G', k) is a yes-instance.*

Based on this lemma, we have the following reduction rule for CO-PATH PACKING.

Reduction-Rule *2. *If there is a degree-two-path $P = v_0v_1 \dots v_{h-1}v_h$ with $h \geq 4$, then return $\text{cPP}(G' = (V \setminus v_2, E \setminus \{v_1v_2, v_2v_3\} \cup \{v_1v_3\}), k)$.*

If Reduction Rules 1 and *2 cannot be applied, we have a property that for any degree-2 vertex v , at least one vertex in $N(v)$ is of degree at least 3.

After applying the three simple reduction rules, we will execute some branching steps. For CO-PATH PACKING, we have two branching rules, which are the same as the two branching rules for CO-PATH/CYCLE PACKING. The correctnesses of these two branching rules for CO-PATH PACKING are similar to the two branching rules for CO-PATH/CYCLE PACKING.

Branching-Rule (B1). *For a vertex v of degree at least 3, branch on it to generate $\binom{|N(v)|}{2} + 1$ branches by either (i) deleting v from the graph and including it in the deletion set, or (ii) for every pair of vertices u and w in $N(v)$, deleting $N(v) \setminus \{u, w\}$ from the graph and including $N(v) \setminus \{u, w\}$ in the deletion set.*

Branching-Rule (B2). *Assume that a vertex v of degree at least 3 dominates a vertex u . Branch on v to generate $1 + (|N(v)| - 1) = |N(v)|$ branches by either (i) deleting v from the graph and including it in the deletion set, or (ii) for each vertex $w \in N(v) \setminus \{u\}$, deleting $N(v) \setminus \{u, w\}$ from the graph and including $N(v) \setminus \{u, w\}$ in the deletion set.*

4.2.2.2. Steps. When we execute one step, we assume that all previous steps are not applicable in the current graph anymore. In this subsection, we present four steps: Step 1, Step 2, Step *3 and Step *4 for CO-PATH PACKING. Steps 1 and 2 are the Steps 1 and 2 for CO-PATH/CYCLE PACKING. Step *3 is a new step designed for CO-PATH PACKING. Step *4 is the Step 5 for CO-PATH/CYCLE PACKING.

Step 1 (Vertices of degree at least 5). If there is a vertex v of $d(v) \geq 5$, then branch on v with Branching-Rule (B1) to generate $\binom{d(v)}{d(v)-2} + 1$ branches.

Step 2 (Degree-4 vertices dominating some vertex of degree at least 3). Assume that there is a degree-4 vertex v that dominates a vertex u_1 , where $d(u_1) \geq 3$. Without loss of generality, we assume that the other three neighbors of v are u_2, u_3 and u_4 , and u_1 is adjacent to u_2 and u_3 (u_1 is further adjacent to u_4 if $d(u_1) = 4$). We branch on v with Branching-Rule (B2) and get $|N(v)|$ branches.

Step *3 (Degree-4 vertices in a triangle). Assume that a degree-4 vertex v is in a triangle. Let u_1, u_2, u_3 and u_4 be the four neighbors of v , where we assume without loss of generality that $\{v, u_1, u_2\}$ is a triangle. We branch on v with Branching-Rule (B1). In the branch of deleting $\{u_3, u_4\}$, we can simply assume that the three vertices v, u_1 and u_2 are not deleted, otherwise this branch can be covered by another branch and then it can be ignored. Since v, u_1 and u_2 form a triangle, we know this case is impossible, so we can ignore this subbranch. We generate the following $\binom{4}{2}$ branches

$$\begin{aligned} & \text{cPP}(G \setminus \{v\}, k - 1), \\ & \text{cPP}(G \setminus (\{u_1, u_i\}), k - |\{u_1, u_i\}|) \text{ for each } i = 2, 3, 4, \\ \text{and } & \text{cPP}(G \setminus (\{u_2, u_i\}), k - |\{u_2, u_i\}|) \text{ for each } i = 3, 4. \end{aligned}$$

The corresponding recurrence is

$$T(k) \leq T(k - 1) + \left(\binom{4}{2} - 1\right) \times T(k - 2) + 1.$$

The branching factor of it is 2.7913.

After Step *3, no degree-4 vertex is in a triangle.

Step *4 (Degree-4 vertices adjacent to some vertex of degree at least 3). Assume there is a degree-4 vertex v adjacent to at least one vertex of degree at least 3. Let u_1, u_2, u_3 and u_4 be the four neighbors of v , where we assume without loss of generality that $d(u_1) \geq 3$. First, we branch on v by either (b1) including it in the solution set or (b2) excluding it from the solution set. Next, we focus on the latter case (b2). In (b2), we further branch on u_1 by (b2.1) either including it in the solution set or (b2.2) excluding it from the solution set. For case (b2.1), there are at least $|N(v)| - 2 = 2$ vertices in $N(v)$ that should be deleted by the same argument for Branching-Rule (B1). Thus, we can generate three subbranches by deleting $\{u_1, u_2\}$, $\{u_1, u_3\}$, and $\{u_1, u_4\}$, respectively. For case (b2.2), there are still at least $|N(v)| - 2 = 2$ vertices in $N(v)$ that should be deleted, which can be one of the following three sets $\{u_2, u_3\}$, $\{u_2, u_4\}$, and $\{u_3, u_4\}$. Furthermore, there are also at least $|N(u_1)| - 2 \geq 1$ vertices in $N(u_1)$ that should be deleted. Note that $v \in N(u_1)$ is not allowed to be deleted now. Thus, at least $|N(u_1) \setminus \{v\}| - 1$ vertices in $N(u_1) \setminus \{v\}$ should be deleted. We will generate $\binom{|N(u_1) \setminus \{v\}|}{|N(u_1) \setminus \{v\}| - 1} = |N(u_1) \setminus \{v\}| = d(u_1) - 1$ branches by decreasing k by

■ **Table 3** The branching factors of each of the steps.

Steps	Step 1	Step 2	Step *3	Step *4
Branching factors	2.5445	2.3028	2.7913	2.8192

$|N(u_1) \setminus \{v\}| - 1 = d(u_1) - 2$. Since after Step *3, the degree-4 vertex v is not in any triangle, we know that $N(u_1) \setminus \{v\}$ is disjoint with $\{u_2, u_3, u_4\}$. For case (b2.2), we will generate $3 \times (d(u_1) - 1)$ subbranches by decreasing k by at least $2 + d(u_1) - 2 = d(u_1)$ in each, where $d(u_1) = 3$ or 4 .

The worst branching factors in the above steps are listed in Table 3. After Step *4, any degree-4 vertex can be only adjacent to vertices of degree at most 2. It is easy to see that the remaining graph after Step *4 is a proper graph. We call the algorithm in Lemma 15 to solve the instance for CO-PATH PACKING in $O^*(2.9241^k)$ time with probability at least $2/3$.

► **Theorem 17.** *CO-PATH PACKING can be solved in $O^*(2.9241^k)$ time with probability at least $2/3$.*

5 Conclusion

In this paper, we show that given a path decomposition of width p , CO-PATH PACKING can be solved by a randomized fpt algorithm running in $O^*(5^p)$ time. Additionally, by combining this algorithm with a branch-and-search algorithm, we show that CO-PATH/CYCLE PACKING can be solved in $O^*(2.8192^k)$ time and CO-PATH PACKING can be solved in $O^*(2.9241^k)$ time with probability at least $2/3$. For CO-PATH/CYCLE PACKING, the new bottleneck in our algorithm is Step 5, which is to deal with degree-4 vertices not in any triangle. For CO-PATH PACKING, the new bottleneck in our algorithm is the dynamic programming phase. The idea of using path/tree decomposition to avoid bottlenecks in branch-and-search algorithms may have the potential to be applied to more problems. It would also be interesting to design a deterministic algorithm for CO-PATH PACKING faster than $O^*(3^k)$.

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