On the Expansion of Monadic Second-Order Logic with Cantor-Bendixson Rank and Order Type Predicates

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Abstract

In this work, we consider two extensions of monadic second-order logic, and study in what cases the classical decidability results are preserved.

The first extension, MSO[CBrank_β], is MSO (over the signature of the binary tree) augmented with the extra ability to express that the subtree over a set X has Cantor-Bendixson rank β, for some fixed countable ordinal β. We show that this extension is decidable over the binary tree if and only if β is finite, which means that it is decidable if and only if it is equivalent in expressiveness to MSO.

The second extension, MSO[otp_α], is MSO (over the signature of order) augmented with the extra ability to express that the suborder induced by a set X has order type α for some fixed countable ordinal α. We show that this extension is decidable over countable ordinals if and only if $\alpha < \omega^\omega$, which means that it is decidable if and only if it is equivalent in expressiveness to MSO.

The first result can be established as a consequence of the second. The second result relies on the undecidability results of the logic BMSO (itself relying on the undecidability of MSO+U) in the case of ω^β for β a limit ordinal, and on entirely new techniques when β is a successor ordinal. We also have some partial extensions of the second result to some uncountable cases.

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1 Introduction

This work studies extensions of monadic second-order logic for which the decidability status was not known before.

Monadic second-order logic (MSO) is the extension of first-order logic with set quantifiers. It plays a key role in the context of automata and verification. The central decidability results in this context are (1) the seminal paper of Büchi [10] that proves the decidability of MSO$(\mathbb{N}, <)$ using automata, (2) the breakthrough [24] where Rabin establishes that MSO is decidable on infinite trees of height ω, again using automata, and from which the decidability of MSO over countable chains can be deduced, and finally (3) the introduction by Shelah [31] of model-theoretic techniques for showing the decidability of the MSO-theory of countable linear orderings, traditionally referred to as the "composition method."

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These historical contributions show the extremely strong decidability properties that enjoy MSO. Their extension beyond MSO and beyond linear orderings and trees has been and still is a strong motivation for new research in the field (see more in Section 1.3). The present work pursues this quest by considering two natural extensions of monadic second-order logic (MSO for short).

1.1 Contributions

Our two contributions, Theorems 1 and 2 share a similar form. We consider in both cases a natural notion of measure of the "complexity" of a set, namely, the Cantor-Bendixson rank in the theory of infinite trees for the first one, and the order type in the theory of well-founded linear orders for the second. We then study the decidability status of MSO extended with a construction of the form:

"the complexity of the set X is α"

for a unique fixed value of α. Our results show that either this new construction was already expressible in MSO, or decidability is lost.

First extension: Cantor-Bendixson rank of trees

We consider here an extension of the monadic theory of trees.

The Cantor-Bendixson rank of a tree is an ordinal that measures its branching complexity (see Section 3 for more on trees and the Cantor-Bendixson rank), and is undefined if the tree contains an induced full binary tree. A subset X of a tree is downward closed if whenever $v \in X$, then all the nodes on the path from the root of T to v are in X. Given an ordinal α, for X some downward closed subset of an infinite tree, let $\mathsf{CBrank}_\alpha(X)$ express that "the tree restricted to universe X has Cantor-Bendixson rank α". We denote by $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ monadic second-order logic extended with the new predicate $\mathsf{CBrank}_\alpha(-)$. We prove:

▶ **Theorem 1.** *For all countable ordinals α, the following properties are equivalent:*

- *the $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$-theory of the full binary tree is decidable,*
- *α is finite,*
- *CBrank_α is MSO-definable.*

It can be summarised as the impossibility to extend the main theorem of Rabin of decidability of MSO over the full binary tree with the ability to express the Cantor-Bendixson rank of trees.

Second extension: the order type of an ordinal

We consider here an extension of the monadic theory of linear orders.

Büchi [11] proved a kind of "a small model property": if an MSO-formula is satisfiable in any countable ordinal, then it is satisfiable in an ordinal $< \omega^\omega$. Hence, ω^ω is MSO-undefinable. On the other hand, for every ordinal $\alpha < \omega^\omega$ one can express "the order type of X is α." A natural question follows: is the extension of MSO by the ability to express "the order type of X is ω^ω" still decidable? We provide a negative answer to this question.

Given an ordinal α, we consider the predicate $\mathsf{otp}_\alpha(X)$ which holds if the order type of the set X is the ordinal α, i.e., if the linear order restricted to universe X is isomorphic to α. We denote by $\mathsf{MSO}[\mathsf{otp}_\alpha]$ the monadic second-order logic of order extended with the new predicate $\mathsf{otp}_\alpha(-)$. Our main result reads as follows.

▶ **Theorem 2.** *For all countable ordinals α (and more generally for all $\alpha \leq \omega_1^{\omega_1}$, where ω_1 is the first uncountavle ordinal), the following properties are equivalent:*
- *the* MSO[otp$_\alpha$]*-theory of α is decidable,*
- $\alpha < \omega^\omega$,
- otp$_\alpha(-)$ *is* MSO-*definable.*

Let us recall that the MSO-theory is known to be decidable for the class of ordinals smaller than ω_2, as well as separately for each ordinal smaller than ω_2 (see [12] for the countable case, [31] for ordinals up to ω_2, and [21] for showing that this question is independent of ZFC at ω_2, where ω_2 is the first ordinal of the cardinality greater than the cardinality of ω_1). It is clear that if α is an MSO-definable ordinal, then MSO[otp$_\alpha$] and MSO are effectively expressively equivalent, and thus MSO[otp$_\alpha$] is decidable. Our result shows that if α is not MSO-definable and smaller than or equals to $\omega_1^{\omega_1}$, then, the logic is strictly more expressive than MSO; however, decidability is lost.

Note that we do not rule out the possibility that there exists some uncountable ordinal α larger or equal to $\omega_1^{\omega_1}$ such that the MSO[otp$_\alpha$]-theory of α is still decidable.

1.2 Overview of the proofs

For both theorems, the difficult part is to prove the undecidability of the theory.

The undecidability part of the first theorem, Theorem 1, is obtained from Theorem 2. The main observation is that when the lexicographically order over the leaves of a binary tree T of Cantor-Bendixson rank α, is an ordinal, then its order type is in the interval $[\omega^\alpha, \omega^{\alpha+1})$. This is the crux of the reduction of the undecidability.

The undecidability part of the second theorem, Theorem 2, amounts to prove the undecidablity of the MSO[otp$_\alpha$]-theory for all non MSO-definable ordinals α up to $\omega_1^{\omega_1}$. In fact, it boils down to treat three different cases.

- In Theorem 24, we prove that the MSO[otp$_{\omega^\beta}$]-theory of ω^β is undecidable for all countable limit ordinals β (or more generally for β of cofinality ω). This covers, in particular, the important case of ω^ω which is the first non MSO-definable ordinal. This proof relies on a simple reduction of the BMSO-theory of ω, which is known to be undecidable (see Theorem 9).
- In Theorem 25, we establish that the MSO[otp$_{\omega^\beta}$]-theory of ω^β can be reduced to the MSO[otp$_{\omega^{\beta+1}}$]-theory of $\omega^{\beta+1}$. This is the most interesting part. It involves using the construct otp$_{\omega^{\beta+1}}$ over an input that has been decomposed into ω-blocks in such a way that in almost all the blocks it faithfully simulates otp$_{\omega^\beta}$.
- The two above results treat the case of all ordinals of the form ω^β that are smaller than $\omega_1^{\omega_1}$ and are not MSO-definable. But some ordinals are not of the form ω^β, such as, for instance, $\omega^\omega + 1$. Extending the result to these cases requires some extra work, but can be achieved without difficulty.

Another argument in the proof is a characterization of MSO-definable ordinals. This is well known in the countable case: a countable ordinal α is MSO-definable if and only if it is smaller than ω^ω. For proving Theorem 2, there is also a need to characterize the MSO-definable ordinals smaller than ω_2. We provide such a result in Theorem 45. This statement is not deep but, as far as we know, had not been mentioned before.

1.3 Historical Background

It was known in the 50s from Robinson that the extensions of MSO with a plus function, MSO$(\mathbb{N}, +)$, or even the doubling function MSO$(\mathbb{N}, <, x \mapsto 2x)$ were undecidable [27]. Elgot and Rabin studied in [17] the MSO theory of structures of the form $(\mathbb{N}, <, P)$, where P is some unary predicate. They give a sufficient condition on P which ensures decidability of the MSO theory of $(\mathbb{N}, <, P)$. In particular, it holds when P denotes the set of factorials, or the set of powers of some integer. The frontier between decidability and undecidability of related theories was explored in numerous later papers [14, 18, 30, 29, 26, 25, 33, 34, 1].

The Büchi decidability theorem result (and the automata method) was extended to the MSO theory of any countable ordinal [11], to ω_1 - the first uncountable ordinal, and to any ordinal less than ω_2 - the first ordinal of the cardinality greater than ω_1, [13]. Gurevich, Magidor and Shelah [21] proved that the decidability status of the MSO theory of ω_2 depends on set-theoretical assumptions. What can be said about MSO theories for linear orderings beyond ordinals? Using automata, Rabin [24] proved decidability of the MSO theory of the binary tree, from which he deduces decidability of the MSO theory of $\mathbb{Q}$, which in turn implies decidability of the MSO theory of the class of countable linear orderings. Shelah [31] improved model-theoretical techniques yielding new decidability proofs over linear orderings, and proved that the MSO theory of the real line $(\mathbb{R}, <)$ is undecidable. The frontier between decidable and undecidable cases was specified in later papers by Gurevich and Shelah [20, 22, 23]); we refer the reader to the survey [19].

A logic that was much studied in recent years, introduced in [4], is MSO+U. The logic MSO+U extends MSO with a new quantifier-like construct $\mathsf{U}X.\varphi(X)$ expressing that there are sets of arbitrary large cardinality for which $\varphi(X)$ holds. Some non-trivial fragments of MSO+U are known to be decidable over ω (if this new construct is not allowed to appear negatively inside iteslf). [5]. It took more than ten years before it was shown undecidable over ω [8]. Works concerned with weak variants of this logic have also been pursued, yielding decidability results, such as in [9], but these cannot be considered as syntactic extensions of MSO.

The logic BMSO is a logic expressing properties of infinite sequences of numbers. It was designed in order to retain the quantitative aspects of MSO+U, while removing the ability to measure the cardinality of sets [2]. It turns out that its theory is intereducible to the one of MSO+U (see [2]), and thus it is also undecidable by [8]. The decidability of some important fragments of this logic still remain open. The undecidability result concerning MSO+U has been extended in [6], and then eventually, the existence of (almost) any significative extension of MSO that would retain decidability over ω has been ruled out in [7]: as soon as any non-regular property is expressible (with some mild closure assumption), then theory is undecidable. All these undecidability results boil down to reductions to the work [8].

The logic *cost-MSO* is another logic inspired by MSO+U, this time exhibiting the ability to express bound properties on the cardinality of sets, but removing the ability to quantify express asymptotic properties on these quantities. This logic is known to be decidable over finite words [15] and trees [16] over infinite words and partially over infinite trees [3]. The decidability status of cost-MSO over the full binary tree is a difficult open problem in the area.

1.4 Structure of the paper

Some definitions and results concerning MSO and BMSO are recalled in Section 2. Section 3 establishes our result over infinite trees, namely Theorem 1. Section 4 presents the proof of Theorem 24 stating that the $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-theory of ω^β is undecidable for all countable limit ordinals β (or more generally β of cofinality ω). In Section 5, we prove that the $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-theory of ω^β can be reduced to the $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$ of $\omega^{\beta+1}$ (Theorem 25). In Section 6, we combine the results of the two previous sections for proving our main result, Theorem 2, for all countable ordinals. Section 7 concludes the paper.

For space considerations all the results beyond the countable case, i.e., for ordinals up to $\omega_1^{\omega_1}$, do not appear in the main body of the submission, and are found in Appendix B.

2 Preliminaries

In this section we recall standard definitions and notions about ordinals (Section 2.1), monadic second-order logic (Section 2.2) and definability (Section 2.3). In Section 2.4, we introduce BMSO, and the undecidability Theorem 9 for it, which is crucial in our proof.

2.1 Ordinals

We shall use the standard terminology over ordinals. Here, an *ordinal* is seen, up to isomorphism, as a set equipped with a well-founded total order $<$. We use the classical notations over ordinals (order, sum, product, exponentiation). A *limit ordinal* is a non-empty ordinal that has no maximal element. A *successor ordinal* is an ordinal that has a maximum.

Given a subset X of an ordinal α, we denote $\alpha|_X$ its restriction to X. The *order type* of X is the ordinal $\alpha|_X$. We shall denote $[x, y)$ the set of $\{z \in \alpha : x \leqslant z < y\}$, and $[x, \infty)$ for the set $\{z \in \alpha : x \leqslant z\}$. An interval of the form $[x, \infty)$ is called a *final non-empty segment*. Given an ordinal α, a set X is *cofinal* (in α) if for all $x \in \alpha$, there is $y \in X$ with $y \geqslant x$. The *cofinality of α* is the least order type of X for X cofinal subset of α. Let us recall that all countable limit ordinals have cofinality ω.

2.2 Monadic second-order logic of order

We use standard notations and terminology about monadic second-order logic of order.

The *monadic second-order logic of order* (or *monadic logic of order* or simply *monadic logic*, abbreviated as MSO) is the extension of first-order logic over the signature $\{<\}$, where $<$ is a binary relational symbol interpreted as a total order, with (a) *monadic (second-order) variables* interpreted as subsets of the universe (usually denoted by uppercase letters X, Y, Z), (b) existential and universal quantifiers over for monadic variables $\exists X.\psi$ and $\forall X.\psi$, and (c) atomic formulas $y \in X$ expressing that y belongs to X.

In this paper, we shall always interpret this logic over partial orders or ordinals. Given a structure $\mathcal{M} := (M, <)$, which is a model over the signature $\{<\}$, a monadic formula $\varphi(X_1, \ldots, X_k, y_1, \ldots, y_\ell)$, sets $U_1, \ldots, U_k \subseteq M$, and elements $v_1, \ldots, v_\ell \in M$, we denote the fact that the formula holds on the model $\mathcal{M}$ with valuations U_i for X_i, and v_j for y_j as

$$\mathcal{M} \models \varphi(U_1, \ldots, U_k, v_1, \ldots, v_\ell) \ .$$

We shall use the classical technique of *relativization* of formulas. The next lemma is obtained easily by a syntactic transformation of the formula.

▶ **Lemma 3** (Relativization). *Let $\varphi(Y_1, \ldots, Y_l)$ be a formula, U a variable not appearing in φ. We can compute a formula $\varphi^U(Y_1, \ldots, Y_l, U)$ such that for every structure $\mathcal{M}$ and every non-empty $D \subseteq M$ and every l-tuple $\bar{P}$ of subsets of D:*

$$\mathcal{M} \models \varphi^U(\bar{P}, D) \text{ if and only if } \mathcal{M}_{\restriction D} \models \varphi(\bar{P}),$$

where $\mathcal{M}|_D$ is the substructure of $\mathcal{M}$ over D.

When this is the case, we say that φ holds in $(\mathcal{M}, \bar{P})$ *relativized* to D.

To ease the notation, we shall use some shorthands such as overlined variables $\bar{X}, \bar{Y}$ to denote tuples of monadic variables. We allow ourselves to write $X \subseteq Y$ to denote the inclusion of sets, and we use more generally any abbreviation if it is clear that it can be translated into MSO syntax. For instance, we shall use formulas such as ("X is cofinal" $\wedge$ "X has order type ω").

The main results concerning the MSO-theory of ordinals are the following:

▶ **Theorem 4** ([12]). *The MSO-theory of the class of countable ordinals is decidable. The MSO-theory of every countable ordinal is decidable.*

▶ **Theorem 5** ([31]). *The MSO-theory of the class of ordinals smaller than ω_2 is decidable. The MSO-theory of every ordinal smaller than ω_2 is decidable.*

▶ **Theorem 6** ([21]). *The MSO-theory of ω_2 is independent of ZFC.*

Given an ordinal β, we consider the extension of monadic logic in which the extra atomic formula $\mathsf{otp}_\beta(X)$ for X a monadic variable can be used, and is interpreted in an ordinal α as "the order type of X is β". It is denoted $\mathsf{MSO}[\mathsf{otp}_\beta]$.

2.3 Definability

We say that a sentence ψ *defines the ordinal* α if α is the unique ordinal such that $\alpha \models \psi$. An ordinal α is *definable in logic* Ł, or simply (*Ł-definable*) if there is a sentence of Ł that defines α. The following important lemma is well known.

▶ **Lemma 7.** *For all countable ordinals α, α is MSO-definable if and only if $\alpha < \omega^\omega$.*

Definability of ordinals below ω^ω is fairly straightforward, by an inductive definition. The undefinability above ω^ω, follows from the small model property for MSO over the countable ordinals.

We say that a formula $\psi(x)$ *defines α inside the ordinal β* if there is a unique $b \in \beta$ such that $\beta \models \psi(b)$ and the $[0, b)$ has order type α. For a logic Ł, *α is Ł-definable inside β* if there is a formula $\psi(x) \in$ Ł that defines α in β.

It is clear that if α is MSO or $\mathsf{MSO}[\mathsf{otp}_\gamma]$-definable by some sentence, then, by relativization of the sentence to $[0, b)$ it is definable inside β with the same logic for every $\beta > \alpha$. However, the other direction fails, as witnessed by the next lemma, to be put in contrast with Theorem 7.

▶ **Lemma 8.** *ω^ω is MSO-definable inside $\beta := \omega^\omega + \gamma$ for every $0 < \gamma < \omega^\omega$.*

Proof. By Theorem 7, there exists an MSO-sentence ψ_γ that defines γ. The formula that says that x is the minimal element such that ψ_γ holds when relativized to $[x, \infty)$. ◀

2.4 BMSO

The BMSO-logic is an extension of MSO that can express the existence of a bound on numerical quantity. Formally, the syntax of BMSO is the same as the one of MSO, extended with a new construct $B(X)$, for X a monadic variable. These formulas are interpreted on *ω-sequences of natural numbers*, that we see as the ordinal ω extended with a map f from ω to $\mathbb{N}$. The construct $B(X)$ is interpreted as "there exists $n \in \mathbb{N}$ such that $f(x) \leqslant n$ for all $x \in X$".

For instance, the formula $u \models \forall X. B(X)$ for an ω-sequence of natural numbers u if and only if u is bounded. The formula $\forall X.(\forall x \in X. \exists y \in X.(x < y)) \to \neg B(X)$ expresses the non-existence of an infinite set which is bounded: in other words, it expresses that f tends to infinity.

▶ **Theorem 9** (consequence of [2, 8]). *Satisfiability of* BMSO *is undecidable over* *ω-sequences of natural numbers.*

Proof. Theorem 2 in [2] states that BMSO is equivalent to another logic, AMSO. Theorem 13 states that the satisfiability of AMSO is equivalent to the one of MSO+U. Finally, Theorem 1.1 in [8] establishes the undecidability of MSO+U. ◀

3 The tree case

In this section we derive Theorem 1 from Theorem 2. This section is organized as follows. First, we recall some standard definitions and results about trees. Then, we recall a definition of Cantor-Bendixson rank of trees and state some well-known facts. Finally, we prove Theorem 1.

3.1 Trees

A *tree* $(T,<)$ is a structure over a signature with a unique a binary relation $<$ such that (1) there is a minimal element (called the *root*), and (2) for every $b \in T$ the set $\{a \mid a < b\}$ is finite and linearly ordered by $<$. Elements of the tree are called *nodes*. A node u is *parent* of a node v (and v is a *child* of u) if $u < v$ and there is no w such that $u < w < v$. A tree is *binary* if every node has at most two children. Nodes u and v are *incomparable* if neither $u \leqslant v$ nor $v \leqslant u$. An *antichain* is a subset of a tree such that all pairs of distinct nodes are incomparable. For a subset A of a tree we denote by $A{\downarrow}$ the downward closure of A, i.e., the set $\{b \mid \exists a \in A (b < a)\}$. For a node u, denote $T^{u\uparrow}$ the tree T restricted to nodes that are above or equal to u. $T^{u\uparrow}$ is itself a tree, and is called the *subtree at u*. Note that if T is a binary tree, then $T^{u\uparrow}$ also is. A tree is *regular* if it has finitely-many subtrees up to isomorphism.

The *full binary tree* is a tree for which (a) there is a partition of the children into *left* and *right children* and (b) all nodes have exactly two children, one being left, and the other right. The full binary tree is considered as a structure for the signature $\{<, \mathit{Left}(), \mathit{Right}()\}$. The *standard representation* of the full binary tree has as the domain the finite strings over $\{L, R\}$; the relation $<$ is interpreted as the prefix relation, and a node is left (respectively, right) if its last letter is L (respectively, R).

The major decidability result is *Rabin's theorem* [24]:

▶ **Theorem 10.** *The* MSO*-theory of the full binary tree is decidable.*

We shall also use the so-called *Rabin's Basis Theorem* [24]:

▶ **Theorem 11.** *If an* MSO*-sentence has a [binary] tree model, it has a [binary] regular tree model.*

We use $<_{\text{lex}}$ for the lexicographical (linear) order on the nodes of the full binary tree. It is definable in the standard way. Rabin proved [24] that there is a definable antichain Q such that $(Q, <_{\text{lex}})$ is order isomorphic to the rationals. As a consequence, he obtained that the MSO-theory of the rationals is decidable. Moreover, since every countable ordinal is embedable into the rationals, and "$(A,<)$ is an ordinal" is MSO-definable, he derived that the MSO-theory of the class of countable ordinals is decidable.

We will use the fact that $<_{\text{lex}}$ is MSO-definable, "A is an antichain" and "$A{\downarrow}$" are MSO-definable.

3.2 Cantor-Bendixson rank

We are going to define the Cantor-Bendixson rank (or CB rank) of a binary tree. There are several equivalent definitions. The only properties of Cantor-Bendixson rank that we need are stated in Theorem 16 and Theorem 17. The reader might skip the definition and use these lemmas as black-boxes.

▶ **Definition 12** (Sum of trees). *Let $T_i = (|T_i|, <_i)$ for $i \in \{0, 1\}$ be trees over disjoint universes. their sum is the tree $T_0 + T_1$ with universe $|T_0| \cup |T_1|$ and its order relation defined as $n_1 \leqslant n_2$ if there is $i \in \{0, 1\}$ such that $n_1, n_2 \in |T_i|$ and $n_1 \leqslant_i n_2$, or n_1 is the root of T_0.*

The ω-sum of trees is defined as follows.

▶ **Definition 13** (ω-sum of trees). *Let $T_i = (|T_i|, <_i)$ for $i \in \omega$ be trees over disjoint universes. We define the tree*

$$\sum_{i\in\omega} T_i$$

as having universe $\bigcup_{i\in\omega} |T_i|$ and its order relation is defined as $n_1 \leqslant n_2$ if there is i such that $n_1, n_2 \in |T_i|$ and $n_1 \leqslant_i n_2$ or n_1 is the root of T_i and $n_2 \in |T_j|$ for $j \geqslant i$.

If the disjointness assumption does not hold, we replace T_i by disjoint isomorphic copies and proceed as above.

In order to simplify notations, we will consider only finitely branching trees. The sets of trees of Cantor-Bendixson rank $\leqslant \alpha$ can be defined by transfinite induction.

▶ **Definition 14** (*Cantor-Bendixson rank* of a tree). *Define two families of trees CBrank_α and CBrank^+_α, where α is a countable ordinal.*

1. *CBrank_0 contains only one element trees.*
2. *CBrank^+_α is the closure of $\cup_{\beta\leqslant\alpha}\mathsf{CBrank}_\beta$ under $+$, i.e. $T \in \mathsf{CBrank}^+_\alpha$ if $T = T_0 + T_1 + \cdots + T_k$ for $T_i \in \cup_{\beta\leqslant\alpha}\mathsf{CBrank}_\beta$.*
3. *$\mathsf{CBrank}_\alpha := \sum_{i\in\omega} T_i$, where $T_i \in \mathsf{CBrank}^+_\beta$ for $\beta < \alpha$.*

If there is no α such that $T \in \mathsf{CBrank}^+_\alpha$, then the Cantor-Bendixson rank is undefined; otherwise we set $\mathsf{CBrank}(T)$ of T to be $\inf\{\alpha \mid T \in \mathsf{CBrank}^+_\alpha\}$, and the tree is called tame.

It is well known that a tree T has a CB rank if there is no embedding of the full binary tree in T, equivalently, T has only countably many branches. We are not going to use this fact.

▶ **Example 15.** Let S_1 be the subtree of the full binary tree T_2 over R^*, S_2 the subtree of T_2 over R^*L^*. Let S_{2i} (respectively, S_{2i+1}) be the subtree of T_2 over $(R^*L^*)^i$ (respectively, over and $(R^*L^*)^iR^*$). Then $\mathsf{CBrank}(S_i) = i$ for $i \geqslant 1$. Let S'_1 be the subtree of T_2 over R^*L; then $\mathsf{CBrank}(S'_1) = 1$.

The following lemma is well-known and is easily proved by induction.

▶ **Lemma 16.** *For every $n \in \mathbb{N}$, the set of binary trees of Cantor-Bendixson rank n is definable, i.e., there is an MSO sentence ϕ_n such that $(T, <) \models \phi_n$ if $(T, <)$ is a binary tree of Cantor-Bendixson rank n.*

The next lemma is proved in the Appendix.

▶ **Lemma 17.** *Let X be an antichain in the full binary tree such that $(X, <_{\text{lex}})$ is isomorphic to an ordinal. Then, $(X\downarrow, <)$ has Cantor-Bendixson rank α if and only if the order type of $(X, <_{\text{lex}})$ belongs to $[\omega^\alpha, \omega^{\alpha+1})$.*

3.3 Theorem 2 implies Theorem 1

Now, relying on Lemma 17, we can express in $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ that the order type of $(X, <_{\text{lex}})$ is ω^α for an antichain X of the full binary tree. The conjunction $\varphi_{\omega^\alpha}(X)$ of (1)-(5) below expresses this property.

1. X is an antichain:

$$\forall y \in X \forall x \in X (y \leqslant x) \to (x = y)$$

2. $(X, <_{\text{lex}})$ is isomorphic to an ordinal, i.e., every non-empty subset has a minimum element:

$$\forall Y \subseteq X\big((Y \neq \emptyset) \to \exists y \in Y(\forall z \in Y(y \leqslant_{\text{lex}} z))\big)$$

3. The downward closure of X has the CB rank α:

$$\mathsf{CBrank}_\alpha(X\downarrow), \text{ and}$$

4. For every final non-empty segment Y of $(X, <_{\text{lex}})$ the CB rank of the downward closure of Y is α:

$$\forall y \in X\big(\mathsf{CBrank}_\alpha(\{z \in X \mid z \geqslant y\}\downarrow)\big).$$

5. For no proper prefix $(Y, <_{\text{lex}})$ of $(X, <_{\text{lex}})$ the CB rank of the downward closure of Y is α.

(1)-(5) are not $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ formulas; however, they can be easily translated into (less readable) $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ formulas.

Next, let ψ be an $\mathsf{MSO}[\mathsf{otp}_{\omega^\alpha}]$ sentence. Let ψ^X be the relativization of ψ to a fresh variable X. Let $\Psi^X \in \mathsf{MSO}[\mathsf{CBrank}_\alpha]$ be obtained from ψ^X when all the occurrences of $\mathsf{otp}_{\omega^\alpha}(Y)$ are replaced by $\varphi_{\omega^\alpha}(Y)$. Finally, let Ψ be $\exists X(\varphi_{\omega^\alpha}(X) \wedge \Psi^X)$. Then, $\omega^\alpha \models \psi$ if and only if Ψ holds in the full binary tree. Hence, we have:

▶ **Lemma 18.** *There is an algorithm that for every $\mathsf{MSO}[\mathsf{otp}_{\omega^\alpha}]$ sentence ψ constructs an $\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ sentence Ψ such that $\omega^\alpha \models \psi$ if and only if Ψ holds in the full binary tree.*

Moreover, in Theorem 18, the algorithm treats α symbolically, i.e., if φ is obtained from ψ when α is replaced by β, then the corresponding translation of φ replaces in Ψ everywhere α by β.

As a corollary of Theorem 18, Theorem 7 and Theorem 2, we obtain:

▶ **Corollary 19.** *$\mathsf{MSO}[\mathsf{CBrank}_\alpha]$ is undecidable for every $\alpha \geqslant \omega$.*

And as a corollary of Rabin's theorem, Theorem 10, we obtain:

▶ **Corollary 20.** *For all $\alpha \geqslant \omega$, the property "the CB rank of a binary tree is α" is not MSO-definable.*

Theorem 1 follows from Theorem 10, Theorem 16, Theorem 19 and Theorem 20.

4 The ω^β case for β an ordinal of cofinality ω

The goal of this section is to establish Theorem 24, stating that the MSO[otp_α]-theory of α is undecidable for α of the form ω^β where β is a countable limit ordinal[1]. This covers in particular the case of ω^ω which is the first non-definable ordinal. Theorem 24 is obtained by reduction from the undecidability of BMSO (Theorem 9).

For the rest of the section, we fix an ordinal β which is limit of an ω-sequence $\widehat{\beta} := 0 = \beta_0 < \beta_1 < \cdots$. This is possible for all countable limit ordinals, and more generally for ordinals that have cofinality ω.

Given an ordinal $0 < \alpha < \omega^\beta$, it is said to be of *$\widehat{\beta}$-rank* k if $\omega^{\beta_k} \leqslant \alpha < \omega^{\beta_{k+1}}$. Let us denote $\widehat{\beta}-\mathrm{rank}(\alpha)$ the $\widehat{\beta}$-rank of α. Recall that $\alpha|_X$ is the substructure of α over X. If X is some subset of an ordinal α, we also denote $\widehat{\beta}-\mathrm{rank}(X)$ for the $\widehat{\beta}-\mathrm{rank}(\alpha|_X)$.

The key argument used in this section is that we can relate the order type of some infinite sums of ordinals to the boundedness of sequences of $\widehat{\beta}$-ranks. This is formalized in the following lemma.

▶ **Lemma 21.** *Given an ω-sequence $(\alpha_i)_{i\in\omega}$ of ordinals smaller than ω^β, then the following properties are equivalent:*

- $\sum_{i\in\omega} \alpha_i = \omega^\beta$,
- *The ω-sequence of natural numbers defined for all $i \in \omega$ as $u_i := \widehat{\beta}-\mathrm{rank}(\alpha_i)$ is unbounded.*

Proof. Assume the ranks of the α_i are unbounded, then there exists an increasing ω-sequence $0 = i_0 < i_1 < \ldots$ such that $j \leqslant \widehat{\beta}-\mathrm{rank}(\alpha_{i_j})$ for all j. As a consequence, we have $\omega^{\beta_j} \leqslant \alpha_{i_j} \leqslant \sum_{i=i_j}^{i_{j+1}-1} \alpha_i$ for all j. We obtain

$$\omega^\beta = \sum_{j\in\omega} \omega^{\beta_j} \leqslant \sum_{j\in\omega} \sum_{i=i_j}^{i_{j+1}-1} \alpha_i = \sum_{i\in\omega} \alpha_i \ .$$

Conversely, assume the $\widehat{\beta}$-ranks of the α_i's would be bounded by some N; this means that $\alpha_i \leq \omega^{\beta_{N+1}}$ for all i. We get $\sum_{i\in\omega} \alpha_i \leqslant \omega^{\beta_{N+1}} \times \omega < \omega^\beta$. ◀

Let $S \subseteq \omega^\beta$ be of order type ω. This means that there exists an increasing ω-sequence $s_0 < s_1 < \cdots$ with $S = \{s_i : i \in \omega\}$. We abbreviate it as $S = \{s_0 < s_1 < \cdots\}$. Given a set $S = \{s_0 < s_1 < \cdots\} \subseteq \omega^\beta$, it is said to *encode the sequence* $u \in \mathbb{N}^\omega$ defined as $u(i) = \widehat{\beta}-\mathrm{rank}([s_i, s_{i+1}))$ for all $i \in \omega$. Finally, given a set $X \subseteq \omega$, the *S-code of X*, written $\overline{X}^S$, is defined as

$$\overline{X}^S := \bigcup_{i\in X} [s_i, s_{i+1}) \ .$$

▶ **Fact 22.** *The key facts concerning these definitions are the following:*

1. *All ω-sequences of natural numbers $u \in \mathbb{N}^\omega$ are encoded by some ω-sequence $S = \{s_0 < s_1 < \cdots\}$. Take, for instance, $s_i = \omega^{\beta_{u(0)}} + \omega^{\beta_{u(1)}} + \cdots + \omega^{\beta_{u(i-1)}}$ for all $i \in \omega$.*
2. *Conversely, every $S \subseteq \omega^\beta$ of order type ω encodes a sequence $u \in \mathbb{N}^\omega$; namely, the one in which $u(i)$ is the β-rank of $[s_i, s_{i+1})$ for all $i \in \omega$.*
3. *For sets S and X, "S is of order type ω", and "X is an S-coded set", are properties definable in first-order logic (X and S are seen as unary predicates).*
4. *If S encodes u, then $u \models \neg B(X)$ if and only if $\omega^\beta \models \mathsf{otp}_{\omega^\beta}(\overline{X}^S)$. This is a direct consequence of Theorem 21 applied to the sequence of α_i, where $X = \{x_0 < x_1 < \cdots\}$ and α_i is the order type of $[s_{x_i}, s_{x_i+1})$.*

[1] This works in the more general case of β being a limit ordinal of cofinality ω.

▶ **Lemma 23.** *Given a formula $\varphi(X_1,\dots,X_k,x_1,\dots,x_l)$ of BMSO, there exists effectively a formula $\varphi^*(S,X_1,\dots X_k,x_1,\dots,x_l)$ of $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$ such that whenever S encodes u, $A_i\subseteq\omega$, and $a_j\in\omega$,*

$$u\models\varphi(A_1,\dots,A_k,a_1,\dots,a_\ell)$$

if and only if

$$\omega^\beta\models\varphi^*(S,\overline{A_1}^S,\dots,\overline{A_k}^S,s_{a_1},\dots,s_{a_\ell})\ .$$

Proof. The translation is defined by structural induction, as in the following table:

$$(x<y)^* := x<y\ ,\qquad (x\in X)^* := x\in X,$$
$$(B(X))^* := \neg\mathsf{otp}_{\omega^\beta}(X)\ ,\qquad (\varphi\wedge\psi)^* := \varphi^*\wedge\psi^*\ ,$$
$$(\forall x\,\psi)^* := \forall x\,(x\in S\to\psi^*)\ ,\qquad (\neg\varphi)^* := \neg\varphi^*\ ,$$
$$(\forall X\psi)^* := \forall X(\text{“}X\text{ is an }S\text{-coded set”}\to\psi^*)\ .$$

The conclusion of the lemma is obtained along the same induction, relying on Theorem 22. ◀

▶ **Corollary 24.** *For all countable limit ordinals β (and more generally for all ordinals of cofinality ω), the $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-theory of ω^β is undecidable.*

Proof. Consider a BMSO sentence φ, and, using the notations in Theorem 23, and the above facts, construct the sentence

$$\psi\quad:=\quad\exists S\ (\text{“}S\text{ has order type }\omega\text{”}\quad\wedge\quad\varphi^*(S))\ .$$

Then, φ has a sequence $u\in\mathbb{N}^\omega$ which models it if and only if ω^β models ψ. Indeed, if $u\models\varphi$, one can take some $S\subseteq\omega^\beta$ encoding u. It is of order type ω by definition, and by Theorem 23, $\omega^\omega\models\varphi^*(S)$. Conversely, if some S is the witness that $\omega^\omega\models\psi$, then S is of order type ω. Thus S encodes some sequence $u\in\mathbb{N}^\omega$, and since $\omega^\beta\models\varphi^*(S)$, by Theorem 23, $u\models\varphi$. In combination with Theorem 9, we get that the $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-theory of ω^β is undecidable. ◀

5 Reduction of $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$ to $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$

We have shown in the previous section the undecidability of $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$ for β an ordinal of cofinality ω. In this section, we show that if $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$ is decidable for some β, then the same goes for $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$. This case requires more work.

We aim at proving the following:

▶ **Lemma 25.** *For all ordinals β, the $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-theory of ω^β is reducible to the the $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$-theory of $\omega^{\beta+1}$.*

In other words, given an $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-sentence φ, our goal is to construct an $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$-sentence ψ such that

$$\omega^\beta\models\varphi\qquad\text{if and only if}\qquad\omega^{\beta+1}\models\psi\ .$$

The principle of this construction is that formula ψ will guess a decomposition of $\omega^{\beta+1}$ into ω intervals, such that almost all of them have order type ω^β. The intuition is then that formula ψ will “simulate” φ independently in each of these blocks. The construction is done in such a way that this “simulation” is “faithful” on almost all the blocks. There, we say that a unary property P holds for *almost all* elements of a set D if P holds on all, but finitely many elements of D.

The first key ingredient for achieving this is Theorem 27. It provides an $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$-formula that allows to chop $\omega^{\beta+1}$ into ω pieces while guaranteeing that almost all of them are of order type ω^β.

We first state some elementary facts about ordinals that will prove useful in the construction.

▶ **Fact 26.** *The following standard facts hold:*

1. *The order type of every final non-empty segment of ω^β has order type ω^β.*
2. *Let $(\alpha_i)_{i\in\omega}$ be an ω-sequences of ordinals smaller than $\omega^{\beta+1}$. Then $\sum_{i\in\omega}\alpha_i = \omega^{\beta+1}$ if and only if $\alpha_j \geqslant \omega^\beta$ for infinitely many j.*
3. *For all ordinals $\alpha < \omega^{\beta+1}$, either $\alpha = \omega^\beta \times k$ for some natural $k > 0$, or α has some final non-empty segment of order type smaller than ω^β.*

▶ **Lemma 27.** *There is effectively an $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$ formula $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(X)$ such that for all $S = \{s_0 < s_1 < \cdots\}$, $\omega^{\beta+1} \models \mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ if and only if the order type of $[s_i, s_{i+1})$ is ω^β for almost all $i \in \omega$.*

Proof. We proceed in three steps, in which we successively describe formulas that approximate each time better the expected behavior of $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}$.

Step 1. The $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$-formula $\mathrm{Split}^{\infty}_{\omega^\beta\times *}(S)$ expressing that

- $\bigcup_{i\in\omega}[a_i, s_{i+1})$ has order type $\omega^{\beta+1}$ for all $(a_i)_{i\in\omega}$ such that $a_i \in [s_i, s_{i+1})$ for all $i \in \omega$,

which holds if and only if there exist infinitely many indices $i \in \omega$ such that $[s_i, s_{i+1})$ is of the order type $\omega^\beta \times k$ for some $k \geqslant 1$.

Indeed, assume that $[s_i, s_{i+1})$ is of the form $\omega^\beta \times k$ with $k \geqslant 1$ for infinitely many $i \in \omega$, then by the first item of Theorem 26, $[a_i, s_{i+1})$ has an order type of the form $\omega^\beta \times \ell$ with $\ell \geqslant 1$ for these i's. Consequently, by the second item of Theorem 26, $\bigcup_{i\in\omega}[a_i, s_{i+1})$ has order type $\omega^{\beta+1}$.

Conversely, assume that $[s_i, s_{i+1})$ is not of the form $\omega^\beta \times k$ with $k \geqslant 1$ for almost all $i \in \omega$. This means by the third item of Theorem 26, that for these i's, there exists $a_i \in [s_i, s_{i+1})$ such that the order type of $[a_i, s_i)$ is smaller than ω^β. This can be completed, by choosing arbitrary a_i in the finitely many other segments, into an ω-sequence of a_i's as in the formula. This time using the second item of Theorem 26, we get that $\bigcup_{i\in\omega}[a_i, s_{i+1})$ has an order type smaller than $\omega^{\beta+1}$.

Step 2. In a second step, we claim that the formula $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}$ that expresses that

- for all infinite sets $S' \subseteq S$, $\mathrm{Split}^{\infty}_{\omega^\beta\times *}(S')$ holds

if and only if $[s_i, s_{i+1})$ is of the form $\omega^\beta \times k$ with $k \geqslant 1$ for almost all $i \in \omega$.

Indeed, if $[s_i, s_{i+1})$ is of the form $\omega^\beta \times k$ with $k \geqslant 1$ for almost all $i \in \omega$, this also holds for every infinite subsequence, and as a consequence, $\mathrm{Split}^{\infty}_{\omega^\beta\times *}(S')$ for all infinite $S' \subseteq S$.

Conversely, assume that $[s_i, s_{i+1})$ is not of the form $\omega^\beta \times k$ with $k \geqslant 1$ for infinitely many $i \in \omega$. In this case, it is possible to extract a subsequence $S' = \{s'_0 < s'_1 < \cdots\} \subseteq S$ such that the order type of $[s'_i, s'_{i+1})$ is not of the form $\omega^\beta \times k$ with $k \geqslant 1$, for all $i \in \omega$. According to the previous step, $\mathrm{Split}^{\infty}_{\omega^\beta\times *}(S')$ does not hold for this choice of S', and hence $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S)$ does not either.

Step 3. Finally, we can define the formula $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ that expresses that

- $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S)$ holds, and
- for all $S' \supseteq S$, if $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S')$ holds then $S' \setminus S$ is not cofinal.

We have to show that it fulfils the conclusion of the lemma, i.e., that $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ holds if and only if the order type of $[s_i, s_{i+1})$ is ω^β for almost all $i \in \omega$.

Indeed, assume that $[s_i, s_{i+1})$ has order type ω^β for almost all $i \in \omega$, and consider some $S' = \{s'_0 < s'_1 < \cdots\} \supseteq S$. If $S' \setminus S$ is cofinal, then for such sufficiently large j, we would have that $s_i < s'_j < s_{i+1}$ for some i such that $[s_i, s_{i+1})$ has order type ω^β. But in this case, $[s_i, s'_j)$ has an order type smaller than ω^β and since $s'_{j-1} \geqslant s_i$, the interval $[s'_{j-1}, s'_j)$ also does. Overall, we have constructed infinitely many intervals $[s'_{j-1}, s'_j)$ of order type smaller than ω^β, and thus $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S')$ does not hold. This is a contradiction, and hence $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ is satisfied.

Conversely, assume by contradiction that $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ holds, and that $[s_i, s_{i+1})$ has an order type different than ω^β for infinitely many $i \in \omega$. Since $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S)$ holds, this means that $[s_i, s_{i+1})$ has order type $\omega^\beta \times k$ with $k > 1$ for infinitely many $i \in \omega$. But each such interval $[s_i, s_{i+1})$ can be decomposed into $[s_i, s'_i)$ and $[s'_i, s_{i+1})$, both of them of order type $\omega^\beta \times k$ for some $k \geqslant 1$. Thus, the set S' obtained by adding to S all such elements s'_i would make $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta\times *}(S')$ satisfied. Since $S' \setminus S$ is infinite, this contradicts the fact that $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ holds. ◀

From now on, we assume that $S = \{s_0 < s_1 < \cdots\}$ is such that $\mathrm{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$ holds. Let α_i be the order type of $[s_i, s_{i+1})$ be the corresponding sequence of ordinals. For the sake of simplicity, we shall not mention the first order variables in formulas in the rest of the proof, since these can be seen as a special case of monadic variables that would be interpreted as singletons (something definable).

▶ **Lemma 28.** *There is an algorithm which given an* $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$*-formula*

$$\varphi(X_1, \ldots X_k),$$

constructs an $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$*-formula*

$$\varphi^*(S, F, X_1, \ldots X_k)$$

such that for all $A_1, \ldots, A_k \subseteq \omega^{\beta+1}$ *and all infinite* $F \subseteq \omega$*,*

$$\omega^{\beta+1} \models \varphi^*(S, \overline{F}^S, A_1, \ldots, A_k)$$

if and only if

$$\omega^{\beta+1}|_{[s_i, s_i+1)} \models \varphi(A_1 \cap [s_i, s_{i+1}), \ldots, A_k \cap [s_i, s_{i+1}))$$

for almost all $i \in F$*.*

Proof. We shall define the formula φ^* by structural induction on φ, and establish the conclusions of the lemma at the same time. The case of existential quantifier, the case of conjunction, and the case of MSO predicates are elementary. The crucial point is the negation.

For the sake of simplicity, we shall write $\omega^{\beta+1} \models^{S,F} \varphi(\bar{A})$ in order to express that $\omega^{\beta+1}|_{[s_i, s_i+1)} \models \varphi(A_1 \cap [s_i, s_{i+1}), \ldots, A_k \cap [s_i, s_{i+1}))$ for almost all i such that $s_i \in F$.

Case of a conjunction, i.e., $\varphi(\bar{X}) = (\varphi_1(\bar{X}) \wedge \varphi_2(\bar{X}))$. We define

$$\varphi^*(S, F, \bar{X}) := \varphi_1^*(S, F, \bar{X}) \wedge \varphi_2^*(S, F, \bar{X}) \ .$$

The induction hypothesis holds: indeed, $\omega^\beta \models \varphi^*(S, F, \bar{A})$ if and only if $\omega^\beta \models \varphi_1^*(S, F, \bar{A})$ and $\omega^\beta \models \varphi_2^*(S, F, \bar{A})$, if and only if (by induction hypothesis) $\omega^{\beta+1} \models^{S,F} \varphi_1(\bar{A})$ and $\omega^{\beta+1} \models^{S,F} \varphi_2(\bar{A})$, if and only if $\omega^{\beta+1} \models^{S,F} \varphi_1(\bar{A}) \wedge \varphi_2(\bar{A})$, if and only if $\omega^{\beta+1} \models^{S,F} \varphi(\bar{A})$.

Case of an existential set quantifier, i.e., $\varphi(\bar{X}) = \exists Y.\varphi_1(\bar{X}, Y)$. We define

$$\varphi^*(S, F, \bar{X}) := \exists Y.\ \varphi_1^*(S, F, \bar{X}, Y)\ .$$

Correctness is also straighforward.

Case of an MSO-formula $\varphi(\bar{X})$, and in particular of the atomic formulas $x \in Y$ and $x < y$ (recall that in this case, we see first-order variable as singleton sets). In this case, we simply set $\varphi^*(S, \overline{F}^S, \bar{X})$ to express $\omega^{\beta+1} \models^{S,F} \varphi(\bar{X})$. This is easily definable using relativization and the fact that "almost all" can be expressed in MSO.

Case of an order type predicate, i.e., $\varphi(\bar{X}) := \mathsf{otp}_{\omega^\beta}(X_m)$. For simplicity, we shall treat the case of $\varphi(\bar{X}) := \neg\mathsf{otp}_{\omega^\beta}(X_m)$, and leave the question of removing the negation to the negation case below. We set:

$$\varphi^*(S, F, \bar{X}) := \neg\mathsf{otp}_{\omega^{\beta+1}}(X_m \cap \overline{F}^S)\ .$$

The correctness of this construction relies on the second item of Theorem 26. Indeed, "$\mathsf{otp}_{\omega^{\beta+1}}(A_m \cap \overline{F}^S)$ does not hold" means that $A_m \cap [s_i, s_{i+1})$ has order type smaller than ω^β for almost all $i \in F$, which is the same as $\omega^{\beta+1} \models^{S,F} \neg\mathsf{otp}_{\omega^\beta}(X_m)$.

Case of a negation, i.e., $\varphi(\bar{X}) = \neg\varphi_1(X_1, \ldots, X_k)$. We set

$$\varphi^*(S, F, \bar{X}) := \forall F' \subseteq F.\ (\text{"}F' \text{ infinite"}) \to \neg\varphi_1^*(S, F', \bar{X})\ .$$

Let us prove that the induction hypothesis holds. Let us assume that $\omega^{\beta+1} \models^{S,F} \neg\varphi(\bar{A})$. This is equivalent to the fact that $\neg\varphi(\bar{A} \cap [s_i, s_{i+1}))$ holds for almost all i with $s_i \in F$. Hence, this is also true for almost all i such that $s_i \in F'$ when $F' \subseteq F$ is infinite. Thus $\omega^{\beta+1} \models \varphi^*(S, F, \bar{A})$. Conversely, assume that $\omega^{\beta+1} \models^{S,F} \neg\varphi(\bar{A})$ does not hold. This means that there are infinitely many i such that s_i in F and $\varphi(\bar{A} \cap [s_i, s_{i+1}))$ holds. Let us chose $F' \subseteq F$ infinite to contain these indices. Then, this F' is a witness that $\omega^{\beta+1} \models \varphi^*(S, F, \bar{A})$ does not hold either. ◀

As a consequence of Theorem 28 and of Theorem 27, we obtain:

▶ **Corollary 29.** *For every $\mathsf{MSO}[\mathsf{otp}_{\omega^\beta}]$-sentence φ, there exists effectively an $\mathsf{MSO}[\mathsf{otp}_{\omega^{\beta+1}}]$-sentence ψ such that*

$$\omega^\beta \models \varphi \qquad \text{if and only if} \qquad \omega^{\beta+1} \models \psi\ .$$

Proof. Set ψ to be the formula

$$\exists S\ \text{"}S \text{ has order type } \omega\text{"} \wedge \text{"}S \text{ is cofinal"} \wedge \mathsf{Split}^{\mathrm{aa}}_{\omega^\beta}(S) \wedge \varphi^*(S, S)\ ,$$

in which φ^* is the formula produced by Theorem 28.

First implication. Let us assume that $\omega^\beta \models \varphi$. We have to prove that $\omega^{\beta+1} \models \psi$. For this, let us choose S to be $\{s_i := \omega^\beta \times i \mid i < \omega\}$. This set S is of order type ω and cofinal in $\omega^{\beta+1}$. It is also such that $[s_i, s_{i+1})$ as order type ω^β for all i. Hence, by Theorem 27, $\omega^{\beta+1} \models \mathsf{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$. Since $\omega^\beta \models \varphi$, we get by Theorem 28 that $\omega^{\beta+1} \models \varphi^*(S, S)$, and hence $\omega^{\beta+1} \models \psi$.

Conversely, let us assume that $\omega^{\beta+1} \models \psi$. This means that there exists S of order type ω and cofinal that satisfies $\mathsf{Split}^{\mathrm{aa}}_{\omega^\beta}(S)$. Let $S = \{s_0 < s_1 < \ldots\}$. By Theorem 27, this means that $[s_i, s_{i+1})$ has order type ω^β for almost all i. Furthermore, $\omega^{\beta+1} \models \varphi^*(S, S)$. Hence, by Theorem 28, this means that $\omega^{\beta+1}|_{[s_i, s_{i+1}} \models \varphi$ for almost all $i \in \omega$. This means that it holds for at least one $i < \omega$ such that $\omega^{\beta+1}|_{[s_i, s_{i+1})} = \omega^\beta$. Hence, $\omega^\beta \models \varphi$. ◀

6 Countable ordinals

By combining Theorems 24 and 25 from the previous sections, we have proved that the MSO[otp_α]-theory of α is undecidable for all countable ordinals α of the form ω^β. The next step is to show it for all countable $\alpha \geqslant \omega^\omega$:

▶ **Lemma 30.** *For all countable ordinals $\alpha \geqslant \omega^\omega$, the MSO[$\mathsf{otp}_\alpha$]-theory of α is undecidable.*

This part of the proof does not involve new interesting arguments. It is presented below for the completeness.

Recall some elementary facts about ordinal arithmetic. Cantor proved that every ordinal α can be uniquely expressed as a finite sum

$$\alpha = \omega^{\beta_n} + \cdots + \omega^{\beta_1} + \omega^{\beta_0} \ ,$$

for ordinals $\beta_n \geqslant \cdots \geqslant \beta_1 \geqslant \beta_0$. This is called the *Cantor normal form* of α.

▶ Notations. In order to avoid multi level subscripts, we will sometimes write MSO[α] instead of MSO[otp_α]

▶ **Lemma 31.** *Let $\alpha = \omega^{\beta_n} + \cdots + \omega^{\beta_1} + \omega^{\beta_0}$ where $\beta_n \geqslant \cdots \geqslant \beta_1 \geqslant \beta_0$. If the MSO[$\alpha$]-theory of α is decidable, the MSO[ω^{β_n}]-theory of ω^{β_n} is also decidable.*

Proof. First note that there is an MSO[α] formula $\psi_{=\omega^{\beta_n}}(x)$ such that $\alpha \models \psi_{=\omega^{\beta_n}}(b)$ if and only if $b = \omega^{\beta_n}$. Indeed, this formula says that x is the minimal element such that the set $\{y \ : \ y \geqslant x\}$ does not have the order type α.

Now, for every MSO[ω^{β_n}] sentence φ, we can construct an MSO[α] sentence φ^* such that $\omega^{\beta_n} \models \varphi$ if and only if $\alpha \models \varphi^*$. Indeed, it is sufficient to relativize the quantifiers to β_n, and this is possible according to the above remark, and replace every occurrence of the predicate $\mathsf{otp}_{\omega^{\beta_n}}(X)$ by $\mathsf{otp}_\alpha(X \cup \{z \mid z > \omega^{\beta_n}\})$. ◀

We are now ready to complete the proof of our theorem in the countable case.

Proof of Theorem 30. Let $\alpha \geqslant \omega^\omega$ be a countable ordinal. Its Cantor normal form $\beta_n \geqslant \cdots \geqslant \beta_1 \geqslant \beta_0$ is such that β_n is countable infinite. Hence, by Theorems 24 and 25, the MSO[$\mathsf{otp}_{\omega^{\beta_n}}$]-theory of ω^{β_n} is undecidable. By Theorem 31, this implies that the MSO[otp_α]-theory of α is undecidable. ◀

To sum up we have the following corollary:

▶ **Corollary 32.** *Let α be a countable ordinal. TFAE*

1. $\alpha \geqslant \omega^\omega$.
2. *α is not MSO definable.*
3. *The MSO[α] theory of α is undecidable.*
4. *The MSO[α] theory of any class of ordinals that contains an ordinal $\geqslant \alpha$ is undecidable.*

Proof. We have already proved the equivalence between (1)-(3). The implication (4)⇒(3) is immediate.

Let us proof that (3) ⇒(4). First note that there is an MSO[α] formula $\Psi_\alpha(x)$ which defines α inside every $\beta > \alpha$, i.e., for every $\beta > \alpha$ there is a unique b such that $\beta \models \Psi_\alpha(b)$ and the substructure of β over the prefix $\{a \mid a < b\}$ is isomorphic to α. Now, using this formula, for every MSO[α] sentence Φ it is easy to construct an MSO[α] sentence Φ^α such that $\alpha \models \Phi$ if and only if $\beta \models \Phi^\alpha$ for every (equivalently some) $\beta > \alpha$. ◀

7 Conclusion

This paper belongs to the body of works that aims at extending the expressive power of monadic second-order logic, while retaining decidability results. More precisely, we have studied the question of decidability of monadic second-order logic of ordinals when extended with predicates about the order type of sets. Since the order type of some ordinals is non definable (in particular for all countable ordinals from ω^ω upward), such extensions of MSO are strictly more expressive than MSO. Our main result for ordinals, Theorem 2, shows that up to ordinal $\omega_1^{\omega_1}$, there is nothing to be gained: if extending MSO with an order type predicate is decidable, then this order type was already definable in plain MSO, and thus the obtained logic is expressively equivalent to MSO.

The proof techniques involve a reduction of non-decidability of the satisfiability BMSO, which itself relies on the deep result of undecidability of the satisfiability of MSO+U over ω. This has to be combined with extra involved arguments for catching all the ordinals $< \omega_1^{\omega_1}$. However, when reaching $\omega_1^{\omega_1}$ all these tools seems to become useless, and the main open question we are left with is the following:

▶ **Problem 33.** *Is it true that α is MSO-definable if and only if the MSO[otp_α]-theory of α is decidable for all $\alpha < \omega_2$?*

Another problem is:

▶ **Problem 34.** *What is the degree of undecidability of the MSO[otp_α]-theory of α?*

We provided a reduction from the satisfiability of BMSO to the satisfiability problem of MSO[otp_α]. The satisfiability problem for BMSO is not in RE and not in Co-RE. We do not know whether there is a reduction in the other direction. Similar problems are about the degree of undecidability of the MSO[CBrank_α]-theory of the full binary tree.

The *Hausdorff rank* (sometimes called VD-rank or F-rank) is naturally definable for linear orders [28]. In particular, a linear order $(L,<)$ has a Hausdorff rank if and only if it is scattered, equivalently if and only if there is no order preserving embedding of the rationals in $(L,<)$. Given an ordinal α, let $\mathsf{Hrank}_\alpha(X)$ express that X has Hausdorff rank α. We denote by MSO[Hrank_α] the monadic second-order logic of order extended with the new predicate $\mathsf{Hrank}_\alpha(-)$.

It is well known that "a linear order has Hausdorff rank n" is definable for every $n \in \mathbb{N}$.

▶ **Theorem 35.** *For every countable ordinals $\alpha \geqslant \omega$, the MSO[Hrank_α]-theory of the class of countable linaer orders is undecidable.*

The proof is a reduction of Theorem 35 to Theorem 2, and it is based on the following observations:

1. An ordinal γ has Hausdorff rank $\leqslant \alpha$ if and only if $\gamma \leqslant \omega^\alpha$.
2. Hence, for an infinite ordinal γ: $OTP(\gamma) = \omega^\alpha$ can be expresses as $\mathsf{Hrank}(\gamma) = \alpha$ and $\mathsf{Hrank}(\{y \mid y \geqslant x\}) = \alpha$ for every $x < \gamma$.

Therefore, there is an MSO[Hrank_α] formula $\varphi_\alpha(X)$ that expresses "X is a well-order and the order type of X is ω^α." Hence, by Theorem 2, we obtain that the MSO[Hrank_α]-theory of the linear orders of Hausdorff rank α as well as the MSO[Hrank_α]-theory of the class of countable linear orders is undecidable.

We can prove a stronger result:

▶ **Theorem 36.** *For every countable ordinals $\beta \geq \alpha \geq \omega$, and every countable linear order $(L,<)$ of Hausdorff rank β, the MSO[Hrank_α]-theory of $(L,<)$ is undecidable.*

Our proof of Theorem 36 is direct. The proof techniques are similar to those of Theorem 2, however, we have not found a reduction of Theorem 36 to Theorem 2.

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A Proof of Lemma 17

Note that $T_0 + T_1$ is a binary tree only if T_0 has at most one child. Since we are dealing with binary trees where every child is either left or right (these trees are considered as structures for the signature $\{<, \mathit{Left}(), \mathit{Right}()\}$) we have further refine sum and ω-sum operations.

First we characterize when a tree T of rank α is isomorphic to the downward closure of an antichain X of the full binary tree such that $(X, <_{\text{lex}})$ is (isomorphic to) an ordinal. Then we state properties of these trees and finally, we prove Theorem 17.

▶ **Definition 37** (*BWT* trees). *Let* BWT *be the set of binary trees such that* $T \in BWT$ *if the children of* T *are partitioned into the left and right children and every node has a leaf as a descendant, and the lexicographic order on the leaves is a well-order.*

▶ **Lemma 38.** *Let* X *be an antichain in the full binary tree such that* $(X, <_{\text{lex}})$ *is isomorphic to an ordinal. Then* $T := (X\downarrow, <)$ *is in* BWT.

▶ **Lemma 39.** *Let* $\pi = u_1 u_1 \cdots \in \{L, R\}^\omega$ *be an* ω*-branch of* $T \in BWT$.

Let T_i *(for* $i \in \omega$*) be the subtree of* T *over* $X_i := \{v \mid v \geqslant u_1 \ldots u_i \text{ and } \neg(v \geqslant u_1 \ldots u_{i+1})\}$. *Then*

1. $T_i \in BWT$.
2. *Infinitely many* T_i *have more than one element.*
3. *There is* i_R *such that for every* $i \geqslant i_R$*: if* T_i *has more than one element, then* $u_{i+1} = R$.

Proof.

(1) $T_i \in BWT$. Indeed (a) every node in X_i has a leaf descendant in X_i, (b) the leaves are well-ordered by $<_{\text{lex}}$ and (c) children are partitioned into left/right as in T.

(2) If T_i is singleton for all $i > j$, then for all $i > j$ the i-th node on π has no leaf as descendant. Contradiction.

(3) Let $F := \{v \mid vL \in \pi$ and there is a leaf above $vR\}$. We claim that F is finite. Indeed if $i \in F$ and u_i is a leaf above vR then $vL <_{\text{lex}} u_i$ and every descendant u of vL is $<_{\text{lex}} u_i$. In particular, $u_{i+1} <_{\text{lex}} u_i$ for all $i \in F$. Hence, if F is infinite, then $<_{\text{lex}}$ is not well-order on the leaves. Contradiction. ◀

The $+$ and ω-sum operations on trees are refined by sums and infinite sums of BWT trees.

▶ **Definition 40** (Sums of *BWT* trees). *Let* $T_i = (|T_i|, <_i)$ *for* $i \in \{0, 1\}$ *be* BWT *trees over disjoint universes. If the root of* T_0 *does not have right (respectively left) child define* $T_0 +_R T_1$ *(respectively,* $T_0 +_L T_1$*) to be a tree with universe* $|T_0| \cup |T_1|$ *and its order relation defined as* $n_1 \leqslant n_2$ *if there is* $i \in \{0, 1\}$ *such that* $n_1, n_2 \in |T_i|$ *and* $n_1 \leqslant_i n_2$*, or* n_1 *is the root of* T_0*; the root of* T_1 *becomes right (respectively, left) child of the root of* T_0*, and for other nodes their left/right status is inherited from* T_0 *and* T_1.

The infinite sums of BWT trees is defined as follows.

▶ **Definition 41** (Infinite sums of *BWT* trees). *Let* $T_i = (|T_i|, <_i)$ *for* $i \in \omega$ *be* BWT *trees over disjoint universes, and let* $u_1 u_2 \cdots \in \{L, R\}^\omega$ *be an* ω*-string.* $\sum^u_{i\in\omega} T_i$ *is defined if*

1. *The root of* T_i *does not have* u_{i+1} *child* ($i \in \omega$).
2. *Infinitely many of* T_i *have more than one element.*
3. *There is* i_R *such that for* $i \geqslant i_R$*: if* T_i *has more than one element, then* $u_{i+1} = R$.

We define the tree

$$\sum^u_{i\in\omega} T_i$$

as having universe $\bigcup_{i\in\omega} |T_i|$ *and its order relation is defined as* $n_1 \leqslant n_2$ *if there is* i *such that* $n_1, n_2 \in |T_i|$ *and* $n_1 \leqslant_i n_2$*, or* n_1 *is the root of* T_i *and* $n_2 \in |T_j|$ *for* $j \geqslant i$.

The root of T_{i+1} *becomes the right (respectively, left) child of the root of* T_i *if* $u_{i+1} = R$ *(respectively,* $u_{i+1} = L$*), and for other nodes of* T_i *their left/right status is inherited from* T_i.

Note the requirement that infinitely many of T_i have more than one element ensures that every node in the ω-sum has a leaf as a descendant. The third requirement ensures that the lexicographic order on the leaves of $\sum_{i\in\omega}^{u} T_i$ is well-order.

If the disjointness assumption does not hold in the above definitions, we replace T_i by disjoint isomorphic copies and proceed as above.

▶ **Lemma 42.**
1. *If T_i are in BWT and $T_0 +_R T_1$ is defined, then $T_0 +_R T_1$ is in BWT.*
2. *If T_i are in BWT and $T_0 +_L T_1$ is defined, then $T_0 +_L T_1$ is in BWT.*
3. *If T_i are in BWT and $\sum_{i\in\omega}^{u} T_i$ is defined, then $\sum_{i\in\omega}^{u} T_i$ is in BWT.*

For $T \in BWT$, we denote by $OTP(T)$ the order type of the lexicographic order on the leaves of T.

▶ **Lemma 43.**
1. $OTP(T_0 +_R T_1) = OTP(T_0) + OTP(T_1)$.
2. $OTP(T_0 +_L T_1) = OTP(T_1) + OTP(T_0)$.
3. *Assume that if T_i have more than one element, then $u_{i+1} = R$. Then $OTP(\sum_{i\in\omega}^{u} T_i) = \sum_{i\in\omega} OTP(T_i)$.*

Define $BWT_\alpha := \mathsf{CBrank}_\alpha \cap BWT$ and $BWT_\alpha^+ := \mathsf{CBrank}_\alpha^+ \cap BWT$, where α is a countable ordinal. Note that BWT_0 are one element trees and BWT_0^+ are finite binary trees with a partition of children into left and right.

▶ **Lemma 44.** *For $\alpha > 0$.*
1. *$T \in BWT_\alpha$ if and only if $T = \sum_{i\in\omega}^{u} T_i$ for $T_i \in \cup_{\beta<\alpha} BWT_\beta^+$.*
2. *$T \in BWT_\alpha^+$ if and only if there are $T'_0, \ldots, T'_k \in \cup_{\beta\leqslant\alpha} BWT_\beta$ and $v_1 \ldots v_k \in \{L, R.\}$ such that $T = (((T'_0 +_{v_1} T'_1) +_{v_2} T'_2) \cdots +_{v_k} T'_k)$.*
3. *Assume that $\mathsf{CBrank}(T) = \alpha$ and $T \in BWT_\alpha$. Let u and T_i be as in 1. Then*
   a. *If α is limit, then for every $\beta < \alpha$ there is $T_i \notin BWT_\beta^+$.*
   b. *If $\alpha = \gamma + 1$, then infinitely often $T_i \in BWT_\gamma^+ \setminus \cup_{\beta<\gamma} BWT_\beta^+$.*

Proof. (1) $\Leftarrow$ direction is easily follows by the induction on α.

$\Rightarrow$-direction. $T \in \mathsf{CBrank}_\alpha$, therefore there is an ω-branch $\pi = u_1u_2\cdots \in \{L, R\}^\omega$ such that $T = \sum_{i\in\omega} T_i$, where $T_i \in \cup_{\beta<\alpha}\mathsf{CBrank}_\beta^+$ is the subtree of T over $X_i := \{v \mid v \geqslant u_1 \ldots u_{i-1}$ and $\neg(v \geqslant u_1 \ldots u_i)\}$. Since $T \in BWT$, it follows that $T_i \in BWT$ and therefore, $T_i \in \cup_{\beta<\alpha} BWT_\beta^+$. It is also clear that $T = \sum_{i\in\omega}^{\pi} T_i$.

(2) and (3) easily follows from the definitions. ◀

Proof Theorem 17. Let X be an antichain in the full binary tree such that $(X, <_{\text{lex}})$ is isomorphic to an ordinal. We have to prove that $T := (X\downarrow, <)$ has Cantor-Bendixson rank α if and only if the order type of $(X, <_{\text{lex}})$ belongs to $[\omega^\alpha, \omega^{\alpha+1})$.

Note that T is a BWT by Theorem 38. The proof proceeds by induction on α using Theorem 43 and Theorem 44.

The base case $\alpha = 0$ is immediate.

Assume that Lemma holds for all $\beta < \alpha$.

Let $T \in BWT_\alpha$. By Theorem 44, $T = \sum_{i\in\omega}^{u} T_i$ for $T_i \in \cup_{\beta<\alpha} BWT_\beta^+$. By the inductive hypothesis $OTP(T_i) < \omega^\alpha$.

Let $u^k := u_{k+1}u_{k+2}\ldots$. Then $u := u_1 \ldots u_k u^k$. By Theorem 39(3), we can choose k such that if T_{k+i} have more than one element, then $u_{k+i+1} = R$. Hence, by Theorem 43(3), $OTP(\sum_{i\in\omega}^{u^k} T_{k+i}) = \sum_{i\in\omega} OTP(T_{i+k}) < \omega^\alpha \times \omega$. $T := (T_0 +_{u_1} +(T_1 +_{u_2} +_{u_3}(\cdots +_{u_k} T'))))$, where $T' := \sum_{i\in\omega}^{u^k} T_{k+i}$. We proved that $OTP(T_i) < \omega^\alpha$ and $OTP(T') < \omega^\alpha \times \omega = \omega^{\alpha+1}$. Hence, by Theorem 43(1)-(2), $OTP(T) < \omega^{\alpha+1}$.

Let us show that $OTP(T) \geqslant \omega^\alpha$.

By Theorem 44(3), if α is limit then there is no $\beta < \alpha$ such that all $T_i \in BWT^+_\beta$ for $i > k$. Therefore, there is no $\beta < \alpha$ such that $OTP(T_i) < \omega^\beta$ for $i > k$. Hence $OTP(\sum_{i\in\omega}^{u^k} T_{k+i}) \geqslant \omega^\alpha$.

By Theorem 44(3), if $\alpha = \gamma + 1$ is a successor, then infinitely often $T_i \in BWT^+_\gamma \setminus \cup_{\beta<\gamma} BWT^+_\beta$. Hence, by the inductive hypothesis, infinitely often $OTP(T_i) \geqslant \omega^\gamma$. Hence, $OTP(\sum_{i\in\omega}^{u'} T_{k+i}) \geqslant \omega^\gamma \times \omega = \omega^\alpha$. Therefore, $OTP(T) \geqslant \omega^\alpha$. This completes the proof for the case when $T \in BWT_\alpha$.

Now assume that $T \in BWT^+_\alpha$ and $\mathsf{CBrank}(T) = \alpha$. Then T is a sum of trees in BWT_α where at least one of the summands has CBrank equal to α. Therefore, the OTP of this summand is at least ω^α, and hence, the OTP of the sum is at least ω^α. ◀

B Beyond countable ordinals

In this section, we consider the decidability questions concerning $\mathsf{MSO}[\mathsf{otp}_\alpha]$ for α smaller than $\omega_1^{\omega_1}$. It essentially relies, as for the countable case, on the techniques in Sections 4–6, but it requires a bit more care. Also, some of the arguments go beyond $\omega_1^{\omega_1}$, but not all.

Thus, our first task is to understand precisely what are the MSO-definable ordinals beyond the countable. Let ω_1 be the first uncountable ordinal and ω_2 be the initial ordinal of the cardinal $\aleph_2$.

Let us first note that ω_1 is MSO-definable, indeed, this is the least limit ordinal which is not of cofinality ω. It is easy to express in an MSO that an ordinal is limit and that an ordinal has a cofinal ω-sequence. In the same way, ω_2 is MSO-definable since it is the least ordinal which is not of cofinality ω_1 or ω.

In Section B.1 we characterize MSO-definable ordinals in $[\omega_1, \omega_2)$. In Section B.2 we consider ordinals of the form ω_1^β where β is countable. We prove that for these ordinals $\mathsf{MSO}[\mathsf{otp}_{\omega_1^\beta}]$ is decidable if and only if ω_1^β is MSO definable. In Section B.3 we extend the equivalence between decidability and definability to all ordinals $< \omega_1^{\omega_1}$. The ordinal $\omega_1^{\omega_1}$ is undefinable, unfortunately we do not know whether $\mathsf{MSO}[\mathsf{otp}_{\omega_1^{\omega_1}}]$ is decidable.

B.1 Definable ordinals $< \omega_2$

In this subsection we characterize the MSO-definable ordinals $< \omega_2$. We use the following variant of the Cantor normal form: If $\alpha \in (0, \omega_2)$, then α has a unique decomposition of the form

$$\alpha = \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_0} \times \gamma_0 \ ,$$

where $\omega_2 > \beta_n > \cdots > \beta_1 > \beta_0 \geqslant 0$ and γ_i is a non-zero countable ordinal for all i. Let us call it the *ω_1-representation of α*.

▶ **Proposition 45** (MSO-definable ordinals). *An ordinal $\alpha < \omega_2$ is MSO-definable if and only if its ω_1-representation $\omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_0} \times \gamma_0$ is such that $\beta_i < \omega$ and $\gamma_i < \omega^\omega$ for all i.*

Proof. $\Leftarrow$ direction.

Let us recall that MSO-definable ordinals are closed under sum and multiplication. Therefore, every finite power of ω_1 is MSO-definable, and since every $\gamma < \omega^\omega$ is MSO-definable, we obtain that all the ordinals that have an ω_1-representation as in the statement, are MSO-definable.

Our proof of the other direction use elements of the compositional methods [32].

For every $n \in \mathbb{N}$ we say that ordinals α and β are $\equiv_n$-equivalent (notation $\alpha \equiv_n \beta$) if for every MSO sentence Φ of the quantifier depth at most n: $\alpha \models \Phi$ if and only if $\beta \models \Phi$. We will use the following well-known facts:

▶ **Fact 46** ($\equiv_n$ is a congruence). *The relation $\equiv_n$ is an equivalence relation and it is a congruence with respect to $+$ and $\times$. If $\alpha \equiv_n \beta$ then for every γ:*
1. $\alpha + \gamma \equiv_n \beta + \gamma$ *and* $\gamma + \alpha \equiv_n \gamma + \beta$.
2. $\alpha \times \gamma \equiv_n \beta \times \gamma$

▶ **Fact 47.** *There is a function $M : \mathbb{N} \to \mathbb{N}$ such that for every n if $\alpha \equiv_{M(n)} \beta$ then $\gamma \times \alpha \equiv_n \gamma \times \beta$.*

▶ **Fact 48** (cf Theorem 3.5(B) [31]).
1. *For every $\delta < \omega_2$ there is $\delta(k) < \omega_1^\omega$ such that $\delta \equiv_k \delta(k)$.*
2. *For every $\delta < \omega_1$ there is $\delta(k) < \omega^\omega$ such that $\delta \equiv_k \delta(k)$.*

Now we are ready to prove the $\Rightarrow$ direction of Theorem 45.
Assume that the ω_1-representation of α is

$$\alpha = \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_i} \times \gamma_i + \cdots + \omega_1^{\beta_0} \times \gamma_0 \ ,$$

If $\beta_i \geqslant \omega$ for some i, then $\alpha \geqslant \omega_1^\omega$. Therefore, if α satisfies a sentence Φ of quantifier depth k, by Fact 48(1), there is $\alpha(k) < \omega_1^\omega$ that satisfies Φ. Hence, α is not MSO definable.

Hence, if α is MSO definable then all $\beta_i < \omega$. Now assume that $\gamma_i > \omega^\omega$ for some i. Toward a contradiction, let α be definable by a sentence Φ of quantifier depth k. By Fact 48(2) there is $\gamma_i' < \omega^\omega$ such that $\gamma \equiv_{M(k)} \gamma_i'$, where M is a function from Fact 47. Therefore, by Fact 47, $\omega_1^{\beta_i} \times \gamma_i \equiv_k \omega_1^{\beta_i} \times \gamma_i'$. Hence, by Fact 46, α is $\equiv_k$-equivalent to α' which has the following ω_1-representation:

$$\alpha' = \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_i} \times \gamma_i' + \cdots + \omega_1^{\beta_0} \times \gamma_0$$

Therefore, $\alpha' \models \Phi$. But $\alpha' \neq \alpha$ (by uniqueness of ω_1 representation) and this contradicts that α is definable by Φ. ◀

B.2 ω_1^β for countable β

In this subsection we consider ordinals of the form ω_1^β where β is countable. We prove that for these ordinals MSO[$\mathrm{otp}_{\omega_1^\beta}$] is decidable if and only if ω_1^β is MSO definable.

Theorem 24 states that if β is of cofinality ω, the MSO[$\mathrm{otp}_{\omega^\beta}$]-theory of ω^β is undecidable.

Note that $\omega_1 = \omega^{\omega_1}$ and $\omega_1^\beta = \omega^{\omega_1 \times \beta}$. Observe that if β is ω-cofinal then $\omega_1 \times \beta$ is ω-cofinal. Hence,

▶ **Corollary 49.** *If β is ω-cofinal, then the MSO[$\mathrm{otp}_{\omega_1^\beta}$]-theory of ω_1^β is undecidable.*

Now we are going to show how to reduce MSO[ω_1^β] to MSO[$\omega_1^{\beta+1}$]. From this reduction and Theorem 49 we deduce undecidability of MSO[ω_1^β] for all $\beta \in [\omega, \omega_1)$.

The following standard facts hold:

▶ **Fact 50.**
1. *Let $(\alpha_i)_{i\in\omega}$ be an ω-sequence such that $\alpha_i < \omega^\beta$ for all i. Then,*

$$\sum_{i\in\omega} \alpha_i \leqslant \omega^\beta.$$

2. *Let $(\alpha_i)_{i\in\omega}$ be an ω-sequences of ordinals smaller than $\omega^{\beta+1}$.*
$$\sum_{i\in\omega}\alpha_i=\omega^{\beta+1} \text{ if and only if } \alpha_j\geqslant\omega^{\beta} \text{ for cofinally many } j.$$
3. *Let $(\alpha_i)_{i\in\omega_1}$ be an ω_1-sequences of ordinals smaller than $\omega_1^{\beta+1}$.*
$$\sum_{i\in\omega_1}\alpha_i=\omega_1^{\beta+1} \text{ if and only if } \alpha_j\geqslant\omega_1^{\beta} \text{ for cofinally many } j.$$
4. *Let $(\alpha_i)_{i\in\omega_1}$ be an ω_1-sequence such that $\alpha_i<\omega_1^{\beta}$ for all i. Then,*
$$\sum_{i\in\omega_1}\alpha_i\leqslant\omega_1^{\beta}.$$

Theorem 50(1)-(2) was used in the reduction of MSO$[\omega^{\beta}]$ to MSO$[\omega^{\beta+1}]$. We will use Theorem 50(3)-(4) in our reduction of MSO$[\omega_1^{\beta}]$ to MSO$[\omega_1^{\beta+1}]$.

Note that for $\beta\geq\omega$, it is impossible to express in MSO$[\omega_1^{\beta+1}]$ "an interval is of length ω_1^{β}." Even there is no MSO$[\omega_1^{\beta+1}]$ formula $B(x)$ such that $\omega_1^{\beta+1}\models B(b)$ if and only if the interval $[0,b):=\{c\mid c<b\}$ has the order type ω^{β}.

Our first aim is to chop $\omega_1^{\beta+1}$ into ω_1 disjoint intervals all of length ω_1^{β}, except a countable many.

▶ **Lemma 51.** *There is an MSO$[\omega_1^{\beta+1}]$ formula Chop(X_B,X_E) such that for every $B,E\subseteq\omega_1^{\beta+1}$ we have $\omega_1^{\beta+1}\models$ Chop(B,E) if and only if*
1. *B and E have order type ω_1 and $b_i<e_i<b_{i+1}$ for $i\in\omega_1$. Hence, for $i\neq j$ the intervals (b_i,e_i) and (b_j,e_j) are disjoint.*
2. *For all but countable many i, the order type of $[b_i,e_i)$ is ω_1^{β}.*

Proof. Let $B:=(b_i)_{i\in\omega_1}$ and $E:=(e_i)_{i\in\omega_1}$ be increasing ω_1 sequences such that $b_i<e_i<b_{i+1}$ for all $i\in\omega_1$. (This can be formalized in MSO.) We want to ensure that OTP$([b_i,e_i))=\omega_1^{\beta}$ for all $i\in\omega_1$, except countable many one.

Let us formalize it as follows:

Requirements.
(1) $\bigcup_{i\in I}[b_i,e_i)$ has order type $\omega_1^{\beta+1}$ for every cofinal subset I of ω_1, and
(2) for all $C:=(c_i)_{i\in\omega_1}$ such that $c_i\in[b_i,e_i)$ the order type of $\bigcup[b_i,c_i)$ is not $\omega_1^{\beta+1}$ (it should be $<\omega_1^{\beta+1}$).

It is easy to formalize (1) and (2) by an MSO$[\omega_1^{\beta+1}]$ formula.

We claim that (1) and (2) hold if and only if for all, but countable many i: OTP$([b_i,e_i))=\omega_1^{\beta}$.

Indeed, if for all i: OTP$([b_i,e_i))=\omega_1^{\beta}$, then, by Theorem 50(4), for all $C:=(c_i)_{i\in\omega_1}$ such that $c_i\in[b_i,e_i)$ we have OTP$((\bigcup[b_i,c_i))\leqslant\omega_1^{\beta}$. Therefore, if for all but countable many i: OTP$([b_i,e_i))=\omega_1^{\beta}$, then OTP$((\bigcup[b_i,c_i))<\omega_1^{\beta+1}$. Therefore, (1) and (2) hold.

For the other direction. If (1) holds, then by Theorem 50(4) there are at most countable many i such that OTP$([b_i,e_i))<\omega_1^{\beta}$. If (2) holds, then, by Theorem 50(3), there are at most countable many i such that OTP$([b_i,e_i))>\omega_1^{\beta}$. Hence, there are at most countable many i such that OTP$([b_i,e_1))\neq\omega_1^{\beta}$. We proved the Lemma ◀

Now we will reduce MSO$[\omega_1^{\beta}]$ to MSO$[\omega_1^{\beta+1}]$. This is exactly like the reduction of Theorem 28, but for negation we should replace[2] "infinite" by ω_1.

[2] Instead of the filter of co-finite subset of ω, we use the filter of co-countable subsets of ω_1.

▶ **Lemma 52.** *There is an algorithm which given an* MSO$[\omega_1^\beta]$*-formula* $\varphi(X_1, \ldots X_k)$ *constructs an* MSO$[\omega_1^{\beta+1}]$*-formula* $\varphi^*(X_B, X_E, X_1, \ldots X_k)$ *such that for all* $A_1, \ldots, A_k \subseteq \omega_1^{\beta+1}$ *and all* B, E *which satisfy* $\omega_1^{\beta+1} \models \mathit{Chop}(B, E)$*:*

$$\omega_1^{\beta+1} \models \varphi^*(B, E, A_1, \ldots, A_k)$$

if and only if

$$\omega_1^{\beta+1}|_{[b_i,e_i)} \models \varphi(A_1 \cap [b_i, e_i), \ldots, A_k \cap [b_i, e_i))$$

for all but countable many i*.*

Proof. We shall define the formula φ^* by structural induction on φ, and establish the conclusions of the lemma at the same time. The case of existential quantifier, the case of conjunction, and the case of MSO predicates are elementary. The crucial point is the negation.

Case of a conjunction, i.e., $\varphi(\bar{X}) = (\varphi_1(\bar{X}) \wedge \varphi_2(\bar{X}))$. We define

$$\varphi^*(X_B, X_E, \bar{X}) := \varphi_1^*(X_B, X_E, \bar{X}) \wedge \varphi_2^*(X_B, X_E, \bar{X}) \ .$$

Correctness follows from the fact that co-countable subsets of ω_1 are closed under the intersection.

Case of an existential set quantifier, i.e., $\varphi(\bar{X}) = \exists Y. \varphi_1(\bar{X}, Y)$. We define

$$\varphi^*(X_B, X_E, \bar{X}) := \exists Y. \ \varphi_1^*(X_B, X_E, \bar{X}, Y) \ .$$

Correctness is also straightforward.

Case of an MSO-formula $\varphi(\bar{X})$, and in particular of the atomic formulas $x \in Y$ and $x < y$ is easy and very similar to Theorem 28.

Case of an order type predicate, i.e., $\varphi(\bar{X}) := \mathbf{otp}_{\omega_1^\beta}(X_m)$. For simplicity, we shall treat the case of $\varphi(X) := \neg\mathsf{otp}_{\omega_1^\beta}(X_m)$, and leave the question of removing the negation to the negation case below. We set:

$$\varphi^*(X_B, X_E, X_m) := \mathit{Chop}(X_B, X_E) \wedge \neg\mathsf{otp}_{\omega_1^{\beta+1}}(X_m \cap F), \ .$$

where $F := \{z \mid \exists x \in X_B \exists y \in X_E (x \leqslant z < y \wedge (x, y) \cap X_B = \emptyset \wedge (x, y) \cap X_E = \emptyset)\}$. The correctness of this construction relies on Theorem 50.

Case of a negation, i.e., $\varphi(\bar{X}) := \neg\varphi_1(X_1, \ldots, X_k)$. We set $\varphi^*(X_B, X_E, \bar{X})$ to be the conjunction of

A $\mathit{Chop}(X_B, X_E)$, and

B $\neg\varphi_1^*(X'_B, X'_E, \bar{X})$ holds for every X'_B, X'_E which are cofinal subsets of X_B and X_E such that $\mathit{Chop}(X'_B, X'_E)$.

Let us assume that $\omega^{\beta+1} \models \mathit{Chop}(B, E)$. Then $B := \{b_i \mid i \in \omega_1\}$ and $E := \{e_i \mid i \in \omega_1\}$, where b_i and e_i are increasing ω_1-sequences.

Condition **B** is equivalent to "$\neg\varphi_1(\bar{A} \cap [b_i, e_i))$ holds for all but countable many i.

Hence, the inductive hypothesis holds. ◀

As a consequence of Theorem 49 and Theorem 52 we obtain:

▶ **Corollary 53.** *The* MSO$[\omega_1^\beta]$ *is undecidable on* ω_1^β *for* $\beta \in [\omega, \omega_1)$*.*

B.3 Definability is equivalent to Decidability for $\alpha < \omega_1^{\omega_1}$

We are ready to extend Theorem 2 up to $\omega_1^{\omega_1}$.

▶ **Theorem 54.** *For all ordinals $\alpha < \omega_1^{\omega_1}$, the MSO[$\mathsf{otp}_\alpha$]-theory of α is decidable if and only if α is MSO-definable.*

Proof. For countable α it was proved in Theorem 30.

If α is definable, then MSO[α] is equivalent to MSO. Since, the MSO theory of every $\alpha < \omega_2$ is decidable, we obtain that the MSO[α] theory of α is decidable.

It remains to show that if an uncountable $\alpha < \omega_1^{\omega_1}$ is undefinable, then MSO[α] is undecidable.

Let $\alpha = \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_0} \times \gamma_0$, where $\omega_2 > \beta_n > \cdots > \beta_1 > \beta_0 \geqslant 0$ and γ_i is a non-zero countable ordinal for all i. Assume that $\alpha < \omega_1^{\omega_1}$ and α is undefinable.

By Proposition 45, there is i such that (A) $\beta_i \geqslant \omega$ or (B) $\gamma_i \geqslant \omega^\omega$.

If (A) holds then $\beta_n \in [\omega, \omega_1)$. In this case there is an MSO[α] formula $B(x)$ such that $\alpha \models B(b)$ if and only if the interval $[0, b)$ has the order type $\omega_1^{\beta_n}$. Indeed, let $[x, \infty)$ (respectively, $[0, x)$) be the set $\{y \mid y \leqslant x\}$ (respectively, $\{y \mid y < x\}$) and let $B(x)$ says that x is the minimal element such that $\neg\alpha([x, \infty))$. It is clear that $\alpha \models B(b)$ if the interval $[0, b)$ has the order type $\omega_1^{\beta_n}$. Now, similarly to the proof of Theorem 31, for every MSO[$\omega_1^{\beta_n}$] sentence C we can (effectively) construct an MSO[α] sentence C^* such that $\omega_1^{\beta_n} \models C$ if and only if $\alpha \models C^*$. Since, MSO[$\omega_1^{\beta_n}$] is undecidable by Corollary 53, we derive that MSO[α] is undecidable.

If (A) does not holds and (B) holds we have that all $\beta_i < \omega$ and therefore $\omega_1^{\beta_i}$ are definable for all $i \leqslant n$, and there are $\gamma_i \geqslant \omega^\omega$. Let i be the maximal index such that $\gamma_i \geqslant \omega^\omega$.

There is an MSO formula $B(x)$ such that $\alpha \models B(b)$ if and only if the order type of $[0, b)$ is $\delta := \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_{i+1}} \times \gamma_{i+1}$ indeed δ is a definable ordinal, hence such B exists. There is an MSO formula $E(y)$ such that $\alpha \models E(e)$ if the order type of $[0, e)$ is $\delta_1 := \omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_i} \times \gamma_i$. Indeed $E(x)$ states x is the minimal such that that the order type of $[x, \infty)$ is less than $\omega_1^{\beta_i}$. The order type of $[b, e)$ is $\mu := \omega_1^{\beta_i} \times \gamma_i$.

Now we will use two reductions. The first one reduces MSO[γ_i] to MSO[$\omega_1^{\beta_i} \times \gamma_i$]

▷ Claim 55. For every $k < \omega$ and an $MSO[\gamma]$ sentence A there is an MSO[$\omega_1^k \times \gamma$] sentence A^* such that $\gamma \models A$ if and only if $\omega_1^k \times \gamma \models A^*$.

Proof. Since an ordinal ω_1^k is MSO definable, there is a formula $Mult(X)$ which defines the set of multiples of ω_1^k, i.e., for every ordinal α: $\alpha \models Mult(S)$ if and only if $S := \{a \in \alpha \mid [0, a)$ is a multiple of $\omega_1^k\}$. Let $A^\star$ be defined as $\exists S(Mult(S) \wedge B)$, where B is the relativisation of A on S.

For an MSO (or MSO[γ]) sentence A: $\gamma \models A$ if and only if $\omega_1^k \times \gamma \models A^\star$. If A is an MSO[γ] sentence, then $A^\star$ is an MSO[γ] sentence, and we have to express γ-order type predicates "$X \subseteq S$ has order type γ" by $\omega_1^k \times \gamma$ order type predicates.

For $x = \omega_1^k \times \beta$ (a multiple of ω_1^k), we denote by $[x]$ an interval $[\omega_1^k \times \beta, \omega_1^k \times (\beta + 1))$. For a set X of multiples of ω_1^k, we denote by $[X]$ the set $\cup_{x \in X}[x]$. Note that $[X]$ is MSO definable from X.

Hence, our desired A^* can be defined from $A^\star$ by replacing the subformulas "X has the order type γ" by $\exists Y(Y = [X] \wedge$"Y has the order type $\omega_1^k \times \gamma$"). ◁

Since, $\gamma_i \geqslant \omega^\omega$ and it is countable, MSO[γ_i] is undecidable, hence, we obtain by Theorem 55 that MSO[$\omega_1^{\beta_i} \times \gamma_i$] is undecidable.

The second reduction reduces MSO[$\omega_1^{\beta_i} \times \gamma_i$] to MSO[$\alpha$].

▷ Claim 56. For every MSO$[\omega_1^{\beta_i} \times \gamma_i]$ sentence A there is an MSO$[\alpha]$ sentence A^* such that $\omega_1^{\beta_i} \times \gamma_i \models A$ if and only if $\alpha \models A^*$.

Proof. Recall that there are MSO formulas $B(x)$ and $E(y)$ such that $\alpha \models B(b) \wedge E(e)$ if and only if $[0, b)$ has the order type $\omega_1^{\beta_n} \times \gamma_n + \cdots + \omega_1^{\beta_{i+1}} \times \gamma_{i+1}$ and $[b, e)$ has the order type $\omega_1^{\beta_i} \times \gamma_i$.

As A^* we can take $\exists xy(B(x) \wedge E(y) \wedge C)$, where C is obtained from A first by relativizing A to the interval $[x, y)$. and then replacing atomic subformulas "X has order type $\omega_1^{\beta_i} \times \gamma_i$" by "$X \cup Z$ has the order type α," where $Z := \{z \mid z < x \vee z \geqslant y\}$. ◁

Since MSO$[\omega_1^{\beta_i} \times \gamma_i]$ is undecidable, we obtain by Theorem 56 that MSO$[\alpha]$ is undecidable. and this completes our proof of Theorem 54. ◀