Parameterised Distance to Local Irregularity

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Abstract

A graph G is *locally irregular* if no two of its adjacent vertices have the same degree. The authors of [Fioravantes et al. Complexity of finding maximum locally irregular induced subgraph. *SWAT*, 2022] introduced and provided some initial algorithmic results on the problem of finding a locally irregular induced subgraph of a given graph G of maximum order, or, equivalently, computing a subset S of $V(G)$ of minimum order, whose deletion from G results in a locally irregular graph; S is called an *optimal vertex-irregulator of* G. In this work we provide an in-depth analysis of the parameterised complexity of computing an optimal vertex-irregulator of a given graph G. Moreover, we introduce and study a variation of this problem, where S is a subset of the edges of G; in this case, S is denoted as an *optimal edge-irregulator of* G. We prove that computing an optimal vertex-irregulator of a graph G is in FPT when parameterised by various structural parameters of G, while it is W[1]-hard when parameterised by the feedback vertex set number or the treedepth of G. Moreover, computing an optimal edge-irregulator of a graph G is in FPT when parameterised by the vertex integrity of G, while it is $\mathcal{NP}$-hard even if G is a planar bipartite graph of maximum degree 6, and W[1]-hard when parameterised by the size of the solution, the feedback vertex set or the treedepth of G. Our results paint a comprehensive picture of the tractability of both problems studied here.

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1 Introduction

A fundamental problem in graph theory is "given a graph G, find an induced subgraph H of G, of maximum order, that belongs in the family of graphs verifying a property Π", in which case we say that $H \in \Pi$:

Largest Induced Subgraph with Property Π (ISP-Π)[19]
Input: A graph $G = (V, E)$, an integer k, a property Π.
Task: Does there exist a set $S \subseteq V$ such that $|S| \leq k$ and $G - S \in \Pi$?

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There is a plethora of classical problems that fall under this general setting. Consider, for example, the Vertex Cover and the Feedback Vertex Set, where Π is the property "the graph is an independent set" and "the graph is a forest", respectively.

In this paper we study the ISP-Π problem where Π is the property "the graph is locally irregular", recently introduced in [16]. A graph $G = (V, E)$ is called *locally irregular* if no two adjacent vertices in V have the same degree. We extend the work of [16], by more thoroughly investigating the parameterised behaviour of the problem. In addition, we take the first step towards the problem of finding large locally irregular (not necessarily induced) subgraphs of a given graph G. In particular, we introduce the problem where the goal is to find a subset of edges of G of minimum order, whose removal renders the graph locally irregular. Our results allow us to paint a rather clear picture concerning the tractability of both problems studied here in relation to various standard graph-structural parameters (see Figure 1 for an overview of our results).

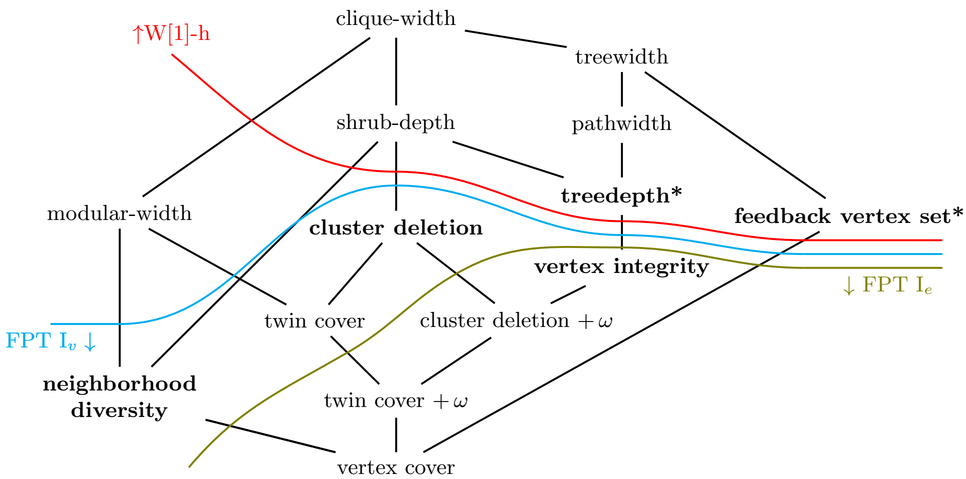

Figure 1 Overview of our results. A parameter A appearing linked to a parameter B with A being below B is to be understood as "there is a function f such that $f(A) \geq f(B)$". The bold font is used to indicate the parameters that we consider in this work. The asterisks are used to indicate that the corresponding result follows from observations based on the work in [16]. In light blue (olive resp.) we exhibit the FPT results we provide for finding an optimal vertex (edge resp.) irregulator. In red we exhibit the W[1]-hardness results we provide for both problems. The clique number of the graph is denoted by ω.

ISP-Π and hereditarity. The ISP-Π problem has been extensively studied for *hereditary* properties. That is, a property Π is hereditary if, for any graph G verifying it, any induced subgraph of G also verifies that property. The properties "the graph is an independent set" or "the graph is a forest" are, for example, hereditary. It was already shown in [29] that ISP-Π is an $\mathcal{NP}$-hard problem for any non-trivial hereditary property. On the positive side, the ISP-Π problem always admits an FPT algorithm, when parameterised by the size of the solution, if Π is a hereditary property [8, 25]. This is an important result, as it allows us to conceive efficient algorithms to solve computationally hard problems, as long as we restrict ourselves to graphs verifying such properties.

It is also worth mentioning the work in [17], which provides a framework that yields exact algorithms that are significantly faster than brute-force to solve a more general version of the ISP-Π problem: given a universe, find a subset of maximum cardinality which verifies some hereditary property. On a high level, the algorithm proposed in [17] builds the solution which is a subset H of maximum cardinality with the wanted property, by continuously extending a partial solution $X \subseteq H$. Note that this approach only works if Π is indeed a hereditary property. More recently, this approach was generalised by the authors of [14], who provide a framework that yields exponential-time approximation algorithms.

However, not all interesting properties are hereditary. E.g., "all vertices of the induced subgraph have odd degree", and "the induced subgraph is d-regular", where d is an integer given in the input (recall that a graph is *d-regular* if all of its vertices have the same degree d), are non-hereditary properties. The authors of [5] studied the ISP-Π problem for the former property, showing that it is an $\mathcal{NP}$-hard problem, and providing an FPT algorithm that solves it when parameterised by the rank-width. Also, the authors of [1, 3, 31] studied the ISP-Π problem for the latter property. It is shown in [3] that finding an induced subgraph of maximum order that is d-regular is $\mathcal{NP}$-hard to approximate, even on bipartite or planar graphs. The authors of [3] also provide a linear-time algorithm to solve this problem for graphs with bounded treewidth. Lastly, it is also worth mentioning [7], where the authors consider the non-hereditary property "the induced subgraph is k-anonymous", where a graph G is k-anonymous if for each vertex of G there are at least $k-1$ other vertices of the same degree.

An important observation is that, in the case of non-hereditary properties, the ISP-Π problem does not necessarily admit an FPT algorithm parameterised by the size of the solution. Indeed, the authors of [31] proved that when considering Π as "the induced subgraph is regular", the ISP-Π problem is W[1]-hard when parameterised by the size of the solution. This indicates the importance of considering graph-structural parameters for conceiving efficient algorithms for such problems. This is exactly the approach followed in [18, 27], where the authors consider a generalisation of Vertex Cover, the ISP-Π problem where Π is "the graph has maximum degree k", for an integer k given in the input.

Distance from local irregularity. In some sense, the property that interests us lies on the opposite side of the one studied in [1, 3, 31]. Recall that a graph G is locally irregular if no two of its adjacent vertices have the same degrees. This notion was formally introduced in [4], where the authors take some steps towards proving the so-called 1-2-3 Conjecture proposed in [23] and recently proven in [24]. Roughly, this conjecture is about functions assigning weights from $[k] = \{1, \dots, k\}$ to the edges of a graph, called proper k-labellings, so that all adjacent vertices have different weighted degrees; the conjecture states that for any non-trivial graph, this is always achievable for $k \leq 3$.

The authors of [16] introduced the problem of finding a locally irregular induced subgraph of a given graph G of maximum order (a non-hereditary property). Equivalently, find a set of *vertices* of minimum cardinality whose deletion renders the given graph locally irregular; such sets are named *optimal vertex-irregulators*. The main focus of [16] was to study the complexity of computing optimal vertex-irregulators. It was shown that this problem is $\mathcal{NP}$-hard even for subcubic planar bipartite graphs, W[2]-hard parameterised by the size of the solution and W[1]-hard parameterised by the treewidth of the input graph. Moreover, for any constant $\varepsilon < 1$, there cannot be a polynomial-time $\mathcal{O}(n^{1-\varepsilon})$-approximation algorithm (unless $\mathcal{P}$=$\mathcal{NP}$). On the positive side, there are two FPT algorithms that solve this problem, parameterised by the maximum degree of the input graph plus either the size of the solution or the treewidth of the input graph. Note that the notion of vertex-irregulators proved to be fruitful in the context of proper labellings. Indeed, the authors of [6] observed a connection between finding large locally irregular induced subgraphs and constructing proper k-labellings that also maximise the use of weight 1 on the edges of the given graph.

Apart from improving the results of [16], in this paper we also introduce the novel problem of computing a subset of a graph's *edges*, of minimum order, whose deletion renders the graph locally irregular; such sets are named *optimal edge-irregulators*. This problem is introduced as a first step towards understanding the problem of finding large locally irregular (not necessarily induced) subgraphs of a given graph. Problems concerned with finding maximum subgraphs verifying a specific property have also been extensively studied (*e.g.*, [9, 10, 2]). One might expect that finding edge-irregulators could be easier than finding vertex-irregulators as it is often the case with graph theoretical problems concerned with subsets of edges, whose versions considering subsets of vertices are intractable (recall, *e.g.*, Edge Cover, Feedback Edge Set and even Min Weighted Lower-Upper-Cover [33]). As it turns out, however, finding small edge-irregulators is also a computationally hard problem.

Our contribution. In this paper we study the complexity of computing optimal vertex and edge-irregulators. We identify the parameters for which the tractability of the former problem changes, considering a multitude of standard graph-structural parameters. We also take steps towards the same goal for the latter problem. In Section 2 we introduce the needed notation and provide some first results. In particular, we observe that computing optimal vertex-irregulators is W[1]-hard when parameterised by the treedepth or the feedback vertex set of the given graph. Section 3 provides FPT algorithms for the problem of finding optimal vertex-irregulators. The considered parameters are the neighborhood diversity, the vertex integrity, or the clusted deletion number of the input graph. In Section 4, we focus on the problem of finding optimal edge-irregulators. First, we prove that this problem is $\mathcal{NP}$-hard, even when restricted to planar bipartite graphs of maximum degree 6. We also show that the problem is W[1]-hard parameterised by the size of the solution or the feedback vertex set of the input graph. Lastly, we modify the FPT algorithm for computing an optimal vertex-irregulator parameterised by the vertex integrity in order to provide an FPT algorithm that solves the edge version of the problem (once more parameterised by the vertex integrity). We close the paper in Section 5, where we propose some directions for further research.

2 Preliminaries

We follow standard graph theory notations [12].

Let $G = (V, E)$ be a graph and $G' = (V', E')$ be a subgraph of G (*i.e.*, created by deleting vertices and/or edges of G). Recall first that the subgraph G' is *induced* if it can be created only by deleting vertices of G; in this case we denote G' by $G[V']$. That is, for each edge $uv \in E$, if $u, v \in V'$, then $uv \in E'$. For any vertex $v \in V$, let $N_G(v) = \{u \in V : uv \in E\}$ denote the *neighbourhood* of v in G and $d_G(v) = |N_G(v)|$ denote the *degree* of v in G. Note that, whenever the graph G is clear from the context, we will omit the subscript and simply write $N(v)$ and $d(v)$. Also, for $S \subseteq E$, we denote by $G - S$ the graph $G' = (V, E \setminus S)$. That is, G' is the graph resulting from the deletion of the edges of S from the graph G.

Let $G = (V, E)$ be a graph. We say that G is *locally irregular* if, for every edge $uv \in E$, we have $d(u) \neq d(v)$. Now, let $S \subseteq V$ be such that $G[V \setminus S]$ is a locally irregular graph; any set S that has this property is denoted as a *vertex-irregulator of* G. Moreover, let $\mathrm{I}_v(G)$ be the minimum order that any vertex-irregulator of G can have. We will say that S is an *optimal* vertex-irregulator of G if S is a vertex-irregulator of G and $|S| = \mathrm{I}_v(G)$. Similarly, we define an *edge-irregulator of* G to be any set $S \subseteq E$ such that $G - S$ is locally irregular. Moreover, let $\mathrm{I}_e(G)$ be the minimum order that any edge-irregulator of G can have. We will say that S is an *optimal* edge-irregulator of G if S is an edge-irregulator of G and $|S| = \mathrm{I}_e(G)$.

We begin with some simple observations that hold for any graph $G = (V, E)$.

▶ **Observation 1.** *If G contains two vertices u, v such that $uv \in E$ and $d(u) = d(v)$, then any edge-irregulator of G contains at least one edge incident to u or v. Also, any vertex-irregulator of G contains at least one vertex in $N(u) \cup N(v)$.*

▶ **Observation 2.** *If G contains two vertices $u, v \in V$ that are twins,* i.e., *$N(u) \setminus \{v\} = N(v) \setminus \{u\}$, such that $uv \in E$, then any vertex-irregulator of G contains at least one vertex in $\{u, v\}$.*

The importance of upcoming Lemma 3 lies in the fact that we can repeatedly apply it, reducing the size of the graph on which we are searching for a vertex-irregulator. This is a core argument behind the algorithms of Theorems 7 and 11.

▶ **Lemma 3.** *Let $G = (V, E)$ be a graph and $u, v \in V$ be a pair of adjacent twins. Let $G' = (V', E')$ be the graph resulting from the deletion of either u or v from G. Then, $\mathrm{I}_v(G) = \mathrm{I}_v(G') + 1$.*

Proof. Assume w.l.o.g. that $u \notin V'$. We first prove that $\mathrm{I}_v(G) \leq \mathrm{I}_v(G') + 1$. Indeed, assume that $\mathrm{I}_v(G) > \mathrm{I}_v(G') + 1$ and let S' be an optimal vertex-irregulator of G'. Next, consider the graph $\tilde{G} = G[V \setminus (S' \cup \{u\})]$. From the construction of G', it follows that $\tilde{G} = G'[V' \setminus S']$. Since S' is a vertex-irregulator of G', we obtain that $\tilde{G}$ is locally irregular. In other words, the set $S' \cup \{u\}$ is a vertex-irregulator of G and $|S' \cup \{u\}| = I_v(G') + 1$, a contradiction.

Next, assume that $\mathrm{I}_v(G) < \mathrm{I}_v(G') + 1$ and let S be an optimal vertex-irregulator of G. It follows from Observation 2 that $|\{u, v\} \cap S| \geq 1$. Assume w.l.o.g. that $u \in S$. Thus, and by the construction of G', we have that $G'[V' \setminus (S \setminus \{u\})] = G[V \setminus S]$ and the set $S \setminus \{u\}$ is a vertex-irregulator of G'. In other words, $\mathrm{I}_v(G') \leq |S| - 1 = \mathrm{I}_v(G) - 1$, a contradiction. ◀

We close this section with some observations on the proof that computing $\mathrm{I}_v(G)$ is W[1]-hard parameterised by the treewidth of G, initially presented in [16], which allows us to show that this result holds even if we consider more "restricted" parameters, such as the treedepth or the feedback vertex set number (*i.e.*, size of a minimum feedback vertex set) of the input graph. Recall that the *treedepth* of a graph $G = (V, E)$ can be defined recursively: if $|V| = 1$, then G has treedepth 1. Then, G has treedepth k if there exists a vertex $v \in V$ such that every connected component of $G[V \setminus \{v\}]$ has treedepth at most $k - 1$. Given a graph G and a tree T rooted at a vertex u, by *attaching* T on a vertex v of G, we mean the operation of adding T to G and identifying u with v, *i.e.*, $V(T) \cap V(G) = \{u\} = \{v\}$.

▶ **Observation 4.** *Let G be a graph with vertex cover number (*i.e.*, size of a minimum vertex cover) k_1 and T be a rooted tree of depth k_2. Let G' be the graph after attaching an arbitrary number of copies of T directly on vertices of G. Then G' has treedepth $\mathcal{O}(k_1 + k_2)$ and feedback vertex set number $\mathcal{O}(k_1)$.*

The reduction presented in [16, Theorem 16] starts with a graph G which is part of an instance of the List Colouring problem, and constructs a graph G' by attaching some trees of depth at most 3 on each vertex of G. The List Colouring problem was shown to be W[1]-hard in [15] when parameterised by the vertex cover number of the input graph. Thus, by Observation 4, we obtain the following:

▶ **Corollary 5.** *Given a graph G, it is* W[1]*-hard to compute $\mathrm{I}_v(G)$ parameterised by either the treedepth or the feedback vertex set number of G.*

3 FPT algorithms for vertex-irregulators

In this section we present two FPT algorithms that compute an optimal vertex-irregulator of a given graph G, when parameterised by the neighbourhood diversity or the vertex integrity of G. The latter algorithm is then used to show that this problem is in FPT also when parameterised by the cluster deletion number of G. We begin by recalling the needed definitions.

The *twin equivalence* of G is the relation on the vertices of V where two vertices belong to the same equivalence class if and only if they are twins.

▶ **Definition 6** ([26]). *A graph G has* neighbourhood diversity *k (denoted as $nd(G) = k$) if its twin equivalence has k classes.*

Let $G = (V, E)$ be a graph with $nd(G) = k$ and let $V_1, \dots, V_k$ be the partition of V defined by the twin equivalence of G. Observe that for any $i \in [k]$, we have that $G[V_i]$ is either an independent set or a clique.

▶ **Theorem 7.** *Given a graph $G = (V, E)$ such that $nd(G) = k$, there exists an algorithm that computes $\mathrm{I}_v(G)$ in FPT-time parameterised by k.*

Proof. Let $V_1, \dots, V_k$ be the partition of V defined by the twin equivalence of G. Note that this partition can be computed in linear time [26]. We begin by constructing an induced subgraph $G' = (V', E')$ of G by applying the following procedure: for each $i \in [k]$, if $G[V_i]$ is a clique on at least two vertices, delete all the vertices of V_i except one; let D be the set of vertices that were deleted and $d = |D|$. This procedure terminates after k iterations and, thus, runs in polynomial time. Moreover, it follows from Lemma 3 that $\mathrm{I}_v(G) = \mathrm{I}_v(G') + d$. Thus, it suffices to solve the problem on G'. For every $i \in [k]$, let $V'_i = V_i \cap V'$.

Observe that for every locally irregular graph H, there exists a prime number p such that $d_H(u) - d_H(v) \neq 0 \bmod p$ for every $uv \in E(H)$. In our case, since for every $uv \in E'$ we have that $u \in V'_i$ and $v \in V'_j$ for $i < j \leq [k]$, it follows that there can be at most $\binom{k}{2}$ possible differences modulo p between the degrees of adjacent vertices in G^*, where $G^* = (V^*, E^*)$ is a locally irregular induced subgraph of G'.

We claim that $p \leq (k^2 \log n + 1)(\log(k^2 \log n + 1) + \log\log(k^2 \log n + 1) - \frac{1}{2})$. Indeed, since each one of the differences we considered in the previous paragraph is at most n, each one of them has at most $\log n$ prime divisors. Thus, p is at most the $k^2 \log n + 1^{th}$ prime number. This, in conjunction with the classical results from [32] gives us the claimed inequality.

For every $i \in [k]$, let $V^*_i = V'_i \cap V^*$. For every such prime p, we consider all the possible cases for $p_i = |V^*_i| \bmod p$, for every $i \in [k]$; there are at most p^k such instances. Let us consider any such instance such that $d_{G^*}(u) - d_{G^*}(v) \neq 0 \bmod p$ for every $uv \in E^*$. Checking this inequality is straightforward from the p_is. We store the maximum orders of the V^*_is such that $|V^*_i| \bmod p = p_i$ for every $i \in [k]$. Having repeated this procedure for all such instances, we are certain to have computed a locally irregular induced subgraph of G' of maximum order. In total, this procedure takes time $p^{k+1} n^{\mathcal{O}(1)}$ which is in FPT due to the upper bound on p by [32] and since $\log^k n \leq f(k)n$, for some computable function f [22]. ◀

We now present an FPT algorithm to compute an optimal vertex-irregulator of an input graph G when parameterised by the vertex integrity of G, which can be computed in FPT-time [13].

▶ **Definition 8.** *A graph $G = (V, E)$ has* vertex integrity *k if there exists a set $U \subseteq V$ such that $|U| = k' \leq k$ and all connected components of $G[V \setminus U]$ are of order at most $k - k'$.*

▶ **Theorem 9.** *Given a graph $G = (V, E)$ with vertex integrity k, there exists an algorithm that computes $\mathrm{I}_v(G)$ in FPT-time parameterised by k.*

Proof. Let $U \subseteq V$ be such that $|U| = k' \leq k$ and $C_1, \ldots, C_m$ be the vertex sets of the connected components of $G[V \setminus U]$ such that $|C_j| \leq k - k'$, $j \in [m]$. Assume that we know the intersection of an optimal vertex-irregulator S of G and the set U, and let $S' = S \cap U$ and $U' = U \setminus S$ (there are at most $2^{|U|} \leq 2^k$ possible intersections S' of U and S). Notice that the graph $G[V \setminus S']$ has an optimal vertex-irregulator that contains only vertices from $\bigcup_{i\in[m]} C_i$. Indeed, assuming otherwise contradicts that S' is the intersection of an optimal vertex-irregulator and U. Thus, in order to find an optimal vertex-irregulator S of G, it suffices to compute $S^* \subseteq \bigcup_{i\in[m]} C_i$, which is an optimal vertex-irregulator of $G[V \setminus S']$, for every set $S' \subseteq U$. Then, we return the set $S^* \cup S'$ of minimum order. We compute S^* through an ILP with bounded number of variables. To do so, we define types and sub-types of graphs $G[U' \cup C_j]$, $j \in [m]$.

Informally, the main idea is to categorise the graphs $G[U' \cup C_j]$, $j \in [m]$, into types based on their structure (formally defined later), whose number is bounded by some function of k. Each type i is associated with a number no_i that represents the number of the subgraphs $G[U' \cup C_j]$, $j \in [m]$, that belong in that type. Then, for each type i, we will define sub-types based on the induced subgraphs $G[(U' \cup C_j) \setminus S_q]$, for $S_q \subseteq C_j$. We also define a variable $no_{i,q}$ that is the number of the subgraphs $G[U' \cup C_j]$, $j \in [m]$, that are of type i and of sub-type q in $G[V \setminus S]$. Note that knowing the structure of these types and sub-types, together with $no_{i,q}$, is enough to compute the order of S^*. Finally, for any $j \in [m]$, the graph $G[U' \cup C_j]$ is of order at most k. Thus, the number of types, sub-types and their corresponding variables, is bounded by a function of k. We will present an ILP formulation whose objective is to minimise the order of S^*.

We begin by defining the types. Two graphs $G[U' \cup C_i]$ and $G[U' \cup C_j]$, $i, j \in [m]$, are of the same type if there exists a bijection[1] $f : C_i \cup U' \to C_j \cup U'$ such that $f(u) = u$ for all $u \in U'$ and $N_{G[U' \cup C_i]}(u) = \{f^{-1}(v) \mid v \in N_{G[U' \cup C_j]}(f(u))\}$ for all $u \in C_i$. Note that if such a function exists, then $G[U' \cup C_i]$ is isomorphic to $G[U' \cup C_j]$.

Let p be the number of different types. Notice that p is bounded by a function of k as any graph $G[U' \cup C_i]$ has order at most k. Also, we can decide if two graphs $G[U' \cup C_i]$ and $G[U' \cup C_j]$, $i, j \in [m]$, are of the same type in FPT-time parameterised by k. For each type $i \in [p]$, set no_i to be the number of graphs $G[U' \cup C_j]$, $j \in [m]$, of type i. Furthermore, for each type $i \in [p]$ we select a C_j, $j \in [m]$, such that $G[U' \cup C_j]$ is of type i, to represent that type; we will denote this set of vertices by R_i.

We are now ready to define the sub-types. Let $i \in [p]$ be a type represented by R_i and $S^i_1, \ldots, S^i_{2^{|R_i|}}$ be an enumeration of the subsets of R_i. For any $q \in [2^{|R_i|}]$, we define a sub-type (i, q) which represents the induced subgraph $G[(U' \cup R_i) \setminus S^i_q]$. Let $no_{i,q}$ be the variable corresponding to the number of graphs represented by $G[U' \cup R_i]$, $i \in [p]$, that is of type (i, q) in $G[V \setminus S^*]$, for a vertex-irregulator S^* such that $S^* \cap R_i = S^i_q$.

Notice that, given a vertex-irregulator $S^* \subseteq \bigcup_{j\in[m]} C_j$ of $G[V \setminus S']$, there exists a sub-type (i, q), $i \in [p]$, $q \in [2^{|R_i|}]$, for each $j \in [m]$, such that the graph $G[(U' \cup C_j) \setminus S^*]$ is of sub-type (i, q). Also, assuming that we know the order of $|S^i_q|$ and the number $no_{i,q}$ for all $i \in [p]$, $q \in [2^{|R_i|}]$, then $|S^*| = \sum_{i\in[p]} \sum_{q\in[2^{|R_i|}]} no_{i,q}|S^i_q|$.

[1] Recall that a function $f : A \to B$ is a *bijection* if, for every $a_1, a_2 \in A$ with $a_1 \neq a_2$, we have that $f(a_1) \neq f(a_2)$ and for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Recall also that the *inverse* function of f, denoted as f^{-1}, exists if and only if f is a bijection, and is such that $f^{-1} : B \to A$ and for each $b \in B$ we have that $f^{-1}(b) = a$, where $f(a) = b$.

Before giving the ILP formulation whose goal is to find a vertex-irregulator S^* while minimising the above sum, we guess the (i,q) such that $no_{i,q} \neq 0$. Let S_2 be the set of pairs (i,q), $i \in [p]$ and $q \in [2^{|R_i|}]$, such that there are two vertices $u, v \in R_i \setminus S_q^i$ where $uv \in E(G[(U' \cup C_i) \setminus S_q^i])$ and $d_{G[(U' \cup R_i) \setminus S_q^i]}(u) = d_{G[(U' \cup R_i) \setminus S_q^i]}(v)$. For every $(i,q) \in S_2$, we have that $no_{i,q} = 0$. Indeed, assuming otherwise contradicts the fact that S^* is a vertex-irregulator. We guess $S_1 \subseteq \{(i,q) \mid i \in [p], q \in 2^{|R_i|}\} \setminus S_2$ such that $no_{i,q} \neq 0$ for all $(i,q) \in S_1$. Observe that the number of different sets that are candidates for S_1 is bounded by some function of k.

Constants

no_i	$i \in [p]$	number of components of type i
$e_{uv} \in \{0,1\}$	$u, v \in U'$	set to 1 iff $uv \in E(G[U'])$
$e_{u,v}^{i,q} \in \{0,1\}$	$i \in [p], q \in [2^{\lvert R_i \rvert}]$, $u \in U'$ and $v \in R_i \setminus S_q^i$	set to 1 iff $uv \in E(G[(U' \cup R_i) \setminus S_q^i])$
$b_u^{i,q} \in [n]$	$i \in [p], q \in [2^{\lvert R_i \rvert}]$ and $u \in U'$	set to $d_{G[(\{u\} \cup R_i) \setminus S_q^i]}(u)$
$d_u^{i,q} \in [n]$	$i \in [p], q \in [2^{\lvert R_i \rvert}]$ and $u \in R_i \setminus S_q^i$	set to $d_{G[(U' \cup R_i) \setminus S_q^i]}(u)$

Variables $no_{i,q}$ $\quad i \in [p], q \in [2^{|R_i|}]$ $\quad$ number of graphs of types (i,q)

Objective

$$\min \sum\nolimits_{i \in [p]} \sum\nolimits_{q \in [2^{|R_i|}]} no_{i,q} |S_q^i| \tag{3.1}$$

Constraints

$$no_{i,q} = 0 \qquad \text{iff } (i,q) \notin S_1 \tag{3.2}$$

$$\sum_{q \in [2^{|R_i|}]} no_{i,q} = no_i \qquad \forall i \in [p] \tag{3.3}$$

$$\sum_{w \in U'} e_{wv} + \sum_{i \in [p]} no_{i,q} b_v^{i,q} \neq \sum_{w \in U'} e_{wu} + \sum_{i \in [p]} no_{i,q} b_u^{i,q} \qquad \forall u, v \in U', e_{uv} = 1 \tag{3.4}$$

$$d_v^{i,q} \neq \sum_{w \in U'} e_{wu} + \sum_{i \in [p]} no_{i,q} b_u^{i,q} \qquad \forall e_{u,v}^{i,q} = 1 \text{ and } (i,q) \in S_1 \tag{3.5}$$

Assume that we have found the values $no_{i,q}$ for (i,q), $i \in [p]$, $q \in [2^{|R_i|}]$. We construct an optimal vertex-irregulator of $G[V \setminus S']$ as follows. Start with an empty set S^*. For each $i \in [p]$ take all components C_j of type i. Partition them into $2^{|R_i|}$ sets $\mathcal{C}_q^i$ such that any set $q \in [2^{|R_i|}]$ contains exactly $no_{i,q}$ of these components. For any component $C \in \mathcal{C}_q^i$, select all vertices represented by the set S_q^i (as it was defined before) and add them to S^*. The final S^* is an optimal vertex-irregulator for $G[V \setminus S']$.

Let $S = S' \cup S^*$. We show that S is a vertex-irregulator of G. To do so, it suffices to verify that in the graph $G[V \setminus S]$ there are no two adjacent vertices with the same degree. Let u, v be a pair of adjacent vertices in a component represented by $R_i \setminus S$, which is of type (i,q). If $d_{G[V \setminus S]}(u) = d_{G[V \setminus S]}(v)$, then $(i,q) \in S_2$. Therefore, $no_{i,q} = 0$ and we do not have such a component in $G[V \setminus S]$. Thus, it suffices to focus on adjacent vertices such that at least one of them is in U'. Notice that, in $G[V \setminus S]$, the degree of vertex $u \in U'$ is equal to $\sum_{w \in U'} e_{wv} + \sum_{i \in [p]} no_{i,q} b_v^{i,q}$. In other words, no two adjacent vertices in U' have the same degree due to the constraint 3.4. Lastly, the constraint 3.5 guarantees that no vertex in U' is adjacent to a vertex in $C_i \setminus S$ (for some $i \in [p]$) such that both of them have the same degree in $G[V \setminus S]$. Moreover, both S' and S^* are constructed to be minimum such sets. Thus, S is an optimal vertex-irregulator of G. Finally, since the number of variables in the model is bounded by a function of k, we can obtain S^* in FPT time parameterised by k (by running for example the Lenstra algorithm [28]). ◀

The previous algorithm can be used to find an optimal vertex-irregulator of a graph G in FPT-time when parameterised by the cluster deletion number of G. Note that the cluster deletion number of a graph can be computed in FPT-time parameterised by k [21].

▶ **Definition 10** ([21]). *A graph $G = (V, E)$ has* cluster deletion number k *if there exists a set $S \subseteq V$ such that all the connected components of $G[V \setminus S]$ are cliques, and S is of order at most k.*

▶ **Theorem 11.** *Given a graph $G = (V, E)$ with cluster deletion number k, there exists an algorithm that computes $\mathrm{I}_v(G)$ in FPT-time parameterised by k.*

Proof. Let S be such that $|S| = k$ and $G[V \setminus S]$ is a disjoint union of cliques $C_1, \dots C_m$ for $m \geq 1$. Our goal is to reduce the size of these cliques so that each one of them has order at most 2^k. We achieve this through the the following procedure. Let $i \in [m]$ be such that the clique $C_i = (V_{C_i}, E_{C_i})$ has $|V_{C_i}| > 2^k$. Let $V_1, \dots, V_p$ be the partition of V_{C_i} defined by the twin equivalence of $G[V_{C_i} \cup S]$. That is, two vertices $u, v \in V_{C_i}$ belong in a V_j, $j \in [p]$, if and only if u and v are twins. Note that $p \leq 2^k$. Observe that, since C_i is a clique, the graphs $C_i[V_j]$, $j \in [p]$, are also cliques. In other words, for each $j \in [p]$, all the vertices of V_j are adjacent twins. We delete all but one vertex of V_j, for each $j \in [p]$, and repeat this process for every $i \in [m]$ such that $|V_{C_i}| > 2^k$. Let $G' = (V', E')$ be the resulting subgraph of G and $d = |D|$, where D is the set of vertices that were removed throughout this process. It follows from Lemma 3 that $\mathrm{I}_v(G) = \mathrm{I}_v(G') + d$. Observe also that $S \subseteq V'$ and that each connected component of $G'[V' \setminus S]$ is a clique of at most 2^k vertices. In other words, G' has vertex integrity at most $2^k + k$. To sum up, to compute $\mathrm{I}_v(G)$ it suffices to compute $\mathrm{I}_v(G')$, which can be done in FPT-time by running the algorithm presented in Theorem 9. ◀

4 Edge-irregulators

In this section we begin the study of finding an optimal edge-irregulator of a given graph. It turns out that the decision version of this problem is $\mathcal{NP}$-complete, even for quite restrictive classes of graphs (see Theorem 12). Furthermore, it is also W[1]-hard parameterised by the size of the solution.

▶ **Theorem 12** ($\star$). *Let G be a graph and $k \in \mathbb{N}$. Deciding if $\mathrm{I}_e(G) \leq k$ is $\mathcal{NP}$-complete, even if G is a planar bipartite graph of maximum degree 6.*

Sketch of Proof. The problem is clearly in $\mathcal{NP}$. We focus on showing it is also $\mathcal{NP}$-hard. This is achieved through a reduction from the PLANAR 3-SAT problem which is known to be $\mathcal{NP}$-complete [30]. In that problem, a 3CNF formula ϕ is given as an input. We say that a bipartite graph $G' = (V, C, E)$ *corresponds* to ϕ if it is constructed from ϕ in the following way: for each literal x_i (resp. $\neg x_i$) that appears in ϕ, add the *literal vertex* v_i (resp. v'_i) in V (for $1 \leq i \leq n$) and for each clause C_j of ϕ add a *clause vertex* c_j in C (for $1 \leq j \leq m$).

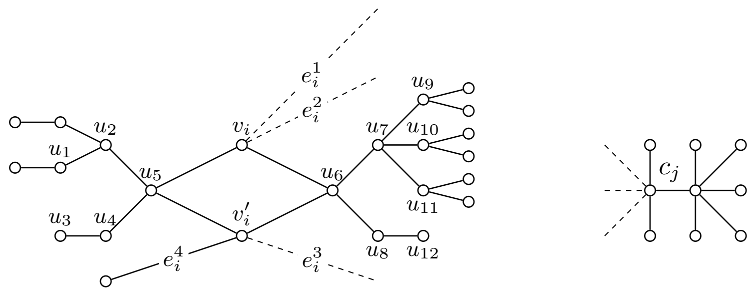

Figure 2 The construction in the proof of Theorem 12. The dashed lines are used to represent the edges between the literal and the clause vertices.

Then the edge v_ic_j (resp. $v_i'c_j$) is added if the literal x_i (resp. $\neg x_i$) appears in the clause C_j. Finally, we add the edge v_iv_i' for every i. A 3CNF formula ϕ is valid as input to the Planar 3-SAT problem if the graph G' that corresponds to ϕ is planar. Furthermore, we may assume that each variable appears in ϕ twice as a positive and once as a negative literal [11]. The question is whether there exists a truth assignment to the variables of X satisfying ϕ. Starting from a 3CNF formula ϕ, we construct a graph G such that $\mathrm{I}_e(G) \leq 3n$ if and only if ϕ is satisfiable.

Construction. We start with the graph G' that corresponds to the formula ϕ. Then, for each $1 \leq i \leq n$, we remove the edge v_iv_i', and attach the gadget illustrated in Figure 2 to v_i and v_i'. Let E_i denote the edges of the gadget attached to v_i and v_i' plus the edges e_i^1, e_i^2 and e_i^3. Finally, for each $1 \leq j \leq m$, we attach two leaves to c_j and then we add the star with 7 vertices and identify one of its leaves as the vertex c_j. Note that the edges e_i^1, e_i^2 correspond to the two positive appearances of the literal x_i, while the edge e_i^3 corresponds to the one negative appearance of the same literal. We stress that the edge e_i^4 is a "simple" edge leading to a vertex of degree 1, and does no correspond to any appearance (positive or negative) of x_i. Observe that the resulting graph G is planar, bipartite and $\Delta(G) = 6$. This finishes the construction.

The rest of the reduction is based on some useful observations about the constructed graph. Indeed, we prove that for every edge-irregulator S of G, for every $1 \leq i \leq n$, we have that $|S \cap E_i| \geq 3$. Also, if S is such that $|S| \leq 3n$, then, for every $1 \leq i \leq n$, we have that if $|S \cap \{e_i^1, e_i^2\}| \geq 1$ then $|S \cap \{e_i^3, e_i^4\}| = 0$ and if $|S \cap \{e_i^3, e_i^4\}| \geq 1$ then $|S \cap \{e_i^1, e_i^2\}| = 0$. Finally, we show that there exists a satisfying truth assignment of ϕ if and only if $\mathrm{I}_e(G) \leq 3n$, where G is the graph constructed from the input formula ϕ as explained above. Starting from a satisfying truth assignment of ϕ, we insert $\{v_i'u_6, e_i^1, e_i^2\}$ ($\{v_iu_6, e_i^3, e_i^4\}$ resp.) into the constructed edge-irregulator S if x_i is set to true (false resp.). For the reverse direction, we set the variable x_i to true if and only if $S \cap \{e_i^1, e_i^2\} \neq \emptyset$. ◀

▶ **Theorem 13** ($\star$). *Let G be a graph and $k \in \mathbb{N}$. Deciding if $\mathrm{I}_e(G) \leq k$ is* W[1]*-hard parameterised by k.*

The proof of this theorem is based on a reduction from k-Multicoloured Clique.

Additionally, this problem exhibits a similar behaviour to finding optimal vertex-irregulators, as it also remains intractable even for "relatively large" structural parameters.

▶ **Theorem 14.** *Let G and $k \in \mathbb{N}$. Deciding if $\mathrm{I}_e(G) \leq k$ is* W[1]*-hard parameterised by either the feedback vertex set number or the treedepth of G.*

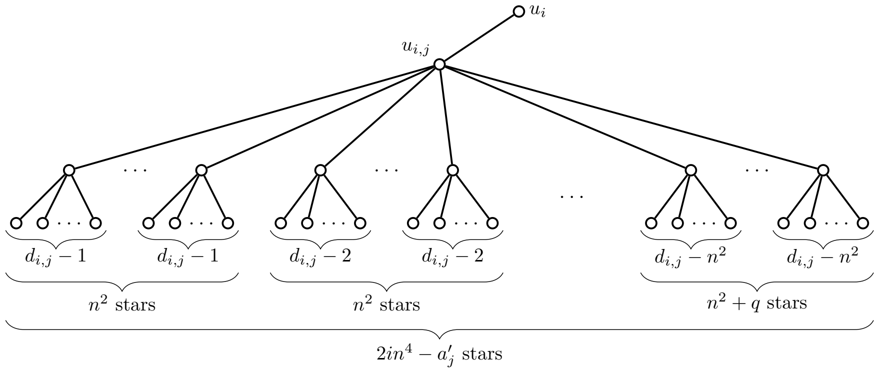

Figure 3 The tree $T_{i,j}$ that is attached to the vertex u_i, where j is such that $a_j \in \overline{L}(u_i)$, in the proof of Theorem 14. The value of q is such that, in total, $d(u_{i,j}) = 2in^4 - a'_j + 1$.

Proof. The reduction is from the General Factor problem:

> General Factor
> **Input:** A graph $H = (V, E)$ and a list function $L : V \to 2^{\Delta(H)}$ that specifies the available degrees for each vertex $u \in V$.
> **Task:** Does there exist a set $S \subseteq E$ such that $d_{H-S}(u) \in L(u)$ for all $u \in V$?

This problem is known to be W[1]-hard when parameterised by the vertex cover number of H [20].

Starting from an instance (H, L) of General Factor, we construct a graph G such that $\mathrm{I}_e(G) \le n^2$, where $n = |V(H)|$, if and only if (H, L) is a yes-instance. Moreover, the constructed graph G will have treedepth and feedback vertex set $\mathcal{O}(vc)$, where vc is the vertex cover number of H. For every vertex $u \in V(H)$, let us denote by $\overline{L}(u)$ the set $\{0, 1, \dots, d_H(u)\} \setminus L(u)$. In the case where $\{0, 1, \dots, d_H(u)\} \setminus L(u) = \emptyset$, we set $\overline{L}(u) = \{-1\}$. On a high level, the graph G is constructed by adding some trees on the vertices of H. In particular, for each vertex $u \in V(H)$ and for each element a in $\overline{L}(u)$, we will attach a tree to u whose purpose is to prevent u from having degree a in $G - S$, for any optimal edge-irregulator S of G. We proceed with the formal proof.

Construction. We begin by defining an arbitrary order on the vertices of H. That is, $V(H) = \{u_1, u_2, \dots, u_n\}$. Next, we describe the trees we will use in the construction of G. In particular, we will describe the trees that we attach to the vertex u_i, for every $1 \le i \le n$. First, for each $a_j \in \overline{L}(u_i)$, define the value $a'_j = d_H(u_i) - a_j$. Also, for each j, let $d_{i,j} = 2in^4 - a'_j$. For each "forbidden degree" a_j in the list $\overline{L}(u_i)$, we will attach a tree $T_{i,j}$ to u_i. We define the tree $T_{i,j}$ as follows.

First, for every $0 \le k \le n^2 - 1$, create n^2 copies of $S_{d_{i,j}-k}$ (the star on $d_{i,j} - k$ vertices) and q additional copies of $S_{d_{i,j}-n^2+1}$ (the exact value of q will be defined in what follows). Then, choose one leaf from each one of the above stars, and identify them into a single vertex denoted as $u_{i,j}$; the value of q is such that $d(u_{ij}) = d_{i,j} - 1 = 2in^4 - a'_j - 1$. Let $T_{i,j}$ be the resulting tree and let us say that $u_{i,j}$ is the root of $T_{i,j}$ (see Figure 3).

Let us now describe the construction of G. For each vertex $u_i \in V(H)$ and for each $a_j \in \overline{L}(u_i)$, add the tree $T_{i,j}$ to H and the edge $u_{i,j}u_i$. Then, for each vertex $u_i \in V(H)$, for any j such that $u_{i,j}$ is a neighbour of u_i, add p_i additional copies of the tree $T_{i,j}$, as well as the edges between u_i and the roots of the additional trees, so that $d_G(u_i) = 2in^4$. The resulting graph is G. Note that, for each vertex of $V(H)$, we are adding at most $\mathcal{O}(n)$ trees, each one containing at most $\mathcal{O}(n^{10})$ vertices. Thus, the construction of G is achieved in polynomial time.

Reduction. Assume first that (H, L) is a yes-instance of General Factor, and let $S \subseteq E$ be such that $d_{H-S}(u) \in L(u)$ for all $u \in V(H)$. We claim that S is also an edge-irregulator of G. By the construction of G, and since S only contains edges from H, there are no two adjacent vertices in $G[V(G) \setminus V(H)]$ that have the same degree in $G - S$. Thus, it remains to check the pairs of adjacent vertices x, y such that, either both x and y belong to $V(H)$, or, w.l.o.g., $x \in V(H)$ and $y \in V(G - H)$. For the first case, let $x = u_i$ and $y = u_{i'}$, for $1 \leq i < i' \leq n$. Then, assuming that $d_{G-S}(u_i) = d_{G-S}(u_{i'})$, we get that $2in^4 - p = 2i'n^4 - p'$, where S contains $0 \leq p \leq n^2$ and $0 \leq p' \leq n^2$ edges incident to u_i and $u_{i'}$ respectively. Thus, $2n^4(i - i') = p - p'$, a contradiction since $-n^2 \leq p - p' \leq n^2$, $i < i'$ and $-n \leq i - i' \leq n$. For the second case, for every i, let $d_{G-S}(u_i) = 2in^4 - p$, where the set S contains $1 \leq p \leq n^2$ edges of H incident to u_i. Also, by the construction of G and since S only contains edges from H, we have that for every j, $d_{G-S}(u_{i,j}) = d_G(u_{i,j}) = 2in^4 - a'_j$, where, recall, $a'_j = d_H(u_i) - a_j$ for $a_j \in \overline{L}(u_i)$ (see Figure 3). Assume now that there exist i, j such that $d_{G-S}(u_i) = d_{G-S}(u_{i,j})$. Then, $2in^4 - p = 2in^4 - d_H(u_i) + a_j$ and thus $d_H(u_i) - p = a_j$. But then $d_{H-S}(u_i) = a_j$, which is a contradiction since $a_j \in \overline{L}(u_i)$. Thus, S is an edge-irregulator of G and $|S| \leq n^2$ since S only contains edges of $E(H)$.

For the reverse direction, assume that $\mathrm{I}_e(G) \leq n^2$ and let S be an optimal edge-irregulator of G. We will show that S is also such that $d_{H-S}(u_i) \in L(u_i)$, for every i. Let us first prove the following claim.

▷ Claim 15. Let S be an optimal edge-irregulator of G. Then either $S \subseteq E(H)$ or $|S| > n^2$.

Proof. Assume there exist i, j such that $|S \cap E_{i,j}| = x \geq 1$ and $x \leq n^2$. Among those edges, there are $x_1 \geq 0$ edges incident to u and $x_2 \geq 0$ edges incident to children of u (but not to u), with $x_1 + x_2 = x \leq n^2$.

Assume first that $x_1 = 0$. Then $x = x_2$ and there is no edge of $S \cap E_{i,j}$ that is incident to u. Then $d_{G-S}(u) = d_G(u)$ and observe that $d_G(u)$ is strictly larger than that of any of its children (by the construction of G). It follows that $S \setminus E_{i,j}$ is also an edge-irregulator of G, contradicting the optimality of S. Thus $x_1 \geq 1$. It then follows from the construction of G that there exist at least n^2 children of u, denoted by $z_1, \ldots, z_{n^2}$, such that $d_{G-S}(u) = d_G(z_k)$, for every $1 \leq k \leq n^2$. Since $x \leq n^2$, there exists at least one $1 \leq k \leq n^2$ such that $d_{G-S}(u) = d_{G-S}(z_k)$, contradicting the fact that S is an edge-irregulator. Thus $x > n^2$. ◁

It follows directly from Claim 15 that S contains only edges of $E(H)$. Assume that there exist i, j such that $d_{H-S}(u_i) = a_j$ and $a_j \in \overline{L}(u_i)$. Then $d_{G-S}(u_i) = 2in^4 - a'_j$. Also, by the construction of G, u_i is adjacent to a vertex $u_{i,j}$ for which (since S contains only edges of $E(H)$) we have that $d_{G-S}(u_{i,j}) = d_G(u_{i,j}) = 2in^4 - a'_j$. This is contradicting the fact that S is an edge-irregulator of G. Thus, for every i, j, we have that if $d_{H-S}(u_i) = a_j$, then $a_j \in L(u_i)$, which finishes our reduction.

Finally, if H has vertex cover number vc, then, by Observation 4, we have that G has treedepth and feedback vertex set $\mathcal{O}(vc)$. ◀

We close this section by observing that the proof of Theorem 9 can be adapted for the case of edge-irregulators. Indeed, it suffices to replace the guessing of vertices and the variables defined on vertices, by guessing of edges and variables defined on the edges of the given graph. Finally, the definition of the sub-types is done through subgraphs produced only by deletion of edges. This leads us to the following:

▶ **Corollary 16.** *Given a graph G with vertex integrity k, there exists an algorithm that computes $\mathrm{I}_e(G)$ in FPT-time parameterised by k.*

5 Conclusion

In this work we continued the study of the problem of finding optimal vertex-irregulators, and introduced the problem of finding optimal edge-irregulators. In the case of vertex-irregulators, our results are somewhat optimal, in the sense that we almost characterise which are the "smallest" graph-structural parameters that render this problem tractable. The only "meaningful" parameter whose behaviour remains unknown is the modular-width of the input graph. The parameterised behaviour of the case of edge-irregulators is also somewhat understood, but there are still some parameters for which the problem remains open. Another interesting direction is that of approximating optimal vertex or edge-irregulators. In particular it would be interesting to identify parameters for which either problem becomes approximable in FPT-time (recall that vertex-irregulators are not approximable within any decent factor in polynomial time [16]). Finally, provided that the behaviour of edge-irregulators is better understood, we would also like to propose the problem of finding locally irregular minors, of maximum order, of a given graph G.

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