Reachability for Multi-Priced Timed Automata with Positive and Negative Rates

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Abstract

Multi-priced timed automata (MPTA) are timed automata with observer variables whose derivatives can change from one location to another. Observers are read-once variables: they do not affect the control flow of the automaton and their value is output only at the end of a run. Thus MPTA lie between timed and hybrid automata in expressiveness. Previous work considered observers with non-negative slope in every location. In this paper we treat observers that have both positive and negative rates. Our main result is an algorithm to decide a gap version of the reachability problem for this variant of MPTA. We translate the gap reachability problem into a gap satisfiability problem for mixed integer-real systems of nonlinear constraints. Our main technical contribution – a result of independent interest – is a procedure to solve such contraints via a combination of branch-and-bound and relaxation-and-rounding.

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1 Introduction

Timed automata [1] are a widely studied model of real-time systems that extend classical finite state-automata with real-valued variables, called *clocks*, that evolve with derivative one and which can be queried and reset along transitions. *Multi-Priced Timed Automata* (MPTA) [7, 10, 13, 25] further extend timed automata with variables, called *observers*, that have a non-negative slope that can change from one location to another. Such variables can model the accumulation of costs or the use of resources along a computation, such as energy and memory consumption in embedded systems, or bandwidth in communication networks. For this reason MPTA are widely used to model multi-objective real-time optimisation problems [9].

While observers exhibit richer dynamics than clocks, they may not be queried while taking edges. Thus MPTA lie between timed automata (for which reachability is decidable) and linear hybrid automata (for which reachability is undecidable [17]). A natural class of verification problems for MPTA concerns reachability subject to constraints on the observers. A simple variant is the *Domination Problem*, which asks to reach a location subject to upper bounds on each observer. Here one can think of the constraints as representing upper bounds on accumulated costs or resources. The Domination Problem was shown decidable in [21] using well-quasi-orders and was later shown to be PSPACE-complete in [12, Theorem 4].

A more expressive version of the Domination Problem partitions the set of observers into *cost variables* and *reward variables* and asks to reach a location subject to upper bounds on costs and lower bounds on rewards. This variant is, unfortunately, undecidable. However it is shown in [12, Theorem 6] that a gap version of the problem – called the *Gap Domination Problem* – is decidable. In the Gap Domination Problem the input additionally contains a slack $\varepsilon > 0$. The objective is to distinguish the case that the constraints on the observers can be satisfied with slack ε from the case in which they cannot be satisfied at all. In general, gap problems are decision versions of approximation problems [3, Chapter 18.2]. Decidability of the Gap Domination Problem implies that the Pareto curve of undominated reachable cost vectors can be computed to arbitrary precision (cf. [11]).

The objective of this paper is to address a more expressive variant of MPTA than hitherto considered: namely those in which observers can have both positive and negative rates. Alternatively, and equivalently, one can consider MPTA with nonnegative rates, but in which one allows reachability specifications to contain constraints on the *difference* between two observers rather than just threshold constraints that compare observers to constants. Indeed, this extension is motivated by the desire to measure net resource use along computations. In this more general setting, the Domination Problem, of course, remains undecidable; one moreover loses monotonicity properties on which previous positive decidability results rely, including the decision procedure for the Gap Domination Problem given in [12, Theorem 15]. The main result of this paper is to establish decidability (in nondeterministic exponential time) of the Gap Domination Problem in the presence of positive and negative rates via a new decision procedure.

We start by recalling a result of [12] that characterises the set of all reachable observer values for a given MPTA via a system of mixed integer-real nonlinear constraints. Our main technical contribution, which is of independent interest, shows how to solve a gap version of the satisfiability problem for such systems of constraints. Our method involves a combination of relaxation-and-rounding and branch-and-bound that relies on Khinchine's Flatness Theorem from Diophantine approximation. We formulate a relaxation of the system of constraints such that a solution to the relaxed version can be rounded to a solution of the original problem, while unsolvability of the relaxed version permits a branch-and-bound step that eliminates a variable from the original system of constraints.

Systems of non-linear constraints over integer and real variables appear in many different domains and are widely studied, although typically not from the point of view of decidability since most classes of problems with unbounded integer variables are undecidable [16]. Other than [12], we are not aware of previous work on the gap problem considered here. Kachiyan and Porkolab [19] showed that it is decidable whether a convex semialgebraic set contains an integer point; however we work with non-convex sets.

In this paper we consider MPTA with arbitrarily many observers. There is a significant literature and mature tool support concerning the special case of MPTA with a single observer, which are variously called Priced Timed Automata or Weighted Timed Automata. In this case, the optimal cost to reach a given location is computable [2, 6, 20]. In the case of one cost and one reward observer, one can also compute the optimal reward-to-cost ratio in reaching a given location [7]. The preceding results use the so-called *corner-point abstraction*, which is insufficient for multi-objective model checking. Instead, the present paper implicitly relies on the *simplex-automaton abstraction*, introduced in [12], which underlies the non-linear constraint problems that are the subject of our main results. All previously mentioned works involve observers that evolve linearly with time. Observer variables that vary non-linearly with time are considered in [4]. In the non-linear setting the optimal cost reachability problem is undecidable in general. Another variant, this time towards greater simplicity, is to consider observers that are only updated through discrete transitions [26].

2 Automata and Decision Problems

2.1 Multi-Priced Timed Automata

Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers. Given a set $\mathcal{X} = \{x_1, \ldots, x_n\}$ of *clocks*, the set $\Phi(\mathcal{X})$ of *clock constraints* is generated by the grammar

$$\varphi ::= \texttt{true} \mid x \leq k \mid x \geq k \mid \varphi \wedge \varphi\,,$$

where $k \in \mathbb{N}$ is a natural number and $x \in \mathcal{X}$. A *clock valuation* is a mapping $\nu : \mathcal{X} \to \mathbb{R}_{\geq 0}$ that assigns to each clock a non-negative real number. We denote by $\mathbf{0}$ the valuation such that $\mathbf{0}(x) = 0$ for all clocks $x \in \mathcal{X}$. We write $\nu \models \varphi$ to denote that ν satisfies the constraint φ. Given $t \in \mathbb{R}_{\geq 0}$, we let $\nu + t$ be the clock valuation such that $(\nu + t)(x) = \nu(x) + t$ for all clocks $x \in \mathcal{X}$. Given $\lambda \subseteq \mathcal{X}$, let $\nu[\lambda \leftarrow 0]$ be the clock valuation such that $\nu[\lambda \leftarrow 0](x) = 0$ if $x \in \lambda$, and $\nu[\lambda \leftarrow 0](x) = \nu(x)$ otherwise.

A *multi-priced timed automaton* (MPTA) $\mathcal{A} = \langle L, \ell_0, L_f, \mathcal{X}, \mathcal{Y}, E, R\rangle$ comprises a finite set L of *locations*, an *initial location* $\ell_0 \in L$, a set $L_f \subseteq L$ of *accepting locations*, a finite set $\mathcal{X}$ of *clock variables*, a finite set $\mathcal{Y}$ of *observers*, a set $E \subseteq L \times \Phi(\mathcal{X}) \times 2^{\mathcal{X}} \times L$ of *edges*, and a *rate function* $R : L \to \mathbb{Z}^{\mathcal{Y}}$. Here $R(\ell)(y)$ is the derivative of the observer $y \in \mathcal{Y}$ in location ℓ. Denote by $\|\mathcal{A}\|$ the length of the description of $\mathcal{A}$, where all integers are written in binary.

A *state* of $\mathcal{A}$ is a triple (ℓ, ν, t) where ℓ is a location, ν a clock valuation, and $t \in \mathbb{R}_{\geq 0}$ is a *time stamp*. A *run* of $\mathcal{A}$ is an alternating sequence of states and edges

$$\rho = (\ell_0, \nu_0, t_0) \xrightarrow{e_1} (\ell_1, \nu_1, t_1) \xrightarrow{e_2} \ldots \xrightarrow{e_m} (\ell_m, \nu_m, t_m)\,,$$

where $t_0 = 0$, $\nu_0 = \mathbf{0}$, $t_{i-1} \leq t_i$ for all $i \in \{1, \ldots, m\}$, and $e_i = \langle \ell_{i-1}, \varphi, \lambda, \ell_i\rangle \in E$ is such that $\nu_{i-1} + (t_i - t_{i-1}) \models \varphi$ and $\nu_i = (\nu_{i-1} + (t_i - t_{i-1}))[\lambda \leftarrow 0]$ for $i = 1, \ldots, m$. The run is *accepting* if $\ell_m \in L_f$. The *value* of such a run is a vector $\mathrm{val}(\rho) \in \mathbb{R}^{\mathcal{Y}}$, defined by $\mathrm{val}(\rho) = \sum_{i=0}^{m-1} (t_{i+1} - t_i) R(\ell_i)$. We refer to Figure 1 for an example of an MPTA and its operational semantics.

2.2 The Gap Domination Problem

The *Domination Problem* is as follows. Given an MPTA $\mathcal{A}$ with set $\mathcal{Y}$ of observers and a target $\gamma \in \mathbb{R}^{\mathcal{Y}}$, decide whether there is an accepting run ρ of $\mathcal{A}$ such that $\mathrm{val}(\rho) \leq \gamma$ pointwise.

Our formulation of the Domination Problem involves a conjunction of constraints of the form $y \leq c$, where $y \in \mathcal{Y}$ and $c \in \mathbb{Q}$. However such inequalities can encode more general linear constraints of the form $a_1 y_1 + \cdots + a_k y_k \sim c$, where $y_1, \ldots, y_k \in \mathcal{Y}$, $a_1, \ldots, a_k, c \in \mathbb{Z}$ and $\sim \in \{\leq, \geq, =\}$. To do this one introduces a fresh observer to denote each linear term $a_1 y_1 + \cdots + a_k y_k$ (two fresh observers are needed for an equality constraint). For this reduction it is crucial that we allow observers with negative rates.

The Domination Problem is PSPACE-complete for MPTA with positive rates only [12, Theorem 11], but is undecidable if negative rates are allowed [12, Theorem 3]. This motivates us to consider the *Gap Domination Problem* – a variant of the above problem in which

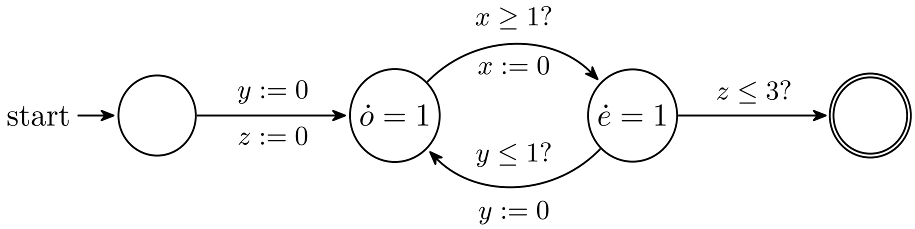

Figure 1 The figure shows an MTPA with three clocks x, y, z and two observer variables o, e, respectively standing for *odd* and *even*. The observer variables have slope 0 unless otherwise indicated; thus o aggregates the total dwell time in the *odd* state and e aggregates the total dwell time in the *even* state. An accepting run is completely determined by a sequence of nonnegative real numbers $d_0, \ldots, d_{2k}$, giving the respective delays between successive transitions. Suppose we wish to reach the accepting state subject to the two objectives $e \geq 2$ and $o \geq 1$. This is achieved, among others, by the run with sequence of time delays $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ and the run with integer sequence of delays $1, 0, 1, 0, 1, 1, 0$ (and any convex combination of the two runs). If the inequalities in the guards on x and y are replaced by equalities then the first run is the unique one realising the two given objectives. In the case of so-called *pure* reachability objectives, i.e., exclusively upper bound constraints or exclusively lower bound constraints on the observers, there is an explicit upper bound on the granularity of the delays in a run witnessing that the objective is realisable ($\frac{1}{3}$ in the present example) [12, Section 6]. This no longer holds in the case of reachability objectives that contain both upper and lower bounds on observers.

the input additionally includes a *slack parameter* $\varepsilon > 0$. If there is some run ρ such that $\mathrm{val}(\rho) \leq \gamma - \varepsilon$ then the output should be "dominated" and if there is no run ρ such that $\mathrm{val}(\rho) \leq \gamma$ then the output should be "not dominated". In case neither of these alternatives hold (i.e., γ is dominated but not with slack ε) then there is no requirement on the output. The Gap Domination Problem is the decision version of the task of computing ε-approximate Pareto curve in the sense of [11].

The following proposition and (a generalisation of [12, Propositions 6 and 7]), concerning the structure of the set of reachable vectors of observer values, allows us to reduce the Gap Domination Problem to a Diophantine problem. Geometrically the proposition says that the set of reachable observer vectors consists of a countable union of simplexes, where each simplex is specified by its vertices – a tuple of integer vectors – and the set of such tuples is semilinear. The proposition is based on the fact that if there are d observers then any reachable observer valuation is a convex combination of $d+1$ valuations that are respectively reached along $d+1$ runs, all taking the same sequence of edges, in which all transitions occur at integer time points (see [12] for details).

▶ **Proposition 1.** *Let $\mathcal{A}$ be an MPTA with set of observers $\mathcal{Y}$ having cardinality d. Then there is a semilinear set $\mathcal{S}_{\mathcal{A}} \subseteq (\mathbb{Z}^{\mathcal{Y}})^{d+1}$ such that for every accepting run ρ of $\mathcal{A}$ there exists $(\gamma_1, \ldots, \gamma_{d+1}) \in \mathcal{S}_{\mathcal{A}}$ for which $\mathrm{cost}(\rho)$ lies in the convex hull of $\{\gamma_1, \ldots, \gamma_{d+1}\}$. Moreover $\mathcal{S}_{\mathcal{A}}$ can be written as a union of a collection of linear sets that can be computed in time exponential in $\|\mathcal{A}\|$ and each of which has a description length polynomial in $\|\mathcal{A}\|$.*

Proof. The proposition was proved in [12] under the assumption that observers have nonnegative slope. The general case follows easily. Indeed, given an arbitrary MPTA $\mathcal{A} = \langle L, \ell_0, L_f, \mathcal{X}, \mathcal{Y}, E, R\rangle$, we define a new MPTA $\mathcal{A}'$, differing from $\mathcal{A}$ only in its set of observers and rate function, such that all observers in $\mathcal{A}'$ have non-negative rates. The set of observers of $\mathcal{A}'$ is $\mathcal{Y}' := \{y_+, y_- : y \in \mathcal{Y}\}$ and the rate function R' is given by

$$R'(y_+)(\ell) := \max(R(y)(\ell), 0) \quad \text{and} \quad R'(y_-)(\ell) := \max(-R(y)(\ell), 0)$$

for all $y \in \mathcal{Y}$ and all $\ell \in L$.

Define $\Phi : \mathbb{Z}^{\mathcal{Y}'} \to \mathbb{Z}^{\mathcal{Y}}$ by $\Phi(\gamma)(y) = \gamma(y_+) - \gamma(y_-)$. If a run ρ of $\mathcal{A}'$ has cost vector γ then ρ has cost vector $\Phi(\gamma)$ considered as a run of $\mathcal{A}$. Thus if we define $\mathcal{S}_{\mathcal{A}} := \Phi(\mathcal{S}_{\mathcal{A}'})$, where Φ has been lifted pointwise to a linear map $\Phi : (\mathbb{Z}^{\mathcal{Y}'})^{d+1} \to (\mathbb{Z}^{\mathcal{Y}})^{d+1}$, then $\mathcal{S}_{\mathcal{A}}$ satisfies the requirements of the proposition. ◀

The following is immediate from Proposition 1.

▶ **Corollary 2.** *Given $\gamma \in \mathbb{R}^{\mathcal{Y}}$, there exists a run ρ with $\mathrm{val}(\rho) \le \gamma$ if and only if the following mixed integer-real system of non-linear inequalities has a solution.*

$$\begin{array}{lcllcl} \lambda_1\gamma_1 + \cdots + \lambda_{d+1}\gamma_{d+1} & \le & \gamma & \quad 1 & = & \lambda_1 + \cdots + \lambda_{d+1} \\ (\gamma_1, \ldots, \gamma_{d+1}) \in \mathcal{S}_{\mathcal{A}} & & & \quad 0 & \le & \lambda_1, \ldots, \lambda_{d+1} \\ \gamma_1, \ldots, \gamma_{d+1} \in \mathbb{Z}^{\mathcal{Y}} & & & \quad \lambda_1, \ldots, \lambda_{d+1} \in \mathbb{R} & & \end{array} \tag{1}$$

In the following two sections we analyse systems of constraints of the above form, obtaining a general result that allows us to solve the Gap Domination Problem.

3 Mixed Integer Bilinear Systems

3.1 The Satisfiability Problem

A *mixed-integer bilinear (MIB) system* is a collection of constraints in integer variables $\boldsymbol{x}$ and real variables $\boldsymbol{y}$ of the form:

$$\begin{array}{ll} \boldsymbol{x}^\top A_i \boldsymbol{y} \le b_i & \quad (i = 1, \ldots, \ell) \\ C\boldsymbol{x} \le \boldsymbol{d} & \\ E\boldsymbol{y} \le \boldsymbol{f} & \\ \boldsymbol{x} \in \mathbb{Z}^m, \boldsymbol{y} \in \mathbb{R}^n \,. & \end{array} \tag{2}$$

We assume that all constants in (2) are integer; thus if the system is satisfiable then there is a satisfying assignment in which $\boldsymbol{y}$ is a rational vector. We say that a satisfying assignment has *slack* $\varepsilon > 0$ if $\boldsymbol{x}^\top A_i \boldsymbol{y} \le b_i - \varepsilon$, for $i = 1, \ldots, \ell$. Note that the slack requirement refers only to the nonlinear constraints.

We say that the system (2) is *bounded* if the polyhedron $\{\boldsymbol{y} \in \mathbb{R}^n : E\boldsymbol{y} \le \boldsymbol{f}\}$ is bounded, i.e., is a polytope. Crucially, the MIB systems arising from multi-priced timed automata in Corollary 2 are bounded. Unfortunately, however, the satisfiability problem for MIB systems is undecidable, even in the bounded case.

▶ **Proposition 3.** *The satisfiability problem for bounded mixed-integer bilinear systems is undecidable.*

Proof. We reduce from the following version of Hilbert's 10th Problem (see [12, Proposition 1]): given a finite system $\mathcal{S}$ of equations in variables $x_1, \ldots, x_n$, with each equation either having the form $x_i = x_j + x_k$ or $x_i = x_j x_k$, determine whether $\mathcal{S}$ has a solution in the set of strictly positive integers.

The reduction involves transforming the system $\mathcal{S}$ into an equisatisfiable MIB system $\mathcal{S}'$ over a set of integer variables $x_0, \ldots, x_n \ge 0$ (i.e, the variables of $\mathcal{S}$ plus a new variable x_0) and real variables $y_1, \ldots, y_n \ge 0$. The construction is such that every solution of $\mathcal{S}$ extends to a solution of $\mathcal{S}'$ and, conversely, every solution of $\mathcal{S}'$ restricts to a solution of $\mathcal{S}$.

The system $\mathcal{S}'$ includes equations $x_0 = 1$ and $x_i y_i = 1$ for $i = 1, \ldots, n$. The linear equations $x_i = x_j + x_k$ from $\mathcal{S}$ are carried over to $\mathcal{S}'$ and, for each equation $x_i = x_j x_k$ in $\mathcal{S}$, we include an equation $(x_j + x_k) y_i = x_0(y_j + y_k)$ in $\mathcal{S}$. The latter is equivalent to

$\frac{x_j+x_k}{x_i} = \frac{1}{x_j} + \frac{1}{x_k}$ in the presence of the equations $x_iy_i = x_jy_j = x_ky_k = 1$ and $x_0 = 1$, which in turn is clearly equivalent to $x_i = x_jx_k$. By adding constraints $0 \leq y_i \leq 1$ for $i = 1, \dots, n$ we furthermore make $\mathcal{S}'$ bounded without affecting the integrity of the reduction. ◀

3.2 The Gap Satisfiability Problem

In light of Proposition 3, we introduce the following gap version of the satisfiability problem for MIB systems. In this variant we seek a procedure that inputs $\varepsilon > 0$ and a MIB system $\mathcal{S}$ in the form (2) and returns either "UNSAT" or "SAT" subject to the following requirements:

1. If $\mathcal{S}$ has a satisfying assignment with slack ε then the output must be "SAT".
2. If $\mathcal{S}$ is not satisfiable then the output must be "UNSAT".

Note that we place no restriction on the output in the case that $\mathcal{S}$ is satisfiable but with no satisfying assignment having slack ε.

In Section 4 we will show that the Gap Satisfiability Problem is decidable for bounded MIB systems. The following proposition shows the necessity of the boundedness hypothesis.

▶ **Proposition 4.** *The Gap Satisfiability Problem is undecidable for (unbounded) MIB systems.*

Proof. The proof is by reduction from the same variant of Hilbert's Tenth Problem as in the proof of Proposition 3. Recall that an instance of this problem comprises a system $\mathcal{S}$ of equations in positive-integer variables $x_1, \dots, x_n$, with each equation having the form either $x_i = x_j + x_k$ or $x_i = x_jx_k$, where $i, j, k \in \{1, \dots, n\}$. Given such a system, we construct an MIB system $\mathcal{S}'$ over integer variables $x_0, \dots, x_{n+1}$ and real variables $y_0, \dots, y_{n+1}$ such that every satisfying assignment of $\mathcal{S}$ extends to a satisfying assignment of $\mathcal{S}'$ with slack $\frac{1}{2}$ and every satisfying assignment of $\mathcal{S}'$ restricts to a satisfying assignment of $\mathcal{S}$.

We include the equations $x_0 = 1$ and $y_0 = 1$ in $\mathcal{S}'$. Each linear equation $x_i = x_j + x_k$ in $\mathcal{S}$ is carried over to $\mathcal{S}'$. For each equation $x_i = x_jx_k$ in $\mathcal{S}$ we include the inequality $|x_iy_0 - x_jy_k| \leq \frac{1}{2}$ in $\mathcal{S}'$. We then add the following collection of constraints to $\mathcal{S}'$ for all $i \in \{1, \dots, n+1\}$ that intuitively force x_i and y_i to be very close together:

1. $|x_iy_0 - x_0y_i| \leq 1$;
2. $|x_{n+1}y_i - x_iy_{n+1}| \leq 1$;
3. $x_{n+1}y_0 \geq 4(x_0 + x_i)(y_0 + y_i) + 1$.

A satisfying valuation of $\mathcal{S}$ can be extended to a valuation that satisfies $\mathcal{S}'$ with slack $\frac{1}{2}$ by setting $x_0 := 1$, $x_{n+1} := 4\max_{i \in \{1,\dots,n\}}(1 + x_i)^2 + 1$, and $y_i := x_i$ for $i = 0, \dots, n+1$.

Conversely, we claim that every satisfying valuation of $\mathcal{S}'$ (with no assumption on the slack) restricts to a satisfying valuation of $\mathcal{S}$. Indeed, by Item 2, above, for all $k \in \{1, \dots, n\}$ we have

$$|x_{n+1}(x_k - y_k) - x_k(x_{n+1} - y_{n+1})| = |x_{n+1}y_k - x_ky_{n+1}| \overset{(2)}{\leq} 1.$$

By Items 1 and 3, this entails that for all $j \in \{1, \dots, n\}$,

$$|x_k - y_k| \leq \frac{x_k|x_{n+1} - y_{n+1}| + 1}{x_{n+1}} \overset{(1)}{\leq} \frac{x_k + 1}{x_{n+1}} \overset{(3)}{\leq} \frac{1}{4(y_j + 1)} \overset{(1)}{\leq} \frac{1}{4x_j}$$

and hence $|x_jx_k - x_jy_k| \leq \frac{1}{4}$. Combined with $|x_i - x_jy_k| \leq \frac{1}{2}$ we conclude that $|x_i - x_jx_k| \leq \frac{3}{4}$ and hence $x_i = x_jx_k$. ◀

It is shown in [12, Theorem 6] how to solve the Gap Satisfiabililty Problem for a subclass of MIB systems, which we here call *positive*. A positive MIB system has the form

$$\begin{array}{ll} \boldsymbol{x}^\top A_i \boldsymbol{y} \leq b_i & (i = 1, \dots, \ell_1) \\ \boldsymbol{x}^\top A_i \boldsymbol{y} \geq b_i & (i = \ell_1 + 1, \dots, \ell_2) \\ C\boldsymbol{x} \leq \boldsymbol{f},\, \boldsymbol{x} \geq \boldsymbol{0} & \\ E\boldsymbol{y} \leq \boldsymbol{f}, \boldsymbol{y} \geq \boldsymbol{0} & \\ \boldsymbol{x} \in \mathbb{Z}^m, \boldsymbol{y} \in \mathbb{R}^n\,. & \end{array}$$

with all coefficients of A_i being non-negative rational for $i = 1, \dots, \ell_2$. This variant can be solved by a naive relaxing and rounding procedure, which does not require the boundedness assumption. However, while suffiicent to handle MPTA with non-negative rates, positive MIB appear insufficient for the case of MPTA with both positive and negative rates.

4 Decidability in the Bounded Case

4.1 Preliminaries

The following proposition on semilinear sets of integers [23, Corollary 1] will be used on several occasions below:

▶ **Proposition 5.** *Consider a set $S := \{\boldsymbol{x} \in \mathbb{Z}^m : A\boldsymbol{x} \leq \boldsymbol{b}\}$, where the entries of A and $\boldsymbol{b}$ are integers of absolute value at most H and the affine hull of S has dimension d. Then there exists a finite set $B \subseteq \mathbb{Z}^m$ and a matrix $P \in \mathbb{Z}^{m \times d}$ such that*

$$S = L(B, P) := \{\boldsymbol{w} + P\boldsymbol{z} : \boldsymbol{w} \in B,\, \boldsymbol{z} \in \mathbb{Z}^d,\, \boldsymbol{z} \geq \boldsymbol{0}\}$$

and the entries of P and $\boldsymbol{w}$ have absolute value at most $(2 + (m + 1)H)^m$.

We will also need the following result [24, Corollary 3.1] on semialgebraic sets of real numbers. We assume that polynomials are written as lists of monomials with all integers, including exponents, written in binary.

▶ **Proposition 6.** *Let $\{f_i\}_{i \in I}$ be a family of polynomials in n variables whose representation has total bit length at most L. Then the set $S := \{\boldsymbol{x} \in \mathbb{R}^n : \bigwedge_{i \in I} f_i \sim_i 0\}$, where $\sim_i \in \{<, =\}$, is either empty or contains a point of distance at most $2^{L^{8n}}$ to the origin.*

For further analysis it will be useful to transform the MIB problem to a *standard form*, shown in (3) below. In standard form the only linear constraints on the integer variables are that they be nonnegative. Correspondingly we enrich the nonlinear constraints, allowing them to contain an extra linear term in $\boldsymbol{y}$.

$$\begin{array}{ll} \boldsymbol{x}^\top A_i \boldsymbol{y} + \boldsymbol{b}_i^\top \boldsymbol{y} \leq c_i & (i = 1, \dots, \ell) \\ D\boldsymbol{y} \leq \boldsymbol{e} & \\ \boldsymbol{x} \geq \boldsymbol{0} & \\ \boldsymbol{x} \in \mathbb{Z}^m, \boldsymbol{y} \in \mathbb{R}^n\,. & \end{array} \tag{3}$$

The transformation of (2) to standard form is based on writing $S := \{\boldsymbol{x} \in \mathbb{Z}^m : C\boldsymbol{x} \leq \boldsymbol{d}\}$ as a semi-linear set $L(B, P)$, following Proposition 5, where $B \subseteq \mathbb{Z}^m$ and $P \in \mathbb{Z}^{m \times d}$ with d the dimension of the affine hull of S. For each vector $\boldsymbol{w} \in B$ we can apply the change of variables $\boldsymbol{x} = P\boldsymbol{z} + \boldsymbol{w}$ to (2) to obtain a problem in standard form: Thus we obtain a finite collection of problems in standard form, whose solutions are in one-one correspondence with the solutions of the original system (2).

4.2 Relaxation and Rounding

In this section we introduce a relaxed version of a bounded MIB system, in which all variables range over the reals. The relaxation is such that a satisfying assignment to the relaxed problem can be rounded to an integer solution of the original system, while unsatisfiability of the relaxed version permits a branch-and-bound step which leads to an equisatisfiable finite collection of MIB instances in one fewer integer variable.

The rounding is based on an application of the Flatness Theorem in Diophantine approximation – Theorem 7, below. To state this result we first recall some standard terminology related to this. Let $K \subseteq \mathbb{R}^n$ be a convex set and let $\boldsymbol{u} \in \mathbb{Z}^n$. Define the *width of K with respect to $\boldsymbol{u}$* to be

$$\mathrm{width}_{\boldsymbol{u}}(K) := \sup\{\boldsymbol{u}^\top(\boldsymbol{x}-\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y} \in K\}\,.$$

The *lattice width* of K is the minimum width in all directions:

$$\mathrm{width}(K) := \min\{\mathrm{width}_{\boldsymbol{u}}(K) : \boldsymbol{u} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}\}\,.$$

▶ **Theorem 7** (Flatness Theorem). *There exists a constant $\omega(n)$, depending only on n, such that every convex polyhedron $K \subseteq \mathbb{R}^n$ with $\mathrm{width}(K) > \omega(n)$ contains an integer point.*

The constant $\omega(n)$ in Theorem 7 is called the *flatness constant*. The best-known upper bound on $\omega(n) = O(n^{3/2})$ [5], although a linear upper bound was conjectured in [18].

We will need the following proposition about definability of lattice width for classes of polyhedral sets.

▶ **Proposition 8.** *There is a quantifier-free formula in the theory of real closed fields, whose free variables respectively represent a matrix $A \in \mathbb{R}^{n\times m}$, vector $\boldsymbol{b} \in \mathbb{R}^n$, and scalar $c > 0$, that expresses the property $\mathrm{width}_{\boldsymbol{u}}(P) \geq c$ where $P := \{\boldsymbol{x} \in \mathbb{R}^m : A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$.*

Proof. A necessary condition that $\mathrm{width}_{\boldsymbol{u}}(P) \geq c$ is that P be non-empty and hence, since it lies in the positive orthant, contain a vertex. Now each vertex of P, being the intersection of n linearly independent bounding hyperplanes, has the form $B^{-1}\boldsymbol{b}'$, where B is a non-singular $n \times n$ sub-matrix of $\begin{pmatrix} A \\ I_n \end{pmatrix}$, where I_n denotes the identity matrix of dimension n, and $\boldsymbol{b}'$ is a corresponding sub-vector of $\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$. Hence the vertices of P are definable by quantifier-free formulas.

Assume that P contains a vertex. Then $\mathrm{width}_{\boldsymbol{u}}(P)$ is infinite if and only if either $\boldsymbol{u}$ or $-\boldsymbol{u}$ lie in the recession cone of P, for which a sufficient and necessary condition is that $A\boldsymbol{u} \geq \boldsymbol{0}$ or $A\boldsymbol{u} \leq \boldsymbol{0}$. If $\mathrm{width}_{\boldsymbol{u}}(P)$ is finite then there exist two vertices $\boldsymbol{x}_0, \boldsymbol{x}_1$ of P such that $\mathrm{width}_{\boldsymbol{u}}(P) = \boldsymbol{u}^\top(\boldsymbol{x}_0 - \boldsymbol{x}_1)$. The proposition follows by combining the above observations. ◀

We now commence the detailed description of the relaxation construction. The input is a bounded MIB program $\mathcal{S}$ in standard form (3) and a slack $\varepsilon > 0$. Assume that $\mathcal{S}$ has at least one non-linear constraint. We start with the observation that for a given $\boldsymbol{y} \in \mathbb{R}^n$ the system (3) admits a solution $\boldsymbol{x} \in \mathbb{Z}^m$ if and only if the polyhedral set

$$P(\boldsymbol{y}) := \{\boldsymbol{x} \in \mathbb{R}^m : \boldsymbol{x} \geq \boldsymbol{0},\, \boldsymbol{x}^\top A_i \boldsymbol{y} + \boldsymbol{b}_i^\top \boldsymbol{y} \leq c_i,\ i = 1, \ldots, \ell\}\,, \tag{4}$$

contains an integer point.

Let H be an upper bound of the absolute value of the integer constants in the system (3). Since $\mathcal{S}$ is bounded, by [14, Lemma 3.1.25] the set $\{\boldsymbol{y} \in \mathbb{R}^n : D\boldsymbol{y} \leq \boldsymbol{e}\}$ is contained in the ball of radius $\kappa_1 := m^{1/2}H^{(m^2+m)}$ centred at the origin.

For a matrix A, let $\|A\|$ denote the spectral norm. Recall that if A has entries of absolute value at most H and has m columns then $\|A\| \leq \sqrt{m}H$. Now write

$$\delta := \min(\delta_0, 1), \quad \text{where } \delta_0 := \min\left\{\frac{\varepsilon}{\|A_i\|\kappa_1} : i = 1, \ldots, \ell\right\} \geq \frac{\varepsilon}{m^{1/2}H\kappa_1} \tag{5}$$

and define $U := \{\boldsymbol{u} \in \mathbb{Z}^m \setminus \{\boldsymbol{0}\} : 2\delta\|\boldsymbol{u}\| < \omega(m)\}$, where $\omega(m)$ is as in Theorem 7. Write $U = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s\}$ and consider the following *relaxed system* $\mathcal{S}'$ of linear and bilinear constraints in exclusively real variables (where the notation $P(\boldsymbol{y})$ is as in (4) and we use Proposition 8 to formulate the constraint $\mathrm{width}_{\boldsymbol{u}_j}(P(\boldsymbol{y})) \geq \omega(m)$):

$$\begin{aligned}
&\boldsymbol{x}^\top A_i \boldsymbol{y} + \boldsymbol{b}_i^\top \boldsymbol{y} \leq c_i - \varepsilon \qquad (i = 1, \ldots, \ell)\\
&\mathrm{width}_{\boldsymbol{u}_j}(P(\boldsymbol{y})) \geq \omega(m) \qquad (j = 1, \ldots, s)\\
&D\boldsymbol{y} \leq \boldsymbol{e},\ \boldsymbol{x} \geq \boldsymbol{1}\\
&\boldsymbol{x} \in \mathbb{R}^m,\ \boldsymbol{y} \in \mathbb{R}^n
\end{aligned} \tag{6}$$

▶ **Proposition 9.** *If the relaxed system $\mathcal{S}'$ is satisfiable, then so is the original system $\mathcal{S}$.*

Proof. Let $\boldsymbol{x}^*, \boldsymbol{y}^*$ be a solution of the system $\mathcal{S}'$, as shown in (6). Consider the set $P(\boldsymbol{y}^*)$ as defined in (4). By construction we have

$$\min_{\boldsymbol{u} \in U} \mathrm{width}_{\boldsymbol{u}}(P(\boldsymbol{y}^*)) \geq \omega(m)\,. \tag{7}$$

But from the fact $\boldsymbol{x}^*$ satisfies each constraint $\boldsymbol{x}^\top A_i \boldsymbol{y}^* + \boldsymbol{b}_i^\top \boldsymbol{y}^* \leq c_i$ with slack ε and that $\boldsymbol{x}^* \geq \boldsymbol{1}$, we see that the ball $B_\delta(\boldsymbol{x}^*)$ is contained in $P(\boldsymbol{y}^*)$, for δ as defined in (5). It follows that

$$\begin{aligned}
\mathrm{width}_{\boldsymbol{u}}(P(\boldsymbol{y}^*)) \quad &\geq \quad 2\delta\|u\|\\
&\geq \quad \omega(m)
\end{aligned}$$

for all $\boldsymbol{u} \notin U$. Together with (7), we have that $\mathrm{width}(P(\boldsymbol{y}^*)) \geq \omega(m)$ and hence, by Theorem 7, $P(\boldsymbol{y}^*)$ contains an integer point. This entails that the original system $\mathcal{S}$ is satisfiable. ◀

▶ **Proposition 10.** *If the relaxed system $\mathcal{S}'$ has no solution then every solution $\boldsymbol{x}^* \in \mathbb{Z}^m$ of the original system $\mathcal{S}$ that has slack ε either has some component equal to zero or satisfies $|\boldsymbol{u}^\top \boldsymbol{x}^*| \leq \kappa_2$ for some $\boldsymbol{u} \in U$, where κ_2 is an explicit constant depending only on $\mathcal{S}$ and ε.*

Proof. Assume that $\mathcal{S}'$ has no solution. Let $\boldsymbol{x}^* \in \mathbb{Z}^m$ and $\boldsymbol{y}^* \in \mathbb{R}^n$ be a solution of $\mathcal{S}$ with slack ε. If some component of $\boldsymbol{x}^*$ is zero then we are done, so we may suppose that $\boldsymbol{x}^* \geq \boldsymbol{1}$. By assumption, $\boldsymbol{x}^*, \boldsymbol{y}^*$ is not a solution of $\mathcal{S}'$ and so it must hold that

$$\min_{\boldsymbol{u} \in U} \mathrm{width}_{\boldsymbol{u}}(P(\boldsymbol{y}^*)) < \omega(m)\,, \tag{8}$$

where $P(\boldsymbol{y}^*)$ is as defined in (4).

Let $\boldsymbol{u} \in U$ be the vector achieving the minimum on the left-hand side of (8). We will exhibit an upper bound on $|\boldsymbol{u}^\top \boldsymbol{x}^*|$ that does not depend on $\boldsymbol{y}^*$.

Assume first that $P(\boldsymbol{y}^*)$ contains the origin. Then by (8),

$$|\boldsymbol{u}^\top \boldsymbol{x}^*| = |\boldsymbol{u}^\top(\boldsymbol{x}^* - \boldsymbol{0})| \leq \omega(m)\,.$$

Assume now that $P(\boldsymbol{y}^*)$ does not contain the origin. Let L be the line segment connecting the origin to $\boldsymbol{x}^*$, and denote by $\boldsymbol{x}$ the point at which L intersects the boundary of $P(\boldsymbol{y}^*)$. Then we have $\boldsymbol{x}^* - \boldsymbol{x} = \lambda \boldsymbol{x}$ for some $\lambda > 0$. Moreover, since $\boldsymbol{x}$ lies on the boundary of $P(\boldsymbol{y}^*)$ there exists $i_0 \in \{1, \dots, \ell\}$ such that

$$\boldsymbol{x}^\top A_{i_0} \boldsymbol{y}^* + \boldsymbol{b}_{i_0}^\top \boldsymbol{y}^* = c_{i_0} , \tag{9}$$

i.e., one of inequalities that define $P(\boldsymbol{y}^*)$ is tight at $\boldsymbol{x}$. But since $\boldsymbol{x}^*, \boldsymbol{y}^*$ satisfies $\mathcal{S}$ with slack ε, we also have that $(\boldsymbol{x}^*)^\top A_{i_0} \boldsymbol{y}^* + \boldsymbol{b}_{i_0}^\top \boldsymbol{y}^* \leq c_{i_0} - \varepsilon$. Subtracting Equation (9) from the previous inequality gives

$$\begin{aligned} -\varepsilon &\geq (\boldsymbol{x}^* - \boldsymbol{x})^\top A_{i_0} \boldsymbol{y}^* \\ &= \lambda(\boldsymbol{x}^\top A_{i_0} \boldsymbol{y}^*) \\ &= \lambda(c_{i_0} - \boldsymbol{b}_{i_0}^\top \boldsymbol{y}^*) \,. \end{aligned}$$

Since $\varepsilon, \lambda > 0$ this entails that $c_{i_0} - \boldsymbol{b}_{i_0}^\top \boldsymbol{y}^* < 0$ and hence

$$\begin{aligned} \lambda^{-1} &\leq \varepsilon^{-1} |c_{i_0} - \boldsymbol{b}_{i_0}^\top \boldsymbol{y}^*| \\ &\leq \varepsilon^{-1} \left(|c_{i_0}| + \|\boldsymbol{b}_{i_0}\| \kappa_1 \right) \end{aligned} \tag{10}$$

We deduce that

$$\begin{aligned} |\boldsymbol{u}^\top \boldsymbol{x}^*| &\leq |\boldsymbol{u}^\top (\boldsymbol{x}^* - \boldsymbol{x})| + |\boldsymbol{u}^\top \boldsymbol{x}| \\ &= |\boldsymbol{u}^\top (\boldsymbol{x}^* - \boldsymbol{x})| \left(1 + \lambda^{-1}\right) \\ &\leq \omega(m) \left(1 + \varepsilon^{-1}(|c_{i_0}| + \|\boldsymbol{b}_{i_0}\| \kappa_1)\right) \qquad \text{by (8) and (10).} \end{aligned}$$

Thus, defining

$$\kappa_2 := \omega(m) \left(1 + H\varepsilon^{-1}(1 + m^{1/2}\kappa_1)\right) , \tag{11}$$

we have $|\boldsymbol{u}^\top \boldsymbol{x}^*| \leq \kappa_2$.

In summary, we have that $|\boldsymbol{u}^\top \boldsymbol{x}^*| \leq \kappa_2$ for every integer point $\boldsymbol{x}^*$ of $P(\boldsymbol{y}^*)$, as required in the proposition. ◀

4.3 Decision Procedure

In this section we describe a decision procedure for the Gap Satisfiability Problem for bounded MIB systems. This is a recursive procedure based on the relaxation construction in the preceding section. We first present a conceptually simple version of the procedure, with no complexity bound, and then give a more detailed treatment from which bounds can be extracted.

▶ **Theorem 11.** *The Gap Satisfiability Problem is decidable for bounded MIB systems.*

Proof. The procedure to solve the Gap Satisfiability Problem is as follows. Consider an instance of the problem, consisting of an MIB system in the form (3) and slack $\varepsilon > 0$. If there are no non-linear constraints then the problem instance is just a system of linear inequalities in real and integer variables, whose satisfiability is straightforward to discern. Thus we may assume that there is at least one non-linear constraint. We construct the associated relaxed system $\mathcal{S}'$, which has the form (6). Using a decision procedure for the existential theory of real-closed fields we determine whether the system $\mathcal{S}'$ is satisfiable.

If $\mathcal{S}'$ is satisfiable then Proposition 9 guarantees that the original MIB system $\mathcal{S}$ is also satisfiable. We can then find a satisfying assignment of $\mathcal{S}$ by enumerating over all values $\boldsymbol{x}^* \in \mathbb{Z}^m$ and solving a linear program to decide whether there exists $\boldsymbol{y}^* \in \mathbb{R}^n$ such that $\boldsymbol{x}^*, \boldsymbol{y}^*$ satisfies $\mathcal{S}$.

If the relaxed problem has no satisfying assignment then Proposition 10 furnishes a finite set $\mathcal{E}$ of linear equations of the form $\boldsymbol{u}^\top \boldsymbol{x} = b$, with coefficients $\boldsymbol{u} \in \mathbb{Z}^m$ and $b \in \mathbb{Z}$, such that for any solution $\boldsymbol{x}^* \in \mathbb{Z}^m, \boldsymbol{y}^* \in \mathbb{R}^n$ of (2) that has slack ε, the integer part $\boldsymbol{x}^*$ satisfies an equation in $\mathcal{E}$. We iterate through all such equations $\boldsymbol{u}^\top \boldsymbol{x} = b$ and in each case we apply Proposition 5 to write

$$\{\boldsymbol{x} \in \mathbb{Z}^m : \boldsymbol{u}^\top \boldsymbol{x} = b,\, \boldsymbol{x} \geq \mathbf{0}\}$$

as a linear set $L(B, P)$ for some finite set $B \subseteq \mathbb{Z}^m$ and matrix $P \in \mathbb{Z}^{m \times m-1}$. Then for each vector $\boldsymbol{w} \in B$, we apply the change of variables $\boldsymbol{x} = \boldsymbol{w} + P\boldsymbol{z}$ to obtain a MIB system in one fewer integer variable to which we can recursively apply the procedure to determine satisfiability. ◀

In the following result we retrace the proof of Theorem 11, this time keeping track of the size of the integers involved. We thereby obtain an upper bound on the smallest satisfying assignment, showing that the gap satisfiability problem can be solved in nondeterministic exponential time.

▶ **Theorem 12.** *Consider a MIB system (3) in which the integer constants have absolute value at most H. If such a system is satisfiable with slack ε then there is a satisfying assignment under which the integer variables have absolute value at most $2^{\kappa_3^{O(m^3(m+n))}}$, where $\kappa_3 := \left(\frac{mH^{m^2}}{\varepsilon}\right)$.*

Proof. We first analyse the effect of a single variable-elimination step on the size of the integers in the system (3). Recall that to eliminate an integer variable we assert a linear equation $\boldsymbol{u}^\top \boldsymbol{x} = b$, where $\|\boldsymbol{u}\| \leq \frac{2w(m)}{\delta}$ and $|b| \leq \kappa_2$. Combining the lower bound $\delta \geq \frac{\varepsilon}{m^{1/2} H \kappa_1}$ from (5), the definition $\kappa_1 := m^{1/2} H^{(m^2+m)}$, the definition of κ_2 in (11), and the bound $\omega(m) = O(m^{3/2})$, we obtain that $\|\boldsymbol{u}\|, |b| = \kappa_3^{O(1)}$, for $\kappa_3 := \left(\frac{mH^{m^2}}{\varepsilon}\right)$.

Employing Proposition 5, the equation $\boldsymbol{u}^\top \boldsymbol{x} = b$, $\boldsymbol{x} \geq \mathbf{0}$, determines a substitution $\boldsymbol{x} = P\boldsymbol{z} + \boldsymbol{w}$ in which the elements of P and $\boldsymbol{w}$ have absolute value at most $\kappa_3^{O(m)}$. Since there are m integer variables, the constants appearing over all MIB instances arising through the process of variable elimination have absolute value at most $\kappa_3^{O(m^2)}$.

Consider a version of the relaxed system (6) in which the integer constants have magnitude at most $\kappa_3^{O(m^2)}$. For the purposes of our complexity analysis we augment the system with a new variable r and constraints $r \geq \|\boldsymbol{x}\| + 1$ and $r \geq \|\boldsymbol{w} \pm w(m)\boldsymbol{u}\|$ for each vertex $\boldsymbol{w}$ of the polyhedron $P(\boldsymbol{y})$ (as defined in (4)) and $\boldsymbol{u} \in U$. The integer constants in the resulting system have absolute value at most $\kappa_3^{O(m^3)}$ by Hadamard's determinant inequality. By construction, if $\boldsymbol{x}^*, \boldsymbol{y}^*$ is a satisfying assignment of (6) then the convex set $\{\boldsymbol{x} \in P(\boldsymbol{y}^*) : \|\boldsymbol{x}\| \leq r\}$ has lattice width at most $w(m)$ and hence contains an integer point. By Proposition 6 an upper bound for r is $2^{\kappa_3^{O(m^3(m+n))}}$, which concludes the proof. ◀

▶ Remark 13. It is evident that the double exponential dependence of the magnitude of the smallest satisfying assignment on the number of variables in Theorem 12 is unavoidable. Indeed, consider the following MIB system:

$$x_i y_i \leq 1 \quad (i = 1, \ldots, n)$$
$$x_{i+1} y_i \geq x_i y_0 \quad (i = 1, \ldots, n-1)$$
$$x_1 = 2, y_0 = 1$$
$$x_1, \ldots, x_n \in \mathbb{Z}_{\geq 0},\ y_0, \ldots, y_n \in \mathbb{R}_{\geq 0}$$

Then any satisfying assignment satisfies $x_{i+1} \geq \frac{x_i}{y_i} \geq x_i^2$ for $I = 1, \ldots, n-1$, whence $x_n \geq 2^{2^{n-1}}$. The system moreover has a satisfying assignment with slack ε for any $\varepsilon > 0$, obtained by successively setting $y_i := \frac{1+\varepsilon}{x_i}$ and $x_{i+1} := \lfloor \frac{x_i+\varepsilon}{y_i} \rfloor$ for $i = 1, \ldots, n-1$.

Proposition 1 and Corollary 2 give an exponential-time Turing reduction of the Gap Domination Problem for MPTA to the Gap Satisfiability Problem for bounded MIB systems, such that resulting instances of the Gap Satisfiability Problem have size polynomial in that of the input MPTA. We thus obtain our second main result.

▶ **Theorem 14.** *The Gap Domination Problem for MPTA is decidable in non-deterministic exponential time.*

5 Conclusion

Our main result shows that pareto curve of undominated reachable observer values of a given MPTA can be approximated to arbitrary precision. This is in contrast with the situation for weighted timed games, where it was recently shown that the optimal value of a weighted timed game with positive and negative rates cannot be computed to arbitrary precision [15].

Throughout this paper we have worked with MPTA with clock guards defined by conjunctions of non-strict inequalities. However, we claim that for an MPTA $\mathcal{A}$ with guards comprising conjunctions of both strict and non-strict inequalities, there exists an MPTA $\mathcal{A}'$ with exclusively closed guards over the same set $\mathcal{Y}$ of observers, such that every observer valuation $\gamma \in \mathbb{R}^{\mathcal{Y}}$ reachable in $\mathcal{A}$ is also reachable in $\mathcal{A}'$ and, conversely, for every valuation $\gamma' \in \mathbb{R}^{\mathcal{Y}}$ reachable in $\mathcal{A}'$ and every $\varepsilon > 0$ there exists a valuation $\gamma \in \mathbb{R}^{\mathcal{Y}}$ reachable in $\mathcal{A}$ such that $|\gamma(c) - \gamma'(c)| < \varepsilon$ for all $c \in \mathcal{Y}$. Indeed, such an MPTA $\mathcal{A}'$ is obtained by directly applying the closure construction for timed automata in [22, Section 4] to MPTA. Then the ability to compute the pareto curve of undominated reachable observer values of $\mathcal{A}'$ to arbitrary precision allows one to achieve the same end for $\mathcal{A}$.

A direction for future work is to consider the feasibility of approximate pareto analysis over infinite runs of MPTA. For double-priced timed automata, that is, MPTA with a single cost and reward observer, it is known how to compute the optimal reward-to-cost ratio over infinite computations using the corner-point abstraction [8]. For more general MPTA it is natural to consider specifications that refer to multiple reward-to-cost ratios.

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