Quantum 2-SAT on Low Dimensional Systems Is QMA_1-Complete: Direct Embeddings and Black-Box Simulation

Dorian Rudolph[1]

Department of Computer Science and Institute for Photonic Quantum Systems (PhoQS), Paderborn University, Germany

Sevag Gharibian

Department of Computer Science and Institute for Photonic Quantum Systems (PhoQS), Paderborn University, Germany

Daniel Nagaj

Institute of Physics, Slovak Academy of Sciences, Bratislava, Slovakia

Abstract

Despite the fundamental role the Quantum Satisfiability (QSAT) problem has played in quantum complexity theory, a central question remains open: At which local dimension does the complexity of QSAT transition from "easy" to "hard"? Here, we study QSAT with each constraint acting on a d_A-dimensional and d_B-dimensional qudit pair, denoted $(d_A \times d_B)$-QSAT. Our first main result shows that, surprisingly, QSAT on qubits can remain QMA_1-hard, in that (2×5)-QSAT is QMA_1-complete. (QMA_1 is a quantum analogue of MA with perfect completeness.) In contrast, (2×2)-QSAT (i.e. Quantum 2-SAT on qu*b*its) is well-known to be poly-time solvable [Bravyi, 2006]. Our second main result proves that $(3\times d)$-QSAT on the 1D line with $d \in O(1)$ is also QMA_1-hard. Finally, we initiate the study of $(2\times d)$-QSAT on the 1D line by giving a frustration-free 1D Hamiltonian with a unique, entangled ground state.

As implied by our title, our first result uses a *direct embedding*: We combine a novel clock construction with the 2D circuit-to-Hamiltonian construction of [Gosset and Nagaj, 2013]. Of note is a new simplified and *analytic* proof for the latter (as opposed to a partially numeric proof in [GN13]). This exploits Unitary Labelled Graphs [Bausch, Cubitt, Ozols, 2017] together with a new "Nullspace Connection Lemma", allowing us to break low energy analyses into small patches of projectors, and to improve the soundness analysis of [GN13] from $\Omega(1/T^6)$ to $\Omega(1/T^2)$, for T the number of gates. Our second result goes via *black-box* reduction: Given an *arbitrary* 1D Hamiltonian H on d'-dimensional qudits, we show how to embed it into an effective 1D $(3\times d)$-QSAT instance, for $d \in O(1)$. Our approach may be viewed as a weaker notion of "analog simulation" (*à la* [Bravyi, Hastings 2017], [Cubitt, Montanaro, Piddock 2018]). As far as we are aware, this gives the first "black-box simulation"-based QMA_1-hardness result.

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[1] corresponding author

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1 Introduction

Boolean satisfiability problems have long served as a testbed for probing the boundary between "easy" (i.e. poly-time solvable) and "hard" (e.g. NP-complete) computational problems. A striking early example of this is the fact that while 3-SAT is NP-complete [12, 33, 28], 2-SAT is in P [38, 15, 31, 18, 5, 36]. Despite this, *MAX*-2-SAT (i.e. what is the *maximum* number of satisfiable clauses of a 2-CNF formula?) remains NP-complete [20]! Thus, the border between tractable and intractable can often be intricate, hiding abrupt transitions in complexity.

In the quantum setting, generalizations of k-SAT and MAX-k-SAT have similarly played a central role, additionally due to their strong physical motivation. The input here is a *k-local Hamiltonian* $H = \sum_i H_i$ acting on n qubits, which is a $2^n \times 2^n$ complex Hermitian matrix (a quantum analogue of a Boolean formula on n bits), given via a succinct description $\{H_i\}$. Here, each H_i is a $2^k \times 2^k$ operator acting on[2] k out of n qubits (i.e. a local quantum clause). Given H, the goal is to compute the smallest eigenvalue $\lambda_{\min}(H)$ of H, known as the *ground state energy*. The corresponding eigenvector, in turn, is the *ground state*. This k-local Hamiltonian problem (k-LH) generalizes MAX-k-SAT, and formalizes the question: If a many-body quantum system is cooled to near absolute zero, what energy level will the system relax into? The complexity of k-LH is well-understood, and analogous to MAX-2-SAT, even 2-LH is complete for[3] Quantum Merlin-Arthur (QMA) [30, 29, 14].

The quantum generalization of k-SAT (as opposed to *MAX*-k-SAT), on the other hand, is generally *less* understood. In contrast to k-LH, one now asks whether there exists a ground state of H which is simultaneously a ground state for *each local term* H_i (analogous to a string satisfying *all* clauses of a k-SAT formula). Formally, in Quantum k-SAT (k-QSAT), each H_i is now a projector, and one asks whether $\lambda_{\min}(H) = 0$. As in the classical setting, it is known that the locality k leads to a complexity transition: On the one hand, Gosset and Nagaj [24] proved that 3-QSAT is[4] QMA_1-complete, while on the other hand, Bravyi gave [8] a poly-time classical algorithm for 2-QSAT (in fact, 2-QSAT is solvable in linear time [4, 16]).

Systems of higher local dimension. In the quantum setting, however, there is an additional, physically motivated direction to probe for complexity transitions for 2-QSAT – *systems of higher local dimension.* Perhaps the most striking example of this is that, while Boolean satisfiability problems in 1D are efficiently solvable via dynamic programming (even for

[2] Formally, if H_i acts on a subset $S \subseteq [n]$ of qubits, to make the dimensions match we consider $(H_i)_S \otimes I_{[n]\setminus S}$.
[3] QMA is the bounded-error quantum analogue of NP, now with poly-size quantum proof and quantum verifier [30].
[4] QMA_1 is QMA but with perfect completeness; see Definition 2.1. Note that while $\mathrm{MA} = \mathrm{MA}_1$ [43], whether $\mathrm{QMA}_1 = \mathrm{QMA}$ remains a major open question.

Table 1 Summary of the known complexity results for $(a\times b)$-QSAT. Rows denote values of d_A, columns values of d_B. New results from this work are bold.

	2	3	4	5	6	7
2	$\in$ P [8]	NP-hard [34]	NP-hard [34]	**$\mathbf{QMA_1}$-comp**	**$\mathbf{QMA_1}$-comp**	$\cdots$
3		NP-hard [34]	**$\mathbf{QMA_1}$-comp**	QMA_1-comp [17]	QMA_1-comp [17]	$\cdots$
4			**$\mathbf{QMA_1}$-comp**	QMA_1-comp [17]	QMA_1-comp [17]	$\cdots$
5				QMA_1-comp [17]	QMA_1-comp [17]	$\cdots$
6					$\cdots$	$\cdots$

d-level systems instead of bits), Aharonov, Gottesman, Irani and Kempe showed [2] that 2-LH on the line remains QMA-complete, *so long as* one uses local dimension $d = 12$! An analogous result for 2-QSAT with $d = 11$ was subsequently given by Nagaj [34]. This raises the guiding question of this work:

What is the smallest local dimension that can encode a QMA_1*-hard problem?*

There are three results we are aware of here. Define $(d_A\times d_B)$-QSAT as 2-QSAT where each constraint acts on a qu-d_A-it and a qu-d_B-it, i.e. on $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}$. (When $d_A \neq d_B$, for this to be well-defined, the interaction graph of the instance must be bipartite.) Not only is 2-QSAT on qubits (i.e. (2×2)-QSAT) in P, Chen, Chen, Duan, Ji, and Zheng showed [11] that a YES instance always has a witness that is a product of single- and two-qubit states. On the other hand, Bravyi, Caha, Movassagh, Nagaj, and Shor gave a frustration-free[5] qutrit construction (i.e. on local dimension $d = 3$) on the 1D line[6] [9] with a *unique*, *entangled* ground state. While this construction does not encode a computation (and thus does not give QMA_1-hardness), it is an important first step in that it shows even such low dimensional systems can encode entangled witnesses (entanglement is necessary, otherwise an NP witness is possible). Together, these works [11] and [9] suggest that qu*b*it systems are a no-go barrier for QMA_1-hardness. Prior to these, Eldar and Regev [17] came closest to establishing a result about qubit systems, showing that (3×5)-QSAT is QMA_1-hard (on general graphs).[7] But the key question remains open – *Can qubit systems support* QMA_1*-hardness for Quantum 2-SAT, i.e. is* $(2\times d)$-QSAT QMA_1*-hard for some* $d \in O(1)$*?*

Our results. We show two main results, along with a third preliminary one.

1. QMA_1*-hardness for qubit systems.* The complexity of $(d_A\times d_B)$-QSAT including our new results is summarized in Table 1. The first main result is as follows.

▶ **Theorem 1.1.** (2×5)-QSAT *is* QMA_1*-complete with soundness* $\Omega(1/N^2)$.

Here, N denotes the number of gates[8] used by the QMA_1-verifier. Thus, qubit systems *can* encode QMA_1-hardness. Let us be clear that the surprising part of this is *not* that this setting is intractable – indeed, even classical (2×3)-SAT is known to be NP-hard (e.g. [34]).

5 A *frustration-free* Hamiltonian is a YES instance of QSAT, i.e. a local Hamiltonian $H \succeq 0$ with $\lambda_{\min}(H) = 0$.

6 "On the line" means $H = \sum_{i=1}^m H_{i,i+1}$, i.e. the qudits can be depicted as a sequence with each consecutive constraint acting on the next pair of qudits in the sequence.

7 QMA_1-hardness of (4×9)-QSAT is claimed without proof in a footnote in [35] (see Reference 12 therein).

8 As an aside, we assume in this paper that the QMA_1 verifier utilizes the standard "Clifford + T" gate set. See also [39] for an in-depth discussion of the "gate set issue" of QMA_1.

What *is* surprising is that one can encode *quantum* verifications in such low dimension, for two reasons. First, *a priori* a 2-dimensional space for 2-local constraints appears too limited to *exactly*[9] encode a computation – a two-dimensional space only appears to suffice to encode a "data qubit", so where does one encode the "clock" tracking the computation? Second, the entanglement of $(2{\times}d)$ systems is generally easier to characterize than that of $(d{\times}d)$ systems. For example, whether a $(2{\times}2)$ or $(2{\times}3)$ system is entangled is detectable via Peres' Positive Partial Transpose (PPT) criterion [37], whereas detecting entanglement for $(d{\times}d)$ systems (for polynomial d) becomes strongly NP-hard [25, 27, 21]. And recall that a "sufficiently entangled" ground space is necessary to encode QMA_1-hardness.

As a complementary result, we show that one can "trade" dimensions in the construction above if one is careful, i.e. the 5 in $(2{\times}5)$ can be reduced to 4 at the expense of increasing 2 to 3.

▶ **Theorem 1.2.** $(3{\times}4)$-QSAT *is* QMA_1*-complete with soundness* $\Omega(1/N^2)$.

One can also consider general Quantum 2-SAT on qu-d-its of identical dimension d, which is QMA_1-complete for $d = 5$ [17]. Theorem 1.2 improves this to $d = 4$.

We remark that obtaining Theorems 1.1 and 1.2 is significantly more involved than the previous best QMA_1-hardness for $(3{\times}5)$-QSAT [17] (details in "Techniques" below), and in particular requires (among other ideas) a simplified analysis of the advanced 2D-Hamiltonian framework of Gosset and Nagaj for $(2{\times}2{\times}2)$-QSAT (i.e. 3-QSAT on qubits) [24][10], a new clock construction, and the Unitary Labelled Graphs of Bausch, Cubitt and Ozols [6]. One of the payoffs is that we obtain a "tight"[11] soundness gap of $\Omega(1/N^2)$ for Theorem 1.1, compared to the $\Omega(1/N^6)$ gap of [24]. This allows us to recover the 3-QSAT hardness results of [24], but with improved soundness[12]:

▶ **Theorem 1.3.** 3-QSAT *is* QMA_1*-complete with soundness* $\Omega(1/N^2)$.

2. Low dimensional systems on the line. Our next main result is the following.

▶ **Theorem 1.4.** $(3{\times}d)$-QSAT *on a line is* QMA_1*-complete with* $d = O(1)$.

This improves significantly on the previous best 1D $(11{\times}11)$-QSAT QMA_1-hardness construction of Nagaj [34], showing that frustration free Hamiltonians on qutrits can encode not just entangled ground states (*cf.* [9]), but also QMA_1-hard computations.

For clarity, there is a trade-off in the parameter, d, which we now elaborate. A key novelty of Theorem 1.4 is its proof via *black-box* reduction (see "Techniques" below), as opposed to a direct embedding of a QMA_1 computation. Specifically, given an *arbitrary* 1D Hamiltonian H on d'-dimensional qudits, we embed it into an effective 1D $(3{\times}d)$-QSAT instance with $d \in O((d')^4)$ (i.e. $d \in O(1)$ for constant d'). On the not-so-positive side, the generality of

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9 An *exact* encoding appears needed by definition of QSAT. This is in strong contrast to 2-LH, where *approximate* encodings are allowed (since all constraints need not be simultaneously satisfiable) via perturbation theory [29, 10].

10 At a first glance, it seems like $(2{\times}2{\times}2)$-QSAT can be reinterpreted as $(2{\times}4)$-QSAT-QSAT by combining two qubits into a qu-4-dit. Indeed, a *single* $(2{\times}2{\times}2)$-constraint can be embedded straightforwardly into a $(2{\times}4)$-constraint. However, the interactions between a *set* of $(2{\times}2{\times}2)$-constraints seems difficult to directly reproduce via a set of $(2{\times}4)$-constraints.

11 By "tight", we mean that it is not currently known for either QSAT or LH how to get a promise gap larger than $\Omega(1/N^2)$ [42].

12 For clarity, this improved soundness is not necessary for our results, but an added bonus of our new analysis.

this approach means that an input 1D Hamiltonian H on d'-dimensional qudits is mapped to a 1D Hamiltonian on constraints of dimension $(3 \times O((d')^4))$, i.e. the first dimension drops to 3 at the expense of the second dimension increasing. Thus, for example, if we plug in the 1D (11×11) QMA_1-hardness construction [34], Theorem 1.4 gives QMA_1-hardness for (2×76176)-QSAT.

On the positive side, however, our technique is the first use of (a weaker notion) of the influential idea of *local simulation* (Bravyi and Hastings [7], Cubitt, Montanaro, and Piddock [13]; see Definition 4.1 of [22] for a simpler statement) in the study of QSAT. Roughly, in such simulations, given a local Hamiltonian H on n qubits, one typically applies a local isometry V to each qubit, i.e. maps $H \mapsto V^{\otimes n} H (V^{\dagger})^{\otimes n}$, blowing up the input space A into a larger, "logical" space B. By cleverly choosing an appropriate Hamiltonian H' on B, one forces the low-energy space of H' to approximate H. Traditionally, the drawback of this approach is its reliance on perturbation theory, which necessarily gives rise to[13] *frustrated* Hamiltonians H'. Here, however, we show for the first time that a weaker[14] version of such local embeddings can be designed even for the frustration-free setting, ultimately yielding Theorem 1.4.

3. Towards qubits on the line. Finally, we initiate the study (2×d) on a line by proving that even a frustration-free Hamiltonian on a line of alternating particles with dimensions 2 and 4 can have a unique entangled ground state.

▶ **Theorem 1.5.** *Consider a line of* $2n$ *particles such that the* i*-th particle has dimension* 2 *for even* i *and* 4 *for odd* i*. There exists a Hamiltonian* $H = \sum_{i=1}^{n} A_{2i-1,2i} + \sum_{i=1}^{n-1} B_{2i,2i+1} + L_{1,2} + R_{2n-1,2n}$*, where* A, B, L, R *are* 2*-local projectors, such that the nullspace of* H *is* $\mathcal{N}(H) = \mathrm{Span}\{|\psi\rangle\}$ *and* $|\psi\rangle$ *is entangled across all cuts.*[15]

Techniques. We focus on our main results, Theorem 1.1 and Theorem 1.4.

QMA_1*-hardness for* (2×5)*-QSAT.* The basic idea behind most (if not all) QMA_1-hardness constructions for QSAT is to embed a so-called *history state* $|\psi_{\text{hist}}\rangle$ encoding the QMA_1 verifier's circuit U into the nullspace of a local Hamiltonian H. Specifically, let $U = U_T \cdots U_1$ be a sequence of 1- and 2-qubit gates. Then, the Feynman-Kitaev history state [19, 30] is given by

$$|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} \big(U_t \cdots U_1 |\psi_{\text{proof}}\rangle_A |0 \cdots 0\rangle_B\big) |t\rangle_C, \tag{1}$$

for register A the proof register, B the ancilla register, and C the clock register. This should be thought of as the quantum analogue of a Cook-Levin tableau, where the steps of the computation are now encoded in *superposition* over time t. While this basic idea is simple, the challenge lies in implementing it when restricted to certain local dimensions, clause/interaction localities, or interaction geometries. Progress for QSAT in particular has largely been difficult, since in contrast to the setting of LH, one does not have access to the tools of perturbation theory, which necessarily create Hamiltonians with frustrated ground spaces (i.e. empty nullspaces). Instead, QSAT hardness constructions must carefully stitch together rather involved gadgets to logically effect the history state idea above.

[13] In words, perturbation theory is only known to be able to show QMA-hardness for k-LH, as opposed to QMA_1-hardness for k-QSAT.

[14] For clarity, the simulations of [7, 13] reproduce the whole target Hamiltonian H, whereas our approach is weaker in that we prove simulation of only H's nullspace. A similar notion of simulation is discussed by Aharonov and Zhou [3], called *gap simulation*, where only the ground space is approximately simulated.

[15] The Schmidt rank is $\Theta(1)$, but we do not explicitly analyze it here.

To this end, we begin by discussing our approach for Theorem 1.1. We wish to encode local Hamiltonian constraints which force any potential null state to repeatedly update the current time in the clock register from t to $t+1$, along with applying the tth gate of U, as sketched above. However, even embedding a clock (never mind the full computation of U!) into the nullspace of a $(2\times d)$-Hamiltonian is non-trivial, as we only have qubit-systems at our disposal to act as "auxiliary particles" (as opposed to qutrits in [17]). Thus, our first challenge is to break this "qutrit barrier". We illustrate our techniques in a toy example.

Our starting point is to embed a simple clock into the null-space of a (3×3)-Hamiltonian, and subsequently logically simulate this clock by pairing "indicator qubits" (i.e. qubit auxiliary particles) with 6-dimensional qudits as follows. It is straightforward to write down a (3×3)-Hamiltonian encoding the simple sequence of clock states (e.g. on 3 qutrits) $(|100\rangle, |200\rangle, |210\rangle, |220\rangle, |221\rangle, |222\rangle)$ in its nullspace, where note that only a single qutrit changes in each step. To break the "qutrit barrier", we thus now implement each qutrit in this clock *logically* in a larger space consisting of a 6-dimensional qudit and a triple of "indicator qubits". Formally, qutrit $|x\rangle \in \{|0\rangle, |1\rangle, |2\rangle\}$ is encoded via

$$|x\rangle \mapsto |x\rangle_\alpha|000\rangle_\beta + |x'\rangle_\alpha|0^x 1 0^{2-x}\rangle_\beta \quad \text{for } x \in \{0,1,2\} \text{ and } x' = x+3 \in \{3,4,5\}, \tag{2}$$

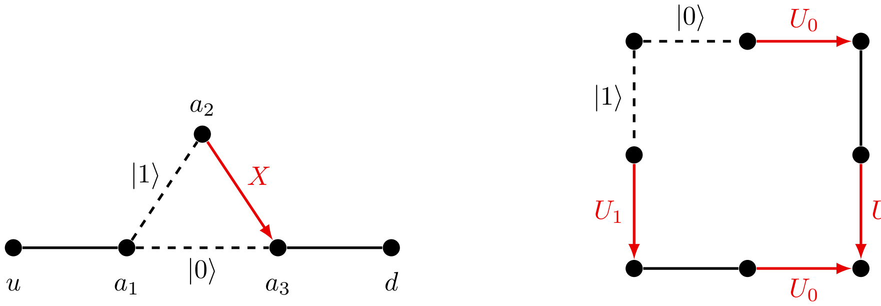
where α is 6-dimensional, and β consists of three qubits labelled $\beta_0, \beta_1, \beta_2$. To illustrate the idea behind this encoding, consider a projector that enforces equality (i.e. applies an identity gate) between the first two clock states (we also call this a $|100\rangle \leftrightarrow |200\rangle$ transition). In (3×3), we can use the projector onto $|10\rangle - |20\rangle$ (i.e. $\frac{1}{2}(|1\rangle - |2\rangle)(\langle 1| - \langle 2|) \otimes |0\rangle\langle 0|$) acting on the first two qutrits, where $|0\rangle\langle 0|$ on the second qutrit ensures that the projector only acts on the first two timesteps. Using our encoding, we can realize this transition with a (2×6)-projector onto $|11\rangle_{\alpha,\beta_0} - |21\rangle_{\alpha,\beta_0}$. This uses two features of the encoding: (1) a transition on a logical qutrit can be implemented with a 1-local projector on qudit α (here $\frac{1}{2}(|1\rangle - |2\rangle)(\langle 1| - \langle 2|)_\alpha$), and (2) a projection onto a standard basis state $|i\rangle \in \{|0\rangle, |1\rangle, |2\rangle\}$ of a logical qudit can be implemented with a 1-local projector onto an indicator qubit β_i (here $|1\rangle\langle 1|_{\beta_i}$).

This qutrit encoding (Equation (2)) generalizes to qudits of any dimension, and combined with the (3×5)-Hamiltonian of [17], gives QMA$_1$-hardness of (2×10)-QSAT. We can remove some unused indicator qubits to improve this to (2×7)-QSAT (see [40]) as the indicator requires an extra qudit dimension ($|x\rangle$ and $|x'\rangle$).[16]

Further improvement to (2×5)-QSAT requires much more work. In order to directly enforce the application of a gate $U_t \neq \mathbb{I}$ in a history state (Equation (1)) with a 2-local projector, we can only use a single qudit from the proof register and a single qudit from the clock register. Hence, we need to be able to project onto clock states with a 1-local qudit projector. For that, Eldar and Regev [17] introduce the notion of "*u*nborn", "*d*ead", and "*a*live" states, where non-trivial transitions only happen on the "alive" states, which uniquely identify a single timestep. Having separate "unborn" and "dead" states allows the clock states to have the general structure $d \cdots d\, a\, u \cdots u$, which is easy to enforce with 2-local constraints.[17] In fact, [17] uses three "alive states" a_1, a_2, a_3 to implement a CNOT gate via a "triangle gadget", which routes states with $|0\rangle$ and $|1\rangle$ in the control register along two different paths, only applying the X gate on the path of $|1\rangle$ (see Figure 1.1a).

[16] We need both $|x\rangle_\alpha$ and $|x'\rangle_\alpha$ so that logical $|x\rangle$ and $|y\rangle$ differ only in a single qudit (on a subspace), which allows for a 1-local transition. Since the indicator qubits are "off" on $|x\rangle_\alpha$, we need a unique $|x'\rangle_\alpha$ to "turn on" the correct indicator via a transition.

[17] A natural question is if we can reduce the alphabet $\{a, u, d\}$ to a simpler binary alphabet $\{0, 1\}$, so that



(a) Effective implementation of a CNOT gate from u to d. Conditioned on a control bit being $|1\rangle$ (respectively $|0\rangle$), the gadget takes transition $|a_1\rangle \leftrightarrow |a_2\rangle$ (respectively $|a_1\rangle \leftrightarrow |a_3\rangle$). Only the former path applies the X gate (indicated by edge $|a_2\rangle \rightarrow |a_3\rangle$).

(b) In this case, when the control bit is $|0\rangle$ (respectively $|1\rangle$), U_1U_0 (respectively U_0U_1) is applied. A suitable choice of non-commuting U_0, U_1 implements CNOT (up to 1-local unitaries) from top left to bottom right. In contrast to the triangle gadget in (a), this gadget in (b) is only the "effective Hamiltonian" and the actual construction is more involved due to geometric limitations.

Figure 1.1 Graphical representation of the 2-local gate gadgets of (a) [17], and (b) [24] as well as this work. Vertices represent clock states, black (solid) edges identity transitions, red (directed) edges unitary transitions, dashed edges conditional identity transitions.

Gosset and Nagaj [24] (see Figure 1.1b) introduce the idea of a 2D-clock with two separate clock registers for the X- and Y-directions. Hence, clock states are vertices in a 2D-grid and therefore can realize separate paths while only requiring transitions between "adjacent" (in one dimension) clock states. The next idea of our construction is to implement this 2D-Hamiltonian of [24] using just two "alive" states in 2-QSAT (compared with three "alive" states in [17] for 2-QSAT; note [24] uses their 2D-clock for 3-QSAT). Figure 1.1b depicts the "effective Hamiltonian" leveraging this idea in order to implement a CNOT gate. Note Figure 1.1b is simplified; the actual implementation with 2-local projectors is significantly more involved (see full version [40]). This is due to complications which arise from geometric limitations: For example, we are not able to restrict the U_0 transition (Figure 1.1b) in the Y-direction (i.e. U_0 would be applied from $x = 2$ to $x = 3$ for all coordinates y).

To implement the ideas of the previous paragraph, we extract the 2D-Hamiltonian of [24] as a generic construction into which we can plug any clock satisfying certain requirements (see Theorem 3.1). Thereby we obtain QMA_1-hardness for (3×4)-QSAT. Then we simulate this (3×4)-clock on a (2×5)-system via the indicator qubit principle (similar to Equation (2)). Note that when implementing a logical 4-dit on a (2×5)-system, we can only use a single indicator qubit. Thus, we need a very carefully crafted (3×4)-clock, and further "technical tricks", which we omit in this introduction.

Finally, to analyze the 2D-Hamiltonian, we prove a novel technical lemma, which we dub the "Nullspace Connection Lemma" (Lemma 3.2). This enables us to split the 2D-Hamiltonian into smaller gadgets (see Figure 1.1b and [40] for the individual gadgets), each of which implements a small part of the history state and is relatively straightforward to analyze. The gadgets are then connected with additional transitions to form the complete Hamiltonian (see [40]), whose nullspace is spanned by superpositions of the gadget's history

clock states look like $\{|100\rangle, |010\rangle, |001\rangle\}$. Unfortunately, there is no set of 2-local projectors whose joint nullspace is the span of $\{|100\rangle, |010\rangle, |001\rangle\}$. The common unary encoding $|100\rangle, |110\rangle, |111\rangle$ cannot be used here because such states cannot be identified 1-locally.

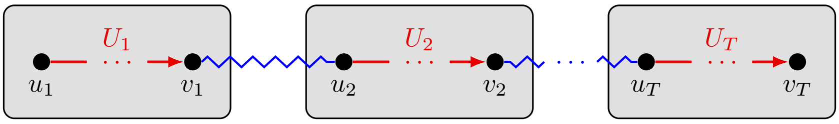


Figure 1.2 Nullspace Connection Lemma: Each box represents a gadget that only acts on a subset of clock states (vertices). Each gadget has an input vertex u_i, and an output vertex v_i. Its nullspace is spanned by history states applying gate U_i from u_i to v_i. Blue zigzag edges connect outputs v_i to inputs u_{i+1} with identity transitions. We show that the entire Hamiltonian has spectral gap $\Omega(1/N^2)$[18] and its nullspace is spanned by history states applying $U_T \cdots U_2 U_1$ from u_1 to v_T.

states. Figure 1.2 gives a visualization and informal statement of the lemma. The Connection Lemma is quite generic and can also be applied to, e.g., the original circuit Hamiltonian of Kitaev [30] (see Remark 3.3), matching the $\Omega(1/N^2)$ soundness (smallest non-zero eigenvalue) therein. Furthermore, the Connection Lemma is proven directly via the Geometric Lemma [30] and does not require transformation to the Laplacian matrix of a random walk, unlike [30]. Our main application of the Connection Lemma is to give a simplified proof for the 2D-Hamiltonian with improved soundness. Since the Connection Lemma requires modifications to the Hamiltonian of [24], we actually use our own variant of [24]'s gadget, which is slightly more compact (using $6M+1$ clock states as opposed to $9M+3$ for M gates). Finally, we do not rely on numerical methods to derive the nullspaces of the individual gadgets (*cf.* the gadget analysis of [24], which required numerics), and instead give a formal proof based on the theory of *unitary labeled graphs* [6]. The upshot is that our overall approach is very flexible – by combining unitary labelled graphs with the Connection Lemma, one can in principle analyze combinations of Hamiltonian gadgets beyond just our 2D setting with relative ease. Therefore, we believe the Connection Lemma will find uses in future works.

QMA$_1$*-hardness for* $(3\times d)$-QSAT *on the line.* As mentioned earlier, in contrast to the direct embedding for Theorem 1.1, for Theorem 1.4 we instead use a black-box simulation. In other words, we do not give a new circuit-to-Hamiltonian mapping, but instead bootstrap the prior QMA$_1$-completeness result of QSAT on a line of qu-d-its with $d = 11$ due to Nagaj [34]. Specifically, we construct a general embedding of *any* 1D-Hamiltonian H on a line of qudits into a Hamiltonian H' on an alternating line of qu-d'-its and qutrits. We then plug the [34] construction into this embedding. To design our embedding, each qu-d'-it is treated as two logical qu-d''-its with $d'' = \sqrt{d'}$ (see Figure 1.3; Figure 4.2 for technical details). We construct a Hamiltonian H'_{logical} that restricts the $(d''\times 3\times d'')$ systems to a d-dimensional subspace, which acts as a logical qu-d-it. This logical subspace has the key feature that, in a sense, the logical qu-d-it can be "accessed" from either its left or right qu-d''-it. This allows us to encode the 2-local terms of H as 1-local terms acting on the qu-d'-its of H'. The rough idea behind the logical subspace is to treat the $(d''\times 3\times d'')$ system as three bins capable of holding two types of balls (red and black) that are exchanged between the bins to move the information between the left and right qu-d''-it in each triple $(d''\times 3\times d'')$. When one d''-bin is full, the other d''-bin must be empty and the total number i of red balls encodes the basis state $|i/2\rangle$ of the corresponding logical qudit (Figure 4.1 depicts the balls and bins). To encode a d-dimensional system, this requires $O(d)$ red balls and $O(d)$ black balls, so that $d'' = \Theta(d^2)$, and thus $d' = \Theta(d^4)$.

[18] This gap is for gadgets of constant size, which suffice for our purposes. However, Lemma 3.2 is more general and even allows gadgets of non-constant size.

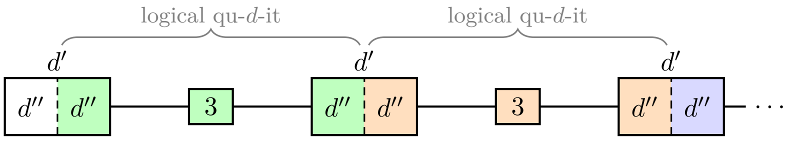


Figure 1.3 Simulation of $(d{\times}d)$-QSAT with $(3{\times}d')$-QSAT on the line. Qu-d'-its are split into two qu-d''-its with $d'' = \sqrt{d'}$. Each $(d''{\times}3{\times}d'')$ system implements a logical qu-d-it.

As previously described, our embedding can be seen as a weaker notion of *simulation* in the sense of [7, 13], in that formally our embedding is achieved via application of *local isometries* (i.e. local observables are mapped to local observables), followed by additional constraints on the logical space (Equations (19a) and (19b)). However, in contrast to [7, 13], we do not analyze the structure of higher energy spaces of H', and only show that H' preserves the nullspace of H as well as its smallest non-zero eigenvalue, up to a linear factor (see Lemma 4.3), as this suffices for our purposes.

Open questions. Although we have shown that qubit systems can support QMA_1-hard problems, the frontier for characterizing the complexity transition of local Hamiltonian problems from low to high local dimension remains challenging. In our setting, in particular, the main open question is whether one can obtain QMA_1-hardness even for $(2{\times}3)$-QSAT? This would complete the complexity characterization for $(d_A{\times}d_B)$-QSAT, as recall $(2{\times}2)$-QSAT (i.e. 2-QSAT) is in P [8]. Getting this down to $(2{\times}3)$-QSAT (even $(3{\times}3)$-QSAT or $(2{\times}4)$-QSAT), however, appears difficult, requiring ideas beyond those introduced here.

As for the 1D line, the best hardness results for 2-LH and 2-QSAT are on 8-dimensional [26] and 11-dimensional [34] systems, respectively. Is 1D 2-QSAT on qudits with $2 < d < 11$ QMA_1-hard? We showed that 1D $(3{\times}d)$-QSAT is QMA_1-hard, but due to our black-box approach we only get $d = 76176$. So it seems likely that this d can be improved significantly. Perhaps more interesting is the question whether 1D $(2{\times}d)$-QSAT is still QMA_1-hard. We showed that even 1D $(2{\times}4)$-dimensional constraints can support a unique globally entangled ground state (Theorem 1.5), but this construction alone does not embed a computation, and thus does not yield QMA_1-hardness. Can this be bootstrapped to obtain QMA_1-hardness for 1D $(2{\times}d)$-QSAT? For 1D 2-LH, the situation is even worse – on qubits, these systems can only be efficiently solved in the presence of a constant spectral gap [32]. In contrast, for inverse polynomial gap, and even with the promise of an NP witness (i.e. via Matrix Product State), the problem is NP-complete [41]. What is the complexity of 1D 2-LH on qudits with $2 < d < 8$?

Organization. Section 2 gives formal definitions for QMA_1, $(d_A{\times}d_B)$-QSAT, and states the Geometric Lemma with various corollaries. Section 3 proves our main result, the QMA_1-completeness of $(2{\times}5)$-QSAT. Section 4 proves the QMA_1-completeness of $(3{\times}d)$-QSAT on a line. Section 5 gives the construction of the $(2{\times}4)$-Hamiltonian on a line with entangled ground space.

2 Preliminaries

In this section, we formally introduce QMA_1 and elaborate on the Geometric Lemma.

2.1 QMA$_1$

The complexity class QMA$_1$ is defined in the same way as QMA, but with the additional requirement of *perfect completeness*, i.e., in the YES-case, there exists a proof that the verifier accepts with a probability of exactly 1. Consequently, QMA$_1$ is not known to be independent of the gate set [24], as approximate decompositions of arbitrary unitaries generally breaks perfect completeness. Therefore, we have to fix a gate set before we define QMA$_1$, and here we follow [24] in choosing the "Clifford + T" gate set $\mathcal{G} = \{\widehat{H}, T, \text{CNOT}\}$, where $\widehat{H}$ denotes the Hadamard gate. Giles and Selinger [23] have proven that a unitary can be synthesized exactly with gate set $\mathcal{G}$ iff its entries are in the ring $\mathbb{Z}[\frac{1}{\sqrt{2}}, i]$.

▶ **Definition 2.1** (QMA$_1$). *A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in QMA$_1$ if there exists a poly-time uniform family of quantum circuits $\{Q_x\}$ with gate set $\mathcal{G}$ such that:*
- *(Completeness) If $x \in A_{\text{yes}}$, then there exists a proof $|\psi\rangle$ with $\Pr[Q_x \text{ accepts } |\psi\rangle] = 1$.*
- *(Soundness) If $x \in A_{\text{no}}$, then then for all proofs $|\psi\rangle$, $\Pr[Q_x \text{ accepts } |\psi\rangle] \leq 1 - 1/\operatorname{poly}(|x|)$.*

▶ **Definition 2.2** ($(d_A \times d_B)$-QSAT). *Consider a system of d_A- and d_B-dimensional particles, denoted $A_i, i \in [n_A]$ and $B_j, j \in [n_B]$, respectively. In the $(d_A \times d_B)$-QSAT problem, the input is a $(d_A \times d_B)$-Hamiltonian $H = \sum_{i \in [n_A], j \in [n_B]} \Pi_{A_i, B_j}$ with 2-local projectors Π_{A_i, B_j} acting non-trivially only on particles A_i and B_j. Decide:*
- *(YES) $\lambda_{\min}(H) = 0$.*
- *(NO) $\lambda_{\min}(H) \geq 1/\operatorname{poly}(n_A + n_B)$.*

Note that the projectors of $(d_A \times d_B)$-QSAT need to have a specific form such that the problem is contained in QMA$_1$ with our chosen gate set (see Section 3.4).

2.2 Geometric Lemma

In our proofs, we frequently apply Kitaev's Geometric Lemma [30] as well as its extension to the frustration-free case due to Gosset and Nagaj [24], where we are interested in lower-bounding the smallest non-zero eigenvalue of a Hamiltonian H, denoted $\gamma(H)$. In the following, we also give further refinements of these statements. As in [24], we use the notation $H|_S = \Pi_S H \Pi_S$ for the restriction of the Hamiltonian H to the subspace S, where Π_S is the projector onto S.[19] $\mathcal{N}(H)$ denotes the nullspace of H.

▶ **Lemma 2.3** (Kitaev's Geometric Lemma [30] as stated in [24]). *Let $H = H_A + H_B$ with $H_A \succeq 0$ and $H_B \succeq 0$. Let $S = \mathcal{N}(H_A)$ and Π_B be the projector onto $\mathcal{N}(H_B)$. Suppose $\mathcal{N}(H) = \{0\}$. Then $\gamma(H) \geq \min\{\gamma(H_A), \gamma(H_B)\} \cdot (1 - \sqrt{c})$, where $c = \max_{|v\rangle \in S: \langle v|v\rangle = 1} \langle v|\Pi_B|v\rangle$.*

▶ **Corollary 2.4** ([24, Corollary 1]). *Let $H = H_A + H_B$ where $H_A \succeq 0$ and $H_B \succeq 0$ each have nonempty nullspaces. Let Γ be the subspace of states in $\mathcal{N}(H_A)$ that are orthogonal to $\mathcal{N}(H)$, and let Π_B be the projector onto $\mathcal{N}(H_B)$. Then $\gamma(H) \geq \min\{\gamma(H_A), \gamma(H_B)\} \cdot (1 - \sqrt{d})$, where $d = \|\Pi_B|_\Gamma\| = \max_{|v\rangle \in \Gamma: \langle v|v\rangle = 1} \langle v|\Pi_B|v\rangle$.*

We give a slightly tighter statement of [24, Corollary 2][20]:

▶ **Corollary 2.5.** *Let $H = H_A + H_B$ where $H_A \succeq 0$ and $H_B \succeq 0$ each have nonempty nullspaces. Let $S = \mathcal{N}(H_A)$ and suppose $H_B|_S$ is not the zero matrix. Then*

$$\gamma(H) \geq \min\{\gamma(H_A), \gamma(H_B)\} \cdot \frac{\gamma(H_B|_S)}{2\|H_B|_S\|}. \tag{3}$$

[19] Note, this is not the standard restriction of linear map to a subspace since H does not necessarily map S to itself.

[20] The only difference is that we have $\|H_B|_S\|$ instead of $\|H_B\|$ in the denominator.

Proof. As stated in [24], $\gamma(H_B|_S) = \min_{v\in\Gamma:\langle v|v\rangle=1}\langle v|H_B|v\rangle$ with Γ as defined in Corollary 2.4. Hence for all unit $|v\rangle \in \Gamma \subseteq S$,

$$\gamma(H_B|_S) \leq \langle v|H_B|v\rangle \leq \langle v|(\mathbb{I}-\Pi_B)|v\rangle \|H_B|_S\| \tag{4}$$

and thus $d \leq 1-\gamma(H_B|_S)/\|H_B|_S\|$. The statement then follows from $1-\sqrt{1-x} \geq \frac{x}{2}$ for $x \in [0,1]$. ◀

▶ **Corollary 2.6.** *Let $H = H_A + H_B$ where $H_A \succeq 0$ and $H_B \succeq 0$ each have nonempty nullspaces. Let $S = \mathcal{N}(H_A)$ and suppose $H_B|_S$ is not the zero matrix. Then*

$$\gamma(H) \geq \min\{\gamma(H_A),\gamma(H_B)\} \cdot \min_{|v\rangle\in\Gamma:\langle v|v\rangle=1}\langle v|(\mathbb{I}-\Pi_B)|v\rangle/2. \tag{5}$$

Proof. Follows from Corollary 2.4 and the fact $1-\sqrt{1-x} \geq \frac{x}{2}$ for $x \in [0,1]$. ◀

▶ **Corollary 2.7.** *Let $H_A \succeq 0, H_B \succeq 0$ be Hamiltonians with $\mathcal{N}(H_A) \subseteq \mathcal{N}(H_B)$. Then $\gamma(H_A+H_B) \geq \min\{\gamma(H_A),\gamma(H_B)\}$.*

Proof. In Corollary 2.4, we have $\Gamma = \mathcal{N}(H_A) \cap \mathcal{N}(H_A+H_B)^\perp = \mathcal{N}(H_A)\cap\mathcal{N}(H_A)^\perp = \{0\}$. Thus $d = \|\Pi_B|_\Gamma\| = 0$ and $\gamma(H_A+H_B) \geq \min\{\gamma(H_A),\gamma(H_B)\}$. ◀

3 (2×5)-QSAT is QMA$_1$-complete

Our proof mainly leverages the 2D-Hamiltonian construction of Gosset and Nagaj [24], which gives a circuit-to-Hamiltonian embedding with a set of "simple" projectors that are constructed from a suitable clock encoding. In Section 3.2, we give the main technical tool which we use to analyze the soundness of the aforementioned Hamiltonian, and in Section 3.3 we give our clock construction for (2×5)-QSAT.

3.1 2D-Hamiltonian

The following theorem extracts the 2D-Hamiltonian construction central to [24] so that we can use it in conjunction with our own clock construction. In the full version [40], we give a complete proof with a simplified Hamiltonian construction and improved analysis that gives soundness $\Omega(\gamma(H^{(N)}_{\text{clock}})/N^2)$, as opposed to $\Omega(\gamma(H^{(N)}_{\text{clock}})/N^6)$ from [24]. Our proof is fully analytic, improving on the partially numeric analysis of [24].

Note, our construction is structurally almost the same as [24], which would also work in conjunction with our clock (see Section 3.3), requiring only a slight modification to the H_{init} and H_{end} terms. However, the application of Lemma 3.2 (necessary for improved soundness) to their Hamiltonian is not as straightforward because there is no separate gadget for 1-local gates. So, our contribution to the following theorem are a simplified proof and improved soundness bounds. The intuition of the construction is given in the introduction.

▶ **Theorem 3.1.** *Suppose we are given Hamiltonian terms as follows:*

(1) *Clock Hamiltonian $H^{(N)}_{\text{clock}}$ acting on Hilbert space $\mathcal{H}_{\text{clock}}$ with nullspace $\mathcal{N}(H^{(N)}_{\text{clock}}) = \text{Span}\{|C_1\rangle,\dots,|C_n\rangle\} =: \mathcal{C}$ for* clock states *$|C_i\rangle$.*

(2) *Projectors $h_{i,i+1}(U)$ acting on $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock}}$, where U is a unitary acting on $\mathcal{H}_{\text{comp}}$, such that*

$$\left(\mathbb{I}\otimes\Pi^{(N)}_{\text{clock}}\right) h_{i,i+1}(U) \left(\mathbb{I}\otimes\Pi^{(N)}_{\text{clock}}\right) = c_1\big((\mathbb{I}\otimes|C_i\rangle\langle C_i| + \mathbb{I}\otimes|C_{i+1}\rangle\langle C_{i+1}|) \\ -(U^\dagger\otimes|C_i\rangle\langle C_{i+1}| + U\otimes|C_{i+1}\rangle\langle C_i|)\big) \tag{6}$$

for constant[21] c_1. We write $h_{i,i+1} := h_{i,i+1}(\mathbb{I})$.

[21] It suffices for our purposes to have the same c for all i. In principle, however, one can also allow different

(3) *Projectors $C_{\geq i}, C_{\leq i}$ acting on $\mathcal{H}_{\text{clock}}$ such that*

$$\Pi^{(N)}_{\text{clock}} C_{\geq i} \Pi^{(N)}_{\text{clock}} = \sum_{j=i}^{N} c_{2,i,j} |C_j\rangle\langle C_j|, \qquad \Pi^{(N)}_{\text{clock}} C_{\leq i} \Pi^{(N)}_{\text{clock}} = \sum_{j=1}^{i} c_{3,i,j} |C_j\rangle\langle C_j|, \tag{7}$$

and $c_{2,i,j}, c_{3,i,j} \geq c_2$ for all $i, j \in [N]$ for some constant c_2.

(4) *All above projectors ($h_{i,i+1}(U), C_{\geq i}, C_{\leq i}$) pairwise commute, besides $h_{i,i+1}(U)$ and $h_{i,i+1}(U')$ for non-commuting U, U'. For all $i \neq j$, $h_{i,i+1}(U) h_{j,j+1}(U') = 0$.*

(5) *If $\Pi_1, \ldots, \Pi_k$ are projectors of the form $C_{\leq i}, C_{\geq i}$, then $\langle C_{j_1}|\Pi_1 \cdots \Pi_k|C_{j_2}\rangle = 0$ for $j_1 \neq j_2$.*

Then any problem in QMA$_1$ *can be reduced to* QSAT *restricted to Hamiltonians H acting on $\mathcal{H}_{\text{comp}} \otimes \mathcal{H}_{\text{clock},1} \otimes \mathcal{H}_{\text{clock},2}$ (operators acting on these spaces are labeled with subscripts Z, X, Y respectively) with terms $(H^{(N)}_{\text{clock}})_X$, $(H^{(N)}_{\text{clock}})_Y$, $\Pi_Z \otimes (h_{i,i+1})_A$, $(C_{\sim j})_A \otimes (h_{i,i+1})_B$, $(h_{i,i+1}(U))_{Z,A}$, $(C_{\leq i})_A \otimes (C_{\geq j})_B$, where $A, B \in \{X, Y\}, A \neq B$, $\sim \in \{\leq, \geq\}$[22], $i, j \in [N]$, $\Pi_Z \in \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is a single-qubit projector acting on $\mathcal{H}_{\text{comp}}$, and U is either a 1-local gate from the* QMA$_1$ *circuit or $U \in \{\sigma^Z, B\}$ with $B = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$, $\sigma^Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The soundness is $\Omega(\gamma(H^{(N)}_{\text{clock}})/N^2)$, where $N = \Theta(g)$ and g is the number of gates used by the* QMA$_1$ *verifier.*

Proof. A rough intuition is given in Section 1 and the formal proof in the full version [40]. ◀

Note, this theorem is not explicitly stated in [24], but is implicitly proven, albeit with a soundness of $\Omega(\gamma(H^{(N)}_{\text{clock}}/N^6))$. Thus, as an immediate first consequence we can (using the clock Hamiltonian of [24]) recover QMA$_1$-hardness of 3-QSAT, but with improved soundness:

▶ **Theorem 1.3.** 3-QSAT *is* QMA$_1$*-complete with soundness* $\Omega(1/N^2)$.

Unfortunately its generality makes Theorem 3.1 somewhat difficult to parse. Hence, we sketch its application to the clock of Kitaev's circuit Hamiltonian [30] as a toy example:

(1) Clock states are given by $|C_i\rangle = |1^i 0^{N-i}\rangle$ with $H_{\text{clock}} = \sum_{i=1}^{N-1} |01\rangle\langle 01|_{i,i+1} + |0\rangle\langle 0|_1$.

(2) $h_{i,i+1}(U)$ essentially applies the gate U from timestep $|C_i\rangle$ to $|C_{i+1}\rangle$. Define for $i \in [N-1]$, $h_{i,i+1}(U) = (\mathbb{I} \otimes |100\rangle\langle 100|_{i,i+1,i+2} + \mathbb{I} \otimes |110\rangle\langle 110|) - (U^\dagger \otimes |100\rangle\langle 110| + U \otimes |110\rangle\langle 100|)$. Equation (6) can easily be verified with $c_1 = 1$. In our actual construction we get $c_1 < 1$ because the clock states are superpositions of multiple standard basis states, of which $h_{i,i+1}$ only acts on one (see Section 3.3).

(3) $C_{\leq i}, C_{\geq i}$ just project onto timesteps $\leq i$ or $\geq i$. Define $C_{\leq i} = |0\rangle\langle 0|_{i-1}$ and $C_{\geq i} = |1\rangle\langle 1|_i$. Equation (7) can easily be verified with $c_2 = 1$.

(4) and **(5)** are only used to prove the soundness lower bound and may be dropped, decreasing soundness by a polynomial factor.

Thus, Theorem 3.1 gives QMA$_1$-completeness of 4-QSAT when applied to Kitaev's unary clock.

3.2 Nullspace Connection Lemma

Abstractly, the 2D-Hamiltonian consists of a sequence of gadgets connected together via the blue zigzag edges, as depicted in Figure 1.2. If we remove the zigzag edges, we can analyze the nullspace and gap of the Hamiltonian easily, as gadgets have only constant size and act on orthogonal clock spaces. One can show that the nullspace of each gadget is spanned by history states on the local clock spaces. The "Nullspace Connection Lemma" below then argues that these local history states can be connected via the zigzag edges.

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constants depending on the choice of index i.

[22] Here we use "$\sim$" as a placeholder for a relation "$\leq$" or "$\geq$".

Our statement is quite general, so that Lemma 3.2 can be applied more broadly. Due to the decomposition into the local gadgets acting on disjoint sets of clock states, one only needs to compute the nullspaces of constant-size gadgets before applying Lemma 3.2. See Remark 3.3 for an application to Kitaev's circuit Hamiltonian. For instance, Rudolph uses Lemma 3.2 to prove correctness of various circuit-to-Hamiltonian embeddings [39] and computes the gadget nullspaces algebraically. In the full version of this work, we compute nullspaces of the 2D-Hamiltonian gadgets via the theory of Unitary Labelled Graphs [6].

▶ **Lemma 3.2** ("Nullspace Connection Lemma"). *Let*

(1) $K = K_1 \uplus \ldots \uplus K_m = [N]$ *be a disjoint partition of the indices of* N *clock states with* $u_i, v_i \in K_i, u_i \neq v_i$ *for all* $i \in [m]$.

(2) $H_1 = \sum_{i=1}^m H_{1,i}$ *be a Hamiltonian on ancilla space* A *and clock space* C*, such that for all* $i \in [m]$:

- **(a)** $\mathcal{N}(H_{1,i}|_{\mathcal{K}_i}) = \mathrm{Span}\{|\psi_i(\alpha_j)\rangle \mid j \in [d]\}$*, where* $\mathcal{K}_i = \mathbb{C}_A^d \otimes \mathrm{Span}\{|v\rangle_C \mid v \in K_i\}$*, and* $|\alpha_1\rangle, \ldots, |\alpha_d\rangle$ *is an orthonormal basis of the ancilla space,*
- **(b)** *the local history states* $|\psi_i(\alpha)\rangle$ *are linear in the input* $|\alpha\rangle$*, i.e., there exists a linear map* L_i *with* $L_i|\alpha\rangle = |\psi_i(\alpha)\rangle$ *and* $L_i^\dagger L_i = \lambda_i \mathbb{I}$ *for some constant* λ_i*,*
- **(c)** H_i *has support only on clock space* $\mathcal{K}_i$*,*
- **(d)** $\||\psi_i(\alpha)\rangle\|^2 =: \delta_i \in [1, \Delta]$*,*
- **(e)** $(\mathbb{I}_A \otimes \langle u_i|_C)|\psi_i(\alpha)\rangle = |\alpha\rangle_A$*,*
- **(f)** $(\mathbb{I}_A \otimes \langle v_i|_C)|\psi_i(\alpha)\rangle = U_i|\alpha\rangle_A$ *for some unitary* U_i*.*

(3) $H_2 = \sum_{i=1}^{m-1} h_{v_i,u_{i+1}}(V_i)$ *with* $h_{v_i,u_{i+1}}(V_i) = \mathbb{I}\otimes|v_i\rangle\langle v_i| + \mathbb{I}\otimes|u_{i+1}\rangle\langle u_{i+1}| - V_i^\dagger\otimes|v_i\rangle\langle u_{i+1}| - V_i \otimes |u_{i+1}\rangle\langle v_i|)$ *for unitaries* V_i*.*

(4) $|\alpha_{ij}\rangle = V_{i-1}U_{i-1}\cdots V_1U_1|\alpha_j\rangle$*.*

Then for $H = H_1 + H_2$*,* $\mathcal{N}(H) = \mathrm{Span}\{\sum_{i=1}^m |\psi_i(\alpha_{ij})\rangle \mid j \in [d]\}$*,* $\gamma(H) = \Omega(\gamma(H_1)/(m^2\Delta))$*.*

Prior to the proof, let us briefly give an intuitive explanation of the parts. (1) just gives a decomposition of the indices of clock states into disjoint components K_i. One can think of these as a generalization of timesteps to a graph, like in the Unitary Labelled Graphs [6]. (2) is the Hamiltonian corresponding to the unitary gadgets without the connection (zigzag) edges. (a) says that the i-th gadget should apply gate U_i, such that it has a "local history state" from input (e) to output (f). The local history state is linear in the "input state" (b). (c) requires the gadgets to only act on their clock states K_i and thus commute. (d) constrains the norm of history states, which is used to compute the gap. (3) is the Hamiltonian corresponding to the connection edges. (4) describes the input states to each gadget.

Proof. Since the $H_{1,i}$ Hamiltonians act on different clock spaces, they commute and it holds $S := \mathcal{N}(H_1) = \mathrm{Span}\{|\psi_i(\alpha_j)\rangle \mid i \in [m], j \in [d]\}$. Next, we derive $\mathcal{N}(H) = S \cap \mathcal{N}(H_2)$. Partition S into orthogonal subspaces $S_1, \ldots, S_d$, where $S_j = \mathrm{Span}\{|\psi_i(\alpha_{ij})\rangle \mid i \in [m]\}$. Orthogonality holds because $\langle\psi_i(\alpha)|\psi_i(\alpha')\rangle = \langle\alpha|L_i^\dagger L_i|\alpha'\rangle = \lambda_i\langle\alpha|\alpha'\rangle$, and $|\psi_i(\alpha)\rangle \in \mathcal{K}_i$, where $\mathcal{K}_i \subseteq \mathcal{K}_{i'}^\perp$ for $i \neq i'$. As H_2 is block diagonal across $S_1, \ldots, S_d$, it suffices to consider the $\mathcal{N}(H_2|_{S_j})$ individually.

Let $|\psi\rangle = \sum_{i=1}^m a_i|\psi_i(\alpha_{ij})\rangle \in S_j$. Then $\langle\psi|H|\psi\rangle = \langle\psi|H_2|\psi\rangle = \sum_{i=1}^{m-1}\langle\psi|h_{v_i,u_{i+1}}(V_i)|\psi\rangle$ with

$$\begin{aligned} h_{v_i,u_{i+1}}(V_i)|\psi\rangle &= a_iU_i|\alpha_{ij}\rangle|v_i\rangle + a_{i+1}|\alpha_{i+1,j}\rangle|u_{i+1}\rangle - a_{i+1}V_i^\dagger|\alpha_{i+1,j}\rangle|v_i\rangle - a_iV_iU_i|\alpha_{ij}\rangle|u_{i+1}\rangle \\ &= a_iU_i|\alpha_{ij}\rangle|v_i\rangle + a_{i+1}|\alpha_{i+1,j}\rangle|u_{i+1}\rangle - a_{i+1}U_i|\alpha_{ij}\rangle|v_i\rangle - a_i|\alpha_{i+1,j}\rangle|u_{i+1}\rangle. \end{aligned}$$

Thus, $\langle\psi|h_{v_i,u_{i+1}}(V_i)|\psi\rangle = a_i^*a_i + a_{i+1}^*a_{i+1} - a_i^*a_{i+1} - a_{i+1}^*a_i = |a_i - a_{i+1}|^2$ and $\mathcal{N}(H_2|_{S_j}) = \mathrm{Span}\{\sum_{i=1}^m|\psi_i(\alpha_{ij})\rangle\}$.

Figure 3.1 Applying the Nullspace Connection Lemma to Kitaev's circuit Hamiltonian.

Next, we apply Corollary 2.5 to show $\gamma(H) \geq \min\{\gamma(H_1), \gamma(H_2)\} \cdot \gamma(H_2|_S)/(2\|H_2\|)$. The terms $\|H_2\|, \gamma(H_2)$ are constant as the $h_{v_i,u_{i+1}}$ projectors act on distinct clock states. Hence, $\gamma(H) = \Omega(\gamma(H_1)\gamma(H_2|_S))$. Since $H_2|_S$ is block diagonal, we have $\gamma(H_2|_S) \geq \min_{j\in[d]} \gamma(H_2|_{S_j})$, where $\gamma(H_2|_{S_j}) = \min_{|v\rangle\in\Gamma_j:\langle v|v\rangle=1}\langle v|H_2|v\rangle$ and $\Gamma_j = S_j \cap \mathcal{N}(H)^\perp$.

Let $|\psi\rangle = \sum_{i=1}^m a_i|\psi_i(\alpha_{ij})\rangle \in \Gamma_j, \langle\psi|\psi\rangle = 1$ and $|\phi\rangle = \sum_{i=1}^m |\psi_i(\alpha_{ij})\rangle \in \mathcal{N}(H)$. Then $0 = \langle\phi|\psi\rangle = \sum_{i=1}^m a_i\delta_i$ and $\langle\psi|H_2|\psi\rangle = \sum_{i=1}^{m-1}|a_i - a_{i+1}|^2 \geq (\sum_{i=1}^{m-1}|a_i - a_{i+1}|)^2/m$. Also, there exists an l such that $|a_l|^2\delta_l \geq 1/m$. Via a global rotation, we may assume without loss of generality $a_l > 0$. Since $\sum_{i=1}^m a_i\delta_i = 0$, there must exist k with $\Re(a_k) < 0$. Thus $|a_l - a_k| = |a_l\delta_l - a_k\delta_l|/\delta_l > |a_l\delta_l|/\delta_l \geq 1/\sqrt{m\Delta}$. By the triangle inequality, $\langle\psi|H_2|\psi\rangle \geq 1/(m^2\Delta)$. ◀

▶ Remark 3.3. The Nullspace Connection Lemma can also used to prove correctness of Kitaev's original circuit-to-Hamiltonian construction [30] directly without transforming H_{prop} to a random walk. Recall $H_{\text{prop}} = \sum_{j=2}^N H_{\text{prop}}^j$ with

$$H_{\text{prop}}^j = -\frac{1}{2}(U_j)_A \otimes |j\rangle\langle j-1|_C - \frac{1}{2}(U_j^\dagger)_A \otimes +\frac{1}{2}I_A \otimes (|j\rangle\langle j| + |j-1\rangle\langle j-1|)_C,$$

where A denotes the ancilla space where the computation takes place, and C the clock space. Assume that $U_j = \mathbb{I}$ for even j. Figure 3.1 then depicts H_{prop} for $N = 7$. The graph is only a line since H_{prop} uses an ordinary (single) clock. To apply Lemma 3.2, let $K_1 = \{0,1\}, K_2 = \{2,3\}, K_3 = \{4,5\}, K_4 = \{6,7\}$, $H_1 = \sum_{j\in\{1,3,5,7\}} H_{\text{prop}}^j$ (red edges in Figure 3.1), and $H_2 = \sum_{j\in\{2,4,6\}} H_{\text{prop}}^j$ (blue zigzag edges in Figure 3.1). It is now easy to verify that all the conditions of Lemma 3.2 are satisfied: For example, $\mathcal{N}(H_{\text{prop}}^1|_{\mathcal{K}_1}) = \text{Span}\{|\psi_1(\alpha_j)\rangle \mid j \in [d]\}$, for $|\psi_1(\alpha_j)\rangle = |\alpha_j\rangle_A|0\rangle_C + U_1|\alpha_j\rangle_A|1\rangle_C$. It follows that $\mathcal{N}(H_{\text{prop}}) = \text{Span}\{|\psi_{\text{hist}}(\alpha_j)\rangle \mid j \in [d]\}$, where $|\psi_{\text{hist}}(\alpha)\rangle = \sum_{i=0}^N U_i \cdots U_1|\alpha\rangle|i\rangle_C$.

The Hamiltonian H_{in} is then used to restrict the ancilla space to $|0\rangle$ on timestep 0, so that $\mathcal{N}(H_{\text{in}} + H_{\text{prop}}) = \text{Span}\{|\psi_{\text{hist}}(0)\rangle\}$.

3.3 Clock Hamiltonian

In this section, we define the clock to prove the QMA$_1$-completeness of (2×5)-QSAT (Theorem 1.1) and (4×3)-QSAT (Theorem 1.2) using Theorem 3.1. The main difficulty was the construction of a suitable clock for Theorem 3.1 using just (2×5)-projectors. The hardness of (3×4)-QSAT is obtained almost for free as part of the proof. Since our construction may seem somewhat arbitrary, we give an intuition in the appendix of [40] by sketching a proof for the QMA$_1$-hardness of (2×7)-QSAT.

We begin by defining several logical states. A 5-dit (whose standard basis states are labeled u, a_1, a_2, d, u') and a qubit are combined to construct a logical 4-dit (labeled $\mathbf{U}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{D}$), as shown on the left of Equation (8) (normalization factors omitted). A 5-dit (labeled u, a, d, x, d') and two qubits are combined to construct a logical qutrit (labeled $\mathbf{u}, \mathbf{a}, \mathbf{d}$), as shown on the right of Equation (8). The labels u, a, d maybe be interpreted as "unborn", "alive", and "dead", respectively, following the convention of [1, 8, 17].

$$\begin{aligned} |\mathbf{U}\rangle &= |u,0\rangle + |u',1\rangle & |\mathbf{u}\rangle &= |u,1,0\rangle + |x,0,0\rangle \\ |\mathbf{A}_1\rangle &= |a_1,0\rangle & |\mathbf{a}\rangle &= |a,0,0\rangle + |x,1,0\rangle \\ |\mathbf{A}_2\rangle &= |a_2,0\rangle & |\mathbf{d}\rangle &= |d,0,0\rangle + |d',0,1\rangle \\ |\mathbf{D}\rangle &= |d,0\rangle \end{aligned} \tag{8}$$

▶ **Lemma 3.4.** *There exist* 2*-local Hamiltonians* $H_{\text{logical}_4}, H_{\text{logical}_3}$ *of* $(2,5)$*-projectors with* $\mathcal{N}(H_{\text{logical}_4}) = \text{Span}\{|\mathbf{U}\rangle, |\mathbf{A}_1\rangle, |\mathbf{A}_2\rangle, |\mathbf{D}\rangle\}$ *and* $\mathcal{N}(H_{\text{logical}_4}) = \text{Span}\{|\mathbf{U}\rangle, |\mathbf{A}_1\rangle, |\mathbf{A}_2\rangle, |\mathbf{D}\rangle\}$.

Proof. In the full version [40]. ◀

The main "feature" of these logical qudits is the fact that a 1-local qubit projector suffices to identify the states $|\mathbf{U}\rangle$, $|\mathbf{d}\rangle$, "$|\mathbf{u}\rangle$ or $|\mathbf{a}\rangle$" (e.g. $|u\rangle\langle u|$), and implement a transition between $|\mathbf{u}\rangle, |\mathbf{a}\rangle$ as $(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)$ (assuming $|\mathbf{d}\rangle$ is already penalized by another projector).

We make use of these properties in the definition of the *clock states* $|C_i\rangle =: |C_{i,1}\rangle + |C_{i,2}\rangle + |C_{i,3}\rangle + |C_{i,4}\rangle$ for $i \in [N]$, where the states $|C_{i,j}\rangle$ are defined as follows:

$$|C_{i,1}\rangle = |\mathbf{dD}\rangle^{\otimes i-2} \otimes |\mathbf{dA}_2, \mathbf{uU}\rangle \otimes |\mathbf{uU}\rangle^{\otimes N-i} \tag{9a}$$
$$|C_{i,2}\rangle = |\mathbf{dD}\rangle^{\otimes i-2} \otimes |\mathbf{dD}, \mathbf{aU}\rangle \otimes |\mathbf{uU}\rangle^{\otimes N-i} \tag{9b}$$
$$|C_{i,3}\rangle = |\mathbf{dD}\rangle^{\otimes i-2} \otimes |\mathbf{dD}, \mathbf{dU}\rangle \otimes |\mathbf{uU}\rangle^{\otimes N-i} \tag{9c}$$
$$|C_{i,4}\rangle = |\mathbf{dD}\rangle^{\otimes i-2} \otimes |\mathbf{dD}, \mathbf{dA}_1\rangle \otimes |\mathbf{uU}\rangle^{\otimes N-i} \tag{9d}$$

Here, the qudits of the clock register are subdivided into groups of five, where the first three (labeled α, β, γ) represent a logical qutrit and the the last two (labeled δ, ε) represent a logical 4-qudit (e.g. the i-th group of (9d) is $|\mathbf{dA}_1\rangle$). With these clock states, a 1-local transition from $|\mathbf{A}_1\rangle$ of $|C_{i,4}\rangle$ to $|\mathbf{A}_2\rangle$ of $|C_{i+1,1}\rangle$ can be implemented on a single 5-dit: $(|a_1\rangle - |a_2\rangle)(\langle a_1| - \langle a_2|)_{\delta_i}|_{\mathcal{C}} = \Omega(1) \cdot (|C_i\rangle - |C_{i+1}\rangle)(\langle C_i| - \langle C_{i+1}|)$ with $\mathcal{C}$ as below.

▶ **Lemma 3.5.** $\mathcal{N}(H_{\text{clock}}) = \text{Span}\{|C_1\rangle, \ldots, |C_N\rangle\} =: \mathcal{C}$ *and* $\gamma(H_{\text{clock}}) = 1$.

Proof sketch. The first step is to enforce the "logical space" using the Hamiltonian $H_{\text{logical}} = \sum_{i=0}^{N-1} \big((H_{\text{logical}_3})_{\alpha_i\beta_i\gamma_i} + (H_{\text{logical}_4})_{\delta_i\varepsilon_i}\big)$. Next, enforce valid clock states using the following Hamiltonian, where the annotations on the left explains the function of each projector.

U implies **u** to the right:
$$H_{\text{clock},1} = \sum_{i=1}^{N-1} \sum_{v \in \{a,d\}} |1,v\rangle\langle 1,v|_{\varepsilon_i\alpha_{i+1}} \tag{10a}$$

d implies **d** to the left:
$$+ \sum_{i=1}^{N-1} |1,d\rangle\langle 1,d|_{\beta_i\alpha_{i+1}} \tag{10b}$$

d implies **D** to the left:
$$+ \sum_{i=1}^{N-1} \sum_{v \in \{u,a_1,a_2\}} |v,1\rangle\langle v,1|_{\varepsilon_i\gamma_{i+1}} \tag{10c}$$

u, **a** implies **U** to the right:
$$+ \sum_{i=1}^{N} \sum_{v \in \{a_1,a_2,d\}} |1,v\rangle\langle 1,v|_{\beta_i\delta_i} \tag{10d}$$

U or $\mathbf{A}_1$ is last:
$$+ |a_2\rangle\langle a_2|_{\delta_N} + |d\rangle\langle d|_{\delta_N} \tag{10e}$$

We then have $\mathcal{N}(H_{\text{logical}} + H_{\text{clock},1}) = \text{Span}(\{|C_{i,j}\rangle \mid i \in [N], j \in [4]\} \cup \{|E_{i,j}\rangle \mid i \in \{2,\ldots,N\}, j \in [4]\})$, where the $|E_{i,j}\rangle$ are similar to the $|C_{i,j}\rangle$ but with $|\mathbf{dA}_1, \mathbf{aU}\rangle$, $|\mathbf{dA}_2, \mathbf{aU}\rangle$, $|\mathbf{dD}, \mathbf{uU}\rangle$ in the middle. The $|E_{i,j}\rangle$ are still in the nullspace because $H_{\text{clock},1}$ cannot yet penalize all logical states that are not valid clock states. Next, we construct $H_{\text{clock},2} = \sum_{i=1}^{N} H_{\text{clock},2,i}$ to enforce the superposition $|C_{i,1}\rangle + \cdots + |C_{i,4}\rangle$, and also penalize the $|E_{i,j}\rangle$ using a similar argument to the *clairvoyance lemma* [2]. The idea is that the transition terms of (11) "evolve" the states $|E_{i,j}\rangle$ to states that are penalized by Equation (10). $H_{\text{clock},2,i}$ is given by

$$(C_{i,1} \leftrightarrow C_{i,2})\ |\mathbf{A}_2, \mathbf{u}\rangle \leftrightarrow |\mathbf{D}, \mathbf{a}\rangle\text{:} \quad \begin{cases} (|x,0\rangle - |x,1\rangle)(\langle x,0| - \langle x,1|)_{\alpha_1\beta_1}, & i = 1 \\ (|a_2,0\rangle - |d,1\rangle)(\langle a_2,0| - \langle d,1|)_{\delta_{i-1}\beta_i}, & i > 1 \end{cases} \tag{11a}$$

$$(C_{i,2} \leftrightarrow C_{i,3})\ |\mathbf{a}, \mathbf{U}\rangle \leftrightarrow |\mathbf{d}, \mathbf{U}\rangle\text{:} \quad + (|a,1\rangle - |d,1\rangle)(\langle a,1| - \langle d,1|)_{\alpha_i\varepsilon_i} \tag{11b}$$

$$(C_{i,3} \leftrightarrow C_{i,4})\ |\mathbf{d}, \mathbf{U}\rangle \leftrightarrow |\mathbf{d}, \mathbf{A}_1\rangle\text{:} \quad + (\sqrt{2}|1,u\rangle - |1,a_1\rangle)(\sqrt{2}\langle 1,u| - \langle 1,a_1|)_{\gamma_i\delta_i}. \tag{11c}$$

Finally, we set $H_{\text{clock}} = H_{\text{logical}} + H_{\text{clock},1} + H_{\text{clock},2}$ and prove $\mathcal{N}(H_{\text{clock}}) = \mathcal{C}$ and $\gamma(H_{\text{clock}}) = \Omega(1)$ in the full version [40]. ◀

The transition and selection operators are then defined as follows, where we give two variants of the $C_{\sim i}$ projectors, one acting on a 5-qudit and the other on a qubit.

$$h_{i,i+1}(U_Z) = \frac{1}{2}\left(\mathbb{I}_Z \otimes |a_1\rangle\langle a_1|_{\delta_i} + \mathbb{I}_Z \otimes |a_2\rangle\langle a_2|_{\delta_i} - U_Z^\dagger \otimes |a_1\rangle\langle a_2|_{\delta_i} - U_Z \otimes |a_2\rangle\langle a_1|_{\delta_i}\right),$$
$$C^{(5)}_{\leq i} = |u\rangle\langle u|_{\delta_i}, \quad C^{(2)}_{\leq i} = |1\rangle\langle 1|_{\varepsilon_i}, \quad C^{(5)}_{\geq i} = |d\rangle\langle d|_{\alpha_i}, \quad C^{(2)}_{\geq i} = |1\rangle\langle 1|_{\gamma_i} \tag{12}$$

Thus, the Hamiltonians from Theorem 3.1 can all be implemented as a (2×5)-projector.

3.4 QMA$_1$-completeness

Our main contribution is certainly the QMA$_1$-hardness (Theorem 1.1), but we still need to discuss containment in QMA$_1$ briefly.

▶ **Lemma 3.6.** *The* (2×5)-QSAT *and* (3×4)-QSAT *instances constructed from* QMA$_1$*-circuits are contained in* QMA$_1$*.*

Proof. To evaluate $(d_A \times d_B)$-QSAT instances with a QMA$_1$-verifier, we embed each qudit into $\lceil \log d \rceil$ qubits, and add additional diagonal projectors to reduce the local dimension as necessary. The lemma then follows from the containment of QSAT in QMA$_1$ [24], which requires projectors of a specific form. Besides (11c), all projectors used in Section 3.3 have entries in $\mathbb{Z}[\frac{1}{\sqrt{2}}, i]$. Measurements with respect to such projectors are implemented with [23]. Recall the projector from (11c), $\Pi = \left(\sqrt{\tfrac{2}{3}}|1,u\rangle - \sqrt{\tfrac{1}{3}}|1,a_1\rangle\right)\left(\sqrt{\tfrac{2}{3}}\langle 1,u| - \sqrt{\tfrac{1}{3}}\langle 1,a_1|\right)_{\gamma_i\delta_i}$. Under the qubit embedding, there exists a permutation P such that

$$P\Pi P^\dagger = \left(\sqrt{\frac{2}{3}}|0000\rangle - \sqrt{\frac{1}{3}}|0001\rangle\right)\left(\sqrt{\frac{2}{3}}\langle 0000| - \sqrt{\frac{1}{3}}\langle 0001|\right). \tag{13}$$

A measurement algorithm with perfect completeness is given in [24, Appendix A] for a 3-local projector of analogous structure, which can easily be extend to larger projectors. ◀

▶ **Theorem 1.1.** (2×5)-QSAT *is* QMA$_1$*-complete with soundness* $\Omega(1/N^2)$*.*

Proof. Follows from Theorem 3.1 and Lemma 3.5. ◀

▶ **Theorem 1.2.** (3×4)-QSAT *is* QMA$_1$*-complete with soundness* $\Omega(1/N^2)$*.*

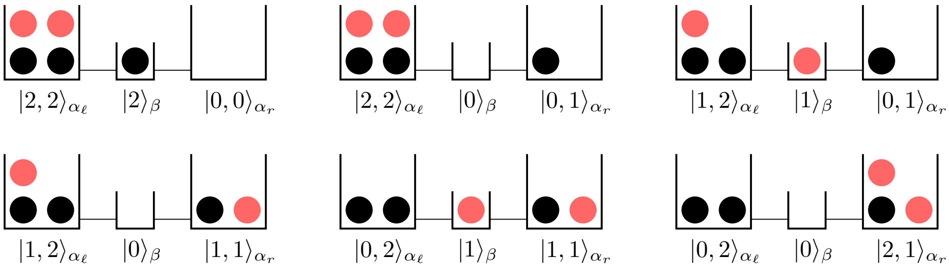

Figure 4.1 Configurations of the balls and bins for $d = 2$. Only configurations in $\mathcal{C}_1$ are depicted ($c_{1,1} = 2$ red balls and $c_{1,2} + 1 = 3$ black balls). The first two are in $\mathcal{C}_1^*$. The configurations evolve according to the transitions of (17b) and (17c). There are $d'' = 15$ possible configurations for the large bins.

Proof. We use the same clock construction operating directly in logical space, though care needs to be taken that everything is realizable with (3×4)-constraints. To implement $H_{\text{clock},1}$ using only (3×4)-terms, we have to replace "**d** implies **d** to the left" with "**D** implies **d** to the left". $H_{\text{clock},2}$ only uses (3×4)-transitions. The $C_{\sim i}$ projectors are implemented as $C^{(4)}_{\leq i} = |U\rangle\langle U|_{\delta_i}, C^{(3)}_{\leq i} = |u\rangle\langle u|_{\alpha_i}, C^{(4)}_{\geq i} = |D\rangle\langle D|_{\delta_{i-1}}, C^{(3)}_{\geq i} = |d\rangle\langle d|_{\alpha_i}$, where α_i, δ_i denote the pairs of (3×4)-qudits as depicted in Equation (9). The computational register is implemented on qutrits restricted to a 2-dimensional subspace. ◀

4 (3×d)-QSAT on a line

▶ **Theorem 1.4.** (3×d)-QSAT *on a line is* QMA_1*-complete with* $d = O(1)$.

To prove this theorem, we give a general construction to embed an arbitrary Hamiltonian $H = \sum_{i=1}^{n-1} H_{i,i+1}$ on n qu-d-its with d into a Hamiltonian H' on an alternating line of $n+1$ qu-d'-its ($d' = (d'')^2$) and n qutrits. The qu-d'-its are treated as two qu-d''-its of dimension $d'' \in O(d^2)$. Then each triple (d''×3×d'') logically stores one qu-d-it, which is "sent" between the outer qu-d''-its with a history-state-like construction. Conceptually, we think of this system as three bins with two kinds of balls (say *red* and *black*). The outer bins (qu-d''-its) may contain up to $B = 2d$ balls, and the middle bin only at most a single ball. A valid configuration of the bins has $B + 1$ balls, and the number of red balls is even and positive. We use transition terms to enforce a superposition of all valid configurations that have the same number of red balls. Hence, the (d''×3×d'') system has a nullspace of dimension d.

The standard basis states of the qu-d''-it are written as $|c_1, c_2\rangle$ with $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ and $c_1 + c_2 \leq B := 2d$. Thus, we get $d'' = \sum_{i=0}^{B}(B + 1 - i) = (B + 1)(B + 2)/2 = (d + 1)(2d + 1)$. Semantically, one may think of c_1 as "number of red balls" and c_2 as "number of black balls" (see Figure 4.1). For $i \in [d]$, let $c_{i,1} = 2i, c_{i,2} = B - 2i$ and define the set of valid configurations corresponding to the state $|i\rangle \in \mathbb{C}^d$ as

$$\mathcal{C}_i =: \left\{ (l_1, l_2, m, r_1, r_2) \;\middle|\; \begin{array}{r} l_1, l_2, r_1, r_2 \in \{0, \ldots, B\} \\ m \in \{0, 1, 2\} \\ l_1 + r_1 + \delta_{m,1} = c_{i,1} \\ l_2 + r_2 + \delta_{m,2} = c_{i,2} + 1 \end{array} \right\}, \tag{14}$$

where $\delta_{x,y}$ denotes the Kronecker delta. The constraints in (14) enforce that in total there are $c_{i,1}$ red balls and $c_{i,2} + 1$ black balls (see also Figure 4.1).

We place the local terms of the original Hamiltonian into the dimensions corresponding to $\mathcal{C}_i^* := \{(l_1, l_2, m, r_1, r_2) \in \mathcal{C}_i \mid (l_1, l_2) = c_i^* \vee (r_1, r_2) = c_i^*\}$, where $c_i^* := (c_{i,1}, c_{i,2})$. These configurations can be identified locally, i.e., one can tell which $\mathcal{C}_i^*$ a configuration corresponds to by only looking at the left or the right bin. Note that $|\mathcal{C}_i^*| = 4$ and $\mathcal{C}_i^* \cap \mathcal{C}_j = \emptyset$ for $j \neq i$. Thus, $\langle c_i^*|_{\alpha_\ell} |\psi_j\rangle = 0$ if $j \neq i$.

A logical $|i\rangle$ is then represented by

$$|\psi_i\rangle = \sum_{x=(l_1,l_2,m,r_1,r_2)\in\mathcal{C}_i} \sqrt{w_x} |l_1, l_2\rangle_{\alpha_\ell} |m\rangle_\beta |r_1, r_2\rangle_{\alpha_r},\ w_x = \begin{cases} \frac{2}{3} \cdot \frac{1}{|\mathcal{C}_i^*|} =: w_i^*, & x \in \mathcal{C}_i^* \\ \frac{1}{3} \cdot \frac{1}{|\mathcal{C}_i \setminus \mathcal{C}_i^*|} =: \overline{w}_i, & x \notin \mathcal{C}_i^* \end{cases}, \tag{15}$$

where α_ℓ, α_r denote the qu-d''-its and β the qutrit. Let

$$V = \sum_{i=1}^{d} |\psi_i\rangle\langle i| \in \mathbb{C}^{3(d'')^2 \times d} \tag{16}$$

be the isometry that maps $|i\rangle$ to $|\psi_i\rangle$. The weights w_x in (15) ensure that the $\mathcal{C}_i^*$ always have the same amplitude ($\sqrt{1/6}$), as the $\mathcal{C}_i$ can have different sizes. The reason for having the additional configurations $\mathcal{C}_i \setminus \mathcal{C}_i^*$ is so that we can use 2-local transitions (see (17b) and (17c)) on the line to enforce a superposition between the $\mathcal{C}_i^*$ states.

Next, we construct a Hamiltonian whose nullspace is spanned by the logical states $|\psi_1\rangle, \ldots, |\psi_d\rangle$. Let $H_{\text{ball}} = (H_{\text{ball}/2})_{\alpha_\ell\beta} + (H_{\text{ball}/2})_{\alpha_r\beta}$, where α_ℓ, α_r denote the qu-d''-its and β the qutrit.

$$H_{\text{ball}/2} = P(|0,0\rangle|0\rangle) + P(|0,B\rangle|2\rangle) + P(|B,0\rangle|1\rangle) + \sum_{c_1+c_2=B, c_1 \text{ odd}} P(|c_1, c_2\rangle|2\rangle) \tag{17a}$$

$$+ \sum_{c_1>0, c_2} \big[T((c_1, c_2, 0), (c_1 - 1, c_2, 1)) + T((c_1, c_2, 2), (c_1 - 1, c_2 + 1, 1))\big] \tag{17b}$$

$$+ \sum_{c_1, c_2>0} \big[T((c_1, c_2, 0), (c_1, c_2 - 1, 1)) + T((c_1, c_2, 1), (c_1 + 1, c_2 - 1, 2))\big], \tag{17c}$$

where $P(|\psi\rangle) := |\psi\rangle\langle\psi|$,

$$T((a_1, a_2, m_a), (b_1, b_2, m_b)) := P(\sqrt{w_b}|a_1, a_2\rangle|m_a\rangle - \sqrt{w_a}|b_1, b_2\rangle|m_b\rangle), \tag{18a}$$

$$(w_a, w_b) = \begin{cases} (w_i^*, \overline{w}_i), & (a_1, a_2) = c_i^* \\ (\overline{w}_i, w_i^*), & (b_1, b_2) = c_i^* \\ (1, 1), & \text{otherwise} \end{cases}. \tag{18b}$$

One may interpret $P(|\psi\rangle)$ as penalizing $|\psi\rangle$ and $T(\cdot)$ as a transition between the two given configurations, where the weights are chosen according to the weights in the $|\psi_i\rangle$ states.

▶ **Lemma 4.1.** $\mathcal{N}(H_{\text{ball}}) = \text{Span}\{|\psi_i\rangle \mid i \in [d]\} =: \mathcal{L}_{\text{ball}}$.

Proof. In the full version [40]. ◀

The logical states $|\psi_1\rangle, \ldots, |\psi_d\rangle$ are orthonormal and can be "identified" by projecting either qudit onto $|c_i^*\rangle$ as $\langle c_i^*|_{\alpha_r} |\psi_i\rangle = \sqrt{1/6}(|0,0\rangle|2\rangle + |0,1\rangle|0\rangle)_{\alpha_\ell\beta} =: \sqrt{1/3}|\eta\rangle$. $|\eta\rangle$ is the residual state of $|\psi_i\rangle$ after projecting onto $|c_i^*\rangle$ and is the same for all $i \in [d]$. In Figure 4.1, $|\eta\rangle_{\alpha_r\beta}$ is the superposition of the right halves of the first two configurations.

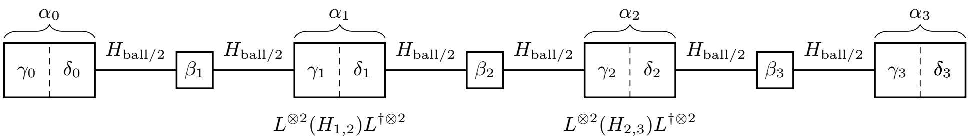

Figure 4.2 Graphical representation of H' embedding an $n = 3$ qudit Hamiltonian. The qu-d'-its $\alpha_0, \ldots, \alpha_3$ are subdivided into qu-d''-its γ_i, δ_i. H' acts trivially on γ_0, δ_3.

We define the isometry $L \coloneqq \sum_{i=1}^{d} |c_i^*\rangle\langle i| \in \mathbb{C}^{d''\times d}$ and finally the complete Hamiltonian H' on qu-d'-its $\alpha_0, \ldots, \alpha_n$ (each logically divided into two qu-d''-its γ_i and δ_i) and qubits $\beta_1, \ldots, \beta_n$ is given by $H' = H'_{\text{logical}} + H'_{\text{sim}}$ with

$$H'_{\text{logical}} = \sum_{i=1}^{n} \underbrace{\left((H_{\text{ball}/2})_{\delta_{i-1}\beta_i} + (H_{\text{ball}/2})_{\gamma_i\beta_i}\right)}_{(H_{\text{ball}})_{\delta_{i-1}\beta_i\gamma_i}} \tag{19a}$$

$$H'_{\text{sim}} = \sum_{i=1}^{n-1} (L\otimes L)(H_{i,i+1})(L\otimes L)^{\dagger}_{\alpha_i}, \tag{19b}$$

where H'_{logical} contains the terms of H_{ball} to enforce the logical subspace, and H'_{sim} embeds the terms of the original Hamiltonian H. Figure 4.2 gives a graphical representation of H'. Note that H' acts as identity on the first half of the first qu-d'-it (γ_0) and the second half of the last qu-d'-it (δ_n). So we can write

$$H' = \mathbb{I}_{\gamma_0} \otimes H''_{\delta_0\alpha_1\beta_1\ldots\alpha_{n-1}\beta_n\gamma_n} \otimes \mathbb{I}_{\delta_n}. \tag{20}$$

The next lemma shows that H' and H are equal inside the nullspace of H'_{logical}, up to an isometry.

▶ **Lemma 4.2.** *$T^\dagger H'' T = \frac{1}{9}H$ with $T = V^{\otimes n}$, for V and H'' as in Equations* (16) *and* (20)*.*

Proof. It suffices to check equality for computational basis states $|x\rangle$ with $x \in [d]^n$. Then $T|x\rangle = |\psi_{x_1}\rangle\cdots|\psi_{x_n}\rangle =: |\psi_x\rangle$. Clearly, $H'_{\text{logical}}|\psi_x\rangle = 0$. Consider now the first summand of H'_{sim}, $(L\otimes L)(H_{1,2})(L\otimes L)^{\dagger}_{\alpha_i} =: M_1$. We have

$$\begin{aligned} V^{\dagger\otimes 2} M_1 |\psi_{x_1}\rangle_{\delta_0\beta_1\gamma_1}|\psi_{x_2}\rangle_{\delta_1\beta_2\gamma_2} &= \frac{1}{3} V^{\dagger\otimes 2}|\eta\rangle_{\delta_0\beta_1} \otimes L^{\otimes 2} H_{1,2}|x_1\rangle_{\varepsilon_1}|x_2\rangle_{\varepsilon_2} \otimes |\eta\rangle_{\gamma_2\beta_2} \\ &= \frac{1}{9} H_{1,2}|x_1x_2\rangle, \end{aligned} \tag{21}$$

where $\varepsilon_1, \varepsilon_2$ denote the qudits $H_{1,2}$ acts on, and the second equality follows from the fact that $V^\dagger(|\eta\rangle_{\alpha_\ell\beta} \otimes L_{\alpha_r}) = \mathbb{I}/\sqrt{3}$. Since M_1 only acts nontrivially on the first two logical qudits, we have $T^\dagger M_1 T|x\rangle = \frac{1}{9}H|x\rangle$. Applying the same argument to the other summands of H'_{sim} yields $T^\dagger H'' T|x\rangle = \frac{1}{9}H|x\rangle$. ◀

▶ **Lemma 4.3.** *Let H be a Hamiltonian on a line of n qu-d-its with $H_{i,i+1} \succeq 0$ and $\gamma(H_{i,i+1}) \in \Omega(1)$ for all $i \in [n-1]$. There exists an efficiently computable Hamiltonian H' on an alternating chain of $n+1$ qu-d'-its and n qutrits with $d' = ((d+1)(2d+1))^2 \in O(d^4)$, such that $\lambda_{\min}(H') = 0$ iff $\lambda_{\min}(H) = 0$ and $\gamma(H') \in \Omega(\gamma(H)/\|H\|)$.*

Proof. If there exists $|\psi\rangle$ such that $H|\psi\rangle = 0$, then $H''T|\psi\rangle = 0$ by Lemma 4.2. To prove $\gamma(H') \in \Omega(\gamma(H)/\|H\|)$, apply Corollary 2.5. We have $\mathcal{N}(H'_{\text{logical}}) = \mathcal{L}_{\text{ball}}^{\otimes n} =: S$ and $\gamma(H'_{\text{logical}}), \gamma(H'_{\text{sim}}) \in \Omega(1)$ since $H'_{\text{logical}}, H'_{\text{sim}}$ are sums of commuting Hamiltonians. Since $T^\dagger \Pi_S = T^\dagger$, it follows that $T^\dagger H'' T = \frac{1}{9}H$ is equal to $H''|_S$ up to change of basis. Hence $\gamma(H'|_S) = \frac{1}{9}\gamma(H)$ and $\|H'|_S\| = \frac{1}{9}\|H\|$. ◀

Proof of Theorem 1.4. 2-QSAT one a line of qu-d-its with $d = 11$ is QMA_1-complete [34]. Using Lemma 4.3, we can embed this QSAT instance into a $(3{\times}d')$-QSAT on a line. ◀

▶ Remark 4.4. Care needs to be taken regarding containment in QMA_1, since the transition terms of $H_{\text{ball}/2}$ include irrational amplitudes (see (15) and (18a)) for which the techniques of [24] (see Lemma 3.6) do not directly apply. If we allow the QMA_1-verifier to use gates with entries from some algebraic field extension (as in [8]), we can easily verify H'. If we are restricted to the "Clifford + T" gate set as in Definition 2.1, we can modify $H_{\text{ball}/2}$ so that the sets $\mathcal{C}_i$ are of equal size for each i. We can do this by adding additional dimensions with transitions to $|c_i^*\rangle$, which increases d' by a constant factor as $|\mathcal{C}_i| = O(d)$. Then we can set all weights in the transitions of H_{ball} to 1 so that the logical states $|\psi_i\rangle$ are just uniform superpositions over the $\mathcal{C}_i$ configurations. Lemma 4.2 then still holds, albeit with a smaller factor that depends on d.

▶ Remark 4.5 (Hamiltonian simulation). Our embedding of the Hamiltonian H into H' is related to the notion of Hamiltonian simulation [7, 13]. In a sense, our embedding is almost a *perfect simulation* [13, Definition 20], but then only the nullspace is really simulated perfectly since H'_{logical} does not commute with H'_{sim}. Our construction takes the form $H' = H'_{\text{logical}} + H'_{\text{sim}}$, such that $T^\dagger H'_{\text{sim}} T = cH$ for some constant c, where T is an isometry with $TT^\dagger = \Pi_{\mathcal{N}(H'_{\text{logical}})}$. This notion of simulation may be helpful for future quantum SAT research.

5 Hamiltonian with unique entangled ground state on a (2×4)-Line

We were only able to prove QMA_1-hardness of quantum SAT on a $(3{\times}d)$-line. This raises the question whether hardness still holds with qubits instead of qutrits. In this section, we show that it is at least possible to construct a (2×4)-QSAT instance on a line with a unique entangled null state. Therefore, the $(2{\times}d)$-QSAT problem on a line does not necessarily have a product state solution like 2-QSAT.

▶ **Theorem 1.5.** *Consider a line of $2n$ particles such that the i-th particle has dimension 2 for even i and 4 for odd i. There exists a Hamiltonian $H = \sum_{i=1}^{n} A_{2i-1,2i} + \sum_{i=1}^{n-1} B_{2i,2i+1} + L_{1,2} + R_{2n-1,2n}$, where A, B, L, R are 2-local projectors, such that the nullspace of H is $\mathcal{N}(H) = \operatorname{Span}\{|\psi\rangle\}$ and $|\psi\rangle$ is entangled across all cuts.*

▶ Remark 5.1. Observe that besides the left and right boundary (projectors L, R), all (4×2)-projectors have the same form A and all (2×4)-projectors have the same form B. Therefore, H may be considered translation invariant, in a weaker sense. The Hamiltonian with a fully entangled ground state on a line of qutrits [9] also has additional projectors on the boundary.

We begin by explicitly writing the unique ground state of this Hamiltonian on line of 6 particles. The dimensions of these particles are $(4, 2, 4, 2, 4, 2)$, although for the first and second to last particle a qutrit would suffice as the the $|0\rangle/|3\rangle$ dimension is not used.

$$\begin{aligned}
&|10\,|\,00\,|\,00\rangle && +|20\,|\,20\,|\,10\rangle \\
+&|21\,|\,00\,|\,00\rangle && +|31\,|\,20\,|\,10\rangle \\
+&|20\,|\,10\,|\,00\rangle && +|20\,|\,31\,|\,10\rangle \\
+&|31\,|\,10\,|\,00\rangle && +|31\,|\,31\,|\,10\rangle \\
+&|20\,|\,21\,|\,00\rangle && +|20\,|\,20\,|\,21\rangle \\
+&|31\,|\,21\,|\,00\rangle && +|31\,|\,20\,|\,21\rangle \\
& && +|20\,|\,31\,|\,21\rangle \\
& && +|31\,|\,31\,|\,21\rangle
\end{aligned} \tag{22}$$

While this state might look complex at a first glance, it can easily be understood semantically. We again think of particles as bins holding balls, but now there is only one kind of ball. A state $|c\rangle$ means that the bin holds c balls. Thus, a qu-d-it can hold at most $d-1$ balls and we have *large* bins of capacity 3 , and *small* bins of capacity 1. Initially, only the first bin contains a ball (first state in the superposition). Then we evolve according to the following rules (also in the reverse):

- If a large bin is not empty and the bin to the right is empty, we can add a ball to both bins ($|c,0\rangle \leftrightarrow |c+1,1\rangle$ for $c \in [1, d-2]$).
- If a small bin contains a ball and the large bin to the right is empty, we can move the ball from the small to the large bin ($|1,0\rangle \leftrightarrow |0,1\rangle$).

We can simplify the transitions by factoring $|*\rangle := \sqrt{1/2}(|20\rangle + |31\rangle)$. To obtain a uniform superposition, we also need to change the amplitudes of the transitions. On $2n$ particles, we obtain the following state.

$$|\phi\rangle := \sum_{i=1}^{n} \begin{pmatrix} |*\rangle^{\otimes i-1} \otimes |10\rangle \otimes |00\rangle^{\otimes n-i} \\ +|*\rangle^{\otimes i-1} \otimes |21\rangle \otimes |00\rangle^{\otimes n-i} \end{pmatrix} \tag{23}$$

Note, $|\phi\rangle$ has a quite similar structure to common clock constructions and can be extended to $2n$ particles. We will show that it is the unique ground state of the following Hamiltonian:

$$H = |0\rangle\langle 0|_1 + |3\rangle\langle 3|_{2n-1} + \sum_{i=1}^{n} A_{2i-1,2i} + \sum_{i=1}^{n-1} B_{2i,2i+1} \tag{24}$$

$$A = \big(|10\rangle - |21\rangle\big)\big(\langle 10| - \langle 21|\big) + \big(|20\rangle - |31\rangle\big)\big(\langle 20| - \langle 31|\big) + |30\rangle\langle 30| + |11\rangle\langle 11| \tag{25}$$

$$B = \big(|10\rangle - \sqrt{2}|01\rangle\big)\big(\langle 10| - \sqrt{2}\langle 01|\big) \tag{26}$$

▶ **Lemma 5.2.** *$|\phi\rangle$ is fully entangled, i.e. $|\phi\rangle \neq |\phi_1\rangle_A \otimes |\phi_2\rangle_B$ for all cuts A/B and $|\phi_1\rangle, |\phi_2\rangle$.*

Proof. Consider the random experiment of measuring $|\psi\rangle$ in standard basis. The outcome is denoted by the string x. Let $S \subset [2n]$ be a subset of particles. If $|\phi\rangle = |\phi_S\rangle|\phi_{\bar{S}}\rangle$, then the random variables x_S and $x_{\bar{S}}$ (substrings of x on particles S and $\bar{S} = [2n] \setminus S$, respectively) are independent. In the following, we argue that this is not the case.

Note for an odd i, $P(x_i = 3, x_{i+1} = 0) = 0$, but $P(x_i = 3)P(x_{i+1} = 0) > 0$. Hence, if there exists odd i such that $|\{i, i+1\} \cap S| = 1$, then $x_S, x_{\bar{S}}$ are not independent.

Otherwise, there exists an odd i such that $|\{i, i+2\} \cap S| = 1$. Again, $x_S, x_{\bar{S}}$ are not independent as $P(x_i = 0, x_{i+2} = 1) = 0$, but $P(x_i = 0)P(x_{i+2} = 1) > 0$. ◀

▶ **Lemma 5.3.** *$|\phi\rangle$ is the unique ground state of H.*

Proof. It is easy to verify $H|\phi\rangle = 0$. Now, assume $H|\psi\rangle = 0$. If there exists a standard basis vector $|x\rangle$ such that $\langle x|\psi\rangle \neq 0$, it corresponds to an illegal state of the ball game (terms of (23) are the legal states). Observe that the transition terms of A and B directly correspond to the allowed moves in the ball game. The illegal states that are not caught directly, are `|10|` or `|21|` not followed by all zeroes. By applying the transition rules, we can go to `|11|`, which is caught by A. Hence, $|\psi\rangle$ and $|\phi\rangle$ overlap the same standard basis vectors. The transition terms ensure the weights are such that $|\phi\rangle$ can be written as in (5.2). ◀

▶ Remark 5.4. $|\phi\rangle$ has only constant entanglement entropy, whereas [9] achieves $\Omega(\log n)$. So it remains an open problem whether logarithmic entanglement entropy can be achieved on the $(2{\times}4)$-line.

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