# A Complete Graphical Language for Linear Optical Circuits with Finite-Photon-Number Sources and Detectors

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**Abstract**

Graphical languages are powerful and useful to represent, rewrite and simplify different kinds of processes. In particular, they have been widely used for quantum processes, improving the state of the art for compilation, simulation and verification. In this work, we focus on one of the main carrier of quantum information and computation: linear optical circuits. We introduce the $\mathbf{LO}_{fi}$-calculus, the first graphical language to reason on the infinite-dimensional photonic space with circuits only composed of the four core elements of linear optics: the phase shifter, the beam splitter, and auxiliary sources and detectors with bounded photon number. First, we study the subfragment of circuits composed of phase shifters and beam splitters, for which we provide the first minimal equational theory. Next, we introduce a rewriting procedure on those $\mathbf{LO}_{fi}$-circuits that converge to normal forms. We prove those forms to be unique, establishing both a novel and unique representation of linear optical processes. Finally, we complement the language with an equational theory that we prove to be complete: two $\mathbf{LO}_{fi}$-circuits represent the same quantum process if and only if one can be transformed into the other with the rules of the $\mathbf{LO}_{fi}$-calculus.

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## 1 Introduction

Quantum computing is a paradigm for processing information [41, 45] that performs computation with quantum states, instead of the classical states of bits. This computational paradigm allows specific computational problems to be solved with quadratic [24] or even exponential speedup [50, 26] compared to their classical counterparts. To encode that quantum data, many technologies have been pursued, such as superconducting circuits [32], trapped ions [8] and cold atoms [23].

One of the prominent candidates for quantum computation is linear optics [36, 42, 47], where the *logical* information is encoded into the quantum states of photons, the *particles* of light. For quantum computation, the logical states are encoded onto the *modes* of the photons, i.e. their degrees of freedom like their *positions* in the circuit, and the desired logical operations are performed with optical components. All scalable quantum computations with linear optics [34, 53, 40, 7, 6, 17] encoding with the positions of the photons use predominantly these following elements.

- Sources: they generate the quantum state, i.e. a vector in a Hilbert space,
- Phase shifters: they change the quantum state by adding a phase to the light passing through them[1],
- Beam splitters: they alter the quantum state by causing photons on two different paths to interfere with each other[2],
- Detectors: they project the quantum state on a subspace.

As ubiquitous as the circuits made of those components are in linear optical quantum computation schemes, as illustrated in Figure 1 and 2, many unanswered questions persist regarding optimality, minimality and an efficient use of those components. We wish to have a framework finding the most appropriate implementation for the desired computation or protocol. The purpose of this work is therefore to propose a formal framework to model and manipulate generic circuits composed of the four previous elements.

**State of the art.** Some main formal frameworks to study, develop or optimize quantum processes are graphical languages [2, 49, 3, 43], representing processes with diagrams and equations between those diagrams. These formalisms have been shown to be very useful for addressing quantum processes in general, such as **ZX**-diagrams [13] with applications in compilation [29, 5, 51], simulation [31, 30, 35] and verification [19, 21]. To completely capture the processes those diagrams model, [28, 25] have introduced a complete set of equations: two equivalent **ZX**-diagrams can always be transformed from one to the other with those equations.

Recently, some works have modeled optical processes with diagrams [4, 12, 39], including notably $\mathbf{LO_v}$ [10], a complete graphical language for linear optical circuits with vacuum sources and detectors, and **QPath** [15], a graphical language to compute amplitudes. Remarkably, both have also led to results beyond the optical realm, as a subfragment of the first led to derive the first complete equational theory for quantum circuits [11], while the second introduced a functor from the **ZX**-calculus [15] and led to a more generic language [16].

However, those two frameworks don't completely capture linear optical circuits with sources and detection schemes. In particular, $\mathbf{LO_v}$ lacks a many-photon semantics and can only cover the single-photon case, while **QPath** uses sums of diagrams in the rewriting process along with generators that are not linear optical components. For instance, we would like to be able to model the photonic implementation of the CZ gate [34, 33], a prominent logical quantum gate, and rewrite it to equivalent forms, as illustrated in Figure 1.

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[1] They are typically implemented using thermo-optic components, where the refractive index of the waveguide is changed by heating the material.

[2] In integrated circuits, beam splitters are implemented using waveguides that split and combine light paths.

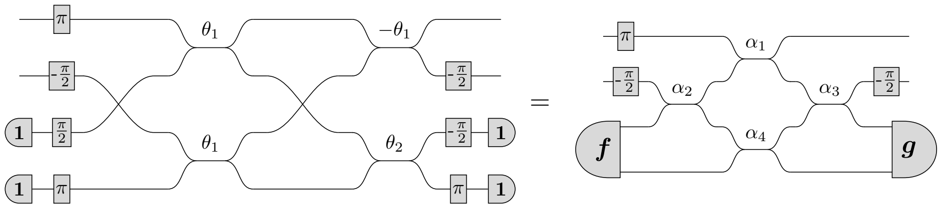



**Figure 1** Optical circuits implementing the CZ two-qubit logic gate with auxiliary sources and detectors. On the left is the original circuit[3] of [33]. There are two auxiliary photon generated on the bottom left: if exactly one photon is detected for each of the two last wires on the bottom right, then we know we have performed the operation $|11\rangle \mapsto -|11\rangle$ on the two first wires. This event has a probability $\frac{2}{27}$ to occur. On the right is an equivalent representation in the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are two-photon states and linear forms.

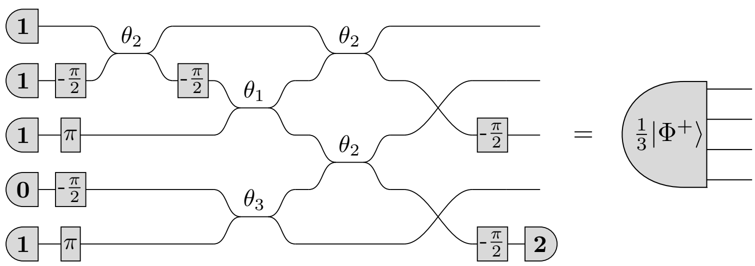



**Figure 2** Linear optical circuit generating with a $\frac{1}{9}$ probability the Bell state $|\Phi^+\rangle = |1010\rangle + |0101\rangle$, with the use of auxiliary sources and detectors. On the left is the original[3] circuit of [20], on the right is an equivalent and modular description. Both are equivalent circuits in the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus.

**Challenges.** In seeking to develop a graphical language for modeling linear optical circuits with a many-photon semantics, there are two main challenges. First, the bosonic Fock space, representing all the states that photons can be in, is an infinite-dimensional Hilbert space: the bosonic Fock space. In particular, some properties and generators of graphical languages with finite-dimensional theories [44, 52, 18] cannot be used. Second, the interaction of photons, even without bringing auxiliary modes and detections into the picture, is described by the permanents of matrices [48, 1], making cumbersome explicit expression and manipulation of all the involved terms.

**Contributions.** In this paper, we propose such a framework, and introduce the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus, the first graphical language defined on the bosonic Fock space, with circuits composed of four core elements of linear optics: the phase shifter, the beam splitter, and auxiliary finite-photon-number sources and detectors. Our main contributions are the following.

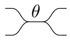
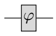
- A complete equational theory for circuits with phase shifters and beam splitters which is simpler than the one in [10], and that we prove to be minimal (Section 2).

[3] Some phases have been added to take into account the different conventions for the semantics of the beam splitters.

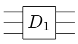
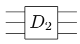
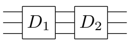
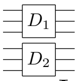
- A new sound and complete equational theory for linear optical circuits that allows all auxiliary finite-photon-number states and detections (Section 3).
- A unique and compact universal form for optical circuits of this kind, obtained through a deterministic rewriting procedure and proven to be unique with new techniques (Section 4).

All the notation introduced in the paper is summarized in Table 1.

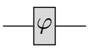
## 2 LOPP: Linear optical quantum circuits with single-photon semantics

A linear optical quantum circuit consists of spatial modes through which photons pass – which may be physically instantiated by optical fibers, waveguides in integrated circuits, or simply by paths in free space (bulk optics) – and operations that act on those spatial modes, including in particular *beam splitters* (  ), and *phase shifters* (  ).

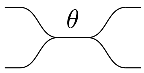
### 2.1 Syntax and single-photon semantics

Similarly to [10], in order to formalize linear optical circuits with diagrams, we use the formalism of PROPs [38]. A PRO is a strict monoidal category whose monoid of objects is freely generated by a single $X$: the objects are all of the form $X \otimes X \otimes \cdots \otimes X$, and simply denoted by $n$, the number of occurrences of $X$. PROs are typically represented graphically as circuits: each copy of $X$ is represented by a wire and morphisms by boxes on wires, so that $\otimes$ is represented vertically and morphism composition $\circ$ is represented horizontally. For instance, $D_1$ and $D_2$ represented as  and  can be horizontally composed as $D_2 \circ D_1$, represented by , and vertically composed as $D_1 \otimes D_2$, represented by . A PROP is the symmetric monoidal analogue of PRO, so it is equipped with a swap. It means the circuits are equivalent through wire deformations and that only the connectivity matters.

▶ **Definition 1.** **LOPP**[4] *is the PROP of* **LOPP***-circuits generated by:*

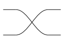


$: 1 \to 1$

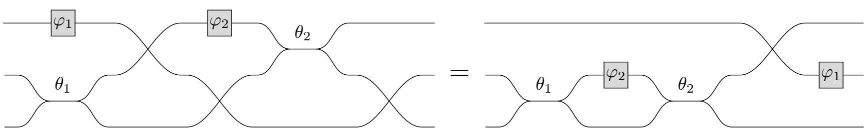


$: 2 \to 2$

*with $\varphi \in \mathbb{R}$ and $\theta \in \mathbb{R}$.*

The convention is to go through from left to right, meaning the inputs (resp. outputs) are on the left (resp. right), and from top to bottom, meaning the first (resp. last) input is the top (resp. last) wire. The identity, the swap and the empty diagrams are noted as follows: , , .

▶ **Example 2.** Here are two examples of **LOPP**-circuits, that are equivalent up to deformation with the rules of PROPs:

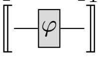



[4] The PROP version of **LOPP** has first been defined in [9], as [10] defined $\mathbf{LO}_{\mathbf{PP}}$ as a PRO.

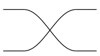
The semantics of linear optical components are usually described by their behavior when there is one single photon passing through those components. Given a circuit of $m$ wires, the single photon can be in a superposition of the $m$ different positions, so its state is a vector in $\mathbb{C}^m$. We consider the standard orthonormal basis $\{|e_i\rangle, i \in [1,m]\}$ where $e_i = |0,\ldots,0,1,0,\ldots,0\rangle$ with a 1 at the $i^{th}$ component. The object of our PROP is therefore $X = \mathbb{C}$, and the vertical composition is interpreted as the direct sum [10, 11]. The semantics is defined as follows.

▶ **Definition 3** (Semantics of **LOPP**). *The single photon semantics of* **LOPP** *is inductively defined as follows:* $[\![C_1 \otimes C_2]\!]_1 = [\![C_1]\!]_1 \oplus [\![C_2]\!]_1$, $[\![C_2 \circ C_1]\!]_1 = [\![C_2]\!]_1 \circ [\![C_1]\!]_1$ *and:*

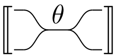
$[\![\ \text{——}\ ]\!]_1 : \mathbb{C} \to \mathbb{C} := |1\rangle \mapsto |1\rangle$




$[\![\ \cdot\ ]\!]_1 : \mathbb{C} \to \mathbb{C} := |1\rangle \mapsto e^{i\varphi}|1\rangle$



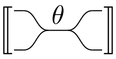
$[\![\ \cdot\ ]\!]_1 : \mathbb{C}^2 \to \mathbb{C}^2 := \begin{array}{lll} |1,0\rangle & \mapsto & |0,1\rangle \\ |0,1\rangle & \mapsto & |1,0\rangle \end{array}$




$[\![\ \cdot\ ]\!]_1 : \mathbb{C}^2 \to \mathbb{C}^2 := \begin{array}{lll} |1,0\rangle & \mapsto & c_\theta |1,0\rangle + is_\theta |0,1\rangle \\ |0,1\rangle & \mapsto & is_\theta |1,0\rangle + c_\theta |0,1\rangle \end{array}$

*where* $c_\theta = \cos(\theta)$ *and* $s_\theta = \sin(\theta)$.

▶ Remark 4. It is also usual to represent those linear operators as matrices, with $[\![C_1]\!]_1 \oplus [\![C_2]\!]_1 = \left(\begin{array}{c|c} [\![C_1]\!]_1 & 0 \\ \hline 0 & [\![C_2]\!]_1 \end{array}\right)$ and for instance




$[\![\ \cdot\ ]\!]_1 = \begin{pmatrix} c_\theta & is_\theta \\ is_\theta & c_\theta \end{pmatrix}$.

## 2.2 Simpler equational theory of LOPP

Two distinct **LOPP**-circuits may represent the same quantum evolution. For instance, shifting the phase of a photon by two phase shifters of phase $\varphi_1$ and $\varphi_2$ is the same as shifting it with one phase $\varphi_1 + \varphi_2$. In order to characterize those equivalences, an equational theory of **LOPP** has been introduced in [10]. In this section, we provide a simpler set of equations in Figure 3. Some of the old equations, given in Figure 4, have been removed, while two Equations (oE2) and (oE3) of Figure 4 have been replaced by the two Equations (E2) and (E3), respectively representing Euler rotations with two and three axes. Previously, those old Euler equations were not directly reversible; while the angles of the right-hand side (RHS) could be uniquely determined by those of the left-hand side (LHS), the inverse was true only with non-trivial constraints, making the equations hardly reversible and not explicitly constructive. More specifically, we made the following simplifications:

- The Equations (b0), (p0) and (pp-b) have been derived and removed from the equational theory.
- A phase has been added in Equation (oE2), so now the LHS can also represent any element of the unitary group $U(2)$. Now the angles of the LHS can be straightforwardly derived without any constraints from the RHS.
- All the phases of Equation (oE3) have been removed. The formulae of the equations are now far simpler, and the equation is now both symmetrical and reversible.

▶ **Definition 5** (**LOPP**-calculus). *Two* **LOPP***-circuits* $D$, $D'$ *are equivalent according to the rules of the* **LOPP***-calculus, denoted* **LOPP** $\vdash D = D'$*, if one can transform* $D$ *into* $D'$ *using the equations given in Figure 3. More precisely,* **LOPP** $\vdash \cdot = \cdot$ *is defined as the smallest congruence which satisfies the equations of Figure 3 and the axioms of PROP.*

▶ **Proposition 6** (Soundness of **LOPP**). *For any* **LOPP***-circuits* $D$ *and* $D'$*, if* **LOPP** $\vdash D = D'$ *then* $[\![D]\!]_1 = [\![D']\!]_1$.

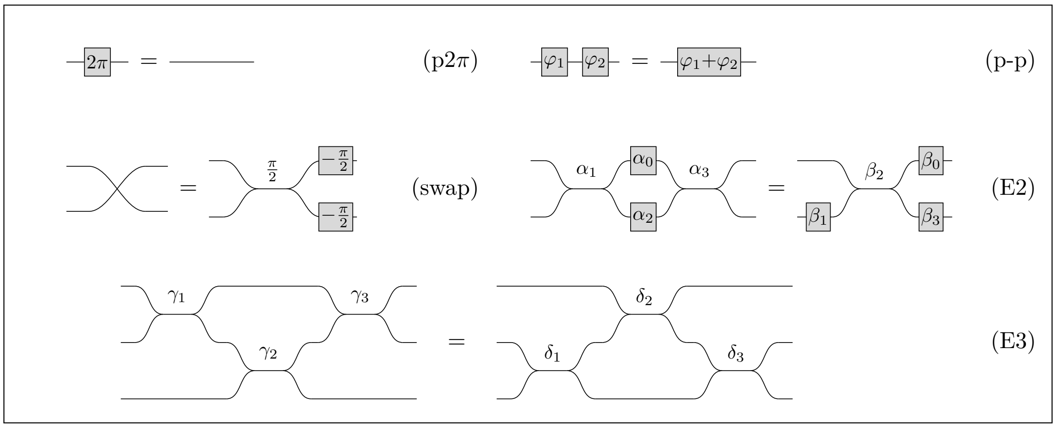
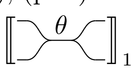
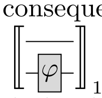



**Figure 3** New and minimal equational theory of the **LOPP**-calculus. For any angle of the LHS (resp. RHS) of the Equation (E2) and (E3), there exist angles for the RHS (resp. LHS) such that the equations are sound. The angles of are unique if we restrict $\alpha_0, \alpha_2, \beta_0, \beta_1, \beta_3 \in [0, 2\pi)$, $\alpha_1 \in [0, \frac{\pi}{2}), \alpha_3 \in [0, \pi)$, $\beta_2 \in [0, \frac{\pi}{2}]$, and by taking $\alpha_1 = 0$ if $\alpha_0 - \alpha_2 = 0 \bmod \pi$ and $\beta_1 = 0$ if $\beta_2 \in \left\{0, \frac{\pi}{2}\right\}$. The rotations associated with Equations (E2) and (E2) are detailed in the proof of Proposition 6.

**Proof.** Since semantic equality is a congruence, it suffices to check that for every equation of Figure 3. The soundness of Equations (swap), (p2π) and (p-p) are direct consequences of Definition 3. We can notice that $R_X(-2\theta) =$  and $e^{i\frac{\varphi}{2}} R_Z(\varphi) =$ , where $R_X$ (resp. $R_Z$) is the rotation operator about the $\hat{x}$ axis (resp. $\hat{z}$ axis) of the Bloch sphere [41]. Any unitary of $U(2)$ can be decomposed into $e^{i\cdot} R_X(\cdot) R_Z(\cdot) R_X(\cdot)$ (resp. $e^{i\cdot} R_Z(\cdot) R_X(\cdot) R_Z(\cdot)$), giving the LHS (resp. RHS) angles of (E2). By transforming the $iY$-axis into the $Y$-axis, the three rotations of the LHS (resp. RHS) of (E3) can be seen as three real rotations along the $z - x - z$ real axes (resp. $x - z - x$). The angles are therefore given by the Euler angles [22]. ◀

▶ **Theorem 7** (Completeness of **LOPP**). *For any two* **LOPP***-circuits $D$ and $D'$, if $[\![D]\!]_1 = [\![D']\!]_1$ then* **LOPP** $\vdash D = D'$.

**Proof.** The equational theory of Figure 4 has been proven to be complete in [10]. All equations of Figure 4 can be derived by those of Figure 3. ◀

▶ **Theorem 8** (Minimality). *The equational theory of Figure 3 is minimal for* **LOPP***-circuits, i.e. none of its equations can be derived from the others.*

**Proof.** We define an alternative interpretation which satisfies all the equations aside from the one we prove to be necessary. In particular:

- (p2π) is the only rule on one wire that changes the sum of the phases.
- (p-p) is the only rule on one wire that can reduce the number of phases different from $2\pi$.
- (swap) is the only rule changing the parity of the number of SWAPs.
- (E2) is the only rule changing the parity of (number of beam splitter + number of SWAPs).

For (E3), here is the sketch of the proof:

- We define an equivalence relation $\sim_\varphi$ on three-wire **LOPP**-circuits.
- We introduce a confluent rewriting procedure that is conserving the relation $\sim_\varphi$, and that is converging to normal forms.

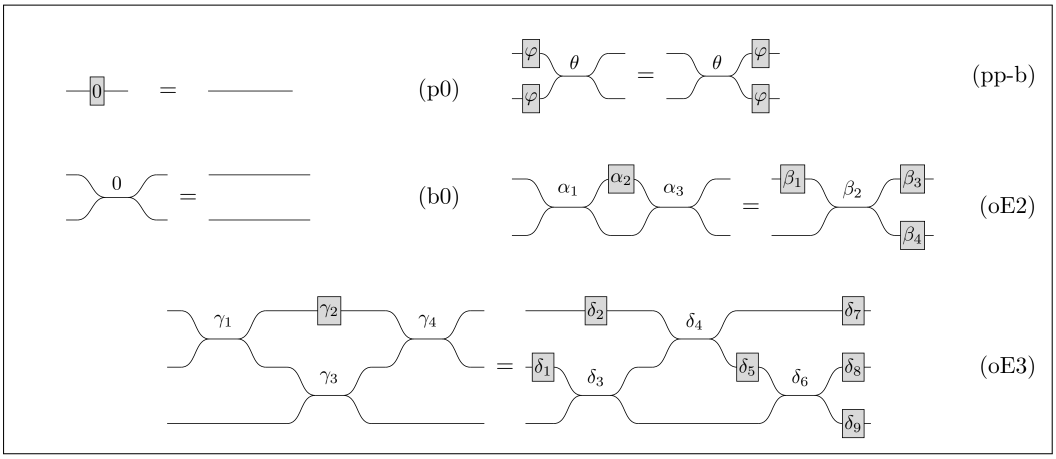


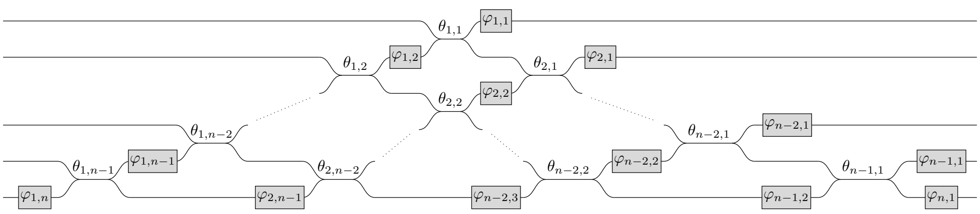

**Figure 4** Old axioms of the **LOPP**-calculus that are not in Figure 3. In Equations (oE2) and (oE3), the LHS circuit has arbitrary parameters which uniquely determine the parameters of the RHS circuit. For any $\alpha_i \in \mathbb{R}$, there exist $\beta_j \in [0, 2\pi)$ such that Equation (oE2) is sound, and for any $\gamma_i \in \mathbb{R}$, there exist $\delta_j \in [0, 2\pi)$ such that Equation (oE3) is sound. We can ensure that the angles $\beta_j$ are unique by assuming that $\beta_1, \beta_2 \in [0, \pi)$ and if $\beta_2 \in \{0, \frac{\pi}{2}\}$ then $\beta_1 = 0$, and we can ensure that the angles $\delta_j$ are unique by assuming that $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in [0, \pi)$. If $\delta_3 \in \{0, \frac{\pi}{2}\}$ then $\delta_1 = 0$, if $\delta_4 \in \{0, \frac{\pi}{2}\}$ then $\delta_2 = 0$, if $\delta_4 = 0$ then $\delta_3 = 0$, and if $\delta_6 \in \{0, \frac{\pi}{2}\}$ then $\delta_5 = 0$. The existence and uniqueness of such $\beta_j$ and $\delta_j$ are given by Lemmas 10 and 11 of [10].

- All the rules of the PROP, (p0), (swap), (p-p) and (E2) also conserve the relation $\sim_\varphi$.
- We conclude that (E3) is necessary, because the LHS and RHS are different normal forms, and therefore can't be transformed from one to the other without (E3). ◀

## 2.3 Useful triangular forms

In this subsection, we introduce three classes of **LOPP**-circuits, with the following inclusions: ${}_{\tilde{n}}\Diamond_n \subset {}_{\tilde{n}}\triangle_{\tilde{m}} \subset \triangle$. Their shape and properties are illustrated and summarized in Table 2. They are of particular interest as $\triangle$-circuits are the normal forms of the **LOPP**-calculus [10], ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuits will be used in the normal forms of the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus (Definition 36), and their uniqueness will be proved thanks to use of ${}_{\tilde{n}}\Diamond_n$-circuits (Section 4).

▶ **Definition 9** ($\triangle$-circuits)**.** *A $\triangle$-circuit is a* **LOPP***-circuit with the following shape:*



*with $\varphi_{i,j} \in [0, 2\pi), \theta_{i,j} \in [0, \frac{\pi}{2}]$ and the following conditions: $\theta_{i,j} = 0 \Rightarrow (\forall j' > j, \varphi_{i,j'} = \theta_{i,j'} = 0)$ and $\theta_{i,j} = \frac{\pi}{2} \Rightarrow \varphi_{i,j} = 0$. $\theta_{i,j}$ is on the $i^{th}$ right (resp. $j^{th}$ left) diagonal, and on the $(i + j - 1)^{th}$ row of beam splitters.*

▶ Remark 10 (Coefficients of $[\![\triangle]\!]_1$). The coefficient $t_{i,j}$ of $[\![\triangle]\!]_1$ is determined by the sum of all the paths from the $j^{th}$ input wire to the $i^{th}$ output wire, where for each path we multiply by a cos (resp. sin) term when the photon is reflected on (resp. transmitted through) a beam splitter, and by a phase when the path crosses a phase shifter. For instance, $t_{1,2} = \cos(\theta_{1,2})e^{i\varphi_{1,2}} i\sin(\theta_{1,1})e^{i\varphi_{1,1}}$. More generally, we have $t_{i,j} = e^{i\varphi_{i,j}}\cos(\theta_{i,j}) \times q_{i,j} + r_{i,j}$ where $q_{i,j}, r_{i,j}$ are terms depending uniquely on the angles with lower indexes.

▶ **Proposition 11** (Universality and Uniqueness of $T$). *For any* **LOPP**-*circuit $D$, there exists a unique circuit $T$ in triangular form of Definition 9 such that* $[\![D]\!]_1 = [\![T]\!]_1$.

**Proof.** The existence is a direct consequence of [46] or the Proposition 26 of [10]. The uniqueness is a consequence of Remark 10 by sequentially showing the uniqueness of $(\varphi_{i,j}, \theta_{i,j})$ in $t_{i,j}$, and by noticing that for any $z$ with $0 < |z| \leq 1$, there are unique $\varphi, \theta \in [0, 2\pi) \times [0, \frac{\pi}{2})$ such that $e^{i\varphi}c_\theta = z$, with $\varphi, \theta = (0,0)$ for the special case of $z = 0$. More details are provided in Appendix B. ◀

▶ Remark 12. A generic $\triangle$-circuit $T : n \to n$ has $\frac{n(n-1)}{2}$ beam splitters and $\frac{n(n+1)}{2}$ phase shifters, having a total of $n^2$ different angles, the dimension of the unitary group $U(n)$.

▶ **Definition 13** (${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuits). *A* **LOPP**-*circuit $\tilde{\triangle} : n + \tilde{n} \to m + \tilde{m}$ is a ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuit if:*

1. *$\tilde{\triangle}$ is a $\triangle$-circuit as defined in Definition 9,*
2. *there is no beam splitter or phase shifter fully and directly connected to the $\tilde{n}$ last input wires, ie. $\varphi_{i,j} = \theta_{i,j} = 0$ if $row_{i,j} = i + j - 1 > n$ and there doesn't exist $(k, \ell)$ such that $k + \ell - 1 = row_{i,j} - 1$, $k < i$ and $\theta_{k,\ell} \neq 0$,*
3. *there is no beam splitter or phase shifter fully and directly connected to the $\tilde{m}$ last output wires, ie. $\varphi_{i,j} = \theta_{i,j} = 0$ if $row_{i,j} = i + j - 1 > m$ and there doesn't exist $(k, \ell)$ such that $k + \ell - 1 = row_{i,j} - 1$, $k \geq i$ and $\theta_{k,\ell} \neq 0$, and*
4. *there exists one nonzero $\theta_{i,j}$ for each of the last $\max(\tilde{n}, \tilde{m})$ rows.*

The Property 4 is an additional constraint that appears in the normal forms defined in Definition 36. Property 2 and 3 imply the only nonzero angles have indexes $(i \leq m, j \leq n)$, leading to the following proposition, direct consequence of Remark 10 and the proof of Proposition 11.

▶ **Proposition 14** (Uniqueness of ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuits on their $m \times n$ submatrix). *For any ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuits $\triangle, \triangle' : n + \tilde{n} \to m + \tilde{m}$, if $[\![\triangle]\!]_1 (1:m, 1:n) = [\![\triangle']\!]_1 (1:m, 1:n)$ then $\triangle = \triangle'$, where $M(1:k, 1:\ell)$ is the $k \times \ell$ matrix composed of the first $k$ rows and $\ell$ columns of $M$.*

▶ **Definition 15** (${}_{\tilde{n}}\lozenge_n$-circuits). *A ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuit $\tilde{\triangle} : n + \tilde{n} \to m + \tilde{m}$ is a ${}_{\tilde{n}}\lozenge_n$-circuit if $\tilde{m} = n$.*

▶ Remark 16. As $\tilde{m} = n$, ${}_{\tilde{n}}\lozenge_n$-circuits have exactly $\tilde{n} \times n$ beam splitters shaped like in Table 2. Furthermore, their angles are necessarily nonzero, as one zero would imply the rest of the right-diagonal to be zero with Definition 9, contradicting the Property 4. That particular shape and those nonzero properties will be useful in the proofs of Section 4.

# 3 $\mathbf{LO}_{fi}$-calculus

## 3.1 Fock space

As described in Section 2.1, the state space of one photon with $m$ spatial modes is $\mathbb{C}^m$, as it can be on a superposition of all the different positions. Photons are particles that can bunch and share the same state, so each mode can be occupied by many photons. Furthermore, to observe quantum effects like interferences, we need the photons to be indistinguishable, meaning we can't know *which photon is in which state*.

For those two reasons, the usual way to represent a state with indistinguishable photons is by using the occupation number representation, where we indicate *"how many photons are there in each state"*. We consider the basis states $\Phi_m \coloneqq \{|n_1, n_2, \dots, n_m\rangle, (n_1, n_2, \dots, n_m) \in \mathbb{N}^m\}$ [1], denoted as *Fock states*. The state $|n_1, n_2, \dots, n_m\rangle$ represents a configuration where $n_i$ photons occupy the $i^{th}$ mode. The space of possible many-photon states over $m$ modes, the *bosonic (symmetrical)* Fock space and denoted as $\mathcal{B}_m$, is defined as follows.

▶ **Definition 17** (Fock space). *We define the Hilbert space $\mathcal{B}_m$ as the Hilbert completion $\ell^2(\Phi_m)$ equipped with the bra-ket inner product $\langle\cdot|\cdot\rangle$, with the convention $\mathcal{B}_0 = \mathbb{C}$.*

▶ Remark 18. $\mathcal{B}_1$ contains states that are an *infinite* superposition of basis states, like the coherent states $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} |k\rangle$. We can note that $\mathcal{B}_m$ is isomorphic to $\ell^2(\mathbb{N}^m)$.

To describe the space of the auxiliary sources, we consider a sub vector space of $\mathcal{B}_m$.

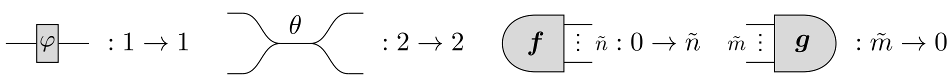
▶ **Definition 19** (Subspace of the Fock space: $\mathcal{B}_m^{\mathrm{pre}}$). *We define the pre-Hilbert space $\mathcal{B}_m^{pre}$ as the linear span of $\Phi_m$ equipped with the bra-ket inner product $\langle\cdot|\cdot\rangle$, with the convention $\mathcal{B}_0^{pre} = \mathbb{C}$.*

▶ Remark 20. $\mathcal{B}_m^{\mathrm{pre}}$ only contain states that are *finite* linear combination of the Fock basis states. In particular, the coherent states are not included. We can note that $\mathcal{B}_1^{\mathrm{pre}}$ is isomorphic to $c_{00}$, i.e. the space of eventually zero sequences.

▶ **Definition 21** ($\mathcal{B}_{\tilde{m}}^{*\mathrm{pre}}$). *We define the pre-Hilbert space $\mathcal{B}_{\tilde{m}}^{*pre}$ as the linear span of $\{\langle n_1, \dots, n_{\tilde{m}}|, (n_1, \dots, n_{\tilde{m}}) \in \mathbb{N}^{\tilde{m}}\}$.*

## 3.2 Syntax and many-photon semantics

▶ **Definition 22** ($\mathbf{LO}_{fi}$-calculus). *$\mathbf{LO}_{fi}$ is the PROP of $\mathbf{LO}_{fi}$-circuits generated by*



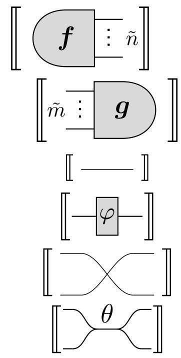

*where $\varphi, \theta \in \mathbb{R}$, and $\boldsymbol{f}$ (resp. $\boldsymbol{g}$) is a state in $\mathcal{B}_{\tilde{n}}^{pre}$ (resp. in $\mathcal{B}_{\tilde{m}}^{*pre}$) with $\tilde{n}, \tilde{m} \in \mathbb{N}^+$.*

▶ Remark 23. In string diagrams, a process $0 \to \tilde{n}$ (resp. $\tilde{m} \to 0$) is called a state (resp. an effect). We will keep the source (resp. detector) terms to be consistent with their physical representation. A process $0 \to 0$ is called a scalar.

▶ Remark 24. The choice of those generators is discussed in Appendix C.

▶ **Definition 25** (Conventions for the notations). *Bold terms will always be vectors. In particular $\boldsymbol{f}, \boldsymbol{f}_k$ (resp. $\boldsymbol{g}, \boldsymbol{g}_\ell$) will always represent a ket (resp. a bra). Bold integers $\boldsymbol{k}$ (resp. $\boldsymbol{\ell}$) will represent $|k\rangle$ (resp. $\langle\ell|$) in the sources (resp. detectors). The summation term $\sum$ will often be omitted, the index being implicitly the sum index. Note that for clarity, the summation term won't be omitted in Figure 6, and for conciseness, they will be omitted in Figure 5. For instance $\boldsymbol{f} = \sum_{k\in\mathcal{K}} |\boldsymbol{f}_k\rangle |k\rangle$ will simply be noted as $\boldsymbol{f}_k \otimes \boldsymbol{k}$. $|.\rangle$ (resp. $\langle.|$) represents an arbitrary ket (resp. bra) on one mode. $|...\rangle$ (resp. $\langle...|$) represents an arbitrary ket (resp. bra) for an arbitrary number of modes, representing an arbitrary scalar when the number of modes is zero. Those are used to omit terms when the specific value of those terms are not of interest, as in some equations of Figure 5. For the zero vector $\boldsymbol{f} = 0$ (resp. $\boldsymbol{g} = 0$), as there is no term in the sum, we chose to represent it with* (∅)— *(resp. an empty detector* —(∅) *). Note it is different from* (**0**)— *(resp.* —(**0**) *) representing the nonzero vector $|0\rangle$ (resp. $\langle 0|$).*

▶ **Definition 26.** *Let $C\colon n \to m$ a $\mathbf{LO}_{\boldsymbol{fi}}$-circuit, let $[\![C]\!] : \mathcal{B}_n \to \mathcal{B}_m$ be the linear map inductively defined as $[\![C_2 \circ C_1]\!] = [\![C_2]\!] \circ [\![C_1]\!]$, $[\![C_1 \otimes C_2]\!] = [\![C_1]\!] \otimes [\![C_2]\!]$ and:*




| | |
|---|---|
| $0 \to \mathcal{B}_{\tilde{n}}$ | $\boldsymbol{f} \in \mathcal{B}_{\tilde{n}}^{pre}$ |
| $\mathcal{B}_{\tilde{m}} \to 0$ | $\boldsymbol{g} \in \mathcal{B}_{\tilde{m}}^{*pre}$ |
| $\mathcal{B}_1 \to \mathcal{B}_1$ | $\lvert k\rangle \mapsto \lvert k\rangle$ |
| $\mathcal{B}_1 \to \mathcal{B}_1$ | $\lvert k\rangle \mapsto P_\varphi(\lvert k\rangle)$ |
| $\mathcal{B}_2 \to \mathcal{B}_2$ | $\lvert k_1, k_2\rangle \mapsto \lvert k_2, k_1\rangle$ |
| $: \mathcal{B}_2 \to \mathcal{B}_2$ | $\lvert k_1, k_2\rangle \mapsto B_\theta(\lvert k_1, k_2\rangle)$ |

*where* $B_\theta(\lvert k_1, k_2\rangle) \coloneqq \sum\limits_{\ell_1+\ell_2=k_1+k_2} \sqrt{\frac{\ell_1!\ell_2!}{k_1!k_2!}} \sum\limits_{\substack{p+q=\ell_1\\ \delta=p-q}} \binom{k_1}{p}\binom{k_2}{q} c_\theta^{k_2+\delta}(is_\theta)^{k_1-\delta} \lvert \ell_1, \ell_2\rangle$ *and* $P_\varphi(\lvert k\rangle) \coloneqq e^{ik\varphi}\lvert k\rangle$*, with the convention* $\binom{k}{k'} = 0$ *for* $k < k'$*.*

We can check that $P_\varphi$ and $B_\theta$ are unitary operators [1] and are therefore well-defined on the Hilbert space by continuity and linearity. One can also look at [36] for a more physical interpretation of where the formulas come from.

▶ Remark 27 (Hermitian conjugate). We have $P_\varphi^\dagger = P_{-\varphi}$ and $B_\theta^\dagger = B_{-\theta}$, where $\dagger$ is the Hermitian conjugate. Therefore, $\langle \ell\rvert P_\varphi = (P_{-\varphi}\lvert \ell\rangle)^\dagger$ and $\langle \ell_1, \ell_2\rvert B_\theta = (B_{-\theta}\lvert \ell_1, \ell_2\rangle)^\dagger$.

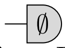
## 3.3 Equational theory of $\mathbf{LO}_{\boldsymbol{fi}}$

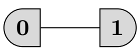
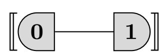
Similarly to Section 2.2, we introduce an equational theory for $\mathbf{LO}_{\boldsymbol{fi}}$ in Figure 5.

▶ **Definition 28** ($\mathbf{LO}_{\boldsymbol{fi}}$-calculus)**.** *Two $\mathbf{LO}_{\boldsymbol{fi}}$-circuits $C, C'$ are equivalent according to the rules of the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus, denoted $\mathbf{LO}_{\boldsymbol{fi}} \vdash C = C'$, if one can transform $C$ into $C'$ using the equations given in Figure 5.*

▶ Remark 29. The Equation (p-p) is not present for it can be derived with the Equations (p2π), (E2) and (s0-0d), alongside with Equation (b0), that can be derived with the rules of the PROP, and the Equations (swap) and (E2). Note that the Equaiton (h2) can be considered an equation of *diagrams with holes.*

▶ **Proposition 30** (Soundness)**.** *For any two $\mathbf{LO}_{\boldsymbol{fi}}$-circuits $C$ and $C'$, if $\mathbf{LO}_{\boldsymbol{fi}} \vdash C = C'$ then $[\![C]\!] = [\![C']\!]$.*

**Proof.** Since semantic equality is a congruence, it suffices to check the soundness for every equation of Figure 5, which follows from Proposition 6 and Definition 26. Informally, Axiom (zero) means that if there is the scalar[5] 0, then the semantics of all the circuit $(X \otimes 0 = 0)$ is the null function. We can therefore nullify the other wires with the zeros  and . This rule is specifically used for Remark 38. Axiom (s-0d) means we can either (from LHS to RHS) project on $\lvert 0\rangle$ on the last mode or (from right to left) add any states $\boldsymbol{f}_k \otimes \lvert k\rangle$ with $k \neq 0$ as they are trivially orthogonal and cancelling. Axiom (h2) means we can *shift* any function $h : \mathcal{B}_2^{\text{pre}} \to \mathcal{B}_2^{\text{pre}}$ from left to right where there are identity wires, direct consequence of the associativity: $\langle \ell_1, \ell_2\rvert (h\lvert k_1, k_2\rangle) = (\langle \ell_1, \ell_2\rvert h)\lvert k_1, k_2\rangle$. The rules (dd), (b-d), (p-d) and (s0-d) are respectively the duals of (ss), (s-b), (s-p) and (s-0d). ◀

[5]  is an impossible event and is one way to represent the scalar $0 = \langle 1|0\rangle =$ .

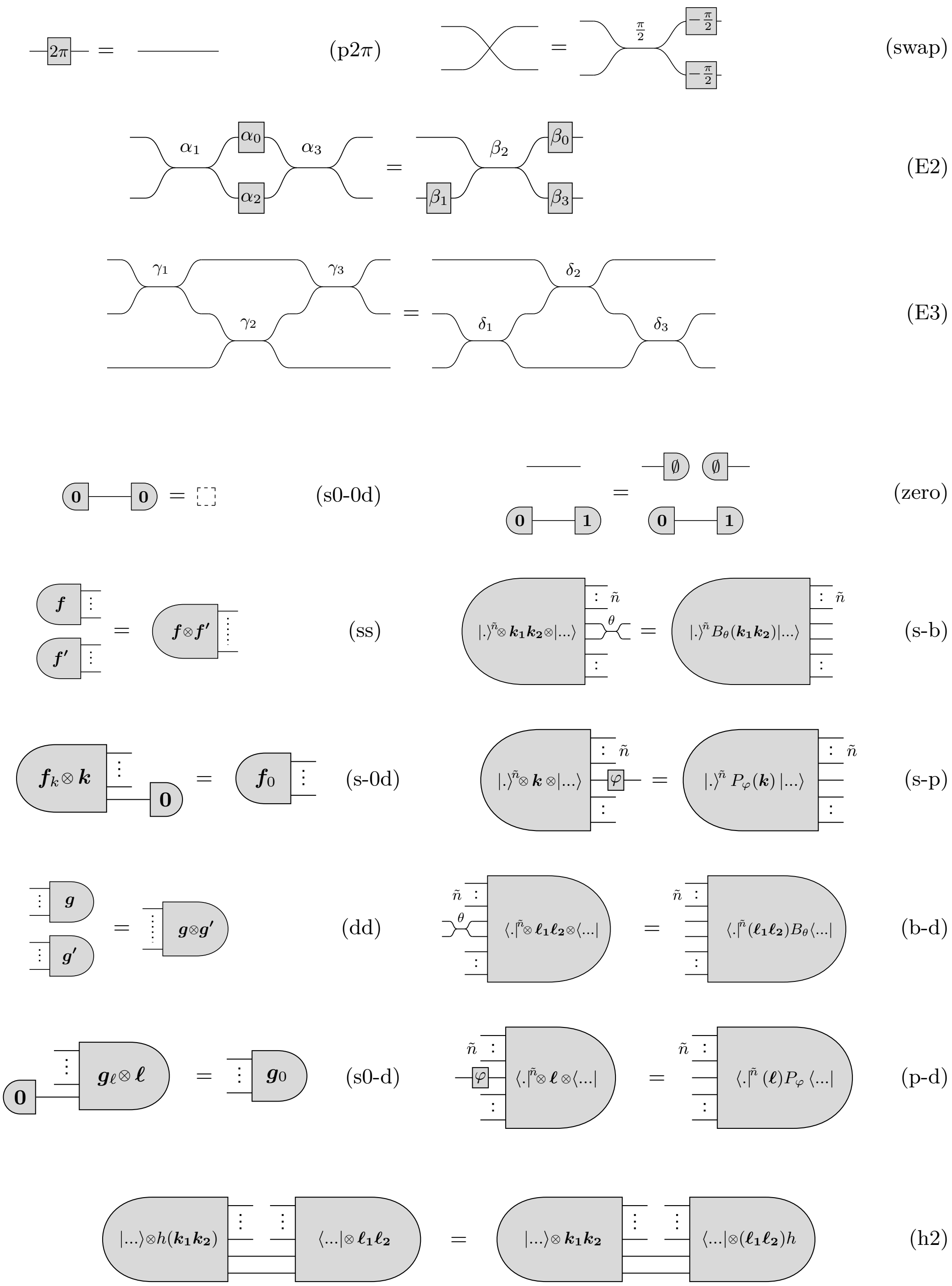



**Figure 5** Axioms of the $\mathbf{LO}_{fi}$-calculus. The angles of (E2) and (E3) are the same as in the axioms of the **LOPP**-calculus (Figure 3). $h$ is any linear function $\mathcal{B}_2^{\mathrm{pre}} \to \mathcal{B}_2^{\mathrm{pre}}$. The conventions for $\{\emptyset, |.\rangle, |...\rangle, \langle.|, \langle...|\}$, and the omitted sums are detailed in Definition 25. The interpretations of the axioms are given in Proposition 30.

▶ **Theorem 31** (Completeness). *For any two* $\mathbf{LO}_{fi}$*-circuits* $C$ *and* $C'$*, if* $[\![C]\!] = [\![C']\!]$ *then* $\mathbf{LO}_{fi} \vdash C = C'$*.*

**Proof.** The proof is in Section 4.4, direct consequence of the uniqueness of the normal forms of Section 4. ◀

# 4 Unique normal forms leading to the completeness of the $\mathbf{LO}_{fi}$-calculus

We introduce a set of oriented rewriting rules in Section 4.1, that converge to a set of $\mathbf{LO}_{fi}$-circuits with specific shape and properties, defined in Section 4.2. The proof of their uniqueness is summarised in Section 4.3. As a direct collorary of the uniqueness of the normal forms, we prove the completeness of the $\mathbf{LO}_{fi}$-calculus in Section 4.4.

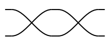
## 4.1 Deterministic rewriting procedure

A strongly normalising rewriting system, i.e. terminating to normal forms, has been introduced in [10] for **LOPP**-circuits. We mainly reuse all the rules, alongside additional rules to now take into account the sources and the detectors.

▶ **Definition 32** (Rewriting system). *Our rewriting system is defined on the PRO*[6] *of* $\mathbf{LO}_{fi}$*-circuits with the rules of Figure 6.*

We can check that all the rules are sound, and have the following meaning:

- The rules 1-11 are either the same or just slightly different from the rules described in [10]. With those rules, the $\mathbf{LOPP}^{PRO}$-circuits will converge to the triangular $\triangle$-circuits defined in Section 2.3.
- The rule 12 removes any vector $|f_{k'}\rangle \otimes |k'\rangle$ in the sources that is trivially cancelling with the detector on the connected last wire, meaning that $\langle g_{k'}| = 0$, i.e. that $k' \notin \mathcal{L}$.
- The rule 13 removes any $\langle g_{\ell'}| \otimes \langle \ell'|$ in the detectors that is trivially cancelling with the source on the connected last wire, meaning that $|f_{\ell'}\rangle = 0$, i.e. that $\ell' \notin \mathcal{K}$.
- The rule 14 allows, without changing the semantics, to transfer the generic coefficients from the detectors to the sources. Specifically, any term of the form $\sum_\ell \xi_\ell \langle \mathcal{N}_{\tilde{m}}(L)| \langle \ell|$ will be rewritten to $\langle \mathcal{N}_{\tilde{m}}(L)| \langle L|$. The coefficients $\xi$ will be in the source, as $|f_L\rangle |L\rangle$ will be rewriten to $\left(\sum_{i\in\mathcal{K}} \xi_i |f_i\rangle\right) |L\rangle$. At the end and by repeating this rule, there won't be any *degree of freedom* in the detectors, and $\boldsymbol{g} = \sum_{\ell\in\mathcal{L}} \langle \mathcal{N}_{\tilde{m}}(\ell)| \langle \ell|$. The condition $(\xi_L \neq 1) \vee (\exists \ell \neq L, \xi_\ell \neq 0)$ is there to ensure that the rule is only used once for each $L$, and only when it's necessary.
- The rule 15 uses the bijection $\mathcal{N}_2 : \mathbb{N} \to \mathbb{N}^2$ to remove one identity wire, by just relabelling the indexes in the sources and detectors. Note that one identity wire will always remain at the end.
- The oriented rule (from left to right) coming from the axioms (ss) and (dd) merge all the sources and detectors together.
- The oriented rule (from left to right) coming from the axioms (s-b), (s-p), (b-d) and (p-d) reduce the number of phase shifters and beam splitters as much as possible, by making them be *absorbed* into the sources and detectors.

[6] This is similar to [10], to prevent any deformation of the form  = .

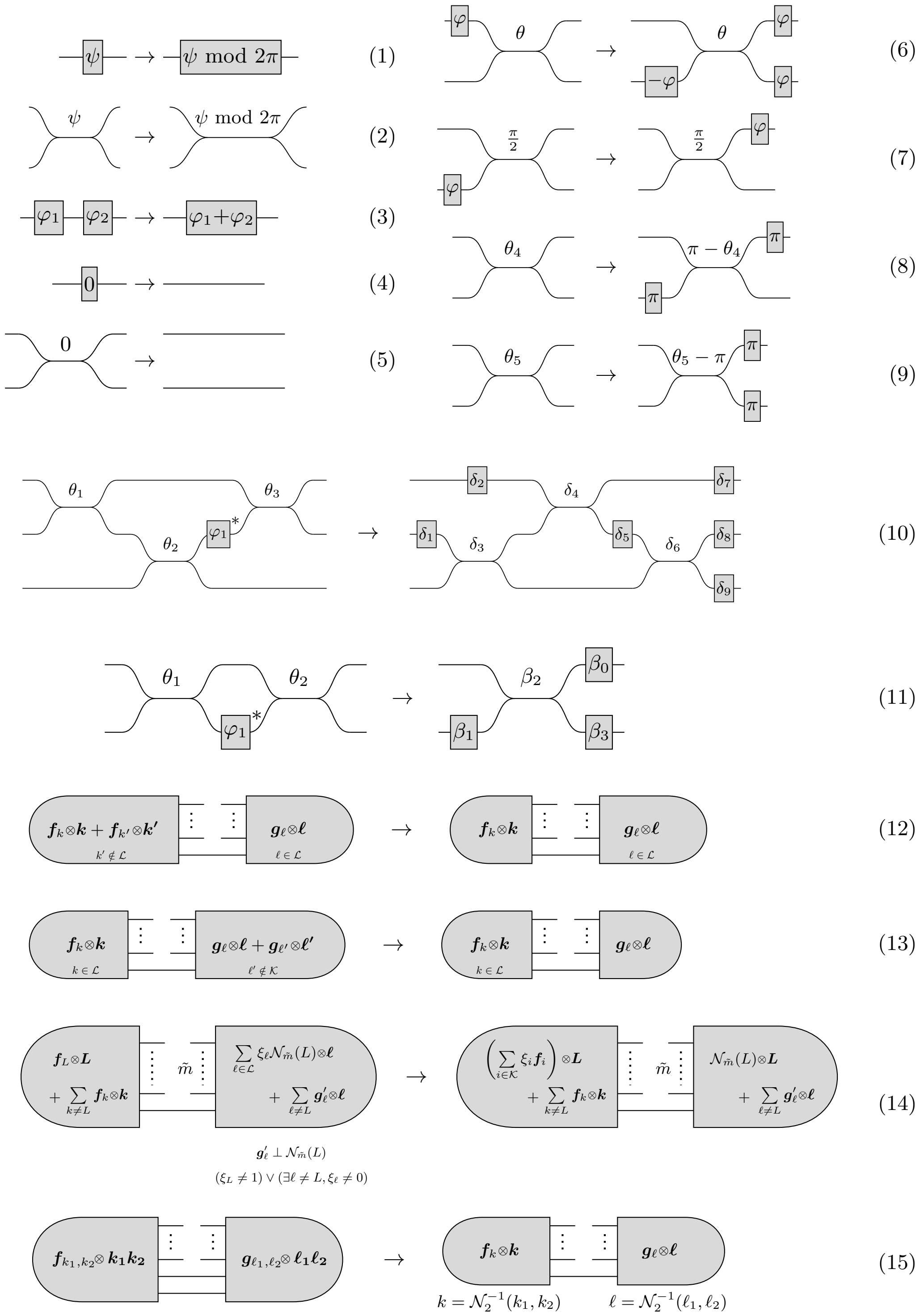



**Figure 6** Rewriting system of the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus, alongside with the oriented version, from the LHS to the RHS, of the axioms (ss), (s-b), (s-p), (dd), (b-d), and (p-d). $\psi \in \mathbb{R} \setminus [0, 2\pi)$, $\varphi, \varphi_1, \varphi_2 \in (0, 2\pi)$, $\theta_4 \in (\frac{\pi}{2}, \pi)$, $\theta_5 \in [\pi, 2\pi)$, $\theta, \theta_1, \theta_2, \theta_3 \in (0, \pi)$. $\overset{*}{\varphi}$ denotes either $\varphi$ or ——— . The angles of the RHS of (11) and (10) are given by [10]. $\mathcal{N}_m : \mathbb{N} \to \mathbb{N}^m$ is a bijection arbitrary chosen to be $\mathcal{N}_m^{-1} := \mathcal{N}_2^{-1} \circ \mathcal{N}_{m-1}^{-1}$ for $m > 2$, where $\mathcal{N}_2^{-1}(\ell, \ell') := \frac{1}{2}(\ell + \ell)(\ell + \ell' + 1) + \ell'$ is the Cantor pairing function and $\mathcal{N}_1$ is the identity. By convention, the summation index is $k \in \mathcal{K}$ for the sources and $\ell \in \mathcal{L}$ for detectors, aside from the rule (14) where the sum is explicit for clarity.

▶ **Definition 33** (Inputs of the rewriting system). *For convenience, the inputs of the rewriting procedures are $\mathbf{LO}_{fi}$-circuits with at least one identity wire connecting sources and generators, and where all the sources (resp. detectors) are on the bottom left (resp. right).*

▶ Remark 34. Note that choice is not restrictive, as the identity wire can always be added with Axiom (s0-0d), and the sources and detectors can be placed at the desired position, without changing the semantics, with the rules of PROP and by adding SWAPs.

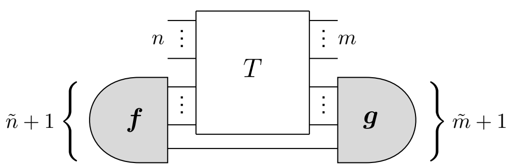
▶ **Lemma 35.** *If $C_1$ rewrites to $C_2$ using the rules of Figure 6, then $\mathbf{LO}_{fi} \vdash C_1 = C_2$.*

**Proof.** As an illustration, we show how we can derive the rule (14) in Appendix D. ◀

## 4.2 Normal forms of the $\mathbf{LO}_{fi}$-calculus

Formally with the rules of Figure 6 and informally with their meaning described in Section 4.1, we can show that an irreducible form is a $\mathbf{LO}_{fi}$-circuit defined as follows:

▶ **Definition 36** (Normal form). *The normal forms of any nonzero $\mathbf{LO}_{fi}$-circuits are denoted $N(T, \boldsymbol{f}) : n \to m$ and are of the form:*




*where:*

- $\boldsymbol{f}$ *is a nonzero generic state of* $\mathcal{B}^{pre}_{\tilde{n}+1}$.
- $\boldsymbol{g} = \sum_{\ell \in \mathcal{K}} \langle \mathcal{N}_m(\ell)| \otimes \langle \ell|$, *where* $\mathcal{N}_m : \mathbb{N} \to \mathbb{N}^m$ *is a bijection defined in Figure 6 and* $\mathcal{K}$ *is the nonempty finite set* $\{k \in \mathbb{N} \mid \boldsymbol{f}_k \neq 0\}$ *of* $\boldsymbol{f} = \sum_{k \in \mathcal{K}} \boldsymbol{f}_k \otimes |k\rangle$, *with the convention* $\mathcal{K} = \{0\}$ *if* $\tilde{n} = 0$ *or* $\tilde{m} = 0$.
- $T : n + \tilde{n} \to m + \tilde{m}$ *is a* ${}_{\tilde{n}}\triangle_{\tilde{m}}$*-circuit as defined in Definition 13.*

▶ Remark 37. If $\tilde{n} = \tilde{m} = 0$, then the normal form is a normal form of **LOPP** (can be [ ] for $n = m = 0$) tensored with the scalar $\alpha|0\rangle$—$\langle 0|$ which has the semantics of a global scalar $\alpha \in \mathbb{C}$.

▶ Remark 38. We could also consider the particular case of $\boldsymbol{f} = 0$, i.e. $\mathcal{K} = \emptyset$, where $[\![N]\!] : \mathcal{B}_n \to \mathcal{B}_m$ is the null function. In that case, $N : n \to m$ can be written to $(\emptyset\!-)^{\otimes m} \circ (-\!\emptyset)^{\otimes n}$, which is a more fitted form for representing the null function.

▶ **Lemma 39** (Strongly normalising). *The rewriting system of Figure 6 is strongly normalising.*

**Proof.** We introduce a ranking function $(x_1, \dots, x_6) \in \mathbb{N}^6$, where each component of the tuple is determined by properties of the circuit, like the number of beam splitter with angles out of $[0, 2\pi)$, the number of sources and detectors, or the number of identity wires connecting them. One nontrivial component is $x_6$, that we explicit here.

Let note the generic terms in the sources as $\boldsymbol{f} = \sum \alpha_{k_1,\dots,k_{\tilde{n}+1}} |k_1, \dots, k_{\tilde{n}+1}\rangle$ and in the detectors as $\boldsymbol{g} = \sum \beta_{\ell_1,\dots,\ell_{\tilde{m}+1}} |\ell_1, \dots, \ell_{\tilde{m}+1}\rangle$. We define:

$$x_6 \coloneqq \sum_{\boldsymbol{f} \in \text{sources}} C_1(\boldsymbol{f}) + \sum_{\boldsymbol{g} \in \text{detectors}} (2C_2(\boldsymbol{g}) - C_3(\boldsymbol{g}))$$

with $C_1(\boldsymbol{f}) := \#\{\alpha_{k_1,\dots,k_{\tilde{n}+1}} \neq 0\}$, $C_2(\boldsymbol{g}) := \#\{\beta_{\ell_1,\dots,\ell_{\tilde{m}+1}} \neq 0\}$, and $C_3(\boldsymbol{g}) := \#\{\beta_{\mathcal{N}_{\tilde{m}}(L),L} = 1, L \in \mathbb{N}\}$. The proof to show that the rule (14) strictly decreases $x_6$ is the following. Let us consider the two cases: $(\xi_L \neq 1) \wedge (\forall \ell \neq L, \xi_\ell = 0)$ and $(\xi_L = 1) \wedge (\exists \ell \neq L, \xi_\ell = 0)$. The first case doesn't change $C_1$ and $C_2$, but the term $-C_3$ strictly decreases by 1. The second case doesn't change $C_3$, and the increase of $C_1$, i.e. the amount of new terms in $\boldsymbol{f}$, is bounded by $\#\{\xi_i \neq 0, i \neq L\}$, the number of terms removed in $\boldsymbol{g}$, which is the exact decrease of $C_2$. Therefore, $C_1 + 2C_2$ decreases by at least $\#\{\xi_i \neq 0, i \neq L\} > 0$. We can conclude that the rule (14) strictly decreases $x_6$. ◀

Now that the normal forms are well-defined, it remains to prove their uniqueness, which is the purpose of the Section 4.3.

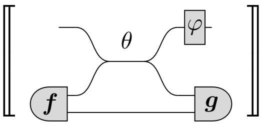
▶ **Lemma 40** (Uniqueness of the Normal Forms). *If two* $\mathbf{LO}_{\boldsymbol{fi}}$*-circuits $N$ and $N'$ in normal forms are such that $[\![N]\!] = [\![N']\!]$, then $N = N'$.*

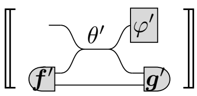
## 4.3 The normal forms are unique: sketch of the proof

Let $N(T, \boldsymbol{f})$ be a normal form. In order to prove the uniqueness of $T$ and $\boldsymbol{f}$, we proceed with the following steps.

1. We first show that $T$ is unique.
2. We introduce a set of operators $\Omega^{\boldsymbol{i},\boldsymbol{j}}(T)$, such that $[\![N]\!] = \sum_{\boldsymbol{i},\boldsymbol{j}} \omega_{\boldsymbol{i},\boldsymbol{j}} \Omega^{\boldsymbol{i},\boldsymbol{j}}(T)$. We show the $\omega_{\boldsymbol{i},\boldsymbol{j}}$ to be canonically and uniquely associated with the coefficients of $\boldsymbol{f}$.
3. We introduce a set of operators $\Delta^{\boldsymbol{u},\boldsymbol{v}}(T)$, that have very convenient properties and that we show to be linearly independent.
4. We give an isomorphism between the $\Omega$ and $\Delta$ operators, therefore proving the linear independence of the $\Omega^{\boldsymbol{i},\boldsymbol{j}}(D)$, and proving the uniqueness of the coefficients of $\boldsymbol{f}$.

▶ **Lemma 41** (Uniqueness of $T$). *For any two normal forms $N(T, \boldsymbol{f})$ and $N'(T', \boldsymbol{f'})$, if $[\![N]\!] = [\![N']\!]$ then $T = T'$.*

**Proof.** For any nonzero $W =$ 

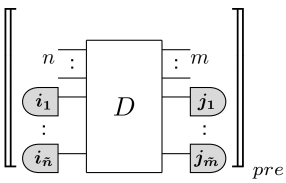
and $W' =$ 

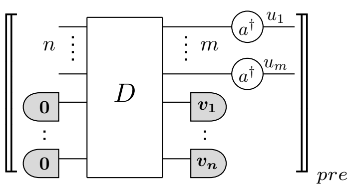
, we first show that:

$$(\theta, \varphi) \neq (\theta', \varphi') \Rightarrow \exists k \in \mathbb{N}, \lim_{n\to\infty} \frac{\langle n+k|\, W\, |n\rangle}{\langle n+k|\, W'\, |n\rangle} \neq 1.$$

As $W = W'$, the limit of the ratio is necessarily equal to 1; the parameters can't be different and are therefore equal. The proof relies purely on the semantics defined in Definition 26. We then prove the uniqueness of $D$ by induction on the min (number of inputs, number of outputs). ◀

▶ **Definition 42** ($\Omega$ and $\Delta$ morphisms). *For any* **LOPP***-circuit $D$, $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{u}, \boldsymbol{v}) \in (\mathbb{N}^{\tilde{n}}, \mathbb{N}^{\tilde{m}}, \mathbb{N}^{\tilde{m}}, \mathbb{N}^{m})$ we define $\Omega^{\boldsymbol{i},\boldsymbol{j}}(D) : n \to m$ and $\Delta^{\boldsymbol{u},\boldsymbol{v}}(D)$ as:*

$\Omega^{\boldsymbol{i},\boldsymbol{j}}(D) :=$ 

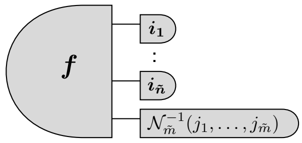
$\Delta^{\boldsymbol{u},\boldsymbol{v}}(D) :=$ 


*where $\hat{a}^\dagger : |k\rangle \mapsto \sqrt{k+1}\,|k+1\rangle$ is the creation operator and $[\![\cdot]\!]_{pre} := [\![\cdot]\!]\,|_{pre}$ is the restriction of $[\![\cdot]\!]$ to $\mathcal{B}^{pre}$.*

▶ Remark 43. All the proofs regarding the $\Omega$ and $\Delta$ morphisms only consider the semantics on $[\![.]\!]_{pre}$. That ensures the soundness of the proofs involving the unbounded operator $\hat{a}^\dagger$, as now all sums will be finite.

We give here two propositions that are the core of the proofs.

▶ **Proposition 44** (Unique $\Omega$-decomposition of the normal forms). *For any ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuit $T : n + \tilde{n} \to m + \tilde{m}$ and finite set $\left\{\omega_{\boldsymbol{i},\boldsymbol{j}}, (\boldsymbol{i}, \boldsymbol{j}) \in (\mathbb{N}^{\tilde{n}}, \mathbb{N}^{\tilde{m}})\right\}$, there exists an unique normal form $N(T, \boldsymbol{f}) : n \to m$, such that $[\![N]\!]_{pre} = \sum_{\boldsymbol{i},\boldsymbol{j} \in (\mathcal{I},\mathcal{J})} \omega_{\boldsymbol{i},\boldsymbol{j}} \Omega^{\boldsymbol{i},\boldsymbol{j}}(T)$.*

**Proof.** It follows from the linearity of $[\![.]\!]_{pre}$ and that $\omega_{\boldsymbol{i},\boldsymbol{j}} =$ 

. ◀

▶ **Proposition 45** (Threshold properties of the $\Delta$-morphisms). *For any ${}_{\tilde{n}}\Diamond_n$-circuit $\tilde{\Diamond} : n+m \to n + m$ and $(\boldsymbol{u}, \boldsymbol{v}) \in (\mathbb{N}^n, \mathbb{N}^m)$, $\langle \boldsymbol{y}| \Delta^{\boldsymbol{u},\boldsymbol{v}}(\tilde{\Diamond}) |\boldsymbol{x}\rangle$ is nonzero for $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{v}, \boldsymbol{u})$ and is zero if $(\boldsymbol{x} \prec_r \boldsymbol{v}) \vee (\boldsymbol{y} \prec_r \boldsymbol{u})$, where $\prec_r$ is the reverse lexicographical order, i.e. $\boldsymbol{y} \prec_r \boldsymbol{v}$ if there exists $k$ such that $y_n = v_n, \ldots, y_{k+1} = v_{k+1}$ and $y_k < v_k$.*

**Proof.** It is a consequence of the shape of the ${}_{\tilde{n}}\Diamond_n$-circuits (Definition 15), and the properties of $\Delta^{\boldsymbol{u},\boldsymbol{v}}$. As there is no photon in the auxiliary sources, the input needs a certain number of photons for them to be detected in the auxiliary detectors. Similarly, as we create photons at the output with the creation operators $\hat{a}^\dagger$, the output needs a certain number of photons. ◀

The linear independence of $\Delta$ will be a consequence of Proposition 45 and a decomposition of the $\Omega$ with $\Delta$ morphisms will give the independence of the $\Omega$, thus the uniqueness of the $\omega_{\boldsymbol{i},\boldsymbol{j}}$, and therefore the uniquess of the normal forms with Proposition 44.

## 4.4 Completeness of the $\mathbf{LO}_{fi}$-calculus: Proof of Theorem 31

Let $C, C'$ two $\mathbf{LO}_{fi}$-circuits such that $[\![C]\!] = [\![C']\!]$. They can be rewritten to normal forms by Lemma 35: $\mathbf{LO}_{fi} \vdash C = N$ and $\mathbf{LO}_{fi} \vdash C' = N'$. By soundness of $\mathbf{LO}_{fi}$, we have $[\![N]\!] = [\![C]\!] = [\![C']\!] = [\![N']\!]$ thus $[\![N]\!] = [\![N']\!]$. By Lemma 35, the normal forms are unique. Therefore, $N = N'$ and we have $\mathbf{LO}_{fi} \vdash\ C = N = N' = C'$, thus $\mathbf{LO}_{fi} \vdash C = C'$, proving the completeness of the $\mathbf{LO}_{fi}$-calculus. ◀

# 5 Outlook

The formalism of the $\mathbf{LO}_{fi}$-calculus helped to find normal forms for linear optical circuits, and the new operators introduced in Section 4 were particularly relevant for proving their uniqueness. It is an open problem to know if those normal forms and new operators can have further applications in simulation, compilation or the synthesis of linear optical circuits, or even broader reach as the **LOPP**-calculus had for quantum circuits [11]. As those normal forms make only sense with finite states, it is also an open problem to determine whether normal forms exist in the infinite case, let alone their uniqueness.

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# A Notations

**Table 1** Notations used in the paper.

| Symbol | Meaning |
|---|---|
| $C, C'$ | $\mathbf{LO}_{fi}$-circuits. |
| $D, D', T, \tilde{\triangle}, \tilde{\Diamond}$ | **LOPP**-circuits, cf Table 2 for the specific classes of circuits. |
| $\varphi, \theta$ | Parameters (angles) of phase shifters and beam splitters |
| $n, m, \tilde{n}, \tilde{m}$ | Integers used for the number of inputs ($n$ or $n+\tilde{n}$) and outputs ($m$ or $m+\tilde{m}$) |
| $i, j, k, \ell, p, q$ | Integers used for indexing. |
| $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y}$ | Fock basis vectors. |
| $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ | Finite set of indexes associated with their lowercase vector. Often omitted in the sums. |
| $\mathcal{B}_m$ | Hilbert space of the bosonic Fock space over $m$ modes, cf Definition 17. |
| $\mathcal{B}_m^{\text{pre}}$ | Pre-Hilbert space of the bosonic Fock space over $m$ modes, cf Definition 19. |
| $\boldsymbol{f}, \boldsymbol{f}'$ | Vectors of $\mathcal{B}^{\text{pre}}$. |
| $\boldsymbol{g}, \boldsymbol{g}'$ | Vectors of $(\mathcal{B}^{\text{pre}})^*$, the dual of $\mathcal{B}^{\text{pre}}$. |
| $\hat{a}_j^\dagger$ | Creation operator over the mode $j$, introduced in Definition 42 |
| $\Omega^{\cdot,\cdot}, \Delta^{\cdot,\cdot}$ | Operators defined in Definition 42. |
| $\prec_r, \preceq_r$ | Reverse lexicographic order on vectors, cf Proposition 45. |
| $\sum, \prod$ | Finite sums and products when the upper bound or the set of indexes is omitted. |

# B Properties of the triangular forms of Section 2.3

## Proof of Proposition 11

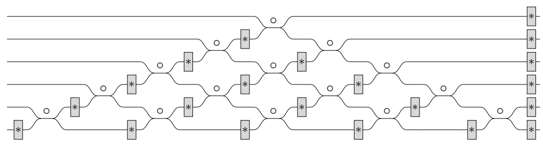
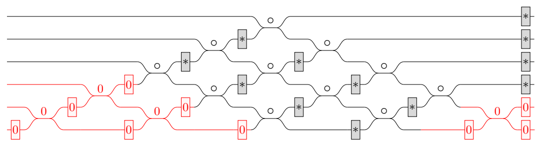
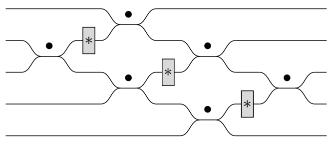
The coefficient $t_{i,j}$ of $[\![\triangle]\!]_1$ is determined by the sum of all the paths from the $j^{th}$ input wire to the $i^{th}$ output wire, where for each path, we multiply by a cos (resp. sin) term when the photon is reflected on (resp. transmitted through) a beam splitter, and by a phase when the path crosses a phase shifter. For instance:

$$\begin{aligned} t_{1,2} &= \cos(\theta_{1,2})e^{i\varphi_{1,2}} i\sin(\theta_{1,1})e^{i\varphi_{1,1}} \text{ and} \\ t_{2,2} &= i\sin(\theta_{1,2})\cos(\theta_{2,2})e^{\varphi_{2,2}} i\sin(\theta_{2,1})e^{i\varphi_{2,1}} + \cos(\theta_{1,2})e^{\varphi_{1,2}} i\sin(\theta_{1,1})\cos(\theta_{2,1})e^{\varphi_{1,2}}. \end{aligned}$$

More generally, we have $t_{i,j} = e^{i\varphi_{i,j}}\cos(\theta_{i,j}) \times q_{i,j} + r_{i,j}$ where $q_{i,j}, r_{i,j}$ are terms depending uniquely on the the angles with lower indexes. We can notice there is at most one path from the $j^{th}$ input wire to the $i^{th}$ output wire involving $\theta_{i,j}$ and $\varphi_{i,j}$ and that $q_{i,j} \neq 0$ if and only if all $\theta_{k<i,j}$ and $\theta_{i,\ell<j}$ are nonzero. If one $\theta_{i,\ell<j}$ is zero, then we have $\varphi_{i,j} = \theta_{i,j} = 0$ by the properties of the $\triangle$-circuits. If there are $K$ values of $\theta_{k<i,j}$ which are zero, then all the $K$ diagonals $\theta_{k,\ell'\geq j}$ are zero. By now considersing the path from the $(j+K)^{th}$ input wire to the $i^{th}$ output wire, we recover the same type of equation with $q_{i,j} \neq 0$. Now, we can substract $r_{i,j}$ and dividing by $q_{i,j}$, so that we have $e^{i\varphi_{i,j}}\cos(\theta_{i,j}) = z_{i,j}$ with $z_{i,j} = (t_{i,j} - r_{i,j})/q_{i,j}$. If $z_{i,j} \neq 0$ then $\theta_{i,j} \in [0, \frac{\pi}{2})$ and $\varphi \in [0, 2\pi)$ are uniquely determined. If $z_{i,j} = 0$, then $\theta_{i,j} = \frac{\pi}{2}$ and by the properties of $\triangle$-circuits, we have $\varphi_{i,j} = 0$.

▶ Remark 46. The existence and the uniqueness have been shown in [10] for very similar circuits that have two minor differences; the phases were on the top left of the beam splitters, and the range of the thetas and phases, expect the last layer, were all in $[0, \pi)$. We can therefore have an alternative proof by changing the strongly normalising and confluent rewriting system so the thetas are always in $[0, \frac{\pi}{2}]$ and the phases stay on the bottom left instead of the top left, without restricting their range.

**Table 2** Shapes and properties of classes of triangle **LOPP**-circuits: $n + \tilde{n} \to m + \tilde{m}$. $(*, \circ)$ are angles in $[0, 2\pi) \times [0, \frac{\pi}{2}]$ that satisfy the properties of Defitions 9 and 13. We emphasis the nonzero angles of $\tilde{\Diamond}$ by noting $\bullet$ an arbitrary angle in $(0, \frac{\pi}{2}]$. The angles which are necessarily zero for the property 3 and 4 of Definition 13 are in red. We have $n = \tilde{n} = 3, m = 4$ and $\tilde{m} = 2$ for the first two figures, and $\tilde{m} = n = 2$ and $\tilde{n} = m = 3$ for the third.

| Shape | Properties |
|---|---|
|  | $\triangle$-circuits (Definition 9)<br><br>Uniquely determined by $[\![\cdot]\!]_1$ (Proposition 11). |
|  | ${}_{\tilde{n}}\triangle_{\tilde{m}}$-circuits (Definition 13)<br><br>Uniquely determined by the submatrix $[\![\cdot]\!]_1\,(1 : m, 1 : n)$ (Proposition 14). Used for the normal forms of $\mathbf{LO}_{\boldsymbol{fi}}$. |
|  | ${}_{\tilde{n}}\Diamond_{n}$-circuits (Definition 15)<br><br>They have exactly $\tilde{n} \times \tilde{m}$ nonzero beam splitters, with no identity wire. They are used in the proofs of Section 4. |

## C Choice of the generators

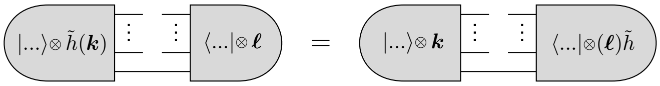
The sources and detectors of the $\mathbf{LO}_{\boldsymbol{fi}}$-calculus allow any and arbitrary finite support state on many modes, which may seem to be too powerful or far from the physical implementation. In that regard, we would like to highlight that:

- Some sources can directly generate more generic states such as a coherent superposition with the vacuum of the 2-photon state [37], or even directly create entangled states [14].
- Linear optical circuits are very modular, and each building block is usually used many times. It would therefore be more convenient to sometimes represent those building blocks directly by specifying what they do, instead of how they are implemented, as illustrated in Figure 2.
- Optical interactions are very combinatorics, thus being unlikely to have a complete equational theory with only single mode sources[7].
- This formalism still allows finding new results for linear optics, like the unique normal forms Section 4.

[7] We can note that [15] bypasses that problem by allowing sums of diagrams

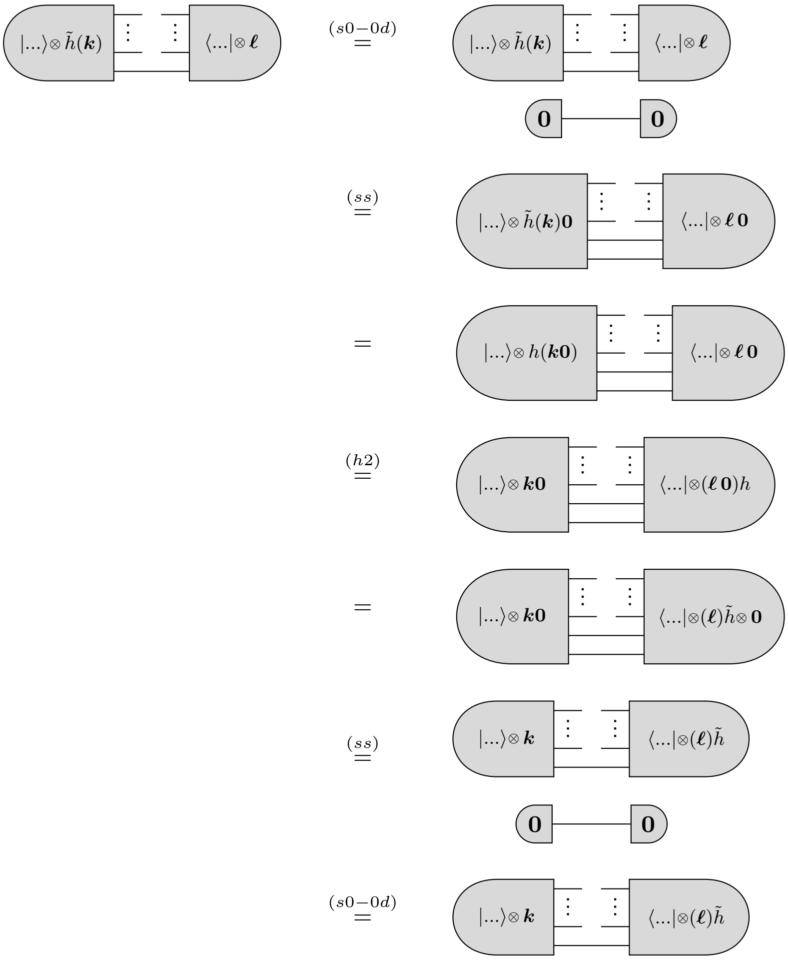
## D Derivation in LOPP of the rule (14)

First, we show that we can derive a similar rule of (h2) but only on one wire:

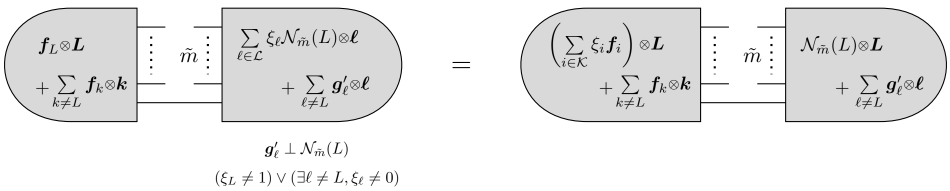



(h1)

To prove it, we consider a linear function $h : \mathcal{B}_2^{\text{pre}} \to \mathcal{B}_2^{\text{pre}}$ such that for every $k \in \mathbb{N}$, $h(|k,0\rangle) = \tilde{h}(|k\rangle)$ and $(\langle k,0|)h = (\langle k|)\tilde{h}$.




▶ **Lemma 47.** *We can derive the equation (14) in the* $\mathbf{LO}_{fi}$*-calculus:*



**Proof.** Let $\langle\psi_L| = \sum_{\ell\in\mathcal{L}} \xi_\ell \langle\ell|$. We have $\boldsymbol{g} = \langle\mathcal{N}_{\tilde{m}}(L)| \langle\psi_L| + \sum_{\ell\in\mathcal{L}\setminus\{L\}} \langle g_\ell| \langle\ell|$. Let $\tilde{h} : \mathcal{B}_1^{\text{pre}} \to \mathcal{B}_1^{\text{pre}}$ be a linear function which is the identity on $|i\rangle$ for every $i \in (\mathcal{K}\cup\mathcal{L})\setminus\{L\}$, such that $\langle L|\,\tilde{h} = \langle\psi_L|$, and zero elsewhere. We can check that $\tilde{h}\,|k\rangle = |k\rangle + \xi_k\,|L\rangle$ for $k \neq L$, and $\tilde{h}\,|L\rangle = \xi_L\,|L\rangle$. We have:

$$\begin{aligned}\boldsymbol{g} &= \langle\mathcal{N}_{\tilde{m}}(\ell)| \langle\psi_L| + \textstyle\sum_{\ell\in\mathcal{L}\setminus\{L\}} \langle g_\ell| \langle\ell| \\ &= \langle\mathcal{N}_{\tilde{m}}(L)| \langle L|\,\tilde{h} + \textstyle\sum_{\ell\in\mathcal{L}\setminus\{L\}} \langle g_\ell| \langle\ell|\,\tilde{h}\end{aligned}$$

The linear function $\tilde{h}$ can therefore be removed with the equation (h1), leading to:

$$\begin{aligned}\boldsymbol{f} &= |f_L\rangle\,\tilde{h}(|L\rangle) + \textstyle\sum_{k\neq L} |f_k\rangle\,\tilde{h}(|k\rangle) \\ &= \xi_L\,|f_L\rangle\,|L\rangle + \textstyle\sum_{k\neq L} |f_k\rangle\,(|k\rangle + \xi_k\,|L\rangle) \\ &= \left(\textstyle\sum_{i\in\mathcal{K}} \xi_i\,|f_i\rangle\right)|L\rangle + \textstyle\sum_{k\neq L} |f_k\rangle\,|k\rangle\end{aligned}$$

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