A Quadratic Upper Bound on the Reset Thresholds of Synchronizing Automata Containing a Transitive Permutation Group

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Abstract

For any synchronizing n-state deterministic automaton, Černý conjectures the existence of a synchronizing word of length at most $(n-1)^2$. We prove that there exists a synchronizing word of length at most $2n^2 - 7n + 7$ for every synchronizing n-state deterministic automaton that satisfies the following two properties: 1. The image of the action of each letter contains at least $n-1$ states; 2. The actions of bijective letters generate a transitive permutation group on the state set.

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1 Introduction

1.1 Synchronizing automata and Černý Conjecture

Let Q be a set. Denote the set of all mappings from Q to itself by $\mathsf{T}(Q)$. For the purposes of this article, an *automaton* $\mathcal{A}$ is a triple (Q, Σ, δ) where Q and Σ are two finite sets, δ is a mapping from Σ to $\mathsf{T}(Q)$. The elements of Q are called *states* of $\mathcal{A}$; the elements of Σ are called *letters* of $\mathcal{A}$; and δ is called the *transition function* of $\mathcal{A}$. For a mapping $f : X \to Y$ and $x \in X$, we denote the value of f at x by $x.f$ or $f(x)$. When the transition function δ is clear from the context, to simplify notations, $q.(\delta(a))$ will be shortened to $q.a$ where $q \in Q$ and $a \in \Sigma$. For subsets $P \subseteq Q$ and $A \subseteq \Sigma$, write $P.A$, or $P.a$ if $A = \{a\}$, for the set $\{p.a : p \in P, a \in A\}$.

Let X be a set. Finite sequences over X (including the empty sequence denoted by ϵ) are called *words*. For each nonnegative integer i, write X^i ($X^{\leq i}$, respectively) for the set of words of length i (at most i, respectively). Denote the set of all words over X by X^*.

The transition function δ extends to the mapping of the set of finite words Σ^* on $\mathsf{T}(Q)$ (still denoted by δ) via the recursion: $q.\epsilon = q$ and $q.(wa) = (q.w).a$ for every $w \in \Sigma^*$, $a \in \Sigma$ and $q \in Q$.

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. A word $w \in \Sigma^*$ is a *reset word* if $|Q.w| = 1$. An automaton that admits a reset word is called a *synchronizing* automaton. The minimum length of reset words for $\mathcal{A}$ is called the *reset threshold* of $\mathcal{A}$, denoted $\mathsf{rt}(\mathcal{A})$. For example, Figure 1 shows a synchronizing automaton $\mathcal{C}_4$ with the state set $\{1, 2, 3, 4\}$ and two letters a and b and transition function δ such that

$$\delta(i, a) = i.a = \begin{cases} 1 & \text{if } i = 4, \\ i & \text{otherwise;} \end{cases} \quad \text{and} \quad \delta(i, b) = i.b = \begin{cases} 1 & \text{if } i = 4, \\ i+1 & \text{otherwise,} \end{cases}$$

for $i \in \{1, 2, 3, 4\}$. The shortest reset word of $\mathcal{C}_4$ is ab^3ab^3a and $\mathsf{rt}(\mathcal{C}_4) = 9$.

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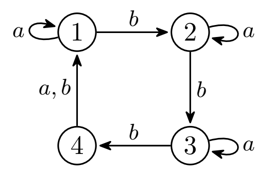

Figure 1 The automaton $\mathcal{C}_4$.

The following conjecture is the most famous conjecture of synchronizing automata.

▶ **Conjecture 1** (Černý-Starke). *Let $\mathcal{A}$ be an n-state synchronizing automaton. Then* $\mathsf{rt}(\mathcal{A}) \leq (n-1)^2$.

This conjecture is usually called Černý Conjecture [6], although it was first published in 1966 by Starke [21]. Regarding the history of Conjecture 1, we recommend [27, Section 3.1].

Černý [5] showed that there exists an n-state automaton with the reset threshold equal to $(n-1)^2$ for every n. That means the upper bound in Conjecture 1 is optimal.

For a long time, the best upper bound of reset thresholds was $\frac{n^3-n}{6}$, obtained by Pin and Frankl [9, 14]. In 2018, Szykuła [24] improved the Pin-Frankl bound. Based on Szykuła's method, Shitov [20] made a further improvement and obtained the new upper bound $cn^3 + o(n^3)$, where the coefficient c is close to 0.1654.

Although Černý Conjecture is widely open in general, it has been shown to be true in many special classes e.g. [7, 11, 25, 26]. For a summary of the state-of-the art around the Černý Conjecture, we recommend the two surveys [12, 27].

1.2 Automata containing transitive groups and our contribution

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. The *defect* of a word $w \in \Sigma^*$ is the integer $|Q| - |Q.w|$. For a non-negative integer i, write Σ_i for the set of letters of defect i.

Let $A \subseteq \Sigma_0$. Observe that δ induces a homomorphism from the free monoid A^* to the *symmetric group* $\mathsf{Sym}(Q)$. We say that $\mathcal{A}$ *contains* the permutation group $\delta(A^*)$ with the generating set $\delta(A)$. A subgroup G of $\mathsf{Sym}(Q)$ is *transitive* if $q.G = \{q.g : g \in G\} = Q$ for each $q \in Q$. We use $\mathbb{ST}$ to denote the family of synchronizing automata that contain a transitive permutation group on its state set. Note that the automaton $\mathcal{C}_4$ displayed in Figure 1 belongs $\mathbb{ST}$.

In this article, we focus on automata in $\mathbb{ST}$. Many subfamilies of $\mathbb{ST}$ have been studied in detail [1, 16, 17, 18, 19, 15, 22]. We introduce two important results that are strongly relevant to this article.

For any $A \subseteq \Sigma_0$, the minimum integer d such that $\delta(A^d) = \delta(A^*)$ is denoted by $\mathsf{d}_{\mathcal{A}}(A)$. Observe that $\mathsf{d}_{\mathcal{A}}(A)$ is the diameter of the Cayley digraph of the group $\delta(A^*)$ with the generating set $\delta(A)$. As an example, a Cayley graph of the symmetric group $\mathsf{Sym}(\{1,2,3\})$ is depicted in Figure 2.

The following theorem is essentially contained in the results of Rystsov [16].

▶ **Theorem 2** (Rystsov). *Let $\mathcal{A} = (Q, \Sigma, \delta) \in \mathbb{ST}$ be an n-state automaton. Then* $\mathsf{rt}(\mathcal{A}) \leq 1 + (n-2)(n-1+\mathsf{d}_{\mathcal{A}}(A))$, *where $A \subseteq \Sigma_0$ such that $\delta(A^*)$ is transitive.*

Araújo, Cameron and Steinberg [1, Theorem 9.2], using representation theory over the field of rationals $\mathbb{Q}$, have improved Rystsov's bound as displayed in Theorem 3. It is worth mentioning that a similar result can be also found in [22, Theorem 3.4].

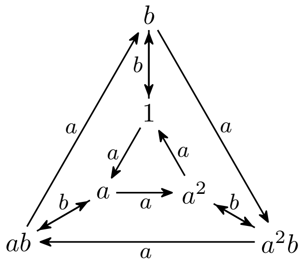

Figure 2 The Cayley digraph of $\mathsf{Sym}(\{1,2,3\})$ with the generating set $\{a,b\}$, where $a=(123)$ and $b=(12)$. Its vertex set is $\mathsf{Sym}(\{1,2,3\})$. For any two vertices x,y and a generator $g\in\{a,b\}$, there exists an arc with label g from x to y if $xg=y$.

▶ **Theorem 3** (Araújo-Cameron-Steinberg)**.** *Let $\mathcal{A}=(Q,\Sigma,\delta)\in\mathbb{ST}$ be an n-state automaton. Then $\mathsf{rt}(\mathcal{A})\le 1+(n-2)(n-m+\mathsf{d}_{\mathcal{A}}(A))$, where $A\subseteq\Sigma_0$ such that $\delta(A^*)$ is transitive and m is the maximum dimension of an irreducible $\mathbb{Q}(\delta(A^*))$-module of $\mathbb{Q}^Q$*

In the case that $\mathsf{d}_{\mathcal{A}}(A)$ is small (a linear function of n), Theorems 2 and 3 bound $\mathsf{rt}(\mathcal{A})$ from above by a quadratic function of n, or even verify Černý Conjecture [16, 1, 22]. However, generally, $\mathsf{d}_{\mathcal{A}}(A)$ is not a linear function of n: in the case that $A=\{(12),(12,\dots,n)\}$, we have $\mathsf{d}_{\mathcal{A}}(A)\approx\frac{3}{4}n^2$ (asymptotically) [28].

In this article, we obtain Theorem 13 which improves Theorem 2 in a different way. As an application of Theorem 13, we obtain the following result.

▶ **Theorem 4.** *Let $\mathcal{A}\in\mathbb{ST}$ be an n-state automaton. If $\Sigma=\Sigma_0\cup\Sigma_1$, then $\mathsf{rt}(\mathcal{A})\le 2n^2-7n+7$.*

1.3 Approach and layout

To prove Theorems 4 and 13, we use the so-called *extension method* which is based on Proposition 5. The proof of Proposition 5 can be found in many papers (e.g. [27, Section 3.4]).

Let $\mathcal{A}=(Q,\Sigma,\delta)$ is an automaton. For a subset $S\subseteq Q$ and a word $w\in\Sigma^*$, write $S.w^{-1}$ for the set $\{q\in Q : q.w\in S\}$, that is, the set of states from which upon reading w, the automaton reaches a state in S. A subset $S\subseteq Q$ is *extended* by a word $w\in\Sigma^*$ if $|S.w^{-1}|>|S|$. A subset $S\subseteq Q$ is called *m-extensible*, if S is extended by a word of length at most m.

▶ **Proposition 5.** *Let $\mathcal{A}=(Q,\Sigma,\delta)$ be a synchronizing automaton. If every nonempty proper subset S of Q is m-extensible, then $\mathsf{rt}(\mathcal{A})\le 1+(n-2)m$.*

The remaining of this article will proceed as follows. In Section 2, combining the extension method and a dimensional argument for a linear structure, we establish a upper bound for the reset thresholds of automata in $\mathbb{ST}$, see Theorem 13. In Section 3, using some graph theoretical techniques, we present a proof of Theorem 4. Using these graph theoretical techniques, we can slightly improve some known results about reset thresholds. At the end, we summarize our results in Section 4.

2 Linear structure

In this section, we will encode some information of an n-state synchronizing automaton into some objects in $\mathbb{Q}^n$. Using the linear structure of $\mathbb{Q}^n$, we will obtain a upper bound for its reset threshold.

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an n-state automaton. We always assume that $Q = \{1, \ldots, n\}$. Fix a subset A of Σ_0 and denote $\delta(A^*)$ by G.

Firstly, let us recall some concepts in linear space. Let $X \subseteq \mathbb{Q}^n$. The linear subspace *spanned* by X, denoted by $\mathsf{span}(X)$, is defined as

$$\mathsf{span}(X) := \left\{ \sum_{x \in X} c_x x : c_x \in \mathbb{Q} \right\}.$$

The *cone* generated by X, denoted $\mathsf{cone}(X)$, is the set

$$\mathsf{cone}(X) := \left\{ \sum_{x \in X} c_x x : c_x \in \mathbb{Q}_{\geq 0} \right\}$$

where $\mathbb{Q}_{\geq 0}$ is the set of non-negative rationals. Write $\langle \cdot, \cdot \rangle$ for the *standard inner product* of $\mathbb{Q}^n$, that is the map such that $\langle x, y \rangle = \sum_{i=1}^n x(i)y(i)$ for every $x, y \in \mathbb{Q}^n$. The *polar cone* of X, denoted X°, is the set

$$X^\circ := \{ y \in \mathbb{Q}^n : \langle x, y \rangle \leq 0, \forall x \in X \}.$$

If X is a linear subspace of $\mathbb{Q}^n$, the *orthogonal complement* of X, denoted $X^\perp$, is defined as

$$X^\perp = \{ y : \langle x, y \rangle = 0, \forall x \in X \}.$$

It clearly holds $X^\circ = X^\perp$ in the case that X is a linear subspace.

Next we will encode some information of an n-state synchronizing automaton into some objects in $\mathbb{Q}^n$. For a subset $S \subseteq Q$, define $\mathbf{1}_S$ to be the $(1 \times n)$-vector over $\mathbb{Q}$ such that its i-th coordinate is

$$\mathbf{1}_S(i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

For any $q \in Q$, to simplify notation, we write $\mathbf{1}_q$ for $\mathbf{1}_{\{q\}}$. Let w be an arbitrary word in Σ^*. Define $[w]$ to be the $(n \times n)$-matrix over $\mathbb{Q}$ such that $\mathbf{1}_q[w] = \mathbf{1}_{q.w^{-1}}$ for all $q \in Q$. It is clear that $\mathbf{1}_S[w] = \mathbf{1}_{S.w^{-1}}$ for all $S \subseteq Q$. Define $\mathbf{k}_w$ for the $(1 \times n)$-vector over $\mathbb{Q}$ such that its i-th coordinate is

$$\mathbf{k}_w(i) = |i.w^{-1}| - 1.$$

For every $g \in G$, set $\mathbf{k}_g$ to be $\mathbf{k}_w$ and $[g]$ to be $[w]$, where $w \in \Sigma^*$ is an arbitrary word such that $\delta(w) = g$.

▶ **Example 6.** Consider the automata $\mathcal{C}_4$ (see Figure 1) and the words a, b and ab. One can calculate that

$$\begin{aligned} 1.a^{-1} = \{1,4\}, \quad 2.a^{-1} = \{2\}, \quad 3.a^{-1} = \{3\}, \quad 4.a^{-1} = \emptyset; \\ 1.b^{-1} = \{4\}, \quad 2.b^{-1} = \{1\}, \quad 3.b^{-1} = \{2\}, \quad 4.b^{-1} = \{1\}; \\ 1.ab^{-1} = \emptyset, \quad 2.ab^{-1} = \{1,4\}, \quad 3.ab^{-1} = \{2\}, \quad 4.ab^{-1} = \{3\}. \end{aligned}$$

And then $\mathbf{k}_a = (1, 0, 0, -1)$, $\mathbf{k}_b = (0, 0, 0, 0)$, and $\mathbf{k}_{ab} = (-1, 1, 0, 0)$.

A sequence $(X_i)_{i \geq 0}$ is called *eventually constant* if there exists an integer $j \geq 0$ such that for all $k > j$, $X_k = X_j$. For an eventually constant sequence $\mathcal{X} = (X_i)_{i \geq 0}$, the minimum integer j such that for all $k > j$, $X_k = X_j$ is called *transient length* of $\mathcal{X}$, denoted by $\mathsf{len}(\mathcal{X})$

and we denote the *limit* of the sequence $\mathcal{X}$ by $\lim \mathcal{X}$ which clearly equals $X_{\mathsf{len}(\mathcal{X})}$. In the following, we will define some eventually constant sequences which play a crucial role in our proof.

Define $\mathcal{T}(\mathcal{A}, A) = (T_i)_{i\geq 0}$ and $\mathcal{K}(\mathcal{A}, A) = (K_i)_{i\geq 0}$, to be the two sequences such that

$$T_i := \left\{\mathbf{k}_w : w \in (\Sigma \setminus \Sigma_0)A^{\leq i}\right\} \quad \text{and} \quad K_i := \mathsf{cone}(T_i),$$

for every $i \geq 0$. To simplify notations, without ambiguity, we write $\mathcal{T}$ and $\mathcal{K}$ for $\mathcal{T}(\mathcal{A}, A)$ and $\mathcal{K}(\mathcal{A}, A)$, respectively.

Since $\langle A\rangle$ is a finite group, both $\mathcal{T}$ and $\mathcal{K}$ are eventually constant. Denote $\lim \mathcal{T}$ and $\lim \mathcal{K}$ by T_∞ and K_∞, respectively. Observe that $K_\infty = \mathsf{cone}(T_\infty)$ and then

$$\mathsf{len}(\mathcal{T}) \geq \mathsf{len}(\mathcal{K}). \tag{1}$$

We begin with some elementary results. According to the definition, it clearly holds that

$$|S.w^{-1}| - |S| = \langle 1_S, \mathbf{k}_w\rangle \tag{2}$$

for every $S \subseteq Q$ and $w \in \Sigma^*$. As a consequence, we have the following lemma.

▶ **Lemma 7.** *Let $S \subseteq Q$.*

1. *The subset S is extended by w if and only if $\langle 1_S, \mathbf{k}_w\rangle > 0$.*
2. *If $1_S \in (K_\infty)^\circ$ then $|S.a^{-1}| = |S|$ for all $a \in \Sigma$.*
3. *For every vector $x \in K_\infty$, $\sum_{i=1}^{n} x(i) = \langle 1_Q, x\rangle = 0$.*

We say that $\mathcal{A}$ is *strongly connected* if for every two states p and q, there exists a word $w \in \Sigma^*$ such that $p.w = q$.

▶ **Lemma 8.** *Assume that $\mathcal{A} = (Q, \Sigma, \delta)$ is synchronizing and strongly connected. Let S be a nonempty proper subset of Q. If $1_S \in (K_\infty)^\circ$, then there exists a word $w \in \Sigma^*$ such that $1_{S.w^{-1}} \notin K^\circ$.*

Proof. Since $\mathcal{A}$ is synchronizing and strongly connected, there exists a word

$$u = u_1 u_2 \cdots u_t \in \Sigma^*$$

such that $S.u^{-1} = Q$. Since $|S| < |Q|$, by Lemma 7 Item 2, there exists an integer $t' < t$ such that $1_{S.v^{-1}} \notin (K_\infty)^\circ$, where $v = u_1 u_2 \cdots u_{t'}$. ◀

Due to Lemma 8, we can define $\ell(S)$ to be the length of a shortest word w such that $1_{S.w^{-1}} \notin (K_\infty)^\circ$ for every nonempty proper subset $S \subsetneq Q$.

▶ **Proposition 9.** *Assume that $\mathcal{A}$ is synchronizing and strongly connected. Let S be a nonempty proper subset of Q. Then S is $(\mathsf{len}(\mathcal{K}) + \ell(S) + 1)$-extensible.*

Proof. Let $k = \mathsf{len}(\mathcal{K})$ and $\ell = \ell(S)$. By Lemma 8, there exists an ℓ-length word $w = (w_1, \ldots, w_\ell) \in \Sigma^*$ such that $1_S .w^{-1} \notin (K_\infty)^\circ$. Since

$$S.(w_2, \ldots, w_\ell)^{-1} \in (K_\infty)^\circ,$$

by Lemma 7 Item 2, $|S.w^{-1}| = |S|$.

Let $P = S.w^{-1}$. Since $1_P \notin (K_\infty)^\circ$, there exists a vector $x \in K_\infty$ such that $\langle x, 1_P\rangle > 0$. Since $K_\infty = \mathsf{cone}(T_k)$, there exists a vector $y \in T_k$ such that $\langle y, 1_P\rangle > 0$. By the definition of T_k, we can find a word $u \in (\Sigma \setminus \Sigma_0)A^{\leq k}$ such that $y = \mathbf{k}_u$. By Lemma 7 Item 1, $|P.u^{-1}| > |P| = |S|$. Hence, uw extends S and then S is $(k + \ell + 1)$-extensible. ◀

If $\mathsf{len}(\mathcal{K})$ and $\ell(S)$ can be bounded by a linear function of n, using Proposition 5, one can bound $\mathsf{rt}(\mathcal{A})$ by a quadratic function of n. In general, it is hard to estimate $\mathsf{len}(\mathcal{K})$ and $\ell(S)$ of an automaton. However, in the next section, we will establish some linear bounds for $\mathsf{len}(\mathcal{K})$ with the assumption $\mathcal{A} \in \mathbb{ST}$ and $\Sigma = \Sigma_0 \cup \Sigma_1$. And, in the rest of this section, we will establish the following linear bound for $\ell(S)$ in the case that K_∞ is a linear subspace of $\mathbb{Q}^n$.

▶ **Proposition 10.** *Assume that $\mathcal{A}$ is synchronizing and strongly connected. If K_∞ is a linear subspace of $\mathbb{Q}^n$, then $\ell(S) \leq n - 1 - \mathsf{dim}(K_\infty)$ for every nonempty proper subset $S \subsetneq Q$.*

Before proving Proposition 10, we show that "G is transitive" implies "K_∞ is a linear subspace of $\mathbb{Q}^n$".

▶ **Lemma 11.** *If G is transitive, then K_∞ is the linear subspace spanned by T_∞.*

Proof. It is sufficient to prove $-x \in K_\infty = \mathsf{cone}(T_\infty)$ for each $x \in T_\infty$. Take an arbitrary $x \in T_\infty$ and let

$$y := \sum_{g \in G} x[g].$$

It is clear that $y \in K_\infty$. Let i and j be two arbitrary integers in $\{1, \ldots, n\}$. Since G is transitive, there exists $h \in G$ such that $i.h = j$. Note that $y[h](i) = y(j)$ and

$$y[h] = \sum_{g \in G} x[g][h] = \sum_{g \in G} x[g] = y.$$

Then $y(i) = y(j)$. By the arbitrariness of i and j, it holds that $y = c\,\mathbf{1}_Q$ for some $c \in \mathbb{Q}$. Since $y \in K_\infty$, by Lemma 7 Item 3, we have $\sum_{i=1}^{n} y(i) = 0$. This implies $c = 0$ and then

$$-x = \sum_{g \in G \setminus \{\mathsf{id}\}} x[g],$$

where id is the identity map in $\mathsf{Sym}(Q)$. Since $x[g] \in T_\infty$ for all $g \in G$, we obtain that $-x \in K_\infty$ as wanted. ◀

Now, we go back to prove Proposition 10. The following dimension argument plays a crucial role in our proof.

▶ **Lemma 12.** *Let L be a subspace of $\mathbb{Q}^n$ and a non-zero vector $x \in L$. If there exists a word $w \in \Sigma^*$ such that $x[w] \notin L$ then there exists a word $w' \in \Sigma^*$ such that $x[w'] \notin L$ and $|w'| \leq \mathsf{dim}(L)$.*

Proof. For every nonnegative integer i, define $L_i := \mathsf{span}(xv : v \in \Sigma^{\leq i})$. Observe that there exists a unique integer j such that

$$L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_j = L_{j+1} = \cdots$$

Let t be the minimum integer such that $L_t \nsubseteq L$. Observing that $t \leq j$, we have

$$t = \mathsf{dim}(L_0) + t - 1 \leq \mathsf{dim}(L_{t-1}) \leq \mathsf{dim}(L).$$

Then there exists a word w' of length $\leq \mathsf{dim}(L)$ such that $x[w'] \notin L$. ◀

Proof of Proposition 10. Since K_∞ is a linear subspace of $\mathbb{Q}^n$, it holds that $(K_\infty)^\circ = (K_\infty)^\perp$ is also a linear subspace of $\mathbb{Q}^n$. Observe that $1_Q \in (K_\infty)^\circ$. Then we can decompose $(K_\infty)^\circ$ as $(K_\infty)^\circ = V_0 \oplus V_1$, where

$$V_0 = \{x \in (K_\infty)^\circ : \langle 1_Q, \mathbf{k}_w \rangle = 0\} \quad \text{and} \quad V_1 = \mathsf{span}(1_Q).$$

For every subset $R \subseteq Q$, define $\mathbf{p}_R := 1_R - \frac{|R|}{n} 1_Q$. For each $R \subseteq Q$, observe that $1_R \in (K_\infty)^\circ$ if and only if $\mathbf{p}_R \in V_0$.

Let S be a nonempty proper subset of Q such that $1_S \in (K_\infty)^\circ$. Since $\mathcal{A}$ is synchronizing and strongly connected, let w' be a reset word such that $Q.w' \subseteq S$. Then

$$\mathbf{p}_S[w'] = 1_S[w'] - \frac{|S|}{n} 1_Q[w'] = \left(1 - \frac{|S|}{n}\right) 1_Q \notin V_0.$$

Let $w = va$ be a shortest word such that $\mathbf{p}_S[w] \notin V_0$. Lemma 12 provides that the length of w is at most $\mathsf{dim}(V_0) = n - 1 - \mathsf{dim}(K_\infty)$.

We will complete the proof by showing $1_{S.w^{-1}} \notin (K_\infty)^\circ$. Note that $\mathbf{p}_S[v] = 1_{S.v^{-1}} - \frac{|S|}{n} 1_Q$. Since $\mathbf{p}_S[v], 1_Q \in (K_\infty)^\circ$, we have $1_{S.v^{-1}} \in (K_\infty)^\circ$. By Lemma 7 Item 2, $|S| = |S.v^{-1}| = |S.w^{-1}|$. Hence,

$$\mathbf{p}_{S.w^{-1}} = 1_{S.w^{-1}} - \frac{|S.w^{-1}|}{n} 1_Q = 1_{S.w^{-1}} - \frac{|S|}{n} 1_Q = \mathbf{p}_S[w] \notin V_0$$

which is equivalent to $1_{S.w^{-1}} \notin (K_\infty)^\circ$. ◀

Combining Propositions 5, 9, and 10 and Lemma 11, we establish the following bound.

▶ **Theorem 13.** *Let $\mathcal{A} = (Q, \Sigma, \delta) \in \mathbb{ST}$ be an n-state automaton. Then $\mathsf{rt}(\mathcal{A}) \leq 1 + (n - 2)(n - \mathsf{dim}(K_\infty) + \mathsf{len}(\mathcal{K}(\mathcal{A}, A)))$, where $A \subseteq \Sigma_0$ such that $\delta(A^*)$ is transitive.*

▶ Remark 14. Note that $\mathsf{dim}(K_\infty) \geq 1$ and $\mathsf{d}_{\mathcal{A}}(A) \geq \mathsf{len}(\mathcal{K}(\mathcal{A}, A))$.

1. Theorem 13 improves Theorem 2.
2. One of Theorem 3 and Theorem 13 cannot deduce the other one. If we only want to establish a quadratic upper bound for reset thresholds of a special class of automata, it is easier to establish a linear upper bound for $\mathsf{len}(\mathcal{K}(\mathcal{A}, A))$ than for $\mathsf{d}_{\mathcal{A}}(A)$. In this sense, Theorem 13 may have more advantages.

3 Rystsov digraphs

This section is divided into two parts:

- In Section 3.1, we establish some results for digraphs.
- In Section 3.2, we derive some directed graphs from automata. Using the results in Section 3.1 and Theorem 13, we prove Theorem 4.

3.1 Digraphs

Firstly, we recall some notations of digraphs. A *digraph* $\Gamma = (V, E)$ is an ordered pair of sets such that $E \subseteq V \times V$. The set V is called the *vertex set* of Γ and the E is called the *arc set* of Γ. We assume that the digraphs in this section are *loop-free*, that is, (v, v) is not an arc for every vertex v. Let u and v be two vertices of Γ. A sequence of vertices $(v_1, \ldots, v_t)$ is called a *path* from u to v if $v_1 = u$, $v_t = v$ and $(v_i, v_{i+1}) \in E$ for every $1 \leq i \leq t - 1$. We say that u and v are *connected* if there exists a sequence of vertices $(v_1, \ldots, v_t)$ such that $v_1 = u$, $v_t = v$ and either $(v_i, v_{i+1}) \in E$ or $(v_{i+1}, v_i) \in E$ for every $1 \le i \le t-1$. The *strong connectivity* of Γ, denoted $\mathsf{sc}(\Gamma)$, is the equivalence relation such that $(u, v) \in \mathsf{sc}(\Gamma)$ if and only if there exist a path from u to v and a path from v to u. The *weak connectivity* of Γ, denoted $\mathsf{wc}(\Gamma)$, is the equivalence relation such that $(u, v) \in \mathsf{wc}(\Gamma)$ if and only if u, v are connected. A strongly connected component C of Γ is called a *sink component* of Γ if there is no arc $(u, v) \in E$ such that $u \in C$ and $v \notin C$; is called a *source component* of Γ if there is no arc $(u, v) \in E$ such that $u \notin C$ and $v \in C$. A strongly connected component is *non-sink* (*non-source*, resp.) if it is not a sink (source, resp.) component of Γ. Write $\mathsf{SCC}(\Gamma)$ ($\mathsf{WCC}(\Gamma)$, $\mathsf{SinkC}(\Gamma)$, resp.) for the set of strongly connected components (weakly connected components, sink components, resp.) of Γ. Equivalence classes of $\mathsf{sc}(\Gamma)$ ($\mathsf{wc}(\Gamma)$ resp.,) are called a *strongly connected components* (*weakly connected components* resp.,) of Γ.

Let $V = \{1, \ldots, n\}$ and $A \subseteq \mathsf{Sym}(V)$. We will say that the sequence $\Gamma_0, \Gamma_1, \ldots$ is an *A-growth* of Γ_0 if

- $\Gamma_0 = (V, E_0)$ is a digraph with vertex set V;
- for every positive integer i, $\Gamma_i = (V, E_i)$ is the digraph such that $E_i = \{(p.w, q.w) : (p, q) \in E_0, w \in A^{\le i}\}$.

This concept, also called Rystsov Digraphs, was firstly used in [17], and appears widely in the research of synchronizing automata [3, 4, 10, 17, 19].

In the rest of Section 3.1, we set $\Gamma_0, \Gamma_1, \ldots$ is an A-growth of Γ_0. Write $G = \langle A \rangle$. Since G is finite, the sequence $\Gamma_0, \Gamma_1, \ldots$ is eventually constant. Denote $\mathsf{lim}(\Gamma_0, \Gamma_1, \ldots)$ by Γ_∞.

▶ **Lemma 15.** *If G is transitive,* $\mathsf{wc}(\Gamma_\infty) = \mathsf{sc}(\Gamma_\infty)$.

Proof. For any element $g \in G$ and two vertices $u, v \in V$, observe that $(u.g, v.g)$ is an arc of Γ_∞ if and only if (u, v) is an arc of Γ_∞. And then every element of G is a graph automorphism of Γ_∞.

For any vertex $v \in V$, let $R(v)$ be the subset of vertices such that $u \in R(v)$ if and only if there exists a path from v to u.

Let (u, v) be an arc of Γ_∞. It is clear that $R(v) \subseteq R(u)$. Since G is transitive, we can pick $g \in G$ such that $u.g = v$. Since g is a graph automorphism, it induces a bijection from $R(u)$ to $R(v)$. Then $R(u) = R(v)$. Hence, if two vertices x and y belong to the same weakly connected component of Γ_∞, it holds $R(x) = R(y)$. This completes the proof. ◀

For all $(p, q) \in V \times V$, the *incidence vector* of (p, q) is the vector $\mathbb{1}_p - \mathbb{1}_q$, denoted $\chi_{(p,q)}$. For a digraph $\Gamma = (V, E)$, write $L(\Gamma)$ for the subspace $\mathsf{span}(\chi_e : e \in E)$ of $\mathbb{Q}^n$. The following result is well-known in algebraic graph theory (see [2, Chapter 4]).

▶ **Lemma 16.** *Assume that $W_1, \ldots, W_d$ are all weakly connected components of a digraph $\Gamma = (V, E)$. Then the subspace $(L(\Gamma))^\perp$ is the d-dimensional subspace* $\mathsf{span}(\mathbb{1}_{W_1}, \ldots, \mathbb{1}_{W_d})$.

▶ **Lemma 17.** *Let $d = |\mathsf{SCC}(\Gamma_\infty)|$. If the arc set of Γ_0 is nonempty and G is transitive, then* $\mathsf{wc}(\Gamma_{n-d-1}) = \mathsf{wc}(\Gamma_\infty) = \mathsf{sc}(\Gamma_\infty)$.

Proof. By Lemma 15, $\mathsf{wc}(\Gamma_\infty) = \mathsf{sc}(\Gamma_\infty)$. It is sufficient to show $\mathsf{wc}(\Gamma_{n-d-1}) = \mathsf{wc}(\Gamma_\infty)$.

Define $\mathcal{L} = (L_i)_{i \ge 0}$ to be the sequence of linear spaces, where $L_i := L(\Gamma_i)$ for every $i \ge 0$. Write $L_\infty = \mathsf{lim}\,\mathcal{L}$. By Lemma 16, $\mathsf{dim}((L_\infty)^\perp) = d$ and then $\mathsf{dim}(L_\infty) = n - d$. For every $i < \mathsf{len}(\mathcal{L})$, since L_i and L_{i+1} are linear subspaces with $L_i \subsetneq L_{i+1}$, we have $\mathsf{dim}(L_i) < \mathsf{dim}(L_{i+1})$. Due to $E_0 \ne \emptyset$, it holds that $\mathsf{dim}(L_0) \ge 1$ and then $L_{n-d-1} = L_\infty$. By Lemma 16, $\mathsf{wc}(\Gamma_{n-d-1}) = \mathsf{wc}(\Gamma_\infty)$. ◀

▶ **Lemma 18.** *Assume that $E_0 \ne \emptyset$ and G is transitive. For every $v \in V$, there exist two vertices $p, q \in V$ such that $(v, p), (q, v) \in E_{n-1}$.*

Proof. Since $E_0 \neq \emptyset$, let x, y be two vertices such that $(x, y) \in E_0$. Let v be an arbitrary vertex in V. Since G is transitive, there exist two words $w, w' \in A^{\leq n-1}$ such that $x.w = y.w' = v$. Then $(v, y.w), (x.w', v) \in E_{n-1}$. ◀

▶ **Lemma 19.** *Let i be a positive integer. If $\mathsf{sc}(\Gamma_{i+1}) = \mathsf{sc}(\Gamma_i) \neq \mathsf{sc}(\Gamma_\infty)$ and G is transitive, then there exists a non-sink strongly connected component C of Γ_i and $a \in A$ such that $C.a$ is a sink component of Γ_i.*

Proof. Since $\mathsf{sc}(\Gamma_{i+1}) = \mathsf{sc}(\Gamma_i)$, every $a \in A$ induces a permutation on $\mathsf{SCC}(\Gamma_i)$, denoted by a', such that $X.a' = X.a \in \mathsf{SCC}(\Gamma_i)$.

Case 1: There exists a non-sink strongly connected component of Γ_i. Assume, for a contradiction, that every non-sink strongly connected component C and $a \in A$ satisfy $C.a \in \mathsf{SCC}(\Gamma_i) \setminus \mathsf{SinkC}(\Gamma_i)$. Let A' be the set $\{a' : a \in A\}$. Since $G = \langle A \rangle$ is transitive on V, the permutation group $\langle A' \rangle$ is transitive on $\mathsf{SCC}(\Gamma_i)$. Then there exist $C \in \mathsf{SCC}(\Gamma_i) \setminus \mathsf{SinkC}(\Gamma_i)$ and $a \in A$ such that $C.a \in \mathsf{SinkC}(\Gamma_i)$.

Case 2: There is no non-sink strongly connected component of Γ_i. In this case, we have $\mathsf{wc}(\Gamma_i) = \mathsf{sc}(\Gamma_i)$. For every $a \in A$ and $X \in \mathsf{SCC}(\Gamma_i)$, since $X.a \in \mathsf{SCC}(\Gamma_i)$, there are no arcs outside strongly connected components of Γ_{i+1}. Then $\mathsf{wc}(\Gamma_{i+1}) = \mathsf{sc}(\Gamma_{i+1})$. Using this argument repeatedly, we have $\mathsf{sc}(\Gamma_{i+1}) = \mathsf{sc}(\Gamma_{i+2}) = \cdots = \mathsf{sc}(\Gamma_\infty)$ which contradicts with $\mathsf{sc}(\Gamma_{i+1}) \neq \mathsf{sc}(\Gamma_\infty)$. ◀

▶ **Lemma 20.** *Let i be a non-negative integer such that $\mathsf{sc}(\Gamma_i) \neq \mathsf{sc}(\Gamma_\infty)$. If G is transitive, then either*

$$|\mathsf{SCC}(\Gamma_i)| > |\mathsf{SCC}(\Gamma_{i+1})|$$

or

$$|\mathsf{SinkC}(\Gamma_i)| > |\mathsf{SinkC}(\Gamma_{i+1})|.$$

Proof. If $|\mathsf{SCC}(\Gamma_i)| > |\mathsf{SCC}(\Gamma_{i+1})|$, we are done. Otherwise, since $\mathsf{sc}(\Gamma_i) \subseteq \mathsf{sc}(\Gamma_{i+1})$, we have $\mathsf{sc}(\Gamma_i) = \mathsf{sc}(\Gamma_{i+1})$ and

$$\mathsf{SinkC}(\Gamma_i) \supseteq \mathsf{SinkC}(\Gamma_{i+1}). \tag{3}$$

By Lemma 19, there exist $C \in \mathsf{SCC}(\Gamma_i) \setminus \mathsf{SinkC}(\Gamma_i)$ and $a \in A$ such that $C.a \in \mathsf{SinkC}(\Gamma_i)$. Since $C \in \mathsf{SCC}(\Gamma_i) \setminus \mathsf{SinkC}(\Gamma_i)$, there exists an arc $(p, q) \in E_i$ such that $p \in C$ and $q \notin C$. Note that $(p.a, q.a) \in E_{i+1}$. Since $p.a \in C.a$ and $q.a \notin C.a$, it holds that $C.a$ is a non-sink component of Γ_{i+1}. By Equation (3), we have $|\mathsf{SinkC}(\Gamma_i)| > |\mathsf{SinkC}(\Gamma_{i+1})|$. ◀

▶ **Lemma 21.** *Let $d = |\mathsf{SCC}(\Gamma_\infty)|$. If G is transitive and $d > \frac{n}{3}$, then $\mathsf{sc}(\Gamma_n) = \mathsf{sc}(\Gamma_\infty)$.*

Proof. Since G is transitive, d divides n. Then $d \in \{\frac{n}{2}, n\}$. Noting that, by Lemma 15, every weakly connected component of Γ_∞ is strongly connected. Since each strongly connected component of Γ_∞ has at most 2 vertices, it is clear that there exist at most n arcs in Γ_∞. Note that $|E_i| < |E_{i+1}|$ for every integer i such that $\Gamma_i \neq \Gamma_\infty$. Then $\mathsf{sc}(\Gamma_n) = \mathsf{sc}(\Gamma_\infty)$. ◀

▶ **Lemma 22.** *Let $d = |\mathsf{SCC}(\Gamma_\infty)|$. If G is transitive and $d \leq \frac{n}{3}$, then $\mathsf{sc}(\Gamma_{2n-3d-1}) = \mathsf{sc}(\Gamma_\infty)$.*

Proof. Let m be the minimum integer such that $\mathsf{sc}(\Gamma_m) = \mathsf{sc}(\Gamma_\infty)$. Define

$$f(i) = |\mathsf{SCC}(\Gamma_i)| + |\mathsf{SinkC}(\Gamma_i)|,$$

for each $i \geq 0$.

Consider Γ_{n-1}. By Lemma 17, Γ_{n-1} has d weakly connected components and $\mathsf{wc}(\Gamma_{n-1}) = \mathsf{sc}(\Gamma_m)$. In the case that $m \leq n-1$, then $m \leq 2n - 3d - 1$ and we are done.

Now we assume that $m > n-1$. Let C be a weakly connected component of Γ_{n-1} but not a strongly connected component of Γ_{n-1}. Define

$a :=$ the number of source components of Γ_{n-1} in C;

$b :=$ the number of non-source non-sink components of Γ_{n-1} in C;

$c :=$ the number of sink components of Γ_{n-1} in C.

Let D be either a source component or a sink component of Γ_{n-1} in C. By Lemma 18, there exists an arc in D and then there are at least two vertices in D. This implies

$$2a + b + 2c \leq |C| = \frac{n}{d}.$$

Since $a \geq 1$, we have

$$(a + b + c) + c \leq \frac{n}{d} - 1. \tag{4}$$

The left hand side of Equation (4) is the sum of the number of strongly connected components and the number of sink components of Γ_{n-1} in C. Since $d \leq \frac{n}{3}$, we have

$$\begin{aligned} f(n-1) &\leq (\frac{n}{d} - 1)x + 2(d - x) && \text{(by Equation (4))} \\ &\leq (\frac{n}{d} - 1)d && \text{(by } n/d - 1 \geq 2) \\ &= n - d, \end{aligned}$$

where $x = |\,\mathsf{WCC}(\Gamma_{n-1}) \setminus \mathsf{SCC}(\Gamma_{n-1})|$.

Since $f(m) = 2d$, using Lemma 20, we have

$$m - (n-1) \leq f(n-1) - f(m) \leq (n-d) - 2d = n - 3d.$$

Hence, $m \leq 2n - 3d - 1$. ◀

▶ Remark 23.

1. In the case that Γ_∞ is strongly connected and $n \geq 3$, Lemmas 21 and 22 show that $\mathsf{sc}(\Gamma_{2n-4}) = \mathsf{sc}(\Gamma_\infty)$. This slightly improves [17, Theorem 2], [10, Lemma 6] and [19, Theorem 2]. And then one can slightly improve the bounds of reset thresholds in [17, Theorem 3], [10, Theorem 7] and [19, Theorem 4].
2. Assume G is transitive. Lemmas 21 and 22 show that $\mathsf{sc}(\Gamma_{O(n)}) = \mathsf{sc}(\Gamma_\infty)$. The following example shows that $\Gamma_{O(n)} = \Gamma_\infty$ is not true.
   A permutation group $G \subseteq \mathsf{Sym}(Q)$ is called 2-*homogeneous* if for every 2-element subsets $X, Y \subseteq Q$, there exists $g \in G$ such that $X.g = Y$. Observe that a 2-homogeneous permutation group is transitive.
   In [10, Section 3], for every odd integer $n \geq 7$, Gonze, Gusev, Gerencsér, Jungers and Volkov constructed two permutations $a, b \in \mathsf{Sym}(n)$ such that
   - $\langle a, b \rangle$ is 2-homogeneous;
   - for any word $w \in \{a, b\}^*$, if $\{2, 4\}.w = \{\frac{n-1}{2}, \frac{n+3}{2}\}$, then $|w| \geq \frac{n^2}{4} + O(n)$.

   Let $\Gamma_0 = (V, E)$ be a digraph such that $V = \{1, \ldots, n\}$ and $E = \{(2, 4)\}$. Let $\Gamma_0, \Gamma_1, \ldots$ be the $\{a, b\}$-growth of Γ_0. Since $\langle a, b \rangle$ is 2-homogeneous, Γ_∞ is a complete digraph. By the second property of these two permutation, if $\Gamma_m = \Gamma_\infty$, then $m \geq \frac{n^2}{4} + O(n)$.
   Meanwhile, the above two permutations also provide a negative answer for [1, Problem 12.39].

3.2 Automata

In this subsection, we will define a sequence of digraphs with respect to an automaton $\mathcal{A} = (Q, \Sigma, \delta)$ and $A \subseteq \Sigma_0$.

For a 1-defect word $w \in \Sigma^*$, the *excluded state* of w is the state such that $\mathsf{excl}(w) \notin Q.w$, denoted $\mathsf{excl}(w)$; the *duplicate state* is the state q such that $|q.w^{-1}| > 1$, denoted $\mathsf{dupl}(w)$.

For $i \geq 0$, define $\Gamma_i := (Q, E_i)$ to be the digraph where

$$E_i := \left\{(\mathsf{excl}(w), \mathsf{dupl}(w)) : w \in \Sigma_1 A^{\leq i}\right\}.$$

▶ **Lemma 24.** *The sequence* $(\Gamma_0, \Gamma_1, \ldots)$ *is the* $\delta(A)$*-growth of* Γ_0.

Proof. We need to prove $E_{i+1} = E_i \cup E_i.\Sigma_0$ for every $i \geq 0$. Let i be an arbitrary nonnegative integer.

Let $(p, q) \in E_i$. Take $w \in \Sigma_1\Sigma_0^{\leq i}$ such that $p = \mathsf{excl}(w)$ and $q = \mathsf{dupl}(w)$. By directly computing, we have $\mathsf{excl}(wa) = \mathsf{excl}(w).a$ and $\mathsf{dupl}(wa) = \mathsf{dupl}(w).a$ for all $a \in \Sigma_0$. Then $(p.a, q.a) \in E_{i+1}$ which implies $E_{i+1} \supseteq E_i \cup E_i.\Sigma_0$.

Let $(x, y) \in E_{i+1}$. Take $w' \in \Sigma_1\Sigma_0^{\leq i+1}$ such that $x = \mathsf{excl}(w')$ and $y = \mathsf{dupl}(w')$. If the length of w' is less than $i + 2$, then $(x, y) \in E_i$. Otherwise, $w' = wa$ where $w \in \Sigma_1\Sigma_0^i$ and $a \in \Sigma_0$. It is clear that $(\mathsf{excl}(w), \mathsf{dupl}(w)) \in E_i$ and $(\mathsf{excl}(w).a, \mathsf{dupl}(w).a) = (x, y)$. Then $E_{i+1} \subseteq E_i \cup E_i.\Sigma_0$. ◀

▶ **Proposition 25.** *Assume that* $\mathcal{A} = (Q, \Sigma, \delta) \in \mathbb{ST}$*. If* K_∞ *is a linear subspace of* $\mathbb{Q}^n$ *and* $\Sigma = \Sigma_0 \cup \Sigma_1$*, then*

$$\mathsf{len}(\mathcal{K}) \leq \begin{cases} n & \textit{if } \mathsf{dim}(K_\infty) = \frac{n}{2}, \\ 3\,\mathsf{dim}(K_\infty) - n - 1 & \textit{otherwise.} \end{cases}$$

Proof. Recall the definition of T_i that $T_i := \{\mathbf{k}_w : w \in \Sigma_{\geq 1}A^{\leq i}\}$ for $i \geq 0$. Since $\Sigma = \Sigma_0 \cup \Sigma_1$, we have $T_i = \{-\chi_e : e \in E_i\}$ for $i \geq 0$. Let m be the minimal integer such that $\mathsf{sc}(\Gamma_m) = \mathsf{sc}(\Gamma_\infty)$. By Lemmas 15 and 24, $\mathsf{sc}(\Gamma_\infty) = \mathsf{wc}(\Gamma_\infty)$ and $\mathsf{sc}(\Gamma_m) = \mathsf{wc}(\Gamma_m)$. This implies that $\mathsf{span}(T_\infty) = \mathsf{cone}(T_\infty)$ and $\mathsf{span}(T_m) = \mathsf{cone}(T_m)$. Let $C_1, \ldots, C_d$ be the strongly connected components of Γ_m. Since $\delta(\Sigma_0)$ is transitive, using Lemma 16,

$$\mathsf{span}(1_{C_1}, \ldots, 1_{C_d}) = \mathsf{span}(T_m)^\perp = \mathsf{cone}(T_m)^\perp = K_m^\circ$$

and

$$\mathsf{span}(1_{C_1}, \ldots, 1_{C_d}) = \mathsf{span}(T_\infty)^\perp = \mathsf{cone}(T_\infty)^\perp = K_\infty^\circ.$$

Then $K_m = K_\infty$ and $\dim(K_\infty) = n - d$. By Lemmas 21 and 22,

$$\mathsf{len}(\mathcal{K}) \leq \begin{cases} n & \text{if } \mathsf{dim}(K_\infty) = \frac{n}{2}, \\ 3\,\mathsf{dim}(K_\infty) - n - 1 & \text{otherwise.} \end{cases}$$ ◀

Proof of Theorem 4. Computer experiments confirmed Černý conjecture for any synchronizing automata with at most 5 states (see [13, Table 2]). One can check directly that $\mathsf{rt}(\mathcal{A}) \leq (n-1)^2 \leq n^2 - 7n + 7$ for $n \leq 5$.

Now, we assume that $n \geq 6$. Using Lemmas 15 and 16, $\frac{n}{2} \leq \mathsf{dim}(K_\infty) \leq n - 1$. Let S be a nonempty proper subset of Q. By Propositions 10 and 25, if $\mathsf{dim}(K_\infty) = \frac{n}{2}$ then

$$\begin{aligned} \mathsf{len}(\mathcal{K}) + \ell(S) + 1 &\leq n + (n - 1 - \frac{n}{2}) + 1 \\ &= 2n - 3 - (\frac{n}{2} - 3) \leq 2n - 3; \end{aligned} \tag{5}$$

if $\mathsf{dim}(K_\infty) > \frac{n}{2}$,

$$\begin{aligned}\mathsf{len}(\mathcal{K}) + \ell(S) + 1 &\leq (3\,\mathsf{dim}(K_\infty) - n - 1) + (n - 1 - \mathsf{dim}(K_\infty)) + 1 \\ &= 2\,\mathsf{dim}(K_\infty) - 1 \leq 2(n-1) - 1 = 2n - 3. \end{aligned} \tag{6}$$

Combining Proposition 9 and Equations (5) and (6), every nonempty proper subset of Q is $(2n-3)$-extensible. Using Proposition 5, we obtain that

$$\mathsf{rt}(\mathcal{A}) \leq 1 + (n-2)(2n-3) = 2n^2 - 7n + 7.$$ ◀

4 Conclusion and discussions

We obtain an upper bound for the reset thresholds of $\mathbb{ST}$-automata which improves Rystsov's bound. Using this upper bound, we prove that there exists a synchronizing word of length at most $2n^2 - 7n + 7$ for every synchronizing n-state $\mathbb{ST}$-automata whose letters of defect at most 1.

While Theorem 4 is about a specific class of automata, the lemmas presented in Section 2 may be useful tools for the broader study of synchronizing words. We conclude the article by discussing two classes of automata for which these tools have potential applications.

4.1 One-cluster automata

An automaton (Q, Σ, δ) is called *one-cluster* if it has a letter with only one simple cycle on the set of states, more precisely, there exists a letter $a \in \Sigma$ which acts on P as a cyclic permutation where $P = Q.a^{|Q|-1}$. Write $\mathbb{OC}$ for the family of one-cluster automata. It is clear that one of $\mathbb{OC}$ and $\mathbb{ST}$ do not include the other one. Meanwhile, $\mathbb{OC}$ and $\mathbb{ST}$ have nonempty intersection (e.g. the automaton $\mathcal{C}_4$, see Figure 1).

Steinberg [23] proved Černý Conjecture for one-cluster automata with prime length cycles. Béal, Berlinkov and Perrin showed the reset threshold of n-state one-cluster automata is at most $2n^2 - 7n + 7$. To establish this upper bound, Béal, Berlinkov and Perrin use a linear algebra approach which is different from the approach in Section 2. Observing that the two upper bounds are the same, this may not be a coincidence and it is worth unifying these two proofs. It is also interesting to obtain a better upper bound by combining these two approach.

4.2 Completely reachable automata

An automaton (Q, Σ, δ) is called *completely reachable* if for every nonempty subset $P \subseteq Q$, there exists a word $w \in \Sigma^*$ such that $P = Q.w$. Ferens and Szykuła [8, Corollary 31] proved that the reset threshold of n-state completely reachable automata is at most $2n^2 - n \ln n - 4n + 2$. It is clear that every completely reachable automaton has at least one letter of defect 1, since subsets of $(n-1)$ states are reachable. It is not hard to show if a completely reachable automaton has exactly one letter of defect 1, then it contains a transitive permutation group. Hence, the overlap of completely reachable automata and $\mathbb{ST}$-automata with letters of defect ≤ 1 is quite substantial. Therefore, we believe that the tools presented in this article may also useful for studying completely reachable automata.

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