Diversity in Evolutionary Dynamics

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Abstract

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1 Results

We consider the dynamics imposed by natural selection on the populations of two competing, sexually reproducing, haploid species. In this model genomes acquire time-varying fitnesses as a result of the changing mix of species in the population; this is in contrast to most previous works in this area, which impose external rules for varying fitnesses over time. Previous work on our model [2] showed that, in the special case where each of the two species exhibits just two phenotypes, genetic diversity is maintained at all times. This finding supported the tenet that sexual reproduction is advantageous because it promotes phenotype diversity.

In the present paper we consider the more realistic case where there are more than two phenotypes available to each species. The conclusions about diversity in general turn out to be very different from the two-phenotype case. For simplicity, we consider two species with n and m genes respectively, each of which may take two allele values, 0 and 1. Game-theoretically, we think of two teams, A and B, with $|A| = n$ and $|B| = m$. The players within each team play a coordination game, but the two teams between them play a competitive, zero-sum game. Thus for each choice of actions $x \in \{0,1\}^n$, $y \in \{0,1\}^m$ by the $m+n$ players, all players in team A receive common payoff $u(x,y)$ and all players in team B receive $-u(x,y)$, where u is a $2^n \times 2^m$ payoff matrix. Players within a team are not allowed to coordinate their actions, so the strategies of each team are constrained to be product distributions.

Writing $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_m)$ for the (time dependent) mixed strategies of teams A, B respectively, where p_i denotes the probability of player i in team A choosing allele 0 (and q_j similarly for player j in team B), the replicator dynamics under weak selection [1] can be written as

$$\dot{p}_i = p_i\big(\widehat{u}_{i,0}(p,q) - \widehat{u}(p,q)\big)\,; \qquad \dot{q}_j = q_j\big(-\widehat{u}_{j,0}(p,q) + \widehat{u}(p,q)\big)\,, \tag{1}$$

where $\widehat{u}(p,q)$ is the average payoff to team A under the current strategies p, q, and $\widehat{u}_{i,0}(p,q)$ (resp., $\widehat{u}_{j,0}(p,q)$) is this same average conditioned on player i of team A (resp., player j of team B) choosing allele 0.

We consider first the following strong notion of preservation of genetic diversity:

▶ **Property A.** *For any initial populations $(p(0), q(0)) \in (0,1)^{n+m}$ (i.e., all genotypes are initially represented),*

$$\exists \varepsilon > 0, t_0 \geq 0 \ s.t. \ \forall t \geq t_0, \quad H(p(t)) + H(q(t)) \geq \varepsilon, \tag{2}$$

where $H(p) = \sum_i H(p_i)$ and $H(p_i) = -p_i \log p_i - (1-p_i)\log(1-p_i)$ is Shannon entropy. I.e., at all times the two-species system maintains diversity (not necessarily within a particular species).

Our first result is negative: namely, we show that sexual reproduction does not guarantee the maintenance of diversity at all times:

▶ **Theorem 1.** *There exist replicator dynamics with no weak pure Nash equilibrium for which Property A fails. In fact, inequality* (2) *holds only on a set of initial conditions of measure zero.*

Our concrete example used in the proof of Theorem 1 consists of species with three phenotypes each. This is in sharp contrast to the result of [2], which shows that Property A does hold with only two phenotypes per species.

Our main result is a complementary positive statement, which says that in any non-degenerate example, diversity *is* maintained in the following weaker, "infinitely often" sense.

▶ **Property B.** *For any initial populations $(p(0), q(0)) \in (0,1)^{n+m}$,*

$$\exists \varepsilon > 0 \ s.t. \ \int_0^\infty \max\{0, H(p(t)) + H(q(t)) - \varepsilon\} \ dt = \infty. \tag{3}$$

I.e., during an infinite span of time the entropy is uniformly bounded away from 0*.*

We also identify the following slightly weaker version of this property:

▶ **Property C.** *For any initial populations $(p(0), q(0)) \in (0,1)^{n+m}$,*

$$\int_0^\infty (H(p(t)) + H(q(t))) \ dt = \infty. \tag{4}$$

The following theorem summarizes the maintenance of diversity in general replicator dynamics, according to whether or not a pure Nash equilibrium exists:

▶ **Theorem 2.** *The following results hold for general replicator dynamics:*

(i) *If the dynamics has no pure Nash equilibrium, then Property B holds.*

(ii) *If the assumption in (i) is weakened to assume only that the dynamics has no* strict *pure Nash equilibrium, then Property C holds.*

(iii) *If the dynamics has a strict pure Nash equilibrium, then Property C (and therefore also Property B) fails on a set of initial populations $(p(0), q(0)) \in (0,1)^{n+m}$ of positive measure.*

In summary, our results refute the supposition that sexual reproduction ensures diversity at all times, but affirm a weaker assertion that extended periods of high diversity are necessarily a recurrent event.

References

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