

Partial Minimum Branching Program Size Problem Is ETH-Hard

Ludmila Glinskih ✉ 

Google, Mountain View, CA, USA

Artur Riazanov ✉ 

EPFL, Lausanne, Switzerland

Abstract

We show that assuming the Exponential Time Hypothesis, the PARTIAL MINIMUM BRANCHING PROGRAM SIZE PROBLEM (MBPSP*) requires superpolynomial time. This result also applies to the partial minimization problems for many interesting subclasses of branching programs, such as read- k branching programs and OBDDs.

Combining these results with the recent unconditional lower bounds for MCSP [6], we obtain an unconditional superpolynomial lower bound on the size of Read-Once Nondeterministic Branching Programs (1-NBP) computing the *total* versions of the minimum BP, read- k -BP, and OBDD size problems.

Additionally we show that it is **NP**-hard to check whether a given BP computing a partial Boolean function can be compressed to a BP of a given size.

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1 Introduction

The MINIMUM CIRCUIT SIZE PROBLEM (MCSP) has been a central problem of study in computational complexity in recent years. Given the truth-table of a Boolean function, and a size parameter s , MCSP asks one to determine whether the function can be computed by a Boolean circuit of size s . As the input length of MCSP is exponential in the input length of the function defined by the given truth table, it is clear that MCSP is contained in **NP**. Although it is not known whether MCSP is **NP**-complete, it is widely believed to be hard to compute. Starting from the work of Kabanets and Cai [11] there has been mounting evidence that $\text{MCSP} \notin \mathbf{P}$ [20]. For example, $\text{MCSP} \in \mathbf{P}$ implies that widely believed cryptographic assumptions do not hold. On the other hand, there are complexity-theoretic barriers to proving **NP**-hardness of MCSP [15, 21].

Even before the work of Kabanets and Cai [11], variations of MCSP were studied in the context where the function is given not as a truth table, but represented in some computational model. One family of models that was extensively studied in this context is *branching programs*. The line of work [2, 17, 23, 22, 1] showed that branching program minimization is **NP**-hard for the natural cases of succinct representation of a function: either



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the function itself is represented as a branching program as in [2, 22, 1], or an input is a *partial function* with few fixed values as in [23]. The results, however, are limited to weak models of branching programs such as OBDDs or read-once branching programs. To our knowledge, no hardness results are known even for minimizing read-twice branching programs.

The potential cross-pollination between these two research areas motivates us to study MINIMUM BRANCHING PROGRAM SIZE PROBLEM (MBPSP): given a function represented as a truth table $\{0, 1\}^{2^n}$ and a size parameter $s \in [2^n]$ determine whether exists a *branching program* of size at most s computing the given function.

On the meta-complexity side, one of the approaches to prove **NP**-hardness of MCSP is to show lower bounds for \mathcal{C} -MCSP problems, in which we ask whether an input Boolean function can be computed by a circuit from the class \mathcal{C} , where \mathcal{C} is a weaker class than general Boolean circuits. There are already results showing **NP**-hardness of DNF-MCSP [14], (DNF \circ XOR)-MCSP [8], and the hardness of DeMorgan Formulas-MCSP (MFSP) assuming the Exponential Time Hypothesis (ETH) [10]. All of these reductions share a common structure.

- First they show hardness of the Partial \mathcal{C} -MCSP (\mathcal{C} -MCSP*) problem in which the input is the truth table of a partial function.
- Then they show a reduction from the \mathcal{C} -MCSP* problem to the \mathcal{C} -MCSP problem.

Hardness results for Partial MCSP (MCSP*) are already known. In [9] Ilango showed that MCSP* requires superpolynomial time to compute¹, assuming the Exponential Time Hypothesis. Recently, Hirahara [7] showed the **NP**-hardness of MCSP* under randomized reductions. So the only missing component for showing hardness of MCSP is to show an efficient reduction from MCSP* to MCSP.

The study of hardness of MBPSP under the ETH continues the line of work above. Expressive power of branching programs is situated between that of Boolean formulas and Boolean circuits [18, 5]. On the other hand our understanding of branching programs seems to be slightly better than our understanding of circuits. One evidence for this is that the strongest lower bound on the BP size for an explicit function is $\tilde{\Omega}(n^2)$ [16] which is much stronger than $\Omega(n)$ lower bound for Boolean circuits [3]. Showing the ETH-hardness of MBPSP would improve the recent ETH-hardness result for MFSP [10] which is the frontier hardness result for total circuit minimization problems and move us one step closer to the ETH-hardness of MCSP itself. Our main result is completing the first step of the common recipe for establishing the hardness of \mathcal{C} -MCSP by showing hardness of BP-MCSP* (that we call MBPSP* for short).

► **Theorem 1.** *Every deterministic algorithm computing MBPSP* on inputs of length N requires time $N^{\Omega(\log \log(N))}$ assuming the Exponential Time Hypothesis.*

Note that the ETH-hardness of MCSP* or MFSP* does not imply the ETH-hardness of MBPSP*. Although on surface it seems that MBPSP (MBPSP*) is just a special case of MCSP (MCSP*), the hardness of the latter does not necessarily imply the hardness of the former. For example, in some special settings of parameters, an explicit construction of a many-one reduction between MCSP and MBPSP yields breakthrough circuit or BP

¹ The same hardness result holds for Partial MFSP (MFSP*).

lower bounds.² On the other hand, ruling out more natural parameter configurations does not seem as trivial: indeed, our main result shows that the identity function is a reduction on a certain subset of the instances.

On the branching program minimization side, the consequences are twofold. First, our result implies that all partial branching program minimization problems require time $N^{\Omega(\log \log N)}$ under ETH for any representation of a function that is at least as efficient as truth table.³ Second, we show that our result implies **NP**-hardness of minimization of a branching program computing a partial function that itself is represented as a branching program (see Section 1.2.3 for details).

In particular, our result implies that the problem of finding the smallest OBDD (a subclass of branching programs) that computes the given partial function is ETH-hard. But surprisingly, there is a polynomial-time algorithm that finds the smallest OBDD computing a *total* function [4]. This means that assuming ETH holds, the potential reduction from MBPSP* to MBPSP must fail on OBDDs.

1.1 Techniques

To show Theorem 1 we adapt the proof of a similar result of Ilango [9] for the hardness of MCSP* and MFSP* problems.

► **Theorem 2** ([9]). *Every deterministic algorithm computing MCSP* on inputs of length N requires time $N^{\Omega(\log \log(N))}$ assuming the Exponential Time Hypothesis holds. The same holds for MFSP*.*

Ilango’s proof of Theorem 2 is a reduction from the $(n \times n)$ -Bipartite Permutation Independent Set Problem ($(n \times n)$ -BPIS). For now we can think of it as an independent set problem on some special set of graphs. This is an **NP**-hard graph problem introduced in [13], where the authors show that any deterministic Turing machine requires $2^{\Omega(n \log n)}$ -time⁴ to solve it if the Exponential Time Hypothesis holds. For an input graph G with vertices labeled by elements of $[n] \times [n]$ Ilango constructs a truth-table of a $3n$ -variable partial Boolean function γ_G , such that

(★) G is a YES-instance of $(n \times n)$ -BPIS iff γ_G can be computed by a DeMorgan formula with at most $3n - 1$ gates.

The key point here is that if we constrain a circuit or a formula to have at most $3n - 1$ (inner) gates, its structure becomes simple enough for a very precise analysis. [9] characterizes the structure of such formulas that compute γ_G which would yield the correspondence between such formulas with independent sets in G .

Although branching programs are situated between formulas and circuits in computational expressivity [5], Theorem 2 does not apply to MBPSP*, since with a sharp constraint on the size, branching programs may become stronger than circuits. We observe in Proposition 7

² Suppose that f is such a reduction from $\text{MCSP}_{s=3n}$ (the language of the truth-tables computable by circuits with at most $3n$ gates) to $\text{MBPSP}_{s=n^{2.1}}$. Then taking any function requiring circuits of size larger than $3n$ [3, 12] and applying f to it yields $n^{2.1}$ lower bound for general branching programs superior to the current state of the art. In a similar way, we get superlinear circuit lower bounds from any reduction from $\text{MBPSP}_{s=n^2}$ to $\text{MCSP}_{s=n^{1+\epsilon}}$.

³ We note that results in this area often establish **NP**-hardness and even hardness of size approximation for the case of concise representation, while our result only establishes ETH-hardness of the exact version of the problem.

⁴ The input to this problem has $\Theta(n^4 \log n)$ bit size.

that this is indeed the case, showing that branching programs are stronger than circuits if we constrain the size parameter of both models to “minimal viable” size: $m - 1$ inner nodes for a DeMorgan circuit and m non-sink nodes for a BP for the case of a Boolean function on m input bits.

Nevertheless, we show that the reduction to MCSP* from Theorem 2 does yield ETH-hardness for MBPSP*. Indeed, in our proof of Theorem 1 we only change the part (\star), keeping the construction of γ_G . The key part of the proof is to understand what structure a branching program has if it has at most $3n$ (non-sink) nodes. As γ_G is sensitive in every input variable, we need to understand a structure of a branching program that has unique variable labels. We call such programs *once-appearance branching programs* (oaBP). We show that γ_G can be computed by a once-appearance branching program iff G is a YES-instance of $(n \times n)$ -BPIS.

1.2 Corollaries

1.2.1 Weaker Classes

BP is not the only model that collapses to oaBP when constrained to have at most n non-sink nodes. The same also holds for OBDD, and k -BP for every positive integer k . From this we get a family of hardness of minimization results for restricted classes of branching programs.

► **Corollary 3.** *Every deterministic algorithm computing \mathcal{C} -MCSP* for $\mathcal{C} \in \{\text{OBDD}, \text{BP}\} \cup \{k\text{-BP} \mid k \geq 1\}$ ⁵ on inputs of length N requires time $N^{\Omega(\log \log(N))}$ assuming the Exponential Time Hypothesis.*

1.2.2 Unconditional Results for a Weaker Model

In the recent work [6] the authors employed the framework of Ilango from [9] to show an unconditional lower bound on the size of the read-once nondeterministic branching program (1-NBP) computing the total version of MCSP. The proof consists of three components.

1. Show that $(n \times n)$ -BPIS unconditionally cannot be computed by a 1-NBP of smaller than $\Omega(n!)$ size.
 2. Show that Ilango’s reduction from $(n \times n)$ -BPIS to MCSP* can be efficiently encoded using BP. This shows that if MCSP* on inputs of length N could be computed by a 1-NBP of size $N^{o(\log \log(N))}$, then $(n \times n)$ -BPIS could be computed by a 1-NBP of size $o(n!)$.
 3. Show that MCSP and MCSP* have the same 1-NBP complexity up to a linear multiplicative factor. This implies that 1-NBP complexity of MCSP is at least $N^{\Omega(\log \log(N))}$.
- Since the proof of Theorem 1 follows similar steps as the proof of the Theorem 2, and analogs of the third property from [6] also hold for BP, k -BP, and OBDD, we get similar unconditional 1-NBP lower bounds for total minimization problems in all these models.

► **Corollary 4.** *Every 1-NBP computing \mathcal{C} -MCSP for $\mathcal{C} \in \{\text{OBDD}, \text{BP}\} \cup \{k\text{-BP} \mid k \geq 1\}$ on inputs of length N requires size at least $N^{\Omega(\log \log(N))}$.*

⁵ Or any other class of branching programs that collapses to oaBP on the n -node programs.

1.2.3 NP-hardness of Compressing BPs

Additionally, we show that given a graph G we can construct a branching program B of polynomial size that exactly computes the partial function γ_G . Namely it outputs $\{0, 1\}$ values on inputs on which γ_G is defined and $*$ on all other inputs. Then B can be compressed to an oaBP which agrees with γ_G on its domain iff G is a YES-instance of $(n \times n)$ -BPIS. Combining this result with a folklore result of **coNP**-hardness of compressing a branching program, we get the following result on the hardness of compressing branching programs.

► **Corollary 5.** *The language of pairs (B, s) where B is a branching program⁶ over the alphabet $\{0, 1, *\}$ and s is in integer, such that an extension of the partial function described by B can be computed by a BP of size at most s , is **NP** and **coNP**-hard.*

1.3 Organization

The structure of the paper is as follows:

- In Section 2 we give definitions of the branching programs and various versions of MCSP we are working with.
- In Section 3 we state our main result: the hardness of MBPSP* under ETH (Theorem 1) and explain the main ideas behind its proof. We show the proof of Theorem 1 and proofs of all required lemmas in Section 4 and Section 5.
- In Section 6 we sketch the proofs of several corollaries of Theorem 1.
- Finally, in Section 7 we explain the future directions of this work and possible barriers we see for strengthening our results.

2 Preliminaries

Throughout this work we denote by $[n]$ the set of n elements $\{1, \dots, n\}$. Let a and b be partial assignments. If the supports of a and b do not intersect, denote by $a \cup b$ the partial assignment that coincides with a on the support of a and with b on the support of b . For a graph G we denote its set of vertices by $V(G)$ and its set of edges by $E(G)$. Throughout the paper a *circuit* is a DeMorgan circuit (\wedge , \vee and \neg gates) with gates of arity 2. A *formula* is a tree-like DeMorgan circuit. A *read-once formula* is a formula such that each input bit feeds into exactly one gate. E.g. $x \vee (\neg y \wedge z)$ is a read-once formula, and $(x \wedge y) \vee (\neg x \wedge z)$ is not. Since every DeMorgan formulas can be transformed in such a way that all \neg -gates are adjacent to the inputs, we identify DeMorgan formulas with trees with *literals* in the leaves and \wedge/\vee in the inner nodes.

A *Branching Program* (BP) is a form of representation of functions. A n -ary function⁷ $f: D^n \rightarrow \{0, 1\}$, where $|D| = d$, is represented by a directed acyclic graph with exactly one source and two sinks. Each non-sink node is labeled with a variable; every internal node has exactly d outgoing edges: labeled with elements of D . One of the sinks is labeled with 1 and the other is labeled with 0. We say that a node *queries* x if it is labeled with a variable x . The *size* of a branching program is the number of nodes in it.

The value of the function for given values of the variables is evaluated as follows: we start in the source and then follow the edge labeled with the value of the variable in the source, we repeat this process until the current node is a sink. The label of this sink is the value of the function. We refer to a path that terminates in a σ -labeled sink as σ -*path*.

⁶ The same is true for all sub-classes of branching programs that simulate 4-BP.

⁷ We will instantiate this definition for $D = \{0, 1\}$ and $D = \{0, 1, *\}$

A *Non-deterministic Branching Program* (NBP) is a Branching Program over the alphabet $\{0, 1\}$ with additional non-labeled *guessing* nodes with two non-labeled outgoing edges. The value computed by a Non-deterministic Branching Program is 1 if there exists at least one 1-path that agrees with the given assignment.

A Branching Program (deterministic or not) is called *read-once* (denoted as 1-BP in the deterministic case and 1-NBP in the non-deterministic case) if any path from the source to a sink contains each variable label at most once.

For a partial assignment $\alpha: [n] \rightarrow \{0, 1, *\}$ and a branching program B with n input bits over the alphabet $\{0, 1\}$ we write $B|_\alpha$ for a branching program obtained by removing every edge of form $x_i \neq \alpha(i)$ and contracting every edge of form $x_i = \alpha(i)$ if $\alpha(i) \in \{0, 1\}$. It is easy to see that if B computes the function f , then $B|_\alpha$ computes the function $f|_\alpha$.

The MINIMUM BRANCHING PROGRAM SIZE PROBLEM ($\text{MBPSP}(f, s)$) gets as input the truth-table of a Boolean function $f: \{0, 1\}^n \rightarrow \{0, 1\}$ of length $N = 2^n$ and a size parameter s , and outputs 1 if there exists a branching program of size at most s that computes f . We use notation $\text{MBPSP}_{s'(n)}(f) = \text{MBPSP}(f, s'(n))$.

The PARTIAL MINIMUM BRANCHING PROGRAM SIZE PROBLEM ($\text{MBPSP}^*(f^*, s)$) gets as input the partial truth-table of a function $f^*: \{0, 1\}^n \rightarrow \{0, 1, *\}$ of length $N = 2^n$ and a size parameter s , and outputs 1 if there exists a substitution of each $*$ in the truth-table to a 0/1, transforming f^* to a Boolean function f , such that $\text{MBPSP}(f, s) = 1$. We use notation $\text{MBPSP}_{s'(n)}^*(f) = \text{MBPSP}^*(f, s'(n))$.

2.1 Once-Appearance Branching Programs

A *once-appearance branching program* (oaBP) is a branching program with distinct node labels. Hence, for a function depending on n variables such oaBP representation contains at most n non-sink nodes. For oaBP we identify the non-sink nodes with their labels, and identify the edges with equalities of form $x = \alpha$ for a variable x and α from the alphabet.

It is easy to see, that oaBPs are strictly stronger than read-once DeMorgan formulas.⁸

► **Proposition 6.** *If f is computable by a read-once DeMorgan formula, then it is computable by an oaBP.*

Proof. We first observe that if g and h are computable by oaBPs, then so are $g \wedge h$ and $g \vee h$ whenever g and h have disjoint set of variables. Indeed oaBP for $g \wedge h$ is constructed by redirecting all edges going to the 1-in the oaBP computing g to the source of the oaBP computing h , and by redirecting all edges to the 0-sink in g to the 0-sink of h . For $g \vee h$ we redirect all edges to the 0-sink of g to the source of the oaBP for h , and all edges to the 1-sink of g to the 1-sink of h . Now to finish the proof observe that all literals $\{x_1, \neg x_1, \dots, x_n, \neg x_n\}$ are trivially computable by an oaBP. ◀

► **Proposition 7.** *Let $g(x, y, z) := (y \wedge x) \vee (z \wedge \neg x)$. Then g is computable by an oaBP but is is not computable by a read-once DeMorgan formula.*

⁸ Notice that for functions sensitive to all variables, read-once DeMorgan formulas are equivalent to general DeMorgan circuits with $n - 1$ inner gates. Thus Proposition 6 and Proposition 7 hold for such circuits as well.

Proof. The oaBP for g is a decision tree with the root x , the edge $x = 0$ going to z , the edge $x = 1$ going to y , the edge $y = \alpha$, $z = \alpha$ going to the α -sink for $\alpha \in \{0, 1\}$.

Consider a read-once formula G computing g . Then $G|_{y=0, z=1}$ computes a function $\neg x$, so G must have a leaf $\neg x$. But $G|_{y=1, z=0}$ computes a function x , so G must have a leaf x . That contradicts the read-once property of G . ◀

We note that this example can be extended to a function that is sensitive in an arbitrary number of variables, say for $g(x, y, z) \vee t_1 \cdots \vee t_n$.

2.2 Bipartite Permutation Independent Set Problem

In [9] Ilango showed that Partial MCSP is ETH-hard. The main idea of the proof is to show a reduction from the $(n \times n)$ -Bipartite Permutation Independent Set problem that was previously shown to be ETH-hard in [13]. In this section and later we assume that n is even.

In $(n \times n)$ -BIPARTITE PERMUTATION INDEPENDENT SET PROBLEM ($(n \times n)$ -BPIS) we are given the adjacency matrix of an undirected graph G over the vertex set $[n] \times [n]$ where every edge is between the sets of vertices $J_1 = [n/2]^2$ and $J_2 = \{n/2 + 1, \dots, n\}^2$. A graph G is a YES-instance iff it contains an independent set $S \subseteq J_1 \cup J_2$ of size n such that the coordinates of vertices in S define a permutation of $[n]$, i.e. $\forall i \in [n] \exists j, k \in [n]: v = (i, j), w = (k, i), v, w \in S$.

Note that a permutation of $[n]$ corresponding to a YES-instance of $(n \times n)$ -BPIS can be viewed as two permutations on disjoint $n/2$ -element subsets. One permutation is defined by $n/2$ vertices from J_1 (corresponding to a permutation of elements $[n/2]$), and another by $n/2$ vertices from J_2 (corresponding to a permutation of elements $\{n/2 + 1, \dots, n\}$). We use each of these interpretations interchangeably.

► **Theorem 8** ([13]). $(n \times n)$ -BPIS cannot be solved in deterministic time $2^{o(n \log n)}$ unless ETH fails.

Ilango in [9] gave a $2^{O(n)}$ -time reduction from $(n \times n)$ -BPIS to MCSP*, hence, showing that MCSP* cannot be solved in deterministic time $2^{o(n \log n)}$ unless ETH fails.

3 Main Theorem

In this section we show that the reduction Ilango constructed from ETH-hard problem $(n \times n)$ -BPIS to a formula minimization problem holds for the BP minimization problem as well. More precisely we show:

► **Theorem 9.** Suppose that $MBPSP^*: \{0, 1, *\}^{2^n} \times [2^n] \rightarrow \{0, 1\}$ is in $\mathbf{TIME}[N^{o(\log \log N)}]$ where $N = 2^n$, then the Exponential Time Hypothesis is false.

Let us recall the reduction $\mathcal{R}: \{0, 1\}^{\binom{n \times n}{2}} \rightarrow \{0, 1, *\}^{2^{3n}} \times [2^{3n}]$ from [9]⁹. Let $G = ([n] \times [n], E)$ be an instance of $(n \times n)$ -BPIS. Then the reduction outputs the pair $(t, 3n - 1)$ where t is the truth table of a partial function $\gamma_G: \{0, 1\}^n \times \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1, *\}$ defined as $\gamma_G(\vec{x}, \vec{y}, \vec{z}) =$

⁹ We ignore all the edges except for ones between the sets $[n/2] \times [n/2]$ and $\{n/2 + 1, \dots, n\} \times \{n/2 + 1, \dots, n\}$, the second argument is represented in binary form.

$$\begin{aligned}
 & \left\{ \begin{array}{ll} \bigvee_{i \in [n]} (y_i \wedge z_i), & \text{if } \vec{x} = 0^n & (1) \\ \bigvee_{i \in [n]} z_i, & \text{if } \vec{x} = 1^n & (2) \\ \bigvee_{i \in [n]} (x_i \vee y_i), & \text{if } \vec{z} = 1^n & (3) \\ 0, & \text{if } \vec{z} = 0^n & (4) \end{array} \right. \\
 = & \left\{ \begin{array}{ll} \bigvee_{i=1}^{n/2} x_i & \text{if } \vec{z} = 1^{n/2} 0^{n/2} \text{ and } y = 0^n & (5) \\ \bigvee_{i=n/2+1}^n x_i & \text{if } \vec{z} = 0^{n/2} 1^{n/2} \text{ and } y = 0^n & (6) \\ 1 & \text{if } \exists (j, k), (n/2 + j', n/2 + k') \in E: (\vec{x}, \vec{y}, \vec{z}) = (\bar{e}_k \bar{e}_{k'}, 0^n, e_j e_{j'}) & (7) \\ * & \text{otherwise} & (8) \end{array} \right.
 \end{aligned}$$

where $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$, $\vec{z} = (z_1, \dots, z_n)$, and $e_i \in \{0, 1\}^{n/2}$ is the vector with 1 in the i th entry and zeroes in all the others, and \bar{e}_i is such that $e_i + \bar{e}_i = 1^{n/2}$. In the rest of the proof, we use e_i as a vector with the i th coordinate equal to 1 and all the rest equal to zero with the appropriate dimension.

The intuition behind this reduction is that for every read-once formula, computing this function has to group triples of bits x_i, y_j , and z_k such that $j = k$, and $i \rightarrow j$ encodes that an element i maps to an element j in the permutation. The cases (1)-(4) ensure that the read-once formula encodes some permutation on $[n]$. The cases (5) and (6) ensure that the permutation maps the first $n/2$ elements into the first $n/2$ elements and the last $n/2$ elements into the last $n/2$ elements. And, finally, the case (7) enforces that for every edge in the original graph, the permutation defined by the read-once formula contains at most one endpoint of this edge. Note, that every endpoint or vertex chosen in the independent set encodes a mapping of one element of $[n]$ to one element of $[n]$ in the permutation.

As was shown in [9], this reduction maps YES-instances of BPIS to YES-instances of $\text{MCSP}_{s=m-1}^*$ where $m = 3n$ is the number of input bits of $\mathcal{R}(G)$:

► **Theorem 10** ([9]). *For the reduction \mathcal{R} as above $G \in (n \times n)$ -BPIS iff $\mathcal{R}(G) \in \text{MCSP}_{s=m-1}^*$.*

To prove Theorem 10, Ilango shows that γ_G extends to a total function computable by a *read-once monotone formula* iff G is a YES-instance of $(n \times n)$ -BPIS. In our work we show that a similar statement holds for *once-appearance branching programs* (oaBP).

► **Theorem 11.** *For the reduction \mathcal{R} as above $G \in (n \times n)$ -BPIS iff $\mathcal{R}(G)$ can be computed by oaBP.*

The plan of the proof is the following:

- In Section 4 we show that the topological order of an oaBP computing γ_G defines a unique permutation $\pi: [n] \rightarrow [n]$.
- In Section 5 we show that for π as above $\{(i, \pi(i)) \mid i \in [n]\}$ is an independent set in G , thus showing the “only if” direction of the equivalence.
- Since by Proposition 6 that oaBP losslessly simulates read-once formulas, we conclude that Theorem 10 implies the “if” direction of the equivalence.

4 Structure of oaBP Computing γ'

For the first step of our plan it is convenient to simplify the definition of γ_G . Let $\gamma'_n: \{0, 1\}^{3n} \rightarrow \{0, 1, *\}$ be equal to γ_G whenever one of (1)-(4) is satisfied, and $*$ otherwise. Here we may choose an arbitrary G as in this domain γ_G does not depend on it.

$$\gamma'_n(\vec{x}, \vec{y}, \vec{z}) = \begin{cases} \bigvee_{i \in [n]} (y_i \wedge z_i), & \text{if } \vec{x} = 0^n \\ \bigvee_{i \in [n]} z_i, & \text{if } \vec{x} = 1^n \\ \bigvee_{i \in [n]} (x_i \vee y_i), & \text{if } \vec{z} = 1^n \\ 0, & \text{if } \vec{z} = 0^n \\ * & \text{otherwise} \end{cases}$$

We usually shorthand γ'_n as γ' . In Section 4 we identify the structure of an oaBP computing γ' . Let us start by constructing an example of such an oaBP. As we mentioned before, whenever a function is computable by a read-once formula, it is computable by oaBP. Ilango [9] shows that all read-once formulas computing γ' have form

$$\bigvee_{i \in [n]} (x_{\pi(i)} \vee (y_i \wedge z_i)) \quad (9)$$

for a permutation $\pi \in S_n$. One way to translate this into an oaBP is to chain together n three-node oaBPs each of them computing one disjunct of (9). A topological order of such a oaBP is a sequence of n triplets of $\{x_{\pi(i)}, y_i, z_i\}$ with variables in triplets permuted in some order. We claim that topological order of *every* oaBP computing γ' has this form¹⁰. We prove this claim in Lemma 12.

We then say that an oaBP B computing γ' *defines* a permutation κ_B mapping $b \mapsto a$ iff $\{x_a, y_b, z_b\} = \{\ell_{3i+1}, \ell_{3i+2}, \ell_{3i+3}\}$ for some $i \in \{0, \dots, n-1\}$. We refer to the triplets of consecutive nodes of form $\{x_a, y_b, z_b\}$ as *xyz-triplets* or *xyz-blocks*.

In this section we prove the following lemma.

► **Lemma 12.** *Suppose B is an oaBP that computes γ' . Then let $\ell_1, \ell_2, \dots, \ell_{3n}$ be labels of vertices of B in a topological order. Then for every $i \in \{0, \dots, n-1\}$ we have $\{\ell_{3i+1}, \ell_{3i+2}, \ell_{3i+3}\} = \{x_a, y_b, z_b\}$ for some $a, b \in [n]$.*

The proof of Lemma 12 is surprisingly delicate. The starting point is the characterization of the structure of oaBPs computing the subfunctions of γ' corresponding to the conditions (1)-(4). The main tool in the proofs of these characterizations is the reduction to query complexity, e.g. $\gamma'|_{\vec{x}=0^n}$ has maximum possible query complexity $2n$, which enforces a very tight constraint on the structure of the oaBP computing it. A potential easy proof of Lemma 12 could have followed a logic similar to the proof above if γ' had maximal query complexity $3n$. This, however, is not the case, query complexity¹¹ of γ' is at most $2n+2$, so such a simple argument is unfortunately ruled out.

4.1 Structure of γ' Conditioned on (1)-(4)

We start by characterizing the structure of oaBP that computes $\gamma'|_{\vec{x}=0^n}$.

¹⁰Notice that we do not show that the oaBP in question has the exact form as the one constructed according to (9).

¹¹Query x_1, x_2, z_1, z_2 , if $x_1 \neq x_2$ or $z_1 \neq z_2$, then we may return any value, as none of the conditions (1)-(4) hold. Otherwise, for every possible combination of values x_1, z_1 we have only two possible conditions among (1)-(4) that might hold. If $x_1 = z_1 = \alpha$ we return α right away, if $x_1 = 0$ and $z_1 = 1$, it suffices to query \vec{y} and \vec{z} to get the answer, if $x_1 = 1$ and $z_1 = 0$, it suffices to query \vec{x} and \vec{z} to get the answer.

► **Lemma 13.** *The structure of an oaBP computing the function $\bigvee_{i \in [n]} (y_i \wedge z_i)$ is the following: all nodes except for the 1-sink form a single $2n$ -edge path $(a_1, b_1, a_2, b_2, \dots, a_n, b_n, 0\text{-sink})$ where $\{a_i, b_i\} = \{y_{\pi(i)}, z_{\pi(i)}\}$ for some permutation $\pi \in S_n$. For each $i \in [n]$, the edge $a_i = 1$ goes to b_i , the edge $b_i = 0$ goes to a_{i+1} , the edge $a_i = 0$ goes to a_{i+1} (for a_n it goes to the 0-sink), and the edge $b_i = 1$ goes to the 1-sink.*

Proof. The query complexity of the function $\bigvee_{i \in [n]} (y_i \wedge z_i)$ is $2n$, hence there should exist at least one path of length $2n$ in this branching program. Next, consider the first two variables queried in this path. Assume they do not correspond to y and z with the same label. Without loss of generality assume the first node is labeled by y_1 and the second node is labeled by z_2 (observe that the cases of the first two labels being y_1 and y_2 or z_1 and z_2 are identical, since the variables in the pairs are symmetric). Now, both edges outgoing from y_1 should be parts of the paths that lead to z_2 . As for both values of $y_1 = 0$ and $y_1 = 1$ exists a 1-assignment that is zero in all variables other than $z_2 = y_2 = 1$. Hence, if after querying y_1 we do not query z_2 , we get a BP that incorrectly computes the function. As we consider the first two nodes of the longest path, we get that both edges from y_1 should directly go to the node labeled by z_2 . But it turns the BP to be insensitive to a change in the value of y_1 that contradicts the sensitivity of $\bigvee_{i \in [n]} (y_i \wedge z_i)$ function. Hence the edge $y_1 = 1$ must lead to the node labeled with z_1 . Then it remains to observe that after applying the substitutions $y_1 = 0$ or $y_1 = 1, z_1 = 0$ the query complexity of the function decreases by exactly 2. Note that paths corresponding to these two substitutions must lead to the third node in the branching program. This node then must compute $\bigvee_{i \in [n] \setminus \{1\}} (y_i \wedge z_i)$, so the rest of the oaBP has the desired structure by the simple induction on n . ◀

Next, we summarize the structures for the conditions (2)–(4).

► **Lemma 14.** *The following structural properties hold.*

- (2) *An oaBP computing the function $\gamma'_n|_{\vec{x}=1^n}(\vec{y}, \vec{z}) = \bigvee_{i \in [n]} z_i$ has the following property. Every node v of this oaBP reachable from the source computes the function $\bigvee_{i \in S(v)} z_i$ where $S(v) \subseteq [n]$ is the set of indices of all z -variables that do not precede v in the topological order.*
- (3) *An oaBP computing the function $\gamma'_n|_{\vec{z}=1^n}(\vec{x}, \vec{y}) = \bigvee_{i \in [n]} (x_i \vee y_i)$ is a Hamiltonian path terminating in the 0-sink with edges of the form $x_i = 0, y_i = 0$ going to the next node in the path, and the edges $x_i = 1, y_i = 1$ going to the 1-sink.*
- (4) *In an oaBP computing the function $\gamma'_n|_{\vec{z}=0^n}(\vec{x}, \vec{y}) \equiv 0$ the 1-sink is not reachable from the source.*

Proof. The structural property for (4) is obvious: if there is a path from the source to the 1-sink, the function is not identically zero.

The structural property for (3) is also easy to see. The query complexity of $\gamma'_n|_{\vec{z}=1^n}$ is $2n$, so the oaBP must have a $2n$ -edge path. The edges in these path are the 0-edges and the path terminates in the 0-sink. Suppose an edge of the form $y_i = 1$ or $z_i = 1$ does not go to the 1-sink, say it goes from a node u to a node v , then the path from the source to u according to the Hamiltonian path, then following the edge uv and then continuing to the 0-sink by the Hamiltonian path contradicts the fact that the oaBP computes the disjunction of all variables.

Now let us show the structural property for (2). The source computes the function $\bigvee_{i \in [n]} z_i$ by definition. For any node z_i computing $\bigvee_{j \in S} z_j$ for a set $S \subseteq [n]$ we have that the edge $z_i = 0$ goes to the node computing $\bigvee_{j \in S \setminus \{i\}} z_j$. For any node y_i that is reachable from the source its edges go to the nodes that compute the same function since otherwise

the function computed by the source depends on y_i . Then take any node z_i . Since the function $\gamma'_n|_{\vec{x}=1^n}$ depends on all bits of \vec{z} , this node is reachable from the source. Hence, by the observations above, it computes a function of the form $\bigvee_{j \in S} z_j$ with $i \in S$.

Observe that every path from the source to z_i must visit all nodes from $\{z_1, \dots, z_n\}$ that precede z_i in the topological order. Indeed, if there is a path from the source to z_i that does not visit z_j that precedes z_i in the topological order, then the oaBP returns 0 given the assignment $e_j = 0^{j-1}10^{n-j}$, which is a contradiction. Hence, if the node z_i computes $\bigvee_{j \in S_i} z_j$ and the node z_k computes $\bigvee_{j \in S_k} z_j$, where z_i precedes z_k in the topological order, then $S_i \supseteq S_k$. Moreover $S_i \ni i$ and $S_k \not\ni i$ so $S_i \neq S_k$. Therefore, the sets S_1, \dots, S_n form a chain by inclusion and are all different, which implies the property given that S_i does not contain indices of nodes that precede z_i in the topological order. ◀

4.2 Proof of Lemma 12

We derive Lemma 12 from the following two lemmas.

► **Lemma 15.** *Let B be an oaBP computing γ' . Then for every topological ordering of the nodes of B there exists a mapping $\pi: [n] \rightarrow [n]$ such that $x_{\pi(i)}$ neighbors y_i in the topological order. Additionally, the following property holds*

(†) *if $x_{\pi(i)}$ precedes y_i and z_i in the topological order, then the edge $x_{\pi(i)} = 1$ does not go to 1-sink.*

► **Lemma 16.** *Suppose an oaBP B computes γ' . Suppose ℓ_1, \dots, ℓ_{3n} are the labels of the nodes in some topological ordering of B . Then for every $i \in [3n - 4]$ and every $a, b, c \in [n]$ we have $\{\ell_i, \ell_{i+1}, \dots, \ell_{i+4}\} \neq \{z_a, y_a, x_b, y_c, z_c\}$.*

Proof of Lemma 12. First we show that Lemma 16 guarantees that π given by Lemma 15 is a bijection between $\{y_1, \dots, y_n\}$ and $\{x_1, \dots, x_n\}$.

Let \prec denote the topological order of the labels of the nodes in B . Consider a modified version of π that we denote π' such that whenever possible $y_i \prec \pi'(y_i) \prec z_i$ or $z_i \prec \pi'(y_i) \prec y_i$. Observe that this modification never introduces additional violations of injectivity since y_j cannot neighbor a node between y_i and z_i in the topological order for $j \neq i$.

Now suppose $\pi'(y_i) = \pi'(y_j)$ for some $i \neq j$. WLOG assume that $i = 1$ and $j = 2$. Then, by construction of π' and π there exists k such that $\ell_k = y_1, \ell_{k+1} = \pi'(y_1), \ell_{k+2} = y_2$. Notice that $\pi'(y_1)$ must be a single element of $\{x_1, \dots, x_n\}$ that neighbors y_1 and y_2 , thus $\ell_{k-1} = z_1$ and $\ell_{k+3} = z_2$. Then we get a contradiction with the statement of Lemma 16. Hence π' is an injective mapping. As π' is an injective mapping that maps each element of a set of the size n into an element of a set of the size n , we get that π' is a bijection.

By Lemma 13 ℓ_1, \dots, ℓ_{3n} has a subsequence of form $a_1, b_1, \dots, a_n, b_n$ where $\{a_i, b_i\} = \{y_{\tau(i)}, z_{\tau(i)}\}$ for a permutation $\tau: [n] \rightarrow [n]$. Observe that $\{\ell_1, \ell_2, \ell_3\}$ must contain x_i for $i \in [n]$, otherwise $\{\ell_1, \ell_2, \ell_3\} = \{a_1, b_1, a_2\}$, contradicting Lemma 15 as $y_{\tau(1)}$ does not neighbor $\pi(y_{\tau(1)})$.

If $\{\ell_1, \ell_2, \ell_3\}$ contains more than one element of $\{x_1, \dots, x_n\}$ then $\{\ell_1, \ell_2, \ell_3\} \ni y_{\tau(1)}$ by surjectivity of π' . Moreover, $\ell_3 = x_b$ and either $\ell_1 = x_a, \ell_2 = y_{\tau(1)}$ or $\ell_1 = y_{\tau(1)}, \ell_2 = x_a$ for some $a, b \in [n], a \neq b$. Then by Lemma 13 we get $\ell_4 \in \{x_1, \dots, x_n, z_{\tau(1)}\}$. Thus x_b does not belong to the image of π , which is a contradiction. Therefore, $\{\ell_1, \ell_2, \ell_3\} = \{\pi'(y_{\tau(1)}), y_{\tau(1)}, z_{\tau(1)}\}$. Iteratively applying this procedure to $\{\ell_{3k+1}, \dots, \ell_{3n}\}$ and π' restricted to $\{y_{\tau(k+1)}, \dots, y_{\tau(n)}\}$ for $k \in [n]$ concludes the proof of the theorem. ◀

4.3 Proof of Lemma 15

Let B_α for $\alpha \in \{0, 1\}$ be the diagram obtained from B by contracting all edges of form $x_i = \alpha$ and removing all edges of form $x_i = 1 - \alpha$. Then the diagram B_0 computes $\gamma'|_{\vec{x}=0^n}$. Then from Lemma 13 and (1) we get that B_0 consists of a path of length $2n$ that is split into consequent blocks with two nodes in each, where the labels of the nodes in each block are $\{y_i, z_i\}$ for $i \in [n]$.

Consider an arbitrary block in B_0 and let u and v be the nodes of B corresponding to the nodes in the block, such that u precedes v in the topological order of B .

4.3.1 Case 1: u and v are labeled with z_i and y_i respectively

u must still be present in B_1 and be reachable from the source (say, with the assignment $\vec{x} = \vec{y} = 1^n$; $\vec{z} = 0^n$). Then consider edges in B_0 labeled with $z_i = 1$ and $y_i = 0$: uv and vw , where w is the first node in the subsequent block of B_0 , these are uniquely defined since B_0 is an oaBP. We claim that at least one of these edges does not appear in B i.e. it is present in B_0 due to a contraction.

Assume towards a contradiction that both of the edges are present in B . Then let us trace a path from the source of B to the 0-sink according to the assignment $\vec{x} = 1^n$; $\vec{y} = 0^n$; $\vec{z} = e_i$. This path must contain u and w , as z_i -node, corresponding to u should be queried on substitution $x = 0^n$, and then, after assigning $z_i = 1$ and $y_i = 0$ this path leads to the node w . By Lemma 14 for (2) the node w in B_1 and our choice of w , computes $\bigvee_{j \in S} z_j$. Since z_i precedes w in the topological order, $S \not\ni i$. Hence, we get $B(1^n, 0^n, e_i) = 0$ which with value $\gamma'(1^n, 0^n, e_i) = 1$ leads to a contradiction.

Now, since at least one of the edges uv and vw does not appear in B , but appears in B_0 , the node v in B is adjacent to a node r labeled with an element of $\{x_1, \dots, x_n\}$. Since uv and vw belong to the Hamiltonian path in B_0 , r must neighbor v in the topological order. Then define $\pi(y_i) = x_j$ where x_j is the label of r . Notice that here the node $x_{\pi(i)}$ never precedes both y_i and z_i , so the property (\dagger) is satisfied.

4.3.2 Case 2: u and v are labeled with y_i and z_i respectively

First, suppose y_i and z_i are the first two nodes in B_0 . By Lemma 13 the edge $y_i = 0$ in B_0 goes to the first node of the next yz -pair, let us denote it by w . The edge $y_i = 1$ goes to the z_i -node in B_0 . Now, assume, towards a contradiction, that the $y_i = 0$ -edge in B goes to w as well. Then, a path in B corresponding to an assignment $s = (1^n, 0^n, e_i)$ skips the node z_i , as y_i is the source of this oaBP, and w is going after z_i in the topological order. Similarly to the proof of Case 1, we get a contradiction with the fact that the subdiagram of B_1 rooted in w computes the function $\bigvee_{j \in S} z_j$ with $S \not\ni i$, which is 0 on s , hence $B(s) = B_1(0^n, e_i) = 0$, whereas $\gamma'(s) = 1$.

Now, assume there is at least one other yz -pair queried before y_i and z_i . Let y_j and z_j be the pair that precedes y_i and z_i in the topological order of B . Note, that in B_0 both edges $y_j = 0$ and $z_j = 0$ go to y_i . And $y_i = 0$ edge in B_0 goes to a node w (skipping z_i), which is the first node in the next yz -block.

\triangleright **Claim 17.** One of the edges among $z_j = 0$ and $y_i = 0$ differ in B_0 and B (that is, the edge with this label connects different pairs of nodes).

Proof. The proof of this claim repeats a similar argument in the Case 1. Assume these two edges in B are the same as in B_0 . Consider a substitution $s = (1^n, 0^n, e_i)$. As B_1 computes the disjunction of z , a path p corresponding to s in B should query all bits of \vec{z} preceding z_i

in the topological order. Hence, p goes through z_j , and $z_j = 0$ to y_i . Now, as $y_i = 0$ skips z_i and goes to w , which by Lemma 14 computes $\bigvee_{t \in S} z_t$ in B_1 . Analogously to the Case 1, $S \not\ni i$, so we get that $B(s) = 0$, which contradicts $\gamma'(s) = 1$. Thus, one of the edges $z_j = 0$ or $y_i = 0$ in B leads to a different node compared to B_0 . \triangleleft

4.3.3 Case 2a: the edge $z_j = 0$ goes to y_i

By Claim 17 $y_i = 0$ does not go to w . But in this case it goes to a node r labeled by an x -variable in B . Then as oaBP B queries all bits in \vec{z} on the assignment $(1^n, 0^n, 0^n)$, we get that r goes before z_i in the topological order. So we get that y_i neighbors an x -node in the topological order.

4.3.4 Case 2b: the edge $z_j = 0$ does not go to y_i

Let x_ℓ be the endpoint of $z_j = 0$ in B (notice that $z_j = 0$ can not go to y_j since then it would go there in B_0 as well). Consider the path from x_ℓ to y_i in B according to the assignment $\vec{x} = 0^n$, by the construction of B_0 all nodes on this path are labeled with x -variables. Let x_k be the last node on this path before y_i . We define $\pi(i) = k$. Now it remains to show the following:

1. x_k is the immediate predecessor of y_i in the topological order.
2. The property (\dagger): $x_k = 1$ in B does not go to the 1-sink.

Let us first prove Item 1. Consider the program $B|_{\vec{z}=1^n}$. By Lemma 14 for (3), $B|_{\vec{z}=1^n}$ consists of a single Hamiltonian path of 0-edges. Then, consider the edge $x_k = 0$ in B . This edge goes to y_i , so $x_k y_i$ is present in $B|_{\vec{z}=1^n}$, so in topological order there are no x -node or y -node between x_k and y_i . Given that z_j precedes x_k in the topological order, x_k is the immediate predecessor of y_i .

Now let us prove Item 2. Consider an assignment $\vec{x} = 0^n$, $\vec{y} = 1^n$ and $\vec{z} = 0^n$. Let us trace the path corresponding to this assignment from the root of B to z_j . Notice that z_j is reachable by this path since it is reachable in B_0 by the path according to $\vec{y} = 1^n$, $\vec{z} = 0^n$. After z_j let us follow the edge $z_j = 0$ to x_ℓ and then to x_k following the 0-edges. Suppose that $x_k = 1$ goes to the 1-sink. Let p be the path from the root to the 1-sink that we have traced. Then consider the program $B|_{\vec{z}=0^n}$. Let $p|_{\vec{z}=0^n}$ be the result of contraction of the edges of form $z_t = 0$ in p . $p|_{\vec{z}=0^n}$ exists in $B|_{\vec{z}=0^n}$, hence the 1-sink is reachable in this program. This contradicts the fact that $\gamma'|_{\vec{z}=0^n}$ is identically zero.

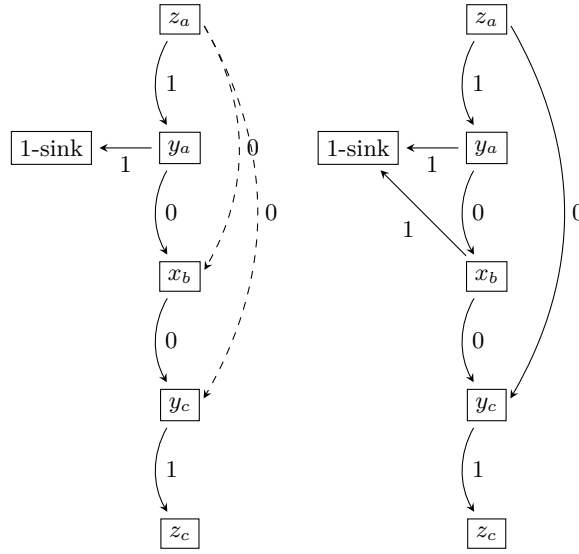
4.4 Proof of Lemma 16

We are going to gradually derive the structure of the edges going out of the nodes $\{z_a, y_a, x_b, y_c, z_c\}$.

By (3) we have $\gamma'|_{\vec{z}=1^n} = \bigvee_{i \in [n]} x_i \vee y_i$, so by Lemma 14 the edges of from $x_i = 0$ and $y_i = 0$ must either go to a variable in $\{z_1, \dots, z_n\}$ or to the next variable in $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ in the topological order. Hence $y_a = 0$ leads to x_b , $x_b = 0$ leads to y_c .

By (1) we have $\gamma'|_{\vec{x}=0^n} = \bigvee_{i \in [n]} z_i \wedge y_i$, the diagram $B|_{\vec{x}=0^n}$ has structure defined by Lemma 13. Then $z_a = 1$ leads to y_a and $y_c = 1$ leads to z_c and $z_a = 0$ leads either to x_b or y_c . Now observe that the edge $y_a = 1$ goes to the 1-sink by the structure of the diagram $B|_{\vec{z}=1^n}$. If it does go to x_b the assignment $\vec{x} = 0^n$, $\vec{y} = 1^n$ and $\vec{z} = e_a$ is evaluated to 0 by B , which is a contradiction with (3). See Figure 1 for the edges we identified so far.

Now consider an edge $x_b = 1$. By Lemma 14 for (3) it must go to the 1-sink in the diagram $B|_{\vec{z}=1^n}$, so in B it goes either to $\{z_1, \dots, z_n\}$ or to the 1-sink. Recall that by (2) we have $B|_{\vec{x}=1^n} = \bigvee_{i \in [n]} z_i$. Suppose $x_b = 1$ goes to z_i for $i \in [n]$. Then consider the assignment



■ **Figure 1** Fixed edges between $\{z_a, y_a, x_b, y_c, z_c\}$.

$\vec{x} = 1^n$, $\vec{z} = e_a$, $\vec{y} = 0^n$. A path p , corresponding to it passes through the node z_a by the structure of $B|_{\vec{x}=1^n}$ given by Lemma 14. Hence, p then leads to x_b and to z_i , which computes $\bigvee_{t \in S} z_t$ for some $S \subseteq [n] \setminus \{a\}$. Thus $B(1^n, 0^n, e_a) = 0$, yet $\gamma'(1^n, 0^n, e_a) = 1$. Hence $x_b = 1$ goes to the 1-sink.

Now we can pinpoint the endpoint of the edge $z_a = 0$. Suppose it goes to x_b . Then we get the contradiction with the assignment $\vec{x} = 1^n$, $\vec{y} = 0^n$ and $\vec{z} = 0^n$. $B(1^n, 0^n, 0^n) = 1$ since we go from z_a to x_b and then to the 1-sink, yet $\gamma(1^n, 0^n, 0^n) = 0$.

Then consider the edge $y_c = 0$, we claim that it goes to z_c . Otherwise consider the assignment $\vec{x} = 1^n$, $\vec{y} = 0^n$ and $\vec{z} = e_c$, $B(1^n, 0^n, e_c) = 0$ since the path of this assignment passes through z_a , then goes to y_c and then to a node strictly below z_c that by Lemma 14 computes $\bigvee_{i \in S} z_i$ for $S \subseteq [n] \setminus \{c\}$. Thus we get a contradiction with $\gamma'(1^n, 0^n, e_c) = 1$.

Then we get the final contradiction as follows: $\gamma'|_{\vec{x}=0^n}$ depends on y_c , but $B|_{\vec{x}=0^n}$ does not, in other words $B(0^n, e_c, e_c) = B(0^n, 0^n, e_c)$, yet $\gamma'(0^n, e_c, e_c) \neq \gamma'(0^n, 0^n, e_c)$.

5 Structure of oaBP Computing γ_G : proof of Theorem 11

The theorem follows from the following lemmas, utilizing the properties (5)-(7) of γ_G .

► **Lemma 18.** *For every oaBP B computing γ_G , the permutation κ_B defined by B maps elements of $[n/2]$ into $[n/2]$, and elements of $\{n/2+1, \dots, n\}$ into elements of $\{n/2+1, \dots, n\}$.*

► **Lemma 19.** *For every oaBP B computing γ_G , the permutation κ_B defined by B corresponds to an independent set in G , that is $\{(i, \kappa_B(i)) \mid i \in [n]\}$ is independent in G .*

We start by deriving the theorem from the lemmas.

Proof of Theorem 11. From Lemma 18 and Lemma 19 we get that an oaBP computing γ_G defines a permutation on $[n]$ which maps elements of the first/second half of $[n]$ to the elements of the first/second half of $[n]$ respectively, and for every edge it contains at most one of the vertices of a BPIS instance. Hence, the permutation defined by any oaBP computing γ_G is a correct certificate for a $(n \times n)$ -BPIS instance. Thus an oaBP computing γ_G exists iff there is a correct permutation defining independent set in an input $(n \times n)$ -BPIS instance. ◀

5.1 Proofs of Lemma 18 and Lemma 19

Both of the lemmas follow from a helper lemma about the structure of oaBP for γ' :

► **Lemma 20.** *Let B be an oaBP computing γ' . Then, if a path p in B goes to the 1-sink from one of the nodes in a triplet x_i, z_j, y_j , then p sets at least two variables among x_i, z_j, y_j to 1.*

We defer the proof of Lemma 20 to Section 5.2. Now we are ready to prove the lemmas.

Proof of Lemma 18. First, as γ_G is a restricted version of γ' , we get that κ_B is well-defined by Lemma 12. Moreover, we know that the topological order of B has the form $a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n$, where for every $h \in [n]$ there exist $k, \ell \in [n]$, such that $\{a_h, b_h, c_h\} = \{x_k, z_\ell, y_\ell\}$, where $\kappa_B(\ell) = k$.

Let x_i, z_j, y_j form a consequent xyz -triplet in the oaBP, with $i = \kappa_B(j)$. Assume towards a contradiction that i and j are from different halves of the set $[n]$. WLOG let $i \in [n/2]$ and $j \in \{n/2 + 1, \dots, n\}$.

Consider a substitution s , in which $z = 1^{n/2}0^{n/2}, y = 0^n, x_i = e_i$. Consider a path p consistent with s . By property (7) it terminates in the 1-sink. Now, consider the last xyz -triplet, that p goes through before getting to the 1-sink. This could either be x_i, z_j, y_j -triplet, or x_k, z_ℓ, y_ℓ , such that $i \neq k; j \neq \ell$. In both cases, at most one of the variables in such xyz -triplets is set to 1. Hence, we get a contradiction with Lemma 20. The proof for the xyz -triplets in the last $n/2$ -elements of $[n]$ goes similarly. ◀

Proof of Lemma 19. Consider an edge connecting (j, k) with (j', k') in G . Assume towards a contradiction that κ_B maps an element j to k , and j' to k' . Then, there are two triplets x_j, y_k, z_k , and $x_{j'}, y_{k'}, z_{k'}$ such that within each xyz -triplet the elements are layered consequently in B . Consider a substitution s , which sets $x_j = x_{j'} = 0$, all other bits of \vec{x} to 1, $z_k = z_{k'} = 1$, all other bits of \vec{z} to 0, and all bits of \vec{y} to 0. $\gamma|_s = 1$, hence there is a path p in B consistent with s which goes from the source to 1-sink. Consider a triplet x_a, y_b, z_b , which is queried the last on the path p . Similarly to the previous case we consider two cases, if this triplet is x_j, y_k, z_k (or $x_{j'}, y_{k'}, z_{k'}$, proof for which is analogous) or not. In the latter case, by our assumptions on the structure of B , all $a \neq j, a \neq j', b \neq k, \text{ and } b \neq k'$.

If $a = j$ and $b = k$, then, by our choice of s and this triplet, we get that p goes to 1-sink after substituting $x_j = 0, z_k = 1, y_k = 0$. Which is a contradiction with Lemma 20.

Finally, assume either $a \neq j$ and $b \neq k$. Then there is a path in B from the nodes x_a, y_b, z_b directly to 1-sink which is consistent with a substitution $x_a = 1, z_b = 0, y_b = 0$. Thus, again, leads to a contradiction with Lemma 20. ◀

5.2 Proof of Lemma 20

Assume the opposite: there exist paths in B , which go to the 1-sink immediately after substituting exactly one 1-value to variables of an xyz -triplet. Consider a path p among these paths, which goes to the 1-sink from the earliest node according to the topological order of B . Let us denote the xyz -triplet, from which p goes to the 1-sink as x_i, z_j, y_j , and let w be the last non-sink node in p .

▷ **Claim 21.** p assigns 1 to w .

Proof. Assume the opposite: the last edge in p going to the 1-sink is $w = 0$.

Consider a case $w = x_i$. We want to show that $x_i = 0$ -edge cannot lead to the 1-sink. To see this observe that oaBP $B|_{\vec{z}=1^n}$ is sensitive in all x -variables. Hence, in the oaBP $B|_{\vec{z}=1^n}$ a path consistent with setting $\vec{x} = \vec{y} = 0^n$ goes through x_i , and goes to the 0-sink. This contradicts our assumption that the edge $x_i = 0$ goes to 1.

54:16 Partial Minimum Branching Program Size Problem Is ETH-Hard

Now consider a case when $w \in \{y_j, z_j\}$. Consider a path p' in the oaBP $B|_{\vec{x}=0^n}$ such that for every $k \in [n]$ $p'(z_k) = 1$ and $p'(y_k) = 0$ if z_k precedes y_k in the topological order, and $p'(z_k) = 0$ and $p'(y_k) = 1$ otherwise. By (1) we have $\gamma'|_{\vec{x}=0^n} = \bigvee_{i \in [n]} (y_i \wedge z_i)$, so every such p' ends in the 0-sink. This contradicts the fact that the edge $w = 0$ goes to the 1-sink, since at least one of such paths p' contains this edge. The claim then follows. \triangleleft

In the remainder of the proof, we show that $w = 1$ cannot be the only 1-value assigned by p to the variables in the last triplet x_i, z_j, y_j . Consider the case when $w = y_j$ or $w = z_j$ and the value of w is assigned to 1 by p . Consider an oaBP $B' = B|_{\vec{x}=0^n}$ corresponding to a partial substitution $\vec{x} = 0^n$. We know the structure of B' from Lemma 13. For each zy -block there is a path to the 1-sink iff we set both $z_i = y_i = 1$. But that contradicts the existence of a path of a form $x_i = 0, y_j = 0, z_j = 1$ or $x_i = 0, y_j = 1, z_j = 0$ from the x_i, z_j, y_j triplet to the 1-sink. Hence, w could not be y or z variable.

Consider the only remaining case, when $w = x_i$, and p sets $x_i = 1$. In the following two claims we show that p should ask either y_j or z_j prior to x_i .

\triangleright **Claim 22.** Either y_j or z_j precedes x_i in the topological order.

Proof. Assume the opposite, x_i is the first node in the triplet. By Lemma 15 y_j goes either right before or right after x_i in the topological order. Hence, we get that nodes in this triplet appear in the order x_i, y_j, z_j in the topological order. But then, by the property (†) in Lemma 15, if x_i is the first node of the triplet then the edge $x_i = 1$ does not go to the 1-sink, contradicting the properties of p . \triangleleft

\triangleright **Claim 23.** Path p queries either a value of y_j or z_j before querying x_i .

Proof. By Claim 22 we know x_i is not the first node of the triplet x_i, y_j, z_j in the topological order. Now assume that p skips querying yz -nodes preceding x_i in this triplet and goes directly to x_i . Let w' be the last node prior to x_i in p . By our assumption w' belongs to an xyz -triplet $\{x_b, y_a, z_a\}$ different from $\{x_i, y_j, z_j\}$.

First let us rule out the case $w' = z_a$ or $w' = y_a$. If p assigns 0 to w' , then the edge $w' = 0$ in $B|_{\vec{x}=0^n}$ goes to the endpoint of the edge $x_i = 0$ in B , which contradicts Lemma 13, since this edge must go to the first variable of the subsequent zy -block, which precedes x_i in the topological order of B . If p assigns 1 to w' , then in $B|_{\vec{x}=0^n}$ the edge $w' = 1$ goes to the 1-sink (as it does not go to the node in the same zy -block), hence in B the edge $x_i = 1$ must go to the 1-sink, which implies that the function computed by B does not depend on x_i , which is a contradiction, since γ' does depend on x_i .

Hence $w' = x_b$. Let us now show that p assigns 1 to x_b . Assume the opposite: x_b is set to 0 in B . Let us then trace the path corresponding to the assignment $\vec{z} = 0^n; \vec{y} = 1^n; \vec{x} = 0^n$ in B until we reach the node x_b . This must happen by Lemma 14 for (3). Now by the Lemma 13 and property (1) the edge $x_b = 0$ must go to the first node of the subsequent zy -block, which is a contradiction with the assumption that $x_b = 0$ ends in x_i . Therefore p assigns $x_b = 1$.

Consider an assignment s of $\vec{z} = 1^n, \vec{y} = 0^n, \vec{x} = e_b$. By (3) we have $\gamma(s) = 1$. But by our assumptions an edge $x_b = 1$ does not go to the 1-sink immediately, it goes to x_i first. A node x_i should also be reachable by a substitution s' which assigns $\vec{z} = 1^n, \vec{y} = 0^n, \vec{x} = 0^n$, as $\gamma'|_{\vec{z}=1^n}$ is sensitive to the value of x_i by (3). But $\gamma(s') = 0$. Hence, we get a contradiction with the fact that paths corresponding to assignments s and s' meet in the node x_i , and follow the same path in B , but one must end in the 0-sink, and another in the 1-sink. Therefore, either z_j or y_j node is not queried by p . \triangleleft

Now, let r be the immediate predecessor of w in p . By Claim 23 we know that either $r \in \{y_j, z_j\}$. First, consider the case of $r = z_j$. Consider a substitution s of $\vec{x} = 1^n$ and $\vec{z} = 0^n$. We know that the oaBP $B|_{\vec{x}=1^n}$ queries all values of z . Hence, $B|_{\vec{x}=1^n}$ contains a path $r = 0, w = 1$, which goes to the 1-sink. But, as the assignment s does not satisfy $B|_{\vec{x}=1^n}$, and is consistent with the substitution $r = 0, w = 1$ we get a contradiction (since the path s passes through r).

Finally, to get a contradiction with the case when $r = y_j$ we show the following two claims.

▷ **Claim 24.** If $r = y_j$, then z_j goes before y_j and x_i in the topological order.

Proof. Assume the opposite. Let z_j go right after x_i in the topological order. So we get that the order of nodes in this triplet is y_j, x_i, z_j . First, note that $y_j = 0$ edge goes to x_i as $B|_{\vec{z}=1^n}$ queries all y s and x s sequentially, and every edge labeled with 0-edge in $B|_{\vec{z}=1^n}$ goes to either the next node or to the 0-sink. We also know that $x_i = 1$ -edge goes to the 1-sink. Now note that that a path consistent with a substitution s defined as $\vec{x} = e_i, \vec{y} = 0^n, \vec{z} = 0^n$ should reach node y_j . But then this path will follow edge $y_j = 0$ to x_i and edge $x_i = 1$ to 1 sink, though $\gamma|_s = 0$. A contradiction. ◁

▷ **Claim 25.** If z_j is the first node in a xyz -triplet then path p goes through z_j .

Proof. Consider all possible edges which get to this triplet. Let x_a, z_b, y_b be the triplet right before this one. From the structure of $B|_{\vec{x}=1^n}$ we know that all edges from z_b and y_b either go to a node within this triplet, or the 1-sink, or to z_j . We also know that $x_a = 0$ cannot skip z_j , as x_a -node is reachable by a path consistent with a substitution $\vec{x} = 0^n, \vec{y} = 0^n, \vec{z} = 1^n$. Hence if $x_a = 0$ skips z_j then $B|_{\vec{x}=0^n}$ would be insensitive to z_j .

It now remains to show that $x_a = 1$ cannot skip z_j either. In $B|_{\vec{z}=1^n}$ every edge $x_a = 1$ should go to 1-sink. Hence edge $x_a = 1$ either goes to z -node or to 1-sink. This finishes the proof. ◁

From these two claims, we get that p should necessarily go through z_j . Hence, by considering $B|_{\vec{x}=1^n}$ we get that a path consistent $z_j = y_j = 0$ should either go to a node in the next triplet, or go to the 0-sink, in case it is the last triplet of B . Which contradicts the fact that p after substituting $z_j = y_j = 0$ and $x_i = 1$ goes to the 1-sink. This finishes the proof of Lemma 20.

6 Corollaries

Multiple corollaries follow from Theorem 11. For all classes of branching programs that degenerate to an oaBP if constrained on having at most n nodes on an n -bit input, we get that the corresponding minimization problem is ETH-hard. In particular it holds for general branching programs (BP), ordered binary decision diagrams (OBDD) and read- k branching programs for any $k \geq 1$ (k -BP). For definitions of the latter two classes we refer the reader to [27]. Formally we have

► **Corollary 26.** *Let \mathcal{C} be a circuit class such that for every $f: \{0, 1\}^n \rightarrow \{0, 1\}$ that depends on all its variables there exists $C \in \mathcal{C}$ with $|C| = n$ computing f if and only if there exists oaBP B computing f .*

*Then, assuming ETH holds we have that \mathcal{C} -MCSP $^*_{s=n}$ requires time $2^{\Omega(n \log n)}$.*

It is easy to see that for $\mathcal{C} \in \{\text{OBDD}, \text{BP}\} \cup \{k\text{-BP} \mid k \geq 1\}$ this statement holds, so in particular Theorem 1 is implied.

Proof of Corollary 26. Let \mathcal{R} be the reduction given by Theorem 11. Observe that every partial function in the image of \mathcal{R} can only be extended to a function that depends on all of its variables.

Hence by the assumption on \mathcal{C} , $\mathcal{R}(G) \in \mathcal{C}\text{-MCSP}|_{s=n}^*$ iff $\mathcal{R}(G)$ is computable by an oaBP. Therefore, Theorem 11 and Theorem 8 imply the statement. \blacktriangleleft

Following on the results from [6], in which the authors show that $(n \times n)$ -BPIS is unconditionally hard for 1-NBPs by showing that reduction of Ilango is computable by a 1-NBPs, we get that similar 1-NBP lower bound holds for all total minimization problems for branching programs.

► **Corollary 27.** *For \mathcal{C} as in Corollary 26, the size of the smallest 1-NBP that computes $\mathcal{C}\text{-MCSP}$ is $N^{\Omega(\log \log(N))}$, where N is the input size of $\mathcal{C}\text{-MCSP}$.*

Proof. By [6, Theorem 6] we have that the sizes of 1-NBPs computing $\mathcal{C}\text{-MCSP}^*$ and $\mathcal{C}\text{-MCSP}$ are linearly related (the theorem is only stated for vanilla MCSP, but the proof does not use this). By [6, Theorem 4], the reduction \mathcal{R} can be implemented in 1-NBP, so by Theorem 11 1-NBP size of $\mathcal{C}\text{-MCSP}^*$ with input size $N = 2^{3n}$ is lower bounded by the 1-NBP size of $(n \times n)$ -BPIS which by [6, Theorem 3] is $2^{\Omega(n \log n)} = N^{\Omega(\log \log N)}$. \blacktriangleleft

Another corollary follows from the fact that \mathcal{R} always outputs a partial functions that can be encoded as a 2-BP.

► **Corollary 28.** *It is NP-hard to check, whether any extension of a partial function expressed by a 2-BP over the alphabet $\{0, 1, *\}$ can be computed with an oaBP.*

Proof. Since $(n \times n)$ -BPIS is an NP-complete problem, it suffices to check that for every G the function γ_G can be represented as a 2-BP of polynomial size. First, notice that a 1-BP can distinguish between the cases (1)-(6) and the case that either (7) is true or the function equals to $*$: we read variables in the order $x_1, y_1, z_1, \dots, x_n, y_n, z_n$ and for each of the patterns among (1)-(6) we maintain if the current prefixes of \vec{x} , \vec{y} , and \vec{z} coincide with the prefix of the pattern. Thus, at each level of our 1-BP we have 2^6 states. For each of the states at the last level, we choose one of the patterns which is satisfied arbitrary, so we can merge the states into 7: 6 corresponding to the conditions (1)-(6) and one to the case where none of the conditions hold. For each of the cases (1)-(6), the restricted function can easily be computed by a 1-BP.

Now it remains to distinguish the case (7) from a star with a 1-BP. In order to do this we split the input into 6 chunks of length $n/2$ and read then sequentially. For each chunk we maintain the set of possible values among $\{0^{n/2}, 1^{n/2}\} \cup \{e_i \mid i \in [n/2]\} \cup \{\bar{e}_i \mid i \in [n/2]\}$ that the chunk might have given the current prefix. Observe, that the number of potential states is linear. After we read all the chunks we have $O(n^6)$ states at the last level, so we know the only tuple j, k, j', k' that may satisfy the predicate in (7). Since the edges are fixed, we can unite each of the states at the last level with the 1-sink or the $*$ -sink. \blacktriangleleft

Finally, to show the coNP-hardness of compressing BPs we use the following fact.

► **Proposition 29** ([24]). *Let (3, 4)-SAT be a language of satisfiable 3-CNF formulas in which every variable occurs in at most 4 clauses. (3, 4)-SAT is NP-complete.*

► **Corollary 30.** *It is coNP-hard to check whether an input 4-BP for a total function can be compressed to a BP of a constant size.*

Proof. By Proposition 29 we get that $\overline{(3,4)\text{-SAT}}$ is coNP -hard. Now, we construct a reduction from $\overline{(3,4)\text{-SAT}}$ to the partial 4-BP minimization as follows. For a 4-CNF formula φ in which every variable appears at most 4 times we construct a branching program B by consequently adding nodes of each clause. Denote the first clause of φ as C_1 , and let x_i, x_j, x_k be the variables in it. We add nodes labeled by x_i, x_j , and x_k to B . We set the node labeled by x_i as a source of B . Then, if x_i was negated in C_1 we direct a 1-edge from the x_i -node to the node labeled by x_j , otherwise we direct a 0-edge from x_i to the x_j -node. Then we similarly define one of the edges of x_j to x_k . Finally, if x_k was negated in C_1 we direct its 1-edge to the 1-sink. Then we add nodes from the next clause of φ , and direct all missing edges for nodes x_i, x_j , and x_k to the first node corresponding to the next node. We repeat this procedure for all clauses, redirecting all unassigned edges for nodes corresponding to variables in the last clause of φ to the 0-sink.

Now, for any substitution s of values of the variables B outputs 1 if φ was not satisfied by s , and 0 otherwise. But note, that if φ was an unsatisfiable formula than B would had to output 1 on all inputs. Hence, it could be compressed to a branching program of size 0 (as we count the number of all nodes other than sinks). On the other hand, if φ is satisfiable and has at least one unsatisfying assignment B has to contain at least one node other than a sink. To account for the fact that the size of B corresponding to φ that is satisfiable by any assignment is also 0 (it is just a 0-sink), we change our reduction to first substitute all 0 assignment to φ . If $\varphi(\bar{0}) = 1$ we set B to be a BP for an identity function on one bit which has the size 1. Otherwise, construct B as described before. ◀

Additionally, as for any 3-SAT instance in which every variable appears in at most 4 clauses we can easily construct an equivalent 3-BP of polynomial size. we get that checking whether this BP can be compressed to a BP of size 1 allows us to check whether the formula is unsatisfiable.

7 Future Work

The main future direction is to strengthen our results to the total version of MBPSP, and to show NP -hardness of MBPSP (or at least MBPSP*). We discuss a couple of barriers for improving results in both of these directions as well as additional open questions motivated by our work.

7.1 Show polynomial-time oracle reduction from MBPSP* to MBPSP

As noted in [10] such reductions are very ad hoc for each specific model. Indeed, the reduction that Ilango showed for the Minimum DeMorgan Formula Size Problem in [10] does not work for branching programs as it relies on the tree-like structure of formulas. The ability of branching programs to reuse the same computation in different branches makes it harder to come up with a suitable reduction. For now we do not even know such a reduction for 1-BPs but the natural next step would be to reduce (1-BP)-MCSP* to (1-BP)-MCSP.

7.2 Prove a stronger conditional lower bound

It would be interesting to find a lower bound stronger than $N^{\Omega(\log \log N)}$ for the time complexity of MBPSP* (or MFSP*). The obstacle here is that we actually show the ETH-hardness of MBPSP* $_{|n}$, for which this time complexity is actually tight, since there are $n^{O(n)} = N^{O(\log \log N)}$ potential branching programs that can compute a function. Hence, in order to have stronger lower bounds we need to consider larger size parameters. The current framework requires us to find explicit functions that require BP-size larger than the

given size parameter. Hence, the best we can hope for without improving the state-of-the-art BP lower bounds is to show that $\text{MBPSP}^*|_{n^2}$ is ETH-hard. Here the trivial upper bound is $n^{O(n^2)} = N^{O(\log N \log \log N)}$, which while still quasipolynomial, is significantly stronger.

7.3 Results for nondeterministic BP minimization

Currently our hardness result only holds for the problem of minimizing deterministic branching programs. But we believe that similar ideas could be used to show that the problem of minimizing nondeterministic branching programs for partial functions is also ETH-hard. Nondeterministic branching programs (NBP) differ from deterministic ones by an additional type of nodes, called *nondeterministic* nodes, that do not have labels. Such nodes correspond to nondeterministic guessing of a branching program. Note, that if we consider a complexity measure that counts *all nodes in an* NBP other than sinks, then our main result immediately extends to the ETH-hardness of minimizing NBP and k -NBP for all k .

However, the standard complexity measure for NBPs is the minimal number of nodes that query a variable (so all nodes other than sinks and nondeterministic nodes) in an NBP computing the function. So one of the approaches to showing the ETH-hardness of the Partial Minimum Nondeterministic Branching Program Size Problem could be in showing that existence of the oaBP with nondeterministic nodes for γ_G is equivalent to the existence of the deterministic oaBP for γ_G . Namely by adding the nondeterministic node we do not make it easier for an oaBP to compute γ_G . Could this result be obtained with an additional careful analysis?

7.4 Could we adapt the NP-hardness proof of Hirahara?

Hirahara in [7] showed a randomized reduction from **NP**-hard problem that is a variation of Minimum Monotone Satisfying Assignment called Collective Minimum Weight Monotone Satisfying Assignment (CMMSA) problem to MCSP*. One of the possible directions of strengthening our result from the ETH-hardness to **NP**-hardness is by adapting this framework for branching programs. Indeed, the reduction from [7] seems to be robust enough to correctly work on all *no*-instances of CMMSA even for branching programs. But the same reduction does not immediately go through for the *yes*-instances. The main obstacle that we see is in showing an analogue of the Uhlig's lemma [25, 26] for branching programs. This lemma states that in order to compute the direct product of the same function (of arbitrary circuit complexity) on subexponentially many inputs, it is sufficient to use one circuit of size $O(2^n/n)$. Such efficient compression is an important step for constructing a small circuit corresponding to the *yes*-instances of CMMSA. It is not known whether an analogue of Uhlig's lemma exists for branching programs. Rao and Sinha [19] showed a variation of the direct-sum theorem for read-once BPs, hence disproving the analogue of Uhlig's lemma in 1-BP setting. We believe that showing an analogue of Uhlig's lemma for general branching programs in the specific setting required by the Hirahara's proof is a promising direction for showing **NP**-hardness of MBPSP*.

References

- 1 Beate Bollig. On the minimization of (complete) ordered binary decision diagrams. *Theory of Computing Systems*, 59(3):532–559, 2016. doi:10.1007/s00224-015-9657-x.
- 2 Beate Bollig and Ingo Wegener. Improving the variable ordering of obdds is np-complete. *IEEE Trans. Computers*, 45(9):993–1002, 1996. doi:10.1109/12.537122.

- 3 Magnus Gausdal Find, Alexander Golovnev, Edward A. Hirsch, and Alexander S. Kulikov. A better-than- $3n$ lower bound for the circuit complexity of an explicit function. In *FOCS*, pages 89–98. IEEE Computer Society, 2016. doi:10.1109/FOCS.2016.19.
- 4 S.J. Friedman and K.J. Supowit. Finding the optimal variable ordering for binary decision diagrams. *IEEE Transactions on Computers*, 39(5):710–713, 1990. doi:10.1109/12.53586.
- 5 Oliver Giel. Branching program size is almost linear in formula size. *Journal of Computer and System Sciences*, 63:222–235, 2001. doi:10.1006/JCSS.2001.1762.
- 6 Ludmila Glinskih and Artur Riazanov. MCSP is hard for read-once nondeterministic branching programs. In Armando Castañeda and Francisco Rodríguez-Henríquez, editors, *LATIN 2022: Theoretical Informatics - 15th Latin American Symposium, Guanajuato, Mexico, November 7-11, 2022, Proceedings*, volume 13568 of *Lecture Notes in Computer Science*, pages 626–640. Springer, 2022. doi:10.1007/978-3-031-20624-5_38.
- 7 Shuichi Hirahara. Np-hardness of learning programs and partial MCSP. In *63rd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2022, Denver, CO, USA, October 31 - November 3, 2022*, pages 968–979. IEEE, 2022. doi:10.1109/FOCS54457.2022.00095.
- 8 Shuichi Hirahara, Igor C. Oliveira, and Rahul Santhanam. Np-hardness of minimum circuit size problem for OR-AND-MOD circuits. In Rocco A. Servedio, editor, *33rd Computational Complexity Conference, CCC 2018, June 22-24, 2018, San Diego, CA, USA*, volume 102 of *LIPICs*, pages 5:1–5:31. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPICs.CCC.2018.5.
- 9 Rahul Ilango. Constant depth formula and partial function versions of MCSP are hard. In *FOCS*, pages 424–433. IEEE, 2020. doi:10.1109/FOCS46700.2020.00047.
- 10 Rahul Ilango. The minimum formula size problem is (ETH) hard. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 427–432. IEEE, 2021. doi:10.1109/FOCS52979.2021.00050.
- 11 Valentine Kabanets and Jin-yi Cai. Circuit minimization problem. In *STOC*, pages 73–79. ACM, 2000. doi:10.1145/335305.335314.
- 12 Jiayu Li and Tianqi Yang. $3 \ln - o(n)$ circuit lower bounds for explicit functions. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2022*, pages 1180–1193, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3519935.3519976.
- 13 Daniel Lokshantov, Dániel Marx, and Saket Saurabh. Slightly superexponential parameterized problems. *SIAM J. Comput.*, 47(3):675–702, 2018. doi:10.1137/16M1104834.
- 14 William J. Masek. Some np-complete set covering problems. *Unpublished Manuscript.*, 1979.
- 15 Cody D. Murray and R. Ryan Williams. On the (non) np-hardness of computing circuit complexity. *Theory of Computing*, 13(4):1–22, 2017. doi:10.4086/toc.2017.v013a004.
- 16 Eduard I. Nechiporuk. On a boolean function. *Dokl. of USSR Academy of Sciences*, 169(4):765–766, 1966. URL: <http://mi.mathnet.ru/dan32449>.
- 17 Arlindo L. Oliveira, Luca P. Carloni, Tiziano Villa, and Alberto L. Sangiovanni-Vincentelli. Exact minimization of binary decision diagrams using implicit techniques. *IEEE Trans. Computers*, 47(11):1282–1296, 1998. doi:10.1109/12.736442.
- 18 Vaughan R. Pratt. The effect of basis on size of boolean expressions. In *FOCS*, pages 119–121. IEEE Computer Society, 1975. doi:10.1109/SFCS.1975.29.
- 19 Anup Rao and Makrand Sinha. A direct-sum theorem for read-once branching programs. In Klaus Jansen, Claire Mathieu, José D. P. Rolim, and Chris Umans, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2016, September 7-9, 2016, Paris, France*, volume 60 of *LIPICs*, pages 44:1–44:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPICs.APPROX-RANDOM.2016.44.
- 20 Alexander A. Razborov and Steven Rudich. Natural proofs. *J. Comput. Syst. Sci.*, 55(1):24–35, 1997. doi:10.1006/JCSS.1997.1494.

- 21 Michael E. Saks and Rahul Santhanam. Circuit lower bounds from np-hardness of MCSP under turing reductions. In Shubhangi Saraf, editor, *35th Computational Complexity Conference, CCC 2020, July 28-31, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 169 of *LIPICs*, pages 26:1–26:13. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPICs.CCC.2020.26.
- 22 Detlef Sieling. The complexity of minimizing and learning obdds and fbdds. *Discrete Applied Mathematics*, 122(1):263–282, 2002. doi:10.1016/S0166-218X(01)00324-9.
- 23 Yasuhiko Takenaga and Shuzo Yajima. Hardness of identifying the minimum ordered binary decision diagram. *Discret. Appl. Math.*, 107(1-3):191–201, 2000. doi:10.1016/S0166-218X(99)00226-7.
- 24 Craig A. Tovey. A simplified np-complete satisfiability problem. *Discret. Appl. Math.*, 8(1):85–89, 1984. doi:10.1016/0166-218X(84)90081-7.
- 25 Dietmar Uhlig. On the synthesis of self-correcting schemes from functional elements with a small number of reliable elements. *Mathematical Notes of the Academy of Sciences of the USSR*, 15.6:558–562, 1974.
- 26 Dietmar Uhlig. Networks computing boolean functions for multiple input values. *Proceedings of the London Mathematical Society Symposium on Boolean Function Complexity*, pages 165–173, 1992.
- 27 Ingo Wegener. *Branching Programs and Binary Decision Diagrams*. Society for Industrial and Applied Mathematics, 2000. doi:10.1137/1.9780898719789.