Routing from Pentagon to Octagon Delaunay Graphs

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Abstract

The standard Delaunay triangulation is a geometric graph whose vertices are points in the plane, and two vertices share an edge if they lie on the boundary of an empty disk. If the disk is replaced with a homothet of a fixed convex shape C, then the resulting graph is called a C-Delaunay graph. We study the problem of local routing in C-Delaunay graphs where C is a regular polygon having five to eight sides. In particular, we generalize the routing algorithm of Chew for square-Delaunay graphs (Chew. *SCG 1986*, 169–177) in order to obtain the following approximate upper bounds of 4.640, 6.429, 8.531 and 4.054 on the spanning and routing ratios for pentagon-, hexagon-, septagon-, and octagon-Delaunay graphs, respectively. The exact expression for the upper bounds of the routing ratio is

$$\Psi(n) := \begin{cases} \sqrt{1 + ((\cos(2\pi/n) + n - 1)/\sin(2\pi/n))^2} & \text{if } n \in \{5, 6, 7\}, \\ \sqrt{1 + ((\cos(\pi/8)\cos(3\pi/8) + 3)/(\cos(\pi/8)\sin(3\pi/8)))^2} & \text{if } n = 8. \end{cases}$$

We show that these bounds are tight for the output of our routing algorithm by providing a point set where these bounds are achieved. We also include lower bounds of 1.708 and 1.995 on the spanning and routing ratios of the pentagon-Delaunay graph.

Our upper bounds yield a significant improvement over the previous routing ratio upper bounds for this problem, which previously sat at around 400 for the pentagon, septagon, and octagon as well as 18 for the hexagon. Our routing ratios also provide significant improvements over the previously best known spanning ratios for pentagon-, septagon- and octagon-Delaunay graphs, which were around 45.

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1 Introduction

A geometric graph is a weighted graph whose vertices are points in the plane and edges are line segments weighted with the Euclidean distance between their endpoints. Two of the main distance-preserving properties of a graph are the spanning ratio and routing ratio. The spanning ratio of a pair of points is the ratio of the shortest path between them in the graph divided by their Euclidean distance, and the spanning ratio of a graph is the maximum spanning ratio over all pairs of points [11]. On the other hand, the routing ratio is defined similarly, except that the path is usually computed locally with only information of the current vertex's neighbourhood. Since a routing ratio is based on an algorithm that finds a path and the spanning ratio is based on the existence of a path, the spanning ratio of any graph is a lower bound on the routing ratio.

In this paper, we consider variants of standard Delaunay triangulations, which are geometric graphs with an edge between two points if there exists a disk with the endpoints on its boundary and no vertices in its interior. The spanning ratio of the Delaunay triangulation is known to be between 1.5932 [14] and 1.998 [13], however the exact value still remains unknown. The gap is even larger for the routing ratio, lying somewhere between 1.70 [2] and 3.56 [1]. Many papers study the related graphs that result from replacing the disk with a homothet of a fixed convex shape C, resulting in C-Delaunay graphs. Chew [8] proved that square-Delaunay graphs have a spanning ratio of at most $\sqrt{10}$ by giving a local routing algorithm. Subsequently, Chew [9] adapted his algorithm to equilateral triangle-Delaunay graphs to find a spanning ratio of 2, however the adapted algorithm was no longer a routing algorithm. In fact, Bose et al. [6] showed that the routing ratio of the equilateral triangle-Delaunay graph is exactly $\frac{5}{\sqrt{3}}$, showing the first separation between the spanning ratio and routing ratio. By generalizing Chew's algorithm, Bose et al. [2] were then able to show that the standard Delaunay triangulation has a routing ratio of at most 5.90 which was an improvement on the previously known upper bound of 15.48 [5]. Currently, the best-known bound is 3.56[1]. In this paper, we show that Chew's algorithm can be further generalized to pentagon-, hexagon-, septagon-, and octagon-Delaunay graphs to obtain routing ratios of 4.640, 6.429, 8.531 and 4.054, respectively.

The hexagon-Delaunay graph is known to have a tight spanning ratio of 2 [12], however less is known about Delaunay graphs based on pentagons, septagons and octagons. With the exception of the hexagon-Delaunay graph, our routing ratio upper bounds yield a significant improvement over the previous best spanning ratio upper bounds. Bose et al. [4] give a spanning ratio upper bound for any C-Delaunay graph, where C is any convex shape. In particular, their bound is based intuitively on the thinness of C, which is essentially measured by the ratio of the perimeter to the width of C. For example, this ratio is π when C is a disk. Furthermore, by the construction of the paths from Bose et al. [4] and Perkovic et al. [12], it is possible to route using the algorithm of Bose and Morin [7] with a constant routing ratio of 9 times the spanning ratio. For each polygon, we compare our contribution to the previous best known upper bound in Table 1. We also prove lower bounds of 1.708 and 1.995 on the spanning and routing ratios of the pentagon-Delaunay graph in the appendix.

Table 1 Comparison to previously best-known upper bounds on the spanning and routing ratio of the C-Delaunay graph.

C	Spanning Ratio	Routing Ratio	Our Routing and Spanning Ratio
Triangle	2[9]	$5/\sqrt{3}$[6]	
Square	$\sqrt{4+2\sqrt{2}}$ [3]	$\sqrt{10}$[8]	
Pentagon	≈ 45[4]	≈ 405[7]	≈ 4.640
Hexagon	2[12]	18[7]	≈ 6.429
Septagon	≈ 45[4]	≈ 405[7]	≈ 8.531
Octagon	≈ 43[4]	≈ 387[7]	≈ 4.054
Circle	≈ 1.998[13]	≈ 3.56[1]	

2 Preliminaries

We denote the line segment between points u, v as uv, and the Euclidean length of uv is denoted $|uv|$. For a path $\mathcal{P}$ in the plane, denote $|\mathcal{P}|$ as the length of the path. If paths $\mathcal{P}, \mathcal{Q}$ share an endpoint, then $\mathcal{P} + \mathcal{Q}$ denotes their concatenation. Next, for $a, b, c \in \mathbb{R}^2$, we define

$\angle abc$ as the angle from ab to bc clockwise around b. The x and y coordinates of $a \in \mathbb{R}^2$ are denoted $x(a), y(a)$ respectively. For two vertices u, v in a geometric graph G, the length of the shortest path from u to v in G is denoted $d_G(u, v)$. Then for a constant $c \geq 1$, G is said to be a c-spanner if for all points u, v in G, we have $d_G(u, v) \leq c|uv|$. The spanning ratio of G is the least c for which G is a c-spanner. The spanning ratio of a class of graphs $\mathcal{G}$ is the least c for which all graphs in $\mathcal{G}$ are c-spanners. A *constant* spanner is a c-spanner where c is a constant.

We make the assumption that the graph is embedded on a polynomial-sized grid and therefore specifying the coordinates of a vertex in $V(G)$ requires $O(\log(|V(G)|))$ bits. Formally, a m-memory local routing algorithm is a function that takes as input $(s, N(s), t, M)$, and outputs some memory M' and a vertex $p \in N(s)$ where s is the current vertex, $N(s)$ is the neighbourhood of s, t is the destination, and both M, M' are bit-strings of length m. An algorithm is said to be c-competitive for a family of geometric graphs $\mathcal{G}$ if the path output by the algorithm for any pair of vertices $s, t \in V(G)$ for $G \in \mathcal{G}$ has length at most $c|st|$. The routing ratio of an algorithm is the least c for which the algorithm is c-competitive for $\mathcal{G}$. Note that the routing ratio is an upper bound on the spanning ratio.

For $n \in \{5, 6, 7, 8\}$, let $\bigcirc_n$ denote a regular n-gon in the plane. Every time we mention an n-gon, it is assumed to be a scaled translate of $\bigcirc_n$. Note that rotations are not permitted. We refer to the boundary of any n-gon C as ∂C, and to the interior as $\text{int}(C)$. We make the *general position* assumptions that no two points are on a line parallel to a side of $\bigcirc_n$, that neither coordinate axis is parallel to a side of $\bigcirc_n$, and that no four points lie on ∂C for some n-gon C. For two points $a, b \in \partial C$, define $\text{Arc}(C, a, b)$ to be the clockwise portion of ∂C from a to b. Let S be a set of points in the plane.

▶ **Definition 1.** *For $a, b \in S$, an edge ab satisfies the empty-$\bigcirc_n$ property with respect to S if there exists an n-gon C with $a, b \in \partial C$ and $S \cap \text{int}(C) = \emptyset$.*

▶ **Definition 2.** *A $\bigcirc_n$-Delaunay graph of S is a maximal planar graph on S such that every edge satisfies the empty-$\bigcirc_n$ property with respect to S. By maximal, we mean that no more edges satisfying the empty-$\bigcirc_n$ property can be added.*

Note that specifying maximality in Definition 2 guarantees that every bounded face is a triangle [4]. Let u, v be two points in the plane that satisfy the general position assumption. Then $\text{Boundary}(u, v)$ denotes the set of n-gons C such that $u, v \in \partial C$. Also for any homothet C, define the point $\text{Center}(C)$ to be the point in C equidistant from all vertices of C. Furthermore, denote $\text{North}(C)$ to be the vertex of C with the largest y-coordinate. Similarly, define $\text{East}(C)$, $\text{South}(C)$ and $\text{West}(C)$. For a set H of homothets of $\bigcirc_n$, let $\text{Center}(H) := \{\text{Center}(C) \mid C \in H\}$. Similarly, we define $\text{West}(H)$. For any homothet C of the n-gon $\bigcirc_n$, we label the vertices clockwise from $\text{West}(C)$ as $C^1, ..., C^n$.

3 Routing Ratio Upper Bound

Recall that

$$\Psi(n) := \begin{cases} \sqrt{1 + ((\cos(2\pi/n) + n - 1)/\sin(2\pi/n))^2} & \text{if } n \in \{5, 6, 7\}, \\ \sqrt{1 + ((\cos(\pi/8)\cos(3\pi/8) + 3)/(\cos(\pi/8)\sin(3\pi/8)))^2} & \text{if } n = 8. \end{cases}$$

The goal of this section is to prove the following theorem.

▶ **Theorem 3.** *The routing ratio of the $\bigcirc_5$-Delaunay graph is at most $\Psi(5) \approx 4.64$.*

3.1 Local Routing Algorithm

We present Algorithm 1, which is a $O(\log(|V(G)|))$-memory local routing algorithm for $\bigcirc_n$-Delaunay graphs generalizing Chew's local routing algorithm [8]. Without loss of generality, we assume that the start vertex s and destination t satisfy $y(s) = y(t)$ and $x(s) < x(t)$. The location of s is stored in memory for each step. In addition, we will assume that all edges of the convex hull are present in the $\bigcirc_n$-Delaunay graph of S. Then in Section 3.2, we describe how Algorithm 1 can be modified to handle routing when the convex hull is not present.

The intuition behind Algorithm 1 is that when the current vertex is p_i, the next vertex p_{i+1} is restricted to one of the vertices of the rightmost triangle T_i of the graph containing vertex p_i and intersecting st. Note that the two neighbours of p_i under consideration are on opposite sides of st. Then, consider the empty n-gon C_i corresponding to T_i. We partition the boundary of C_i into two arcs by splitting at its west point w_i and its rightmost intersection with st, denoted t_i. If p_i is in the upper arc, we choose the clockwise neighbour, otherwise we choose the counterclockwise neighbour. A trace of Algorithm 1 is illustrated in Figure 1. Note that the triangles $T_0, ..., T_k$ are ordered from left to right along st, so the algorithm terminates. We assume that there are $k + 1$ edges in the path output by Algorithm 1. Note that the last edge $p_k t$ in Algorithm 1 may appear to be a separate case from case b in Algorithm 1, but we can avoid analyzing it separately by viewing t as both above and below st. The important detail in the analysis is that t is in the same portion of ∂C_k as p_k (either $p_k, t \in \mathrm{Arc}(C_k, w_k, t_k)$ or $p_k, t \in \mathrm{Arc}(C_k, t_k, w_k)$).

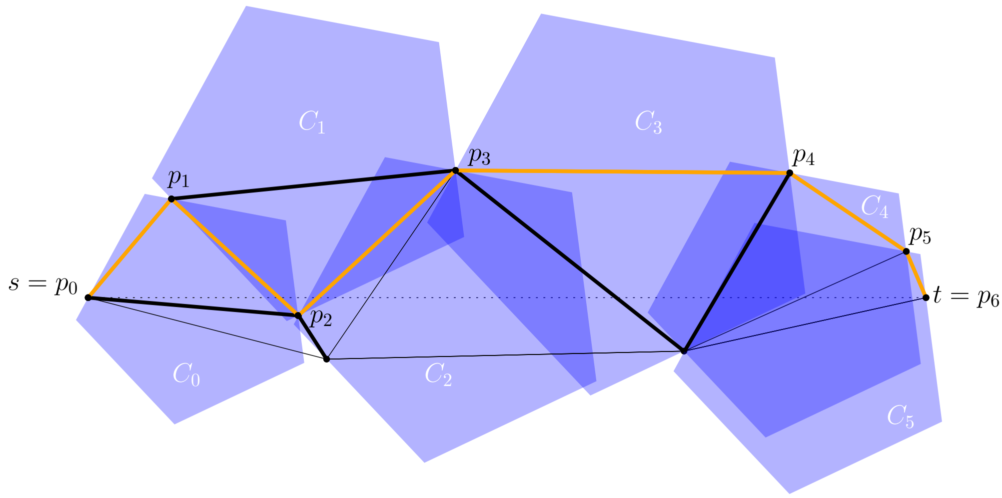

Figure 1 Trace of Algorithm 1. In this case, $k = 5$ and $n = 5$. The orange line is the path chosen by Algorithm 1 and the thick black edges represent the other edge ($p_i a$ or $p_i b$) considered in step b.

When u, v are in general position, the set Center(Boundary(u, v)) is analogous to the perpendicular bisector of uv when the n-gon $\bigcirc_n$ is replaced with a disk. For this reason, we will refer to Center(Boundary(u, v)) as Bisector(u, v). In Lemma 2.2.1.1 of [10], Ma shows that for any regular n-gon, Bisector(u, v) is a polygonal chain completed with two rays at the ends. In this way, Bisector(u, v) partitions $\mathbb{R}^2$ into two half-spaces (see Figure 2).

When $y(v) > y(u)$, then there is a natural ordering of the points in Bisector(u, v) from left to right. By convention, for any $u, v \in \mathbb{R}^2$ in general position, we say that the point Bisector$(u, v) \cap \mathrm{Arc}(C, u, v)$ is to the left of the point Bisector$(u, v) \cap \mathrm{Arc}(C, v, u)$. This is extended to an ordering on all the points of Bisector(u, v). Note that with this convention, Bisector(u, v) does not have the same ordering as Bisector(v, u). For example, this convention

Algorithm 1 Local Routing algorithm in $\bigcirc_n$-Delaunay triangulation.

Data: Two points $s, t \in S$ (w.l.o.g. $y(s) = y(t)$ and $x(s) < x(t)$)
Result: Vertices $s = p_0, ..., t = p_{k+1}$ forming a path in $\bigcirc_n$-Delaunay graph of S
Set $i \leftarrow 0$ and $p_i \leftarrow s$;
while $p_i \neq t$ **do**
- **(a)** If t is a neighbour of p_i, set $p_{i+1} \leftarrow t$.
- **(b)** Otherwise, let T_i be the rightmost triangle in the graph intersecting st with vertex p_i. Let a, b be the other vertices of T_i above and below st, respectively. Let C_i be the empty n-gon with $p_i, a, b \in \partial C_i$. Let $w_i := \text{West}(C_i)$ and t_i be the intersection of C_i with st closest to t.
  - **(1)** If $p_i \in \text{Arc}(C_i, w_i, t_i)$, then set $p_{i+1} \leftarrow a$ and $i \leftarrow i + 1$.
  - **(2)** Else, set $p_{i+1} \leftarrow b$ and $i \leftarrow i + 1$.

end

tells us that in Figure 2, Center(C_{i-1}) is to the left of Center(C_i) on Bisector(q, p_i), whereas Center(C_{i-1}) is to the right of Center(C_i) on Bisector(p_i, q). Then the following remark is based on Ma's plane sweep algorithm [10] which produces the vertices of Bisector(u, v).

▶ Remark 4 ([10]). Let $u, v \in \mathbb{R}^2$ be in general position with $y(v) > y(u)$. Then for any $C, C' \in \text{Boundary}(u, v)$ we have that $\angle\text{South}(C)\text{Center}(C)v \geq \angle\text{South}(C')\text{Center}(C')v$ provided that Center(C) is to the left of Center(C') on Bisector(u, v).

In the following lemma, we describe the structure of the path output by Algorithm 1.

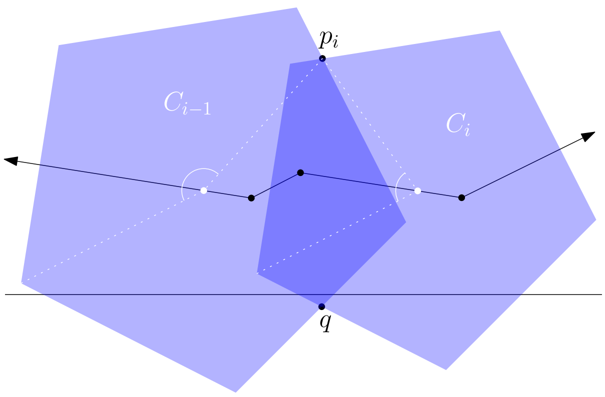

Figure 2 The black polygonal chain is Bisector(q, p_i). The white points are the centers of the 5-gons C_{i-1} and C_i.

▶ **Lemma 5.** *Let $i \in \{1, ..., k\}$. If edges $p_{i-1}p_i$ and p_ip_{i+1} both use case b1 in Algorithm 1, then $\angle w_{i-1}\mathit{Center}(C_{i-1})p_i \geq \angle w_i\mathit{Center}(C_i)p_i$. If instead both edges use case b2 in Algorithm 1, then $\angle p_i\mathit{Center}(C_{i-1})w_{i-1} \geq \angle p_i\mathit{Center}(C_i)w_i$.*

Proof. Assume without loss of generality that edges $p_{i-1}p_i$ and p_ip_{i+1} were chosen using case b1 in Algorithm 1. Then p_i is above st. We will first consider the case where T_{i-1} and T_i share an edge, denoted p_iq. Since p_i is above st, then q is below st. Refer to Figure 2. Both Center(C_{i-1}) and Center(C_i) lie on Bisector(q, p_i), however it remains to establish their relative position. Then by Remark 4, the result follows if we show that Center(C_{i-1}) is to the left of Center(C_i). Define $L := \{\text{Center}(C) \mid C \in \text{Boundary}(p_i, q) \text{ and } p_{i-1} \in \text{int}(C)\}$ and

$R := \{\text{Center}(C) \mid C \in \text{Boundary}(p_i, q) \text{ and } p_{i+1} \in \text{int}(C)\}$. Since $p_{i-1} \in \text{Arc}(C_{i-1}, q, p_i)$, then L propagates from Center(C_{i-1}) to the left. Similarly, R propagates from Center(C_i) to the right since $p_{i+1} \in \text{Arc}(C_i, p_i, q)$.If Center$(C_{i-1})$ is to the right of Center(C_i), then $L \cup R = \text{Bisector}(q, p_i)$. However, Center$(C_{i-1})$ and Center(C_i) are both examples of points in Bisector(q, p_i) but not in $L \cup R$. Therefore Center(C_{i-1}) is to the left of Center(C_i), hence $\angle\text{South}(C_{i-1})\text{Center}(C_{i-1})p_i \geq \angle\text{South}(C_i)\text{Center}(C_i)p_i$. Then we obtain the desired inequality by remarking that p_i is not on Arc$(C_r, \text{South}(C_r), w_r)$ for $r \in \{i-1, i\}$ and also the angle $\angle\text{South}(C)\text{Center}(C)\text{West}(C)$ is constant for all homothets C of $\bigcirc_n$.

If T_i and T_{i-1} do not share an edge, then we use this argument on all the triangles between T_{i-1} and T_i. The result follows since inequality is transitive. ◀

Next, we define the worst-case n-gons, shown in Figure 3.

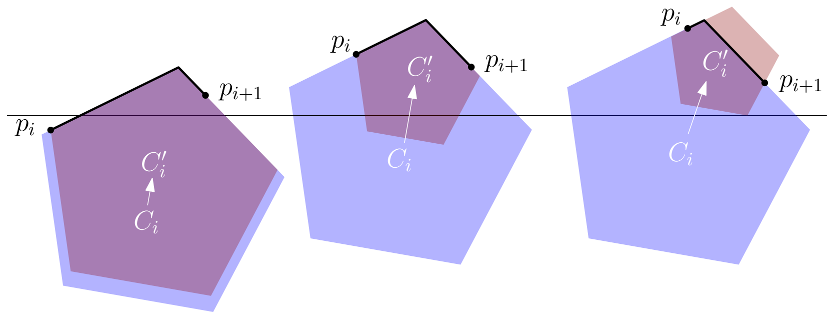

Figure 3 The original 5-gons, C_i, are blue, and the worst-case 5-gons, C_i' from Definition 6, are purple. In all examples, p_{i+1} is above st. Left: West$(C_i') = p_i$. Middle: West$(C_i') = p_i$. Right: South$(C_i'), s, t$ are collinear.

▶ **Definition 6** (Worst-Case n-gons)**.** *Let $i \in \{0, ..., k\}$ and suppose p_{i+1} is above st. Start with an n-gon $C = C_i$, then move Center(C) left along Bisector(p_i, p_{i+1}) while keeping $C \in$ Boundary(p_i, p_{i+1}) until the points South$(C), s, t$ are collinear, or $p_i =$ West(C). The resulting n-gon is denoted by C_i'. If instead p_{i+1} is below st, then move Center(C) right along Bisector(p_i, p_{i+1}) while keeping $C \in$ Boundary(p_i, p_{i+1}) until the points North$(C), s, t$ are collinear, or $p_i =$ West(C). The resulting n-gon is again denoted by C_i'.*

To shorten notation, denote $w_i' := \text{West}(C_i')$ for $i \in \{0, ..., k\}$. A similar statement to that of Lemma 5 can be made about the worst-case n-gons:

▶ **Lemma 7.** *Let $i \in \{1, ..., k\}$. If edges $p_{i-1}p_i$ and p_ip_{i+1} both use case b1 in Algorithm 1, then $\angle w_{i-1}'\text{Center}(C_{i-1}')p_i \geq \angle w_i'\text{Center}(C_i')p_i$. If instead both edges use case b2 in Algorithm 1, then $\angle p_i\text{Center}(C_{i-1}')w_{i-1}' \geq \angle p_i\text{Center}(C_i')w_i'$.*

Proof. Let $i \in \{1, ..., k\}$ and assume edges $p_{i-1}p_i$ and p_ip_{i+1} both use case b1 in Algorithm 1. By Lemma 5, we have $\angle w_{i-1}\text{Center}(C_{i-1})p_i \geq \angle w_i\text{Center}(C_i)p_i$. Since p_i is above st, then the construction of Definition 6 guarantees that $\angle w_{i-1}\text{Center}(C_{i-1})p_i \leq \angle w_{i-1}'\text{Center}(C_{i-1}')p_i$. Similarly, p_{i+1} is above st, meaning that $\angle w_i'\text{Center}(C_i')p_i \leq \angle w_i\text{Center}(C_i)p_i$. Combining inequalities yields the result. Proving the case when both edges use case b2 in Algorithm 1 is similar. ◀

Using the same point set as in Figure 1, the trace of Algorithm 1 is shown in Figure 4 with worst-case n-gons.

Now we define the wedge W_p to be the area swept by stretching $\bigcirc_n$ with its west point on p. More precisely, $W_p := \{v \in \mathbb{R}^2 \mid \exists \text{ homothet } C \text{ of } \bigcirc_n \text{ such that } p = \text{West}(C) \text{ and } v \in C\}$.

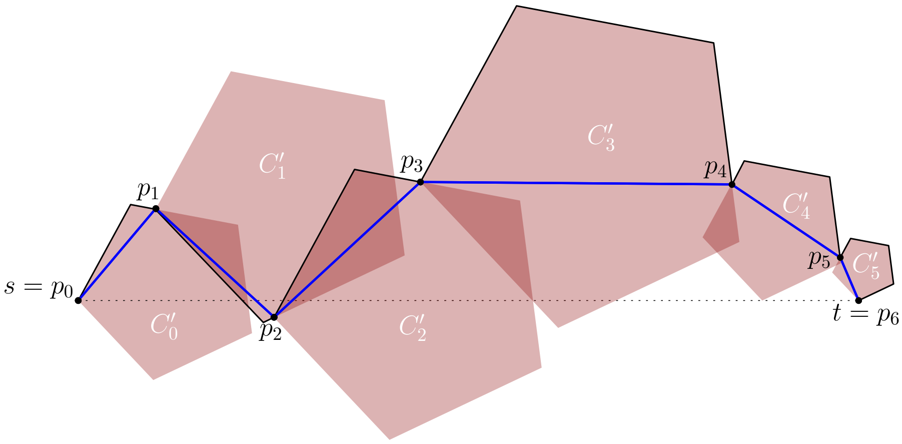

Figure 4 Trace of Algorithm 1 with worst-case 5-gons.

▶ **Lemma 8.** *For $i \in \{1, ..., k\}$, we have $w'_i \in W_{w'_{i-1}}$.*

Proof. Suppose p_i is above st. If $p_i = w'_i$, then w'_i is on the boundary of C'_{i-1}, hence $w'_i \in C_{w'_{i-1}}$. If on the other hand $p_i \neq w'_i$, by Definition 6 we must have $p_i \in \mathrm{Arc}(C'_i, w'_i, \mathrm{South}(C'_i))$ and $\mathrm{South}(C'_i)$ is on st. Suppose for now that $\mathrm{South}(C'_{i-1})$ is also on st. Define the set of homothets

$$L := \{C \mid \mathrm{South}(C), s, t \text{ are collinear and } p_i \in \mathrm{Arc}(C, \mathrm{West}(C), \mathrm{South}(C))\}.$$

Then we will prove the following claim, illustrated in Figure 5.

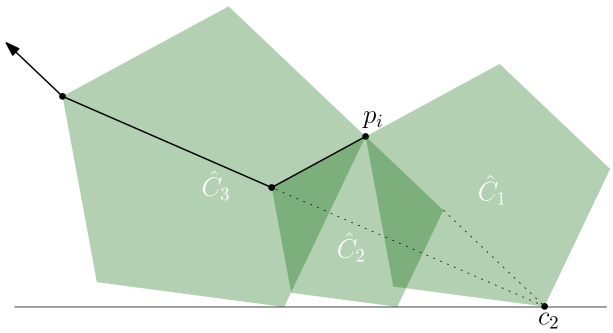

Figure 5 $\mathrm{West}(L)$ is the black line, and c_2 is the homothety center relating $\hat{C}_2$ and $\hat{C}_3$. Segments $\mathrm{West}(\hat{C}_2)\mathrm{West}(\hat{C}_3)$ and $c_2\mathrm{West}(\hat{C}_2)$ lie on the same line. In this example, $\sigma = 4$ and $m = 2$.

▷ Claim. $\mathrm{West}(L)$ is a polygonal chain connected to a ray.

Firstly, suppose $1 < \sigma < n$ such that $C^\sigma = \mathrm{South}(C)$. Then for $j \in \{1, ..., \sigma - 1\}$, define $\hat{C}_j$ to be the unique homothet of $\bigcirc_n$ where $p_i = \hat{C}_j^j$ and $\hat{C}_j^\sigma$ is collinear with st. By definition, for $j \in \{1, ..., \sigma - 1\}$, we have $\mathrm{West}(\hat{C}_j) \in \mathrm{West}(L)$.

Next, for $j \in \{1, ..., \sigma - 2\}$, the n-gons $\hat{C}_j$ and $\hat{C}_{j+1}$ are related by a homothety whose center c_j lies on the intersection of lines given by extending the segments st and $\hat{C}_j^j\hat{C}_j^{j+1}$. Furthermore, for any $C \in L$ with p_i on C^jC^{j+1}, the homothety relating $\hat{C}_j$ and C has the same center, c_j. Therefore the point $\mathrm{West}(C)$ lies on the segment $\mathrm{West}(\hat{C}_j)\mathrm{West}(\hat{C}_{j+1})$. On the other hand, for $C \in L$ with p_i on $C^{\sigma-1}C^\sigma$, the n-gons C and $\hat{C}_{\sigma-1}$ are related by a homothety whose center is South$(\hat{C}_{\sigma-1})$. Therefore the point West(C) must lie on the ray r starting at West$(\hat{C}_{\sigma-1})$ extending directly opposite South$(\hat{C}_{\sigma-1})$. We have shown that West(L) is the polygonal chain (West$(\hat{C}_1), ...,$ West$(\hat{C}_{\sigma-1})$) along with the ray r, so the proof of the claim is completed.

To establish a direction on West(L), which is homeomorphic to a ray by the claim, we choose the convention that p_i is the rightmost point. Then since the vertices C^j are labelled in the clockwise orientation in the claim, we have that for $C, C' \in L$, if West(C) is to the left of West(C') on West(L) then $\angle$West(C)Center$(C)p_i \geq \angle$West(C')Center$(C')p_i$. Therefore by Lemma 5, w'_i must be to the right of w'_{i-1} on West(L).

Finally, we will show that the slope of West(L) remains between the slope of $\bigcirc_n^1\bigcirc_n^n$ and $\bigcirc_n^1\bigcirc_n^2$. Recall the notation $\bigcirc_n^j$ denotes the j-th vertex clockwise around $\bigcirc_n$ where $\bigcirc_n^1 :=$ West$(\bigcirc_n)$. Let $1 < m < n$ such that $\bigcirc_n^m =$ North$(\bigcirc_n)$. We analyze West(L) in three portions.

Segment West$(\hat{C}_j)$West$(\hat{C}_{j+1})$ for $j \in \{1, ..., m-1\}$. Let $j \in \{1, ..., m-1\}$. Then by the homothety relating $\hat{C}_j$ and $\hat{C}_{j+1}$, the segment West$(\hat{C}_j)$West$(\hat{C}_{j+1})$ has slope equal to the slope of segment c_jWest$(\hat{C}_j)$. Since c_j lies to the left of the line by extending West$(\hat{C}_j)\hat{C}_j^2$, then the slope of c_jWest$(\hat{C}_j)$ is positive and less than the slope of $\bigcirc_n^1\bigcirc_n^2$.

Segment West$(\hat{C}_j)$West$(\hat{C}_{j+1})$ for $j \in \{m, ..., \sigma-2\}$. Suppose $j \in \{m, ..., \sigma-2\}$. Then by the homothety relating $\hat{C}_j$ and $\hat{C}_{j+1}$, the segment West$(\hat{C}_j)$West$(\hat{C}_{j+1})$ has slope equal to the slope of segment West$(\hat{C}_j)c_j$. Since c_j lies to the right of South$(\hat{C}_j)$, then the slope of West$(\hat{C}_j)c_j$ is negative and greater than or equal to the slope of $\bigcirc_n^1\bigcirc_n^\sigma$, which is always at least the slope of $\bigcirc_n^1\bigcirc_n^n$.

Ray r. The slope of ray r is West$(\hat{C}_{\sigma-1})c_{\sigma-1}$. Since $c_{\sigma-1}$ is South$(\hat{C}_{\sigma-1})$, then the slope of the ray is equal to the slope of $\bigcirc_n^1\bigcirc_n^\sigma$, which is again in the desired range.

Therefore the slope of each segment of West(L) is within the range given by the cone $W_{w'_{i-1}}$. Since w'_i is to the right on West(L), then we must have $w'_i \in W_{w'_{i-1}}$.

If now South(C'_{i-1}) is instead below st, then we define the homothet C' such that $p_i \in \partial C'$, the points s, South$(C'), t$ are collinear, and Center(C'_{i-1}), Center$(C'), p_i$ are also collinear. Since C' is fully contained in C'_{i-1}, then West$(C') \in W_{w'_{i-1}}$. Also, $\angle$West(C')Center$(C')p_i = \angle$West(C'_{i-1})Center$(C'_{i-1})p_i$, therefore the argument from above can show that $w'_i \in W_{\text{West}(C')}$, hence $w'_i \in W_{w'_{i-1}}$. ◀

Next we define a projection that depends on whether the point is above or below st.

▶ **Definition 9** (West Side Projection). *For a point p above st, let $\overline{p}$ be the intersection of st with the line passing through p with the same slope as $\bigcirc_n^1\bigcirc_n^2$. Similarly, when p is below st, let $\overline{p}$ be the intersection of st with the line passing through p with the same slope as $\bigcirc_n^1\bigcirc_n^n$.*

Using the projections, we define the two paths that we will call *snail paths* and that are used to bound the length of the path output by Algorithm 1.

▶ **Definition 10** (Snail Paths). *Let $a, b \in \mathbb{R}^2$ satisfy $y(a) = y(b)$. When $x(a) < x(b)$, define the n-gon C such that $b =$ South(C) and a, C^1, C^2 are collinear. Then $\mathrel{/\!\!\supset}_{a,b} := aC^1 +$ Arc(C, C^1, b). Similarly, let C' be the n-gon such that $b =$ North(C') and a, C'^1, C'^n are collinear, and then $\mathrel{\backslash\!\!\supset}_{a,b} := aC'^1 +$ Arc(C', b, C'^1). When $x(a) \geq x(b)$, we define $\mathrel{/\!\!\supset}_{a,b}$ and $\mathrel{\backslash\!\!\supset}_{a,b}$ to be empty paths.*

The shape of the snail path is very important as it will directly lead to the routing ratio in the proof of Theorem 3. Notice how the path in Figure 9 is arbitrarily close to $\mathrel{/\!\!\supset}_{s,t}$.

▶ Remark 11. After fixing the orientation of the $\bigcirc_n$, there exists a constant $c > 1$ such that for any $a, b \in \mathbb{R}^2$ with $y(a) = y(b)$ and $x(a) < x(b)$, we have $|\nearrow_{a,b}| = c|\overline{ab}|$. When $x(a) \geq x(b)$, then $|\nearrow_{a,b}| = 0$. The same is true for $|\searrow_{a,b}|$.

When $n = 5$, we prove $\Psi(5)$ is an upper bound on the constant c from Remark 11.

▶ **Lemma 12.** *Let $n = 5$ and $a, b \in \mathbb{R}^2$ with $y(a) = y(b)$ and $x(a) < x(b)$. Then* $\max(|\nearrow_{a,b}|, |\searrow_{a,b}|) \leq \Psi(5)|ab|$.

Proof. Let C be the 5-gon corresponding to $\nearrow_{a,b}$, and let $\theta := \angle \mathrm{West}(C)ab$. For now, assume $C^5 = \mathrm{South}(C)$, meaning $\pi/5 \leq \theta$, and also $\theta \leq \pi/2$. Then the side length $|\mathrm{West}(C)\mathrm{South}(C)| = \frac{|ab|\sin(\theta)}{\sin(2\pi/5)}$ by the law of sines. Similarly, $|a\mathrm{West}(C)| = \frac{|ab|\sin(2\pi/5+\theta)}{\sin(2\pi/5)}$. Since $|\nearrow_{a,b}| = |a\mathrm{West}(C)| + 4|\mathrm{West}(C)\mathrm{South}(C)|$, then we have

$$|\nearrow_{a,b}| = \frac{|ab|\sin(2\pi/5+\theta)}{\sin(2\pi/5)} + 4\frac{|ab|\sin(\theta)}{\sin(2\pi/5)} \leq \Psi(5)|ab|,$$

where the last inequality follows from the analysis in the appendix. See Lemma 16. It is straightforward to verify that the claim still holds when $C^5 \neq \mathrm{South}(C)$. The analysis for $\searrow_{a,b}$ is symmetric. ◀

One useful tool that we will often use is a convex path bound from [2].

▶ **Observation 13** (Convex Path Bound). *Suppose two convex paths $\mathcal{P}_1, \mathcal{P}_2$ have the same endpoints a and b, and $\mathcal{P}_1$ is contained in the region formed by the simple polygon $\mathcal{P}_2 + ab$. Then $|\mathcal{P}_1| \leq |\mathcal{P}_2|$.*

Next, for $i \in \{0, ..., k\}$, we define the path $P_i := \overline{w'_i}w'_i + \mathrm{Arc}(C'_i, w'_i, p_i)$. Paths of this form will be used in the following lemma for pentagons ($n = 5$).

▶ **Lemma 14.** *Let $n = 5$. For $1 \leq i \leq k$, we have*

$$|P_{i-1}| + |p_{i-1}p_i| \leq \Psi(5)|\overline{w'_{i-1}w'_i}| + |P_i|. \tag{1}$$

Furthermore, we have $|P_k| + |p_k t| \leq \Psi(5)|\overline{w'_k}, t|$.

Proof. For $i \in \{1, ..., k\}$, we will show that (1) holds using a case analysis. Without loss of generality, we will assume that p_i is above st for all cases. This is equivalent to assuming that the routing decision is case b1 in Algorithm 1, meaning that $p_{i-1} \in \mathrm{Arc}(C'_{i-1}, w'_{i-1}, p_i)$. The arguments for when p_i is below st are symmetric. We will use the following shorthand: $s'_i := \mathrm{South}(C'_i)$.

Case 1: p_{i-1} is above st and both s'_{i-1}, s'_i lie on st. See Figure 6. We split the snail path of C'_{i-1} up into several parts:

$$|\nearrow_{\overline{w'_{i-1}}, s'_{i-1}}| = |P_{i-1}| + |\mathrm{Arc}(C'_{i-1}, p_{i-1}, p_i)| + |\mathrm{Arc}(C'_{i-1}, p_i, s'_{i-1})|. \tag{2}$$

If $\overline{w'_i}$ is west of s'_{i-1}, then by a convex path bound, we get

$$|\nearrow_{\overline{w'_i}, s'_{i-1}}| \leq |P_i| + |\mathrm{Arc}(C'_{i-1}, p_i, s'_{i-1})|. \tag{3}$$

On the other hand, inequality 3 still holds when $\overline{w'_i}$ is east of s'_{i-1} since $|\nearrow_{\overline{w'_i}, s'_{i-1}}| = 0$. Finally,

$$\begin{aligned} |P_{i-1}| + |\mathrm{Arc}(C'_{i-1}, p_{i-1}, p_i)| &= |\nearrow_{\overline{w'_{i-1}}, s'_{i-1}}| - |\mathrm{Arc}(C'_{i-1}, p_i, s'_{i-1})| - |P_i| + |P_i| && \text{by 2} \\ &\leq |\nearrow_{\overline{w'_{i-1}}, s'_{i-1}}| - |\nearrow_{\overline{w'_i}, s'_{i-1}}| + |P_i| && \text{by 3} \\ &\leq |\nearrow_{\overline{w'_{i-1}}, \overline{w'_i}}| + |P_i| && \text{by Remark 11.} \end{aligned}$$

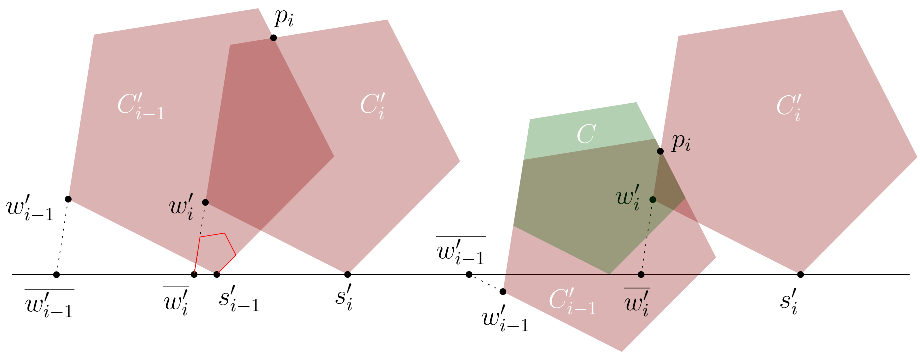

Figure 6 Left: Case 1 when $x(\overline{w'_i}) < x(s'_{i-1})$. The path $\nearrow\!\!\sqsupset_{\overline{w'_i}, s'_{i-1}}$ is red. Right: Case 2 reduces to case 1 by defining n-gon C in green.

Case 2: p_{i-1} is above st and only s'_i lies on st. We will reduce this case to Case 1. See Figure 6 First, define a new n-gon C with $w :=$ West(C) such that $\overline{w} = \overline{w'_{i-1}}$, and South$(C), s, t$ are collinear, and $p_i \in \partial C$. By a convex path bound, we have

$$|P_{i-1}| + |\mathrm{Arc}(C'_{i-1}, p_{i-1}, p_i)| \leq |\overline{w'_{i-1}}w| + |\mathrm{Arc}(C, w, p_i)| \tag{4}$$

Lastly, we proceed with the argument in Case 1, replacing C'_{i-1} with C in order to obtain

$$|\overline{w'_{i-1}}w| + |\mathrm{Arc}(C, w, p_i)| \leq |P_i| + |\nearrow\!\!\sqsupset_{\overline{w'_{i-1}}, \overline{w'_i}}| \tag{5}$$

Combining (4) with (5) yields (1).

Case 3: p_{i-1} is above st and s'_i lies below st. In this case, $p_i = w'_i$, meaning $P_i = \overline{w'_i}p_i$. Let point p be collinear with $\overline{w'_{i-1}}, w'_{i-1}$ such that $y(p) = y(p_i)$. Then by a convex path bound, we have

$$|P_{i-1}| + |\mathrm{Arc}(C'_{i-1}, p_{i-1}, p_i)| \leq |\overline{w'_{i-1}}p| + |\nearrow\!\!\sqsupset_{p, p_i}|$$

Finally, equation (1) follows since the paths $\overline{w'_{i-1}}p$ and $\nearrow\!\!\sqsupset_{p, p_i}$ are translates of P_i and $\nearrow\!\!\sqsupset_{\overline{w'_{i-1}}, \overline{w'_i}}$ respectively.

Case 4: p_{i-1} is below st. In this case, $w'_{i-1} = p_{i-1}$, hence $P_{i-1} = \overline{w'_{i-1}}w'_{i-1}$. Define point p be the intersection of C'_{i-1} closest to s. We will prove the claim, illustrated in Figure 8.

▷ Claim. P_{i-1} is a sub-path of $\searrow\!\!\sqsupset_{\overline{w'_{i-1}}, p}$.

Fix m such that $\bigcirc_n^m :=$ North$(\bigcirc_n)$. Then for $0 \leq j \leq m-2$ let $\hat{C}_j$ be the n-gon such that $\hat{C}_j^{m-j} = p$ and $\overline{\mathrm{West}(\hat{C}_j)} = \overline{w'_{i-1}}$. Note that $\hat{C}_0$ is exactly the n-gon corresponding to $\searrow\!\!\sqsupset_{\overline{w'_{i-1}}, p}$. In addition, p was defined to be collinear with $C'^1_{i-1}C'^2_{i-1}$, therefore $w'_{i-1} =$ West$(\hat{C}_{m-2})$. Then, for $1 \leq j \leq m-2$, the n-gons $\hat{C}_{j-1}, \hat{C}_j$ are related by a homothety with center c_j lying on the intersection of extended segments $\hat{C}_j^1\hat{C}_j^n$ and $\hat{C}_j^{m-j}\hat{C}_j^{m-j+1}$. By construction, we have $\hat{C}_j^{m-j} = \hat{C}_{j-1}^{m-j+1}$, therefore by symmetry (reflection about the extended line c_jCenter$(\hat{C}_j)$), we also have $\hat{C}_j^1 = \hat{C}_{j-1}^n$. Then West$(\hat{C}_j)$ is on the snail path $\searrow\!\!\sqsupset_{\overline{w'_{i-1}}, p}$ if and only if $j < 2$. In particular, the claim holds when $m - 2 < 2$, which is the case for $n \in \{5, 6, 7, 8\}$ since in general $\lfloor \frac{n}{4} \rfloor \leq m - 1 \leq \lceil \frac{n}{4} \rceil$.

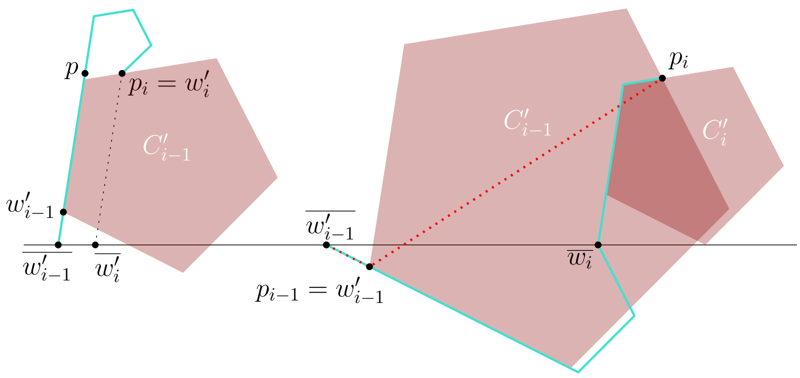

Figure 7 Left: In Case 3, the blue path $\overline{w'_{i-1}}X + \nearrow_{X,p_i}$ is longer than the dotted red path $P_{i-1} + \mathrm{Arc}(C'_{i-1}, p_{i-1}, p_i)$. Right: In Case 4: The blue path $P_{i-1} + p_{i-1}p_i$ is longer than the dotted red path $\searrow_{\overline{w'_{i-1}}, \overline{w'_i}} + P_i$.

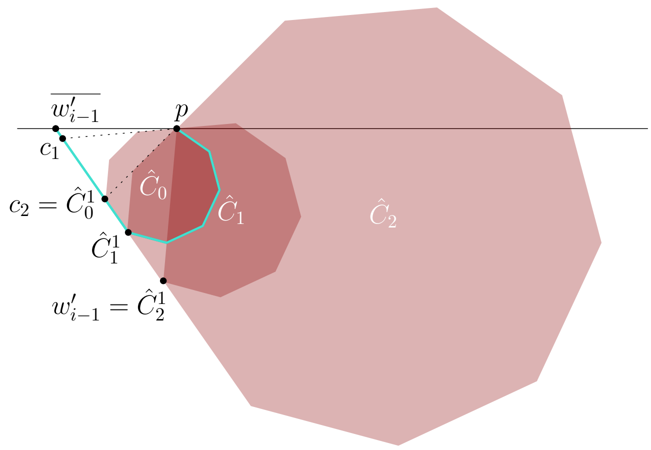

Figure 8 Case 4 claim. Here, $n = 9$, $m = 4$, and therefore $C'_{i-1} = \hat{C}^1_{m-2}$ is not on the blue path $\searrow_{\overline{w'_{i-1}},p}$. If instead $n \in \{5, 6, 7, 8\}$, then $m \in \{2, 3\}$.

Then by Lemma 8, $x(p) \leq x(\overline{w'_i})$, therefore P_{i-1} is a also sub-path of $\searrow_{\overline{w'_{i-1}}, w'_i}$. Inequality (1) follows since the paths $P_{i-1} + p_{i-1}p_i$ and $\searrow_{\overline{w'_{i-1}}, \overline{w'_i}} + P_i$ have the same endpoints, concluding this case.

Finally, we have $|P_k| + |p_k t| \leq \max(|\nearrow_{\overline{w'_k},t}|, |\searrow_{\overline{w'_k},t}|)$ by a convex path bound. Note that for $i = 1$, the edge sp_1 classifies as Case 4. ◀

Putting this all together, we can now prove Theorem 3.

Proof. Consider the path $s = p_0, ..., p_{k+1} = t$ from Algorithm 1 and apply Lemma 14:

$$\begin{aligned}\sum_{i=1}^{k+1} |p_{i-1}p_i| &\leq \Big(\sum_{i=1}^{k} |P_i| + \Psi(5)|\overline{w'_{i-1}w'_i}| - |P_{i-1}|\Big) + (\Psi(5)|\overline{w'_k}t| - |P_k|) \\ &= \Psi(5)|\overline{w'_0}t| - |P_0| = \Psi(5)|st|.\end{aligned}$$

Since Algorithm 1 is a $O(\log(|V(G)|))$-memory local routing algorithm, then $\Psi(5)$ is an upper bound on the routing ratio of $\bigcirc_5$-Delaunay graphs. ◀

Now we present a lemma showing that our analysis is optimal for Algorithm 1.

▶ **Lemma 15.** *Let $n \in \{5, 6, 7, 8\}$. For any $\epsilon > 0$, there is a $\bigcirc_n$-Delaunay graph for which Algorithm 1 has a routing ratio of at least $\Psi(n) - \epsilon$.*

Proof. For simplicity, we present S not in general position, however it is possible to perturb the position of each vertex to force the $\bigcirc_n$-Delaunay graph of S to have the desired edges while being in general position. See Figure 9. Let C be an n-gon, and define σ such that $C^\sigma = \text{South}(C)$. Then let L be a horizontal line above and arbitrarily close to C^σ. Define $t := L \cap C^{\sigma-1}C^\sigma$ and $v_j := C^j$ for $j \in \{2, ..., \sigma - 1\}$. Let v_0 be on $C^\sigma C^{\sigma+1}$ below L. Let v_1 be on C^1C^2 arbitrarily close to C^1. Then let $\hat{C}$ be an n-gon such that $\text{East}(\hat{C}) = v_0$ and $p_1 \in \partial\hat{C}$. Finally, s is the leftmost intersection of L and $\partial\hat{C}$. Let $S := \{s, t, v_0, v_1, ..., v_{\sigma-1}\}$. While there are several valid $\bigcirc_n$-Delaunay graphs of S, we will choose to route on the graph with edges $sv_0, sv_1, v_{j-1}v_j, v_1v_0, v_jv_0, v_0t, v_{\sigma-1}t$ for $j \in \{2, ..., \sigma - 1\}$. When Algorithm 1 routes from s to t in the $\bigcirc_n$-Delaunay graph of S, then each step of case b is sub-case b1, therefore the path is $s, v_1, v_2, ..., v_{\sigma-1}, t$. By construction, this path is arbitrarily close to the snail path $⁄⊐_{s,t}$. Since $\Psi(n)|st|$ represents the maximum length of the snail path $⁄⊐_{s,t}$ over all orientations of $\bigcirc_n$, then the routing ratio of Algorithm 1 for this graph is at least $\Psi(n) - \epsilon$. ◀

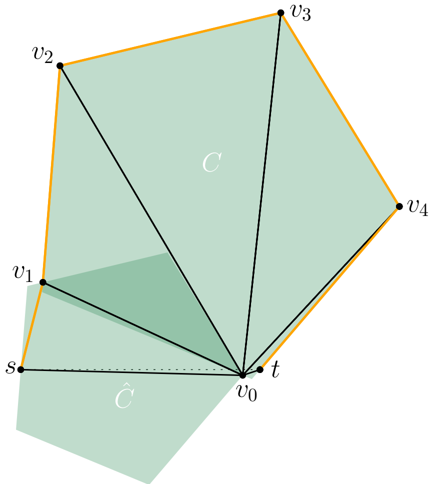

Figure 9 Worst-case point-set construction causes Algorithm 1 to choose the orange path which is arbitrarily close to $|⁄⊐_{s,t}|$ since each step of case b is sub-case b1. Here $n = \sigma = 5$. For more details, see proof of Lemma 15.

3.2 Extending Algorithm 1 to graphs where the convex hull is not present

In order to extend Algorithm 1 to graphs where the convex hull is not present, we add dummy vertices to the graph G to ensure the entire set S is triangulated. In particular, define the set D of dummy vertices as follows:

Let C be the scaled translate of $\bigcirc_n$ with $\text{Center}(C) = s$ and $|\text{West}(C)\text{Center}(C)| = 10|st|$. Then let $\text{Rot}(C)$ denote the rotated n-gon obtained by rotating C about $\text{Center}(C)$ by π radians (or by $\pi - \epsilon$ radians for $\epsilon > 0$ to satisfy the general position assumption). Lastly, let D be the corners of $\text{Rot}(C)$ that are in the unbounded region of the $\bigcirc_n$-Delaunay graph of S.

If Algorithm 1 is used in the $\bigcirc_n$-Delaunay graph of $S \cup D$, then the path from s to t will have length at most $c|st|$ where c is given in Lemma 16. In all cases, $c < 10$, so for all $i \in \{0, ..., k+1\}$, we have that $p_i \notin D$. The choice of D ensures that all p_i are not incident to the unbounded region.

When routing in the $\bigcirc_n$-Delaunay graph of S, Algorithm 1 can be modified to simulate the edges of the $\bigcirc_n$-Delaunay graph of $S \cup D$. Firstly, it can be verified locally whether the current vertex p_i is incident to the unbounded region. Indeed, p_i is in the unbounded region if and only if there exists a $j \in \{1, ..., n\}$ and an n-gon C such that $C^j = p_i$ and no edges incident to p_i intersect C. Note that the size of C is irrelevant. Then if p_i is incident with the unbounded region, it can be verified whether p_i has neighbours in D in the $\bigcirc_n$-Delaunay graph of $S \cup D$ since the algorithm stores the location of s and can calculate the distance $|st|$.

3.3 Extending our result to hexagons, septagons and octagons

So far, we have shown that Algorithm 1 has a routing ratio of at most $\Psi(5) \approx 4.640$ for any $\bigcirc_5$-Delaunay graph. To extend our result to n-gons with $6 \leq n \leq 8$, then Lemma 12 needs to be generalized to Lemma 16. Additionally, Lemma 14 is trivially generalized by replacing each occurrence of $\Psi(5)$ with $\Psi(n)$ for the corresponding n-gon.

▶ **Lemma 16.** *Let $a, b \in \mathbb{R}^2$ with $y(a) = y(b)$ and $x(a) < x(b)$. Then $\max(|\diagup\!\!\sqsupset_{a,b}|, |\diagdown\!\!\sqsupset_{a,b}|) \leq \Psi(n)|ab|$ for $n \in \{5, 6, 7, 8\}$.*

Proof. Let $n \in \{5, 6, 7\}$ and let C be the n-gon corresponding to $\diagup\!\!\sqsupset_{a,b}$. Let $\theta := \angle \mathrm{West}(C)ab$, and for now assume that $\mathrm{South}(C) = C^n$. This means that $\pi/5 \leq \theta \leq \pi/2$. Then by the law of sines, $|\mathrm{West}(C)\mathrm{South}(C)| = \frac{|ab|\sin(\theta)}{\sin(2\pi/n)}$ and $|a\mathrm{West}(C)| = \frac{|ab|\sin(2\pi/n+\theta)}{\sin(2\pi/n)}$. Since $|\diagup\!\!\sqsupset_{a,b}| = |a\mathrm{West}(C)| + (n-1)|\mathrm{West}(C)\mathrm{South}(C)|$, then we have

$$\frac{|\diagup\!\!\sqsupset_{a,b}|}{|ab|} = \frac{\sin(2\pi/n + \theta)}{\sin(2\pi/n)} + (n-1)\frac{\sin(\theta)}{\sin(2\pi/n)}. \tag{6}$$

Focusing on the numerators, we have

$$\begin{aligned}\frac{d}{d\theta}\Big(\sin(2\pi/n+\theta) + (n-1)\sin(\theta)\Big) &= \cos(2\pi/n+\theta) + (n-1)\cos(\theta) \\ &= (\cos(2\pi/n)\cos(\theta) - \sin(2\pi/n)\sin(\theta)) + (n-1)\cos(\theta) \\ &= (\cos(2\pi/n) + n - 1)\cos(\theta) - \sin(2\pi/n)\sin(\theta).\end{aligned}$$

From there, we get that the critical value of θ is

$$\theta^* = \arctan\left(\frac{\cos(2\pi/n) + n - 1}{\sin(2\pi/n)}\right).$$

Therefore the maximum value is

$$\frac{\sin\left(2\pi/n + \theta^*\right) + (n-1)\sin\left(\theta^*\right)}{\sin(2\pi/n)}.$$

When n is 5, 6 or 7, the maximum value of (6) is approximately 4.640, 6.429, or 8.531 respectively. It is straightforward to verify that the claim still holds when $\mathrm{South}(C) \neq C^n$.

When $n = 8$, the analysis is slightly different since we can not have $\mathrm{South}(C) = C^n$. As before, let C be the 8-gon corresponding to $\diagup\!\!\sqsupset_{a,b}$, and let $\theta := \angle \mathrm{West}(C)ab$. Then necessarily, we have $\mathrm{South}(C) = C^7$, meaning that $\angle bC^1a = \frac{3\pi}{8}$. By the law of sines, $|C^1C^7| = \frac{|ab|\sin(\theta)}{\sin(3\pi/8)}$ and $|aC^1| = \frac{|ab|\sin(3\pi/8+\theta)}{\sin(3\pi/8)}$. Also, $|\mathrm{Arc}(C, C^7, C^1)|\cos(\pi/8) = |C^1C^7|$. Since $|\diagup\!\!\sqsupset_{a,b}| = |aC^1| + 3|\mathrm{Arc}(C, C^7, C^1)|$, then we have

$$\frac{|\mathbin{\diagup\!\!\supset}_{a,b}|}{|ab|} = \frac{\sin(3\pi/8+\theta)}{\sin(3\pi/8)} + 3\frac{\sin(\theta)}{\sin(3\pi/8)\cos(\pi/8)} \text{ where } \pi/4 \leq \theta \leq \pi/2,$$

which reaches a maximum of around 4.054 by a similar analysis to above. ◀

4 Routing Ratio Lower Bound for $\bigcirc_5$-Delaunay graphs

▶ **Lemma 17.** *Every local routing algorithm for $\bigcirc_5$-Delaunay graphs must have a routing ratio of at least* 1.995 *for any* $\epsilon > 0$.

Proof. Consider the construction shown in Figure 10.

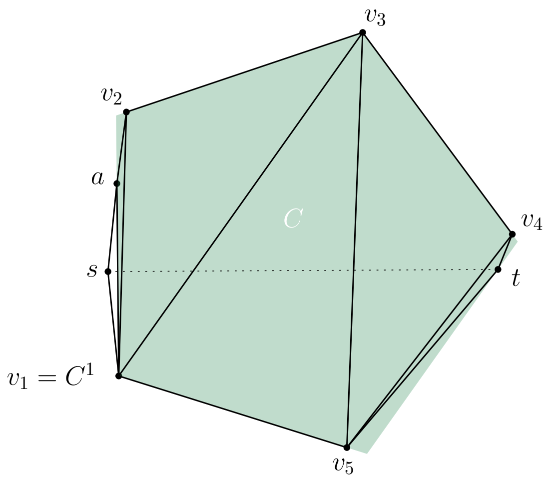

Figure 10 Point set construction obtaining a routing ratio lower bound of approximately 1.995.

Let $\delta > 0$ and let C be a pentagon with $|C^1C^2| = 1 + \delta$. Define $v_1 := C^1$, $v_2 \in C^2C^3$ arbitrarily close to C^2, $v_3 := C^3$, $v_4 \in C^3C^4$ arbitrarily close to C^4, and $v_5 \in C^5C^1$ arbitrarily close to C^5. Finally place point a on C^1C^2 such that $|C^1a| = 1$. Place s outside C equidistant from v_1, a, arbitrarily close to C^1C^2. Define the point $t \in C^4C^5$ on the line perpendicular to C^1C^2 through s. The point set is $S := \{s, t, a, v_1, v_2, v_3, v_4, v_5\}$. While S admits many different triangulations, we choose the triangulation from Figure 10. The edges are $sa, sv_1, av_1, av_2, v_1v_2, v_1v_3, v_1v_5, v_2v_3, v_3v_4, v_3v_5, v_4v_5, v_4t, v_5t$. In particular, the neighbourhood of s is $\{a, v_1\}$, however the shortest path from a to t is approximately along the boundary ∂C. We will analyze the routing ratio of any algorithm that chooses to go to a. Then, any algorithm that chooses v_1 first will perform poorly on a symmetric graph reflected about the line through st.

We consider the clockwise and counterclockwise arcs from point a to point t. The counterclockwise arc has length

$$\begin{aligned}|\mathrm{Arc}(C_1, t, a)| &= |av_1| + |v_1C^5| + |C^5t| \\ &= 1 + (1+\delta) + \frac{\frac{1}{2} + (1+\delta)\sin 18}{\sin 54}.\end{aligned}$$

On the other hand, the clockwise arc from a to t has length

$$\begin{aligned}|\mathrm{Arc}(C_1, a, t)| &= |aC^2| + |C^2C^3| + |C^3C^4| + |C^4t| \\ &= \delta + (1+\delta) + (1+\delta) + (1 + \delta - \frac{\frac{1}{2} + (1+\delta)\sin 18}{\sin 54}).\end{aligned}$$

Finally, we have $|st| = |v_1C^5|\cos 18 + |C^5t|\cos 54 = (1+\delta)\cos 18 + \frac{\frac{1}{2}+(1+\delta)\sin 18}{\tan 54}$. This means that the routing ratio is at least

$$\frac{|sa| + |\text{ShortestPath}(a,t)|}{|st|} = \frac{\frac{1}{2} + \min(|\text{Arc}(C_1,a,t)|, |\text{Arc}(C_1,t,a)|)}{|st|}$$

which reaches approximately 1.995 when $\delta = 0.447$. ◀

5 Spanning Ratio Lower Bound for $\bigcirc_5$-Delaunay graphs

▶ **Lemma 18.** *For any $\epsilon > 0$, there exists a $\bigcirc_5$-Delaunay graph with spanning ratio of at least $\frac{5}{2+3\sin(\frac{\pi}{10})} - \epsilon$.*

Proof. Let $\epsilon > 0$. We will construct a point set for the $\bigcirc_5$-Delaunay graph shown in Figure 11. In particular, let the pentagon C have side length 1, then place a on C^1C^5 with $|C^1a| = \frac{1}{4}$ and b on C^3C^4 with $|C^3b| = \frac{1}{4}$. Next, place $v_1 \in C^1C^2$ arbitrarily close to C^1, $v_2 := C^2$, $v_3 \in C^2C^3$ arbitrarily close to C^3, $v_4 \in C^4C^5$ arbitrarily close to C^4, and $v_5 \in C^4C^5$ arbitrarily close to C^5. We define $S := \{a, b, v_1, v_2, v_3, v_4, v_5\}$ and consider the triangulation with edges $av_1, av_5, v_1v_2, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3b, v_4b, v_4v_5$. The shortest path from a to b is approximately the boundary ∂C. Therefore the length of the shortest path from a to b is approximately $\frac{5}{2}$, whereas $|ab| = 1 + \frac{3}{2}\sin(\frac{\pi}{10})$. This means that the spanning ratio of is approximately 1.708. ◀

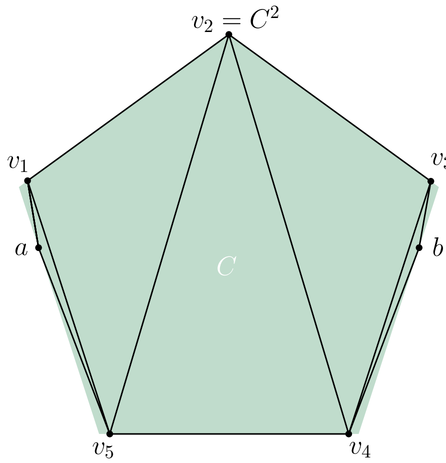

Figure 11 Point set construction obtaining a spanning ratio lower bound of approximately 1.708. The shortest path between a and b is arbitrarily close to the perimeter of the pentagon.

6 Conclusions

We have dramatically improved the upper bound on the spanning ratio when $n \in \{5, 7, 8\}$ and on the routing ratio when $n \in \{5, 6, 7, 8\}$. We also provided matching lower bounds for the paths output by our routing algorithm, showing that our analysis is tight. Furthermore, we prove that no routing algorithm for pentagon-Delaunay graphs can have a routing ratio less than 1.995. Finally, we show that the worst-case spanning ratio is at least 1.708 for the pentagon-Delaunay graph. We conclude with the following three open questions: (1) Can we improve the lower bound on the routing ratio for these graphs? (2) Can we provide a routing algorithm that attains this lower bound? (3) Can our approach be generalized for regular polygons with more than 8 sides?

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