Composition Orderings for Linear Functions and Matrix Multiplication Orderings

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Abstract

We first consider composition orderings for linear functions of one variable. Given n linear functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, the objective is to find a permutation $\sigma : [n] \to [n]$ that minimizes/maximizes $f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}(c)$, where $[n] = \{1, \ldots, n\}$. It was first studied in the area of time-dependent scheduling, and known to be solvable in $O(n \log n)$ time if all functions are nondecreasing. In this paper, we present a complete characterization of optimal composition orderings for this case, by regarding linear functions as two-dimensional vectors. We also show the equivalence between local and global optimality in optimal composition orderings. Furthermore, by using the characterization above, we provide a fixed-parameter tractable (FPT) algorithm for the composition ordering problem with general linear functions, with respect to the number of decreasing linear functions.

We next deal with matrix multiplication as a generalization of composition of linear functions. Given n matrices $M_1, \ldots, M_n \in \mathbb{R}^{m \times m}$ and two vectors $\boldsymbol{w}, \boldsymbol{y} \in \mathbb{R}^m$, where m is a positive integer, the objective is to find a permutation $\sigma : [n] \to [n]$ that minimizes/maximizes $\boldsymbol{w}^\top M_{\sigma(n)} \cdots M_{\sigma(1)} \boldsymbol{y}$. The matrix multiplication ordering problem has been studied in the context of max-plus algebra, but despite being a natural problem, it has not been explored in the conventional algebra to date. By extending the results for composition orderings for linear functions, we show that the matrix multiplication ordering problem with 2×2 matrices is solvable in $O(n \log n)$ time if all the matrices are simultaneously triangularizable and have nonnegative determinants, and FPT with respect to the number of matrices with negative determinants, if all the matrices are simultaneously triangularizable. As the negative side, we prove that three possible natural generalizations are NP-hard. In addition, we derive the existing result for the minimum matrix multiplication ordering problem with 2×2 upper triangular matrices in max-plus algebra, which is an extension of the well-known Johnson's rule for the two-machine flow shop scheduling, as a corollary of our result in the conventional algebra.

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1 Introduction

We first consider composition orderings for linear functions, that is, polynomial functions of degree one or zero. Namely, given a constant $c \in \mathbb{R}$ and n linear functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$, each of which is expressed as $f_i(x) = a_i x + b_i$ for some $a_i, b_i \in \mathbb{R}$, we find a permutation $\sigma : [n] \to [n]$ that minimizes/maximizes $f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}(c)$, where $[n] = \{1, \dots, n\}$ for a positive integer n. Since composition of functions is *not* commutative even for linear functions, i.e., $f_{\sigma(2)} \circ f_{\sigma(1)} \neq f_{\sigma(1)} \circ f_{\sigma(2)}$ holds in general, it makes sense to investigate the problem. For example, let $f_1(x) = -(1/2)x + 3/2$, $f_2(x) = x - 3$, $f_3(x) = 3x - 1$, and $c = 0$, then the identity σ (i.e., $\sigma(1) = 1$, $\sigma(2) = 2$ and $\sigma(3) = 3$) provides $f_3 \circ f_2 \circ f_1(0) = f_3(f_2(f_1(0))) = f_3(f_2(3/2)) = f_3(-3/2) = -11/2$, while the permutation τ with $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(3) = 3$ provides $f_3 \circ f_1 \circ f_2(0) = 8$. In fact, we can see that σ and τ are respectively minimum and maximum permutations for the problem. The composition ordering problem is natural and fundamental in many fields such as combinatorial optimization, computer science, and operations research. This problem, which was introduced by Kawase, Makino and Seimi [17], had been dealt with implicitly in the field of scheduling.

The problem was first studied from an algorithmic point of view under the name of *time-dependent scheduling* (e.g., [8, 9]). We are given n jobs with processing times $p_1, \dots, p_n$. Unlike the classical scheduling, the processing time p_i is *not* constant, depending on the starting time of job i. Here each p_i is assumed to satisfy $p_i(s) \leq t + p_i(s + t)$ for any positive reals s and t, since we should be able to finish processing job i earlier if it starts earlier. The model was introduced to deal with learning and deteriorating effects. As the most fundamental setting of the time-dependent scheduling, we consider the linear model of single-machine makespan minimization, where the makespan denotes the time when all the jobs have been processed, and we assume that the machine can handle only one job at a time and preemption is not allowed. The linear model means that the processing time p_i is linear in the starting time s, i.e., $p_i(s) = \tilde{a}_i s + \tilde{b}_i$ for some constants $\tilde{a}_i$ and $\tilde{b}_i$. Then it is not difficult to see that the model can be regarded as the minimum composition ordering problem with linear functions $f_i(x) = (\tilde{a}_i + 1)x + \tilde{b}_i$, since f_i represents the time to finish job i if the processing of the job starts at time x. Mosheiov [20] showed the makespan is independent of the schedule, i.e., any permutation provides the same composite, if $\tilde{b}_i = 0$ for any $i \in [n]$. Gawiejnowicz and Pankowska [13], Gupta and Gupta [14], Tanaev et al. [23], and Wajs [24] studied the linear deterioration model, that is, $\tilde{a}_i, \tilde{b}_i > 0$ (i.e., $a_i > 1$ and $b_i > 0$) for any $i \in [n]$. Here $\tilde{a}_i$ and $\tilde{b}_i$ are respectively called the *deterioration rate* and the *basic processing time* of job i. It can be shown that a minimum permutation can be obtained by arranging the jobs nonincreasingly with respect to $\tilde{a}_i/\tilde{b}_i \,(= (a_i - 1)/b_i)$. Gawiejnowicz and Pankowska [13] also considered the cases $\tilde{a}_i = 0$ or $\tilde{b}_i = 0$ for some i. Gawiejnowicz and Lin [12] dealt with the linear models with nonnegative coefficients for various criteria. On the other hand, Ho, Leung and Wei [15] considered the linear learning model, that is, $0 > \tilde{a}_i > -1, \tilde{b}_i > 0$ (i.e., $1 > a_i > 0$ and $b_i > 0$) for any $i \in [n]$ and showed that a minimum permutation can be obtained again by arranging the jobs nonincreasingly with respect to $\tilde{a}_i/\tilde{b}_i \,(= (a_i - 1)/b_i)$. Gawiejnowicz, Kurc and Pankowska [11] discussed the relations between the deterioration model and the learning model. Later, Kawase et al. [17] introduced the composition ordering problem, showed that the maximization problem can be formulated as the minimization one, and proposed an $O(n \log n)$-time algorithm if all f_i's are nondecreasing, i.e., $a_i \geq 0$ for any $i \in [n]$. However, it is still open whether it is polynomially computable for general linear functions. Moreover, it is not known even when constantly many functions are decreasing.

We remark that the time-dependent scheduling with the ready time and the deadline can be regarded as the composition ordering problem with piecewise linear functions, and is known to be NP-hard, and Kawase et al. [17] also studied the composition ordering for non-linear functions as well as the related problems such as partial composition and k-composition. We also remark that the free-order secretary problem, which is closely related to a branch of the problems such as the full-information secretary problem [6], knapsack and matroid secretary problems [1, 2, 22] and stochastic knapsack problems [4, 5], can also be regarded as the composition ordering problem [17].

Main results obtained in this paper

We first characterize the minimum composition orderings for increasing linear functions. In order to describe our result, we need to define three important concepts: counterclockwiseness, collinearity, and potential identity.

We view a linear function $f(x) = ax + b$ as the vector $\begin{pmatrix} b \\ 1-a \end{pmatrix}$ in $\mathbb{R}^2$, and its *angle*, denoted by $\theta(f)$, is defined as the polar angle in $[0, 2\pi)$ of the vector, where we define $\theta(f) = \bot$ if the vector of f is the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., f is the identity function. For linear functions $f_1, \ldots, f_n$, a permutation $\sigma : [n] \to [n]$ is called *counterclockwise* if there exists an integer $k \in [n]$ such that $\theta(f_{\sigma(k)}) \leq \cdots \leq \theta(f_{\sigma(n)}) \leq \theta(f_{\sigma(1)}) \leq \cdots \leq \theta(f_{\sigma(k-1)})$, where identity functions f_i (i.e., $\theta(f_i) = \bot$) are ignored and the inequalities are assumed to be transitive. For example, we consider inequalities such as $\theta(f_{\sigma(1)}) \leq \theta(f_{\sigma(3)})$ if $\theta(f_{\sigma(2)}) = \bot$. Linear functions $f_1, \ldots, f_n$ are called *collinear* if the corresponding vectors lie in some line through the origin, i.e., there exists an angle λ such that $\theta(f_i) \in \{\lambda, \lambda + \pi, \bot\}$ for all $i \in [n]$, and *potentially identical* if there exists a counterclockwise permutation $\sigma : [n] \to [n]$ such that the corresponding composite is the identity function, i.e., $f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}(x) = x$. A permutation is called *minimum* (resp., *maximum*) if the corresponding composite is the minimum (resp., maximum). Then we have the following complete characterization of minimum permutations.

▶ **Theorem 1.** *For the minimum composition ordering problem with increasing linear functions $f_1, \ldots, f_n$, one of the following three statements holds.*

(i) *They are collinear if and only if any permutation is minimum.*

(ii) *If they are not collinear, then the following statements are equivalent:*

  (ii-1) *They are potentially identical.*

  (ii-2) *A permutation is minimum if and only if it is counterclockwise.*

(iii) *If they are neither collinear nor potentially identical, then a permutation σ is minimum if and only if it is a counterclockwise permutation such that $\theta(f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}) + \pi \in [\theta(f_{\sigma(t)}), \theta(f_{\sigma(s)})]_{2\pi}$, where s and t denote the first and last integers i such that $f_{\sigma(i)}$ is not the identity function.*

Here we define $[\theta_1, \theta_2]_{2\pi} = \{\theta \in [\lambda_1, \lambda_2] \mid \lambda_1 =_{2\pi} \theta_1,\ \lambda_2 =_{2\pi} \theta_2,\ \lambda_2 - \lambda_1 \in [0, 2\pi)\}$, where for two angles $\theta_1, \theta_2 \in \mathbb{R}$, we write $\theta_1 =_{2\pi} \theta_2$ if they are congruent on the angle, i.e., $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$.

Although a single minimum permutation can be computed efficiently [17], the structure of the minimum permutations has not been clarified. Therefore, it has been difficult to construct an efficient algorithm for the minimum composition ordering problem in general (including decreasing linear functions). Theorem 1 provides an interesting achievement that clarifies the structure. Moreover, we can obtain the characterization of the minimum permutations for nondecreasing linear functions by extending Theorem 1.

We note that Theorem 1 can also characterize maximum permutations by replacing "counterclockwise" by "clockwise", which is obtained from a transformation between minimization and maximization. (See (2) in Section 2 and Remark 16 in Section 3). Incidentally, the lexicographical orderings which Kawase et al. [17] introduced can be interpreted as counterclockwise permutations, and they showed the existence of counterclockwise minimum permutations.

These results enable us to count and enumerate all minimum/maximum permutations efficiently.

▶ **Corollary 2.** *The number of the minimum permutations of the minimum composition ordering problem with increasing linear functions can be computed in polynomial time and there exists a polynomial delay algorithm for enumerating all of them.*

We also show the equivalence between the (global) minimality and the local minimality for increasing linear functions, which is of independent interest from an optimization point of view. To introduce the neighborhood of a permutation, let $\sigma : [n] \to [n]$ be a permutation. For three positive integers ℓ, m and r with $\ell \leq m < r$, define a permutation $\sigma_{\ell,m,r} : [n] \to [n]$ by

$$\sigma_{\ell,m,r}(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i < \ell,\ r < i \leq n, \\ \sigma(i - \ell + m + 1) & \text{if } \ell \leq i < \ell - m + r, \\ \sigma(i + m - r) & \text{if } \ell - m + r \leq i \leq r, \end{cases}$$

which is illustrated in Figure 1.

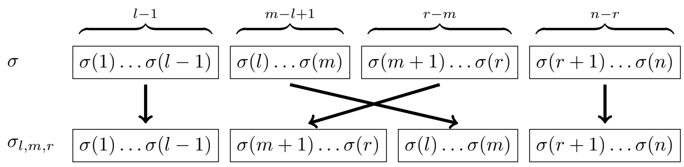

Figure 1 Permutation $\sigma_{l,m,r}$ obtained from σ by swapping two adjacent intervals.

The *neighborhood* $N(\sigma)$ of σ is defined by $N(\sigma) = \{\sigma_{\ell,m,r} \mid \ell \leq m < r\}$, that is, the set of permutations obtained from σ by swapping two adjacent intervals in σ. Note that swapping jobs and considering partial schedules (intervals) can be found in the context of a single machine time-dependent scheduling problem of minimizing the total completion time of linearly deteriorating jobs [10, 21]. A permutation σ is *locally minimum* if $f^\sigma \leq f^\mu$ for any $\mu \in N(\sigma)$, where f^σ is the composite by σ, that is, $f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}$.

▶ **Theorem 3.** *For the minimum composition ordering problem with increasing linear functions, a permutation is (globally) minimum if and only if it is locally minimum.*

The theorem reveals an interesting structural property of composition orderings. We remark that the same results hold if "minimum" is replaced by "maximum" in Corollary 2 and Theorem 3, similarly with Theorem 1. The results also hold if "increasing" is replaced by "nondecreasing".

We then deal with composition orderings for general linear functions. We provide several structural properties of minimum permutations. These, together with the characterization for increasing linear functions, provide a fixed-parameter tractable (FPT) algorithm for the minimum composition ordering problem with general linear functions, with respect to the number of decreasing linear functions.

▶ **Theorem 4.** *A minimum permutation of the minimum composition ordering problem with* n *linear functions can be computed in* $O(k2^k n^6)$ *time, where* $k\,(>0)$ *denotes the number of decreasing linear functions.*

We remark that the FPT algorithm can be modified to count and enumerate all minimum permutations efficiently.

We next consider the multiplication orderings for matrices as a generalization of the composition orderings for linear functions. The problem with matrices is to find a permutation $\sigma : [n] \to [n]$ that minimizes/maximizes $\boldsymbol{w}^\top M_{\sigma(n)} \cdots M_{\sigma(1)} \boldsymbol{y}$ for given n matrices $M_1, \ldots, M_n \in \mathbb{R}^{m\times m}$ and two vectors $\boldsymbol{w}, \boldsymbol{y} \in \mathbb{R}^m$, where m is a positive integer. The problem has been studied in the context of max-plus algebra [3], but despite being a natural problem, to our best knowledge, it has not been explored in the conventional algebra to date.

If we set $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $M_i = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ for any $i \in [n]$, then the matrix multiplication ordering problem is (mathematically) equivalent to the composition ordering problem with linear functions $f_i(x) = a_i x + b_i$, which shows that the matrix multiplication ordering problem is a natural generalization of the composition ordering problem with linear functions.

We obtain the following generalization of the results for linear functions. Matrices $M_1, \ldots, M_n \in \mathbb{R}^{m\times m}$ are called *simultaneously triangularizable* if there exists an invertible matrix $P \in \mathbb{R}^{m\times m}$ such that $P^{-1} M_i P$ is an upper triangular matrix for any $i \in [n]$.

▶ **Theorem 5.** *For the minimum matrix multiplication ordering problem with* n *simultaneously triangularizable* 2×2 *matrices, the following statements hold.*

- **(i)** *If all matrices have nonnegative determinants, then a minimum permutation can be computed in* $O(n \log n)$ *time.*
- **(ii)** *If some matrix has a negative determinant, then a minimum permutation can be computed in* $O(k2^k n^6)$ *time, where* k *denotes the number of matrices with negative determinants.*

As the negative side, we show that all possible natural generalizations turn out to be intractable unless P = NP.

▶ **Theorem 6.**

- **(i)** *The minimum matrix multiplication ordering problem with* 2×2 *matrices is strongly NP-hard, even if all matrices are nonnegative (i.e., all the elements are nonnegative) and have nonnegative determinants.*
- **(ii)** *The minimum matrix multiplication ordering problem with* $m \times m$ *matrices with* $m \geq 3$ *is strongly NP-hard, even if all matrices are nonnegative and upper triangular.*

We also deal with the target version of the matrix multiplication ordering problem, i.e., minimizing the objective function $|\boldsymbol{w}^\top M_{\sigma(n)} \cdots M_{\sigma(1)} \boldsymbol{y} - t|$ for a given target $t \in \mathbb{R}$.

▶ **Theorem 7.** *Given matrices* $M_1, \ldots, M_n$*, two vectors* $\boldsymbol{w}, \boldsymbol{y}$ *and a target* $t \in \mathbb{R}$*, the problem to decide whether there exists a permutation* σ *such that* $|\boldsymbol{w}^\top M_{\sigma(n)} \cdots M_{\sigma(1)} \boldsymbol{y} - t| \leq c_1 \cdot \min_\rho |\boldsymbol{w}^\top M_{\rho(n)} \cdots M_{\rho(1)} \boldsymbol{y} - t| + c_2$ *for any positive* c_1 *and* c_2 *is strongly NP-complete.*

This means that the target version is non-approximable. We can show that the target version is also non-approximable, even if the matrices correspond to increasing linear functions.

We then consider the relationship to matrices in max-plus algebra. Let $\mathbb{R}_{\max}$ be the set $\mathbb{R} \cup \{-\infty\}$ with two binary operations max and + denoted by $\oplus$ and $\otimes$ respectively, i.e., for $a, b \in \mathbb{R}_{\max}$, $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$. The triple $(\mathbb{R}_{\max}, \oplus, \otimes)$ is called *max-plus* *algebra*. We denote by $\mathbb{0}$ the additive identity $-\infty$, and denote by $\mathbb{1}$ the multiplicative identity 0. This notation makes it easier for us to see the correspondence between max-plus algebra and the conventional algebra. The two operations $\oplus$ and $\otimes$ are naturally extended to the matrices on $\mathbb{R}_{\max}$.

Bouquard, Lenté and Billaut [3] dealt with the problem to minimize the objective function

$$\begin{pmatrix} \mathbb{1} & \mathbb{0} & \dots & \mathbb{0} \end{pmatrix} \otimes N_{\sigma(n)} \otimes \cdots \otimes N_{\sigma(1)} \otimes \begin{pmatrix} \mathbb{0} \\ \vdots \\ \mathbb{0} \\ \mathbb{1} \end{pmatrix}, \tag{1}$$

where each N_i is an upper triangular matrix in $\mathbb{R}_{\max}^{m\times m}$. They showed that the problem in the case $m = 2$ is a generalization of the two-machine flow shop scheduling problem to minimize the makespan, and is solvable in $O(n \log n)$ time by using an extension of Johnson's rule [16] for the two-machine flow shop scheduling. Kubo and Nishinari referred to the relationship between the flow shop scheduling and the conventional matrix multiplication [19]. Focusing on this relationship, we show that the following result equivalent to the one of Bouquard et al. is obtained as a corollary of Theorem 5 (i). For a max-plus matrix $N = \begin{pmatrix} a & b \\ \mathbb{0} & d \end{pmatrix}$, where $a, b, d \neq \mathbb{0}$, we introduce $\kappa(N)$ as follows:

$$\kappa(N) = \begin{cases} (-1,\ b-a) & (a > d), \\ (\ \ 0,\ 0) & (a = d), \\ (\ \ 1,\ d-b) & (a < d). \end{cases}$$

▶ **Theorem 8.** *For the minimum max-plus matrix multiplication ordering problem with* $\boldsymbol{w} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \end{pmatrix}^\top$, $\boldsymbol{y} = \begin{pmatrix} \mathbb{0} & \mathbb{1} \end{pmatrix}^\top$, *and* 2×2 *upper triangular matrices, that is, the objective function (1) in the case* $m = 2$*, a minimum permutation can be obtained in the lexicographic order for* κ.

The organization of the paper

The rest of this paper is organized as follows. Section 2 provides some notation and basic properties needed in the paper. In Section 3, we consider composition orderings for increasing linear functions and provide an outline of the proof of Theorem 1. In Section 4 we deal with general linear functions and make the exposition of ideas for an FPT algorithm to prove Theorem 4. In Section 5 we generalize composition of linear functions to matrix multiplication in the conventional algebra and max-plus algebras, and outline the proofs of Theorems 5 and 8.

2 Notation and Basic Properties

In this section, we first fix notation and present several basic properties of linear functions, which will be used in this paper. We then mention that minimum and maximum compositions are polynomially equivalent.

We view a linear function $f(x) = ax + b$ as the vector $\overrightarrow{f} = \begin{pmatrix} b \\ 1-a \end{pmatrix}$ in $\mathbb{R}^2$, and its *angle*, denoted by $\theta(f)$, is defined as the polar angle in $[0, 2\pi)$ of the vector, where we define $\theta(f) = \perp$ if the vector of f is the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., f is the identity function.

For two reals ℓ and r with $\ell < r$, let $[\ell, r] = \{x \in \mathbb{R} \mid l \le x \le r\}$. Similarly, we denote semi-open intervals by $(\ell, r]$ and $[\ell, r)$, and open intervals by (ℓ, r). For a linear function $f(x) = ax + b$, we respectively denote by $\alpha(f)$ and $\beta(f)$ the slope and intercept of $f(x)$, i.e., $\alpha(f) = a$ and $\beta(f) = b$. A linear function f is respectively called *increasing*, *constant*, and *decreasing* if $\alpha(f) > 0$, $\alpha(f) = 0$, and $\alpha(f) < 0$. Since the result of arithmetic operations on angles may take a value outside of $[0, 2\pi)$, we provide some notation to deal with such situations, some of which have already been used in the introduction. For two angles $\theta_1, \theta_2 \in \mathbb{R}$, we write $\theta_1 =_{2\pi} \theta_2$ if they are congruent on the angle, i.e., $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$, and define $[\theta_1, \theta_2]_{2\pi} = \{\theta \in [\lambda_1, \lambda_2] \mid \lambda_1 =_{2\pi} \theta_1,\ \lambda_2 =_{2\pi} \theta_2,\ \lambda_2 - \lambda_1 \in [0, 2\pi)\}$. For example, if $\theta_1 = 3\pi/2$ and $\theta_2 = \pi/3$ then $[3\pi/2, \pi/3]_{2\pi} = \cdots \cup [-\pi/2, \pi/3] \cup [3\pi/2, 7\pi/3] \cup [7\pi/2, 13\pi/3] \cup \cdots$. We similarly define open and semi-open intervals such as $(\theta_1, \theta_2)_{2\pi}$, $[\theta_1, \theta_2)_{2\pi}$, and $(\theta_1, \theta_2]_{2\pi}$. For a non-interval set S, we define $S_{2\pi} = \{\theta \mid \theta =_{2\pi} \lambda \text{ for } \lambda \in S\}$.

We next state four basic properties of linear functions. Note that Lemmas 9, 10, and 11 do not assume increasing linear functions.

▶ **Lemma 9.** *Let g be the identity function, i.e., $g(x) = x$. Then for any function h, we have $h \circ g = g \circ h = h$.*

▶ **Lemma 10.** *For two non-identical linear functions g and h, we have the following two equivalences. The inequality for functions means that the inequality holds for any argument.*

(i) $h \circ g < g \circ h \ \Leftrightarrow\ \theta(h) - \theta(g) \in (0, \pi)_{2\pi}$.

(ii) $h \circ g = g \circ h \ \Leftrightarrow\ \theta(h) - \theta(g) \in \{0, \pi\}_{2\pi}$.

▶ **Lemma 11.** *Let g and h be two linear functions. Then $\overrightarrow{h \circ g} = \vec{h} + \alpha(h)\vec{g}$.*

▶ **Lemma 12.** *For non-identical increasing linear functions g and h, we have the following statements.*

(i) $\theta(h) - \theta(g) \in (0, \pi)_{2\pi} \Leftrightarrow \theta(h \circ g) \in (\theta(g), \theta(h))_{2\pi} \Leftrightarrow \theta(g \circ h) \in (\theta(g), \theta(h))_{2\pi}$.

(ii) $\theta(h) - \theta(g) \in \{0, \pi\}_{2\pi} \Leftrightarrow \theta(h \circ g) \in \{\theta(g), \theta(h), \perp\} \Leftrightarrow \theta(g \circ h) \in \{\theta(g), \theta(h), \perp\}$.

(iii) $\theta(h) = \theta(g) \Rightarrow \theta(h \circ g) = \theta(g \circ h) = \theta(h)\ (= \theta(g))$.

(iv) $\theta(h \circ g) = \perp \Leftrightarrow \theta(g \circ h) = \perp \Rightarrow \theta(h) - \theta(g) =_{2\pi} \pi$.

For linear functions $f_1, \ldots, f_n$ and a permutation $\sigma : [n] \to [n]$, we denote $f_{\sigma(n)} \circ \cdots \circ f_{\sigma(1)}$ by f^σ. Before ending this section, we provide a linear-time transformation between the maximization problem and the minimization problem [17]. For a linear function $f(x) = ax + b$, we define a linear function $\tilde{f}$ by

$$\tilde{f}(x) \ = \ ax - b. \tag{2}$$

Note that the slope of $\tilde{f}$ is the same as that of f. For linear functions $f_1, \ldots, f_n$ and a permutation $\sigma : [n] \to [n]$, we have $\beta(f^\sigma) \ = \ -\beta(\widetilde{f^\sigma})$. Since any permutation $\sigma : [n] \to [n]$ provides $\alpha(f^\sigma) = \alpha(\widetilde{f^\sigma}) = \prod_{i \in [n]} \alpha(f_i)$, we can see that the maximization problem with $f_1, \ldots, f_n$ is equivalent to the minimization problem with $\tilde{f}_1, \ldots, \tilde{f}_n$. Therefore, we mainly deal with the minimization problem with linear functions.

3 Composition of Increasing Linear Functions

In this section, we consider composition orderings for increasing linear functions. Especially, we provide an outline of the proof of Theorem 1.

We first prove Theorem 1 (i), which can be easily obtained from basic properties in Section 2.

Proof of Theorem 1 (i). Let us first show the only-if part. For any $i \in [n-1]$, let $\rho_i : [n] \to [n]$ be the i-th adjacent transposition, i.e., the transposition of two consecutive integers i and $i+1$. Let $\mathrm{id} : [n] \to [n]$ denote the identity permutation. Then we have $f^{\rho_i} = f^{\mathrm{id}}$, since $f_i \circ f_{i+1} = f_{i+1} \circ f_i$ by Lemmas 9 and 10 (ii). It is well-known that any permutation can be obtained by a product of adjacent transpositions and therefore for any permutation σ we obtain $f^\sigma = f^{\mathrm{id}}$, which is minimum.

For the if part, suppose, without loss of generality, that f_1 and f_2 are not collinear. Then we have $f_1 \circ f_2 \neq f_2 \circ f_1$ by Lemma 10 (ii), which implies that $f_1 \circ f_2 \circ (f_n \circ \cdots \circ f_3) \neq f_2 \circ f_1 \circ (f_n \circ \cdots \circ f_3)$, which completes the proof of the if part. ◀

Note that in fact Theorem 1 (i) does not require increasing linear functions, and hence it is true even if f_i's are general linear functions.

The next lemma plays an important role throughout the paper.

▶ **Lemma 13.** *A locally minimum permutation for non-collinear increasing linear functions is counterclockwise.*

By this and the following lemma, we can obtain the proof of Theorem 1 (ii).

▶ **Lemma 14.** *Let $\sigma : [n] \to [n]$ be a counterclockwise permutation for increasing linear functions $f_1, \ldots, f_n$. If it provides the identity, i.e., $f^\sigma(x) = x$, then any of the counterclockwise permutations provides the identity.*

Proof. Let a permutation $\tau : [n] \to [n]$ provide the identity, i.e., $f^\tau(x) = x$. By Lemma 12 (iv), $(f_{\tau(n)} \circ \cdots \circ f_{\tau(k+1)}) \circ (f_{\tau(k)} \circ \cdots \circ f_{\tau(1)}) = (f_{\tau(k)} \circ \cdots \circ f_{\tau(1)}) \circ (f_{\tau(n)} \circ \cdots \circ f_{\tau(k+1)})$ for any $k \in \{0, 1, \ldots, n-1\}$. The permutation producing the right-hand side is $\tau_{1,k,n}$, which we will denote by τ_k and call *k-shift* of τ. The equality means that the composite by τ coincides with the one by its k-shift, that is, $f^\tau = f^{\tau_k}$.

Moreover, for any permutation $\nu : [n] \to [n]$, $\theta(f_{\nu(k)}) = \theta(f_{\nu(k+1)})$ for $k \in [n-1]$ implies that $f^\nu = f^{\nu_{k,k,k+1}}$ by Lemma 10 (ii).

Since any of the counterclockwise permutations is obtained by repeatedly applying adjacent transpositions for the same angles and k-shift of σ, the two claims provide the proof. ◀

Proof of Theorem 1 (ii). (ii-1) $\Longrightarrow$ (ii-2) follows from Lemmas 13 and 14.

For the converse direction, by Lemma 13 we suppose, on the contrary, that all counterclockwise permutations provide the same non-identical function g. Since f_i's are not collinear, there exists a non-identical linear function f_i such that

$$\theta(f_i) \ \notin \{\theta(g), \theta(g) + \pi\}_{2\pi}. \tag{3}$$

Consider a counterclockwise permutation $\sigma : [n] \to [n]$ with $\sigma(1) = i$, and let $h = f_{\sigma(n)} \circ \cdots \circ f_{\sigma(2)}$. Then we have $g = h \circ f_i$. Since $\theta(h) \notin \{\theta(f_i), \theta(f_i) + \pi\}_{2\pi} \cup \{\perp\}$ by (3) and Lemma 11, Lemma 10 (i) implies that $h \circ f_i \neq f_i \circ h$, which contradicts the assumption. ◀

Example 15 demonstrates the optimal condition in Theorem 1 (iii). Since the proof of Theorem 1 (iii) is also involved, we only mention that it relies on the *unimodality* of f^σ for counterclockwise permutations σ.

▶ **Example 15.** Consider the following five increasing linear functions:

$$f_1 = \frac{1}{2}x + 1, \ f_2 = \frac{1}{3}x - 1, \ f_3 = 2x - 2, \ f_4 = 2x - 1, \ \text{and} \ \ f_5 = 3x.$$

Then their vectors are given as follows (See Figure 2):

$$\overrightarrow{f_1} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \ \overrightarrow{f_2} = \begin{pmatrix} -1 \\ 2/3 \end{pmatrix}, \ \overrightarrow{f_3} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \ \overrightarrow{f_4} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \text{ and } \overrightarrow{f_5} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Note that the identity permutation id : $[n] \to [n]$ is counterclockwise for f_i's, and moreover, by Lemma 13, we can see that it is minimum, since

$$f^{\mathrm{id}} = 2x - 23, \ f^{\mathrm{id}_1} = 2x - \frac{27}{2}, \ f^{\mathrm{id}_2} = 2x - \frac{19}{6}, \ f^{\mathrm{id}_3} = 2x - \frac{13}{3}, \ f^{\mathrm{id}_4} = 2x - \frac{23}{3},$$

which also shows that $(f^{\mathrm{id}}, f^{\mathrm{id}_1}, f^{\mathrm{id}_2}, f^{\mathrm{id}_3}, f^{\mathrm{id}_4})$ is unimodal.

We can also see that the identity permutation satisfies $\theta(f^{\mathrm{id}}) + \pi \in [\theta(f_5), \theta(f_1)]_{2\pi}$.

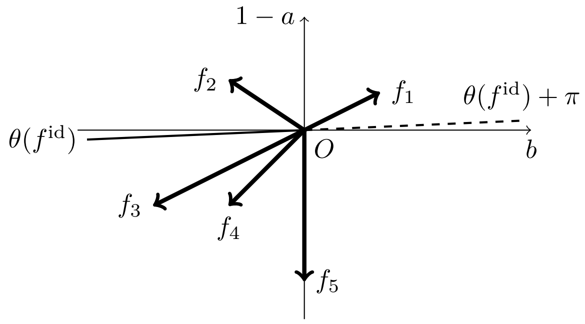

Figure 2 The vector representation for $f_1, \dots, f_5$.

▶ Remark 16. As discussed in Section 2, the maximization for f_i's is equivalent to the minimization for $\widetilde{f_i}$'s given by (2). Thus all the results for increasing functions are applicable for the maximization problem. Since the transformation (2) is the reflection across the $(1-a)$-axis in the vector representation, we can obtain the results by exchanging the term "counterclockwise" by "clockwise".

Corollary 2 is an immediate and direct conclusion of Theorem 1. Theorem 3 is proved by using Theorem 1.

We can generalize increasing linear functions to nondecreasing linear functions in Theorems 1 and 3, and Corollary 2.

4 Composition of General Linear Functions

In this section, we discuss the composition of general linear functions $f_1, \dots, f_n$, where an example of composition for general linear functions is given in Example 17. Let k denote the number of decreasing functions in them, i.e., $k = |\{i \in [n] \mid \alpha(f_i) < 0\}|$. In Section 3 we provided structural characterizations for the minimum permutations when $k = 0$. We present several structural properties for minimum permutations for general linear functions and show ideas for FPT with respect to k for the minimization problem, whose complexity status was open [17].

In the rest of this section, we restrict our attention to the case where no linear function is identity or constant, i.e., $f_i(x) \neq x$ and $\alpha(f_i) \neq 0$ for all $i \in [n]$. Note that the identity function plays no role in minimum composition. For a constant function $f(x) = b$, we consider $f^{(\epsilon)}(x) = \epsilon x + b$ for some $\epsilon > 0$ (we set $f^{(\epsilon)} = f$ for a non-constant function) and can reduce the case containing constant functions to the case of increasing functions. In other words, we can show that the minimality for $f_1^{(\epsilon)}, \dots, f_n^{(\epsilon)}$ implies the one for $f_1, \dots, f_n$, if $|\epsilon|$ is sufficiently small. We remark that our algorithm does not make use of ϵ explicitly, since the orderings of angles $\theta(f_i^{(\epsilon)})$'s are only needed.

▶ **Example 17.** Consider the following seven linear functions:

$$f_1 = \frac{1}{3}x,\ f_2 = \frac{2}{3}x + 1,\ f_3 = x + \frac{1}{2},\ f_4 = -x - 3,\ f_5 = x - 1,\ f_6 = \frac{3}{2}x,\ f_7 = 2x + 1.$$

All but f_4 are increasing. The vector representation is shown in Figure 3.

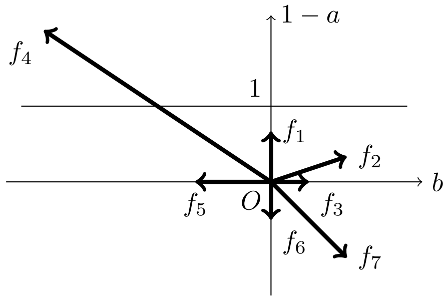

Figure 3 The vector representation for $f_1, \dots, f_7$.

Note that the identity permutation is minimum. Recall that $\theta(f_{\sigma(i+1)}) - \theta(f_{\sigma(i)}) \in [0, \pi]_{2\pi}$ holds for any minimum permutation σ for increasing linear functions by Theorem 3 and Lemma 10. However, this crucial property for increasing linear functions does not hold in general. For example, $\theta(f_2) - \theta(f_1) \in (\pi, 2\pi)_{2\pi}$. Instead, we point out the following properties: $f_3 \circ f_2 \circ f_1$ before f_4 is provided by a maximum permutation for f_1, f_2, and f_3, while $f_7 \circ f_6 \circ f_5$ after f_4 is provided by a minimum permutation for f_5, f_6, and f_7. We also note that f_4 is not suitable for processing time, since both coefficients are negative.

We define two sets L^σ and U^σ of increasing linear functions. For a permutation $\sigma : [n] \to [n]$, let $n_1^\sigma, \dots, n_k^\sigma$ be integers such that $n_1^\sigma < \cdots < n_k^\sigma$ and $\alpha(f_{\sigma(n_j^\sigma)}) < 0$ for all $j \in [k]$. For $j \in \{0, 1, \dots, k\}$, let $I_j^\sigma = \{i \in [n] \mid n_j^\sigma < i < n_{j+1}^\sigma\}$, where $n_0^\sigma = 0$ and $n_{k+1}^\sigma = n + 1$, and define

$$L^\sigma = \bigcup_{k-j:\text{even}} I_j^\sigma \quad \text{and} \quad U^\sigma = \bigcup_{k-j:\text{odd}} I_j^\sigma.$$

By definition, the set of indices of all increasing functions $\{i \in [n] \mid \alpha(f_{\sigma(i)}) \geq 0\}$ is partitioned into L^σ and U^σ. In Example 17, we have $L^{\text{id}} = I_1^{\text{id}} = \{5, 6, 7\}$ and $U^{\text{id}} = I_0^{\text{id}} = \{1, 2, 3\}$.

The following lemma states that L^σ and U^σ are permuted counterclockwisely and clockwisely, respectively, if σ is minimum. Let $L^\sigma = \{\ell_1, \dots, \ell_{|L^\sigma|}\}$ and $U^\sigma = \{u_1, \dots, u_{|U^\sigma|}\}$, where $\ell_1 < \cdots < \ell_{|L^\sigma|}$ and $u_1 < \cdots < u_{|U^\sigma|}$, and let

$$p_i = f_{\sigma(\ell_i)} \quad \text{for} \quad i \in [|L^\sigma|] \quad \text{and} \quad q_i = f_{\sigma(u_i)} \quad \text{for} \quad i \in [|U^\sigma|].$$

▶ **Lemma 18.** *Let $\sigma : [n] \to [n]$ be a minimum permutation for non-constant and non-identical linear functions $f_1, \dots, f_n$. Let p_i $(i \in [|L^\sigma|])$ and q_i $(i \in [|U^\sigma|])$ denote increasing linear functions defined as above. Then we have the following two statements.*

(i) *The identity* $\text{id} : [|L^\sigma|] \to [|L^\sigma|]$ *is counterclockwise for p_i's, unless they are collinear.*

(ii) *The identity* $\text{id} : [|U^\sigma|] \to [|U^\sigma|]$ *is clockwise for q_i's, unless they are collinear.*

Outline of Proof. We only prove the case where k is even and (i), since the odd case or (ii) can be treated similarly.

$$f^\sigma = \overbrace{p_{|L^\sigma|} \circ \cdots \circ p_{|L^\sigma|-|I_k^\sigma|+1}}^{I_k^\sigma} \circ g_{k/2} \circ \cdots \circ g_2 \circ \overbrace{p_{|I_0^\sigma|+|I_2^\sigma|} \circ \cdots \circ p_{|I_0^\sigma|+1}}^{I_2^\sigma} \circ g_1 \circ \overbrace{p_{|I_0^\sigma|} \circ \cdots \circ p_1}^{I_0^\sigma}$$

where $g_j = f_{\sigma(n_{2j}^\sigma)} \circ f_{\sigma(n_{2j}^\sigma - 1)} \circ \cdots \circ f_{\sigma(n_{2j-1}^\sigma)}$ for $j \in \{1, \ldots, \lceil k/2 \rceil\}$,and we set $f_{\sigma(n+1)} = f_{\sigma(0)} = x$. Note that only two linear functions at both ends are decreasing.

Since all the linear functions in the right-hand side are increasing, Theorem 1 implies (i) of the lemma. ◀

Moreover, the following crucial lemma (iii) shows that L^σ and U^σ are partitioned by two angles ψ_1 and ψ_2. For an set $I \subseteq [n]$, let $\theta(I) = \{\theta(f_{\sigma(i)}) \mid i \in I\}$.

▶ **Lemma 19.** *There exists a minimum permutation $\sigma : [n] \to [n]$ for non-constant and non-identical linear functions $f_1, \ldots, f_n$ such that*
- **(i)** $f_{\sigma(\ell)}$ $(\ell \in L^\sigma)$ *are permuted counterclockwisely,*
- **(ii)** $f_{\sigma(u)}$ $(u \in U^\sigma)$ *are permuted clockwisely,*
- **(iii)** $\theta(L^\sigma) \subseteq [\psi_1, \psi_2]$ *and* $\theta(U^\sigma) \subseteq (\psi_2, \psi_1)_{2\pi}$ *for some two angles* $\psi_1 \in (0, \pi)$ *and* $\psi_2 \in (\pi, 2\pi)$*,*
- **(iv)** $\theta(I_s^\sigma) \cap \theta(I_t^\sigma) = \emptyset$ *for any distinct s and t.*

This directly implies that a minimum permutation for linear functions $f_1, \ldots, f_n$ can be computed in $O(k!\, n^{k+4})$ time, where k denotes the number of decreasing f_i's. The reason is as follows. Assume first that no f_i is identity and we utilize $f_i^{(\epsilon)}$'s instead of f_i's. By Lemma 19 (iii), we essentially have n^2 possible angles ψ_1 and ψ_2. Based on such angles, we partition the set of indices of increasing linear functions into $I_0, \ldots, I_k$. By Lemma 19 (i), (ii), and (iv), we have at most n^{k+1} many such partitions. Since there exist $k!$ orderings of decreasing functions, by checking at most $k!\, n^{k+3} (= n^2 \times n^{k+1} \times k!)$ permutations σ, we obtain a minimum permutation for f_i's. Note that each such permutation σ and the composite f^σ can be computed in $O(n)$ time, after sorting $\theta(f_i^{(\epsilon)})$'s. Since $\theta(f_s^{(\epsilon)})$ and $\theta(f_t^{(\epsilon)})$ can be compared in $O(1)$ time for sufficiently small $\epsilon > 0$ without exactly computing their angles, we can sort $\theta(f_i^{(\epsilon)})$'s in $O(n \log n)$ time. Thus in total we require $O(k!\, n^{k+4} + n \log n) = O(k!\, n^{k+4})$ time. If some f_i's are identities, then we can put them into I_0, where I_0 is obtained in the procedure above for the non-identical functions. Therefore, a minimum permutation can be computed in $O(k!\, n^{k+4})$ time.

In order to improve this XP result, namely, to have an FPT algorithm with respect to k, we apply the dynamic programming approach to the following problem.

Problem LU-Ordered Minimum Composition

Input: Two sets of increasing linear functions $L = \{p_1, \ldots, p_{|L|}\}$ and $U = \{q_1, \ldots, q_{|U|}\}$, and decreasing linear functions $g_1, \ldots, g_k$ with $k > 0$.

Output: A minimum permutation σ for linear functions in $L \cup U \cup \{g_1, \ldots, g_k\}$ such that
- **(i)** $L^\sigma = L$ and $U^\sigma = U$,
- **(ii)** the restriction of σ on L produces the ordering $(p_1, \ldots, p_{|L|})$, and
- **(iii)** the restriction of σ on U produces the ordering $(q_1, \ldots, q_{|U|})$.

Note that a minimum permutation for the original problem can be computed by solving **Problem** LU-Ordered Minimu Composition $O(n^4)$ times for $|L| + |U| \le n - k$. Since the problem can be solved in $O(2^k k(|L| + |U| + k)^2)$ time, we obtain Theorem 4.

5 Matrix Multiplication

In this section, we consider matrix multiplication orderings as a generalization of composition orderings for linear functions. We provide outlines of the proofs of Theorems 5 and 8, and refer to the problems we use to prove Theorems 6 and 7.

In order to prove Theorem 5, we first assume that matrices $M_1, \ldots, M_n$ in $\mathbb{R}^{2\times 2}$ are all upper triangular. Then we have the following lemma.

▶ **Lemma 20.** *The minimum matrix multiplication ordering problem with 2×2 upper triangular matrices can be reduced to the one with $\boldsymbol{w}^\top = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\boldsymbol{y}^\top = \begin{pmatrix} 0 & 1 \end{pmatrix}$, and 2×2 upper triangular matrices with positive $(2, 2)$-entries.*

Thus we can assume that a given upper triangular matrix $M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}$ has a positive d_i for $i \in [n]$. We then have

$$\begin{pmatrix} 1 & 0 \end{pmatrix} M^\sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\prod_{i=1}^{n} d_i \right) f^\sigma(0),$$

where $f_i(x) = (a_i/d_i)x + b_i/d_i$ for $i \in [n]$. This implies that the minimum matrix multiplication ordering problem with 2×2 upper triangular matrices can be solved by solving the minimum composition ordering problem with linear functions. We remark that our algorithm concerns the comparison of polar angles $\theta(f_i)$'s, but not of the vectors $\begin{pmatrix} b_i/d_i \\ 1 - a_i/d_i \end{pmatrix}$, and hence we do not need to care about the case where $d_i = \epsilon$. Therefore, we have the following lemma.

▶ **Lemma 21.** *For the minimum matrix multiplication ordering problem with n 2×2 upper triangular matrices, we have the following statements.*

(i) *If all matrices have nonnegative determinants, then a minimum permutation can be computed in $O(n \log n)$ time.*

(ii) *If some matrix has a negative determinant, then a minimum permutation can be computed in $O(k2^k n^6)$ time, where k denotes the number of matrices with negative determinants.*

This immediately implies Theorem 5.

Unfortunately, this positive results cannot be extended to 1) the nonnegative determinant case for $m = 2$, 2) the case of $m \geq 3$, and 3) the target version; see Theorem 6 (i), (ii) and Theorem 7. We use the 3-partition problem to prove Theorems 6 (i) and 7.

Bouquard et al. [3] showed that the problem to minimize (1) for the case $m \geq 3$ is strongly NP-hard by reduction from the three-machine flow shop scheduling problem to minimize the makespan, which is known to be strongly NP-hard [7]. We use the former problem to prove Theorem 6 (ii).

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