Playing with Modalities

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Abstract

In this work, we will explore modalities through dialogical game lenses. Games provide a powerful tool for bridging the gap between intended and formal semantics, often offering a more conceptually natural approach to logic than traditional model-theoretic semantics.

We begin by exploring substructural calculi from a game semantic perspective, driven by intuitions about resource-consciousness and, more specifically, cost-sensitive reasoning. The game comes into full swing as we introduce cost labels to assumptions and a corresponding budget. Different proofs of the same end-sequent are interpreted as strategies for a player to defend a claim, which vary in cost. This leads to a labelled calculus, which can be viewed as a fragment of subexponential linear logic. We conclude this first part with a discussion of cut-admissibility for the proposed system.

In the second part, we show that our games offer an interesting insight also into modal logics. More precisely, we will focus on the modal logic **PNL**, characterised by Kripke frames with two types of disjoint and symmetric reachability relations. This framework is motivated by the study of group polarisation, where the opinions or beliefs of individuals within a group become more extreme or polarised after interaction. Our approach to reasoning about group polarisation is based on **PNL** and highlights a different aspect of formal reasoning about the corresponding models – using games and proof systems. We conclude by outlining potential directions for future research.

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1 Introduction

Modalities, both as formal constructs and as tools for reasoning, have been central to the development of logic and proof theory. In this work, we explore modalities through the lens of dialogical games, emphasising their potential to bridge the gap between formal semantics and conceptual intuition. Games not only offer a dynamic perspective on logical systems but also serve as a unifying framework for analysing the structure of proofs and resource management in a variety of logical settings.

We begin by examining substructural calculi, inspired by resource-sensitive reasoning. We introduce the concept of *prices* for resources (represented by formulas) into the game using the unary operator $!^a$, $a \in \mathbb{R}^+$, which shares some characteristic features with *subexponentials* in linear logic LL (SELL [14, 32]). Intuitively, a formula $!^a A$ represents a *permanent resource*: from $!^a A$, we can derive A as many times as needed, paying the price a each time.

We extend our game to this enriched language by incorporating a *budget* into the game states, which decreases whenever a price is paid. Different strategies for proving the same end-sequent can then be evaluated based on the budget required to execute them safely, *i.e.*, without incurring debt. This approach to resource-consciousness not only enhances the game but also translates naturally into a sequent system, where cost bounds for proofs are expressed as labels attached to sequents. By associating costs with proof steps, we provide a fine-grained analysis of proof strategies and their computational bounds.

We note that, up to this point, the content summarises the work presented in [28], where resources were considered only in *assumptions*. In this setting, sequents are restricted by limiting the occurrences of the modality $!^a$ *negatively*, thereby eliminating the need for a promotion rule.

In Section 2.2, we introduce new perspectives by allowing modalities in positive contexts. This includes the addition of "worse costs," linearisation of the cut formula, and tracking the use of contraction during the cut-elimination process.

In the second part of this paper, we present an overview of our work in [22], going beyond resource-awareness, and showing how games can illuminate modal logics. Specifically, we focus on the positive-negative modal logic (**PNL** [47]), characterised by Kripke frames with two disjoint and symmetric reachability relations. In **PNL**, individuals in a social network are identified with worlds of the frame, and the associated relations represent either "friends" (positive) or as "enemies" (negative). These relationships can be understood in different ways: Instead of genuine friendship or enduring enmity, they may simply mean agreement or disagreement on a particular issue. Our interest in **PNL** stems from its application in modelling phenomena such as group polarisation, where interactions amplify the extremity of opinions within a network. We show how the dialogical game lenses lead to both a semantic game and a provability game for (hybrid) extensions of **PNL**.

In semantic games [25], each instance is played over a formula F and a model $\mathbb{M}$ by two players, traditionally called I (or *Me*) and *You*. At every point in the game, one player acts as the proponent (**P**), while the other acts as the opponent (**O**) of the current formula. The set of actions at each stage is determined by the main connective of the current formula.

In contrast, provability games [29] do not concern truth in a specific model but rather *logical validity*. These games are also played by two participants, *Me* and *You*, and involve attacking assertions of formulas made by the other player and defending against these attacks.

We conclude this summary by showing how to transform the semantic game over single models into a provability game that characterises logical validity. This transformation led to *the first* Gentzen-style systems for variants of **PNL**, which modularly adapt to different frame properties by faithfully capturing the rules for *elementary* games.

Each part concludes with a discussion of future research directions and methodologies for combining and adapting the frameworks presented here to other logics and systems.

2 A game model for costs

Our starting point is a calculus for *affine intuitionistic linear logic* (aILL) [24]. Formulas in aILL are built from the grammar

$$A ::= p \mid \mathbf{0} \mid \mathbf{1} \mid A_1 \,\&\, A_2 \mid A_1 \oplus A_2 \mid A_1 \otimes A_2 \mid A_1 \multimap A_2 \mid\, !\,A.$$

with a denumerable infinite set of propositional variables $\{p, q, r, \ldots\}$, the units $\{\mathbf{0}, \mathbf{1}\}$, the binary connectives for additive conjunction and disjunction $\{\&, \oplus\}$, the multiplicative conjunction $\otimes$, the linear implication $\multimap$, and the exponential !.

Similar to modal connectives, the exponential ! in linear logic is not *canonical*, in the sense that, even having the same scheme for introduction rules, marking the exponentials with different labels does not preserve equivalence. That is, if $i \neq j$ then $!^i A \not\equiv\, !^j A$. Intuitively, this means that we can mark the exponential with *labels* taken from a set $\mathcal{I}$ organized in a pre-order $\preceq$ (*i.e.*, a reflexive and transitive relation), obtaining (possibly infinitely-many) exponentials $!^i$ for $i \in \mathcal{I}$. These are called *subexponentials* [14], and the respective proof system for linear logic with subexponentials is called SELL [33]. As in multi-modal systems, the pre-order determines the provability relation: for a general formula A, $!^b A$ *implies* $!^a A$ iff $a \preceq b$. Pre-ordering the labels (together with an upward closeness requirement) guarantees cut-elimination in SELL [14].

The algebraic structure of subexponentials, combined with their intrinsic structural properties (weakening and contraction) allow for the proposal of rich linear logic based frameworks. This opened a venue for proposing different multi-modal substructural logical systems [46], that encountered a number of different applications (see [37] for a survey).

In this paper, we will use subexponentials to model the notion of *costs*. We will start by considering the particular case where labels will be elements of $\mathbb{R}^+$, the set of non-negative real numbers, with the usual pre-order $\leq$. Formally, we substitute in aILL the exponential ! by the unary modal operators $!^a$ for each $a \in \mathbb{R}^+$.

We shall use A, B, C (resp. Γ, Δ) to range over formulas (resp. multisets of formulas). Sequents have the form $\Gamma \Rightarrow C$ where subformulas $!^a A$ will have a restriction to occur only *negatively* in the sequent.[2] We denote by $!\Gamma$ a set of formulas prefixed with $!^a$ for some (not necessarily the same) $a \in \mathbb{R}^+$.

The rules for the system $\mathcal{C}(\mathbb{R}^+)$ are depicted in Figure 1. Note that the cut rule is not included in our presentation of $\mathcal{C}$ and that weakening is present only implicitly, via the context Γ in the initial sequents. Furthermore, in rule *init*, p is a propositional variable and there is no right rule for ! in $\mathcal{C}(\mathbb{R}^+)$ since this connective only appears in negative polarity. We shall write $\vdash_{\mathcal{C}(\mathbb{R}^+)} S$ if the sequent S is provable in $\mathcal{C}(\mathbb{R}^+)$.

[2] The notion of polarity is the standard one: A subformula occurrence in the antecedent of a sequent is *negative* if it occurs in the scope of an even number (including 0) of contexts $([\cdot] \multimap B)$, and otherwise it is *positive*. For occurrences of a subformula in the consequent, one replaces "even" by "odd". The reason for this restriction will be made clear in Section 2.2.

$$\frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \otimes B \Rightarrow C} \otimes_L \qquad \frac{!\Gamma, \Delta_1 \Rightarrow A \quad !\Gamma, \Delta_2 \Rightarrow B}{!\Gamma, \Delta_1, \Delta_2 \Rightarrow A \otimes B} \otimes_R$$

$$\frac{!\Gamma, \Delta_1 \Rightarrow A \quad !\Gamma, \Delta_2, B \Rightarrow C}{!\Gamma, \Delta_1, \Delta_2, A \multimap B \Rightarrow C} \multimap_L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \multimap_R \qquad \frac{\Gamma, !^a A, A \Rightarrow C}{\Gamma, !^a A \Rightarrow C} !_L$$

$$\frac{\Gamma, A_i \Rightarrow B}{\Gamma, A_1 \,\&\, A_2 \Rightarrow B} \&_{L_i} \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B} \&_R \qquad \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \oplus B \Rightarrow C} \oplus_L \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \oplus A_2} \oplus_{R_i}$$

$$\frac{}{\Gamma, p \Rightarrow p}\, init \qquad \frac{}{\Gamma \Rightarrow \mathbf{1}}\, \mathbf{1}_R \qquad \frac{}{\Gamma, \mathbf{0} \Rightarrow C}\, \mathbf{0}_L$$

Figure 1 The sequent system $\mathcal{C}(\mathbb{R}^+)$.

2.1 Playing with subexponentials

We shall characterize $\mathcal{C}(\mathbb{R}^+)$ proofs as winning strategies (w.s.) in a two-player game, the players denoted **P** and **O**. As usual, we will interpret bottom-up proof search in sequent systems as a game where, at any given state, player **P** first chooses a formula of a sequent and, in the next step:

- if the rule has only one premise: **P** moves to the premise sequent of the corresponding introduction rule;
- if the rule has two premises either
  (i) player **O** chooses a premise sequent in which the game continues; or
  (ii) the game splits into independent subgames, where **P** has to win all of them if she wants to win the game.

The choice between (*i*) and (*ii*) depends on the nature of the rule: branching in *additive rules* is modelled as choices made by **O**, while branching in *multiplicative rules* involves **P** splitting the context into two disjoint parts, which then serve as the corresponding contexts for two subgames played in parallel. Consequently, the state of the game is represented by a *multiset of sequents*, with each sequent belonging to a distinct subgame.

Now, to capture the notion of *costs*, game states include a *budget* (modelled as a real number) that decreases whenever the rule $!_L$ is applied. This implies a cost a is incurred during dereliction, *i.e.*, when unpacking a formula stored within the modality $!^a$. Formally we have the following.

▶ **Definition 1** (The game $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$). *$\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ is a game of two players,* **P** *and* **O**. *Game states are tuples (H, b), where H is a finite multiset of sequents and $b \in \mathbb{R}$ is a "budget". $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ proceeds in rounds, initiated by* **P***'s selection of a sequent S from the current game state. The successor state is determined according to rules that fit one of the two following schemes:*

(1) $(G \cup \{S\}, b) \quad \rightsquigarrow \quad (G \cup \{S'\}, b')$
(2) $(G \cup \{S\}, b) \quad \rightsquigarrow \quad (G \cup \{S^1\} \cup \{S^2\}, b)$

A round proceeds as follows: After **P** *has chosen a sequent $S \in H$ among the current game state, she chooses a rule instance r of $\mathcal{C}(\mathbb{R}^+)$ such that S is the conclusion of that rule. Depending on r, the round proceeds as follows:*

1. *If r is a unary rule different from $!_L$ with premise S', then the game proceeds in the game state $(G \cup \{S'\}, b)$.*
2. Budget decrease: *If $r = !_L$ with premise S' and principal formula $!^a A$, then the game proceeds in the game state $(G \cup \{S'\}, b - a)$.*

3. Parallelism: *If r is a binary rule with premises S_1, S_2 pertaining to a* multiplicative *connective, then the game proceeds as $(G \cup \{S_1\} \cup \{S_2\}, b)$.*
4. **O**-choice: *If r is a binary rule with premises S_1, S_2 pertaining to an* additive *connective, then* **O** *chooses $S' \in \{S_1, S_2\}$ and the game proceeds in the game state $(G \cup \{S'\}, b)$.*

A winning state *(for* **P***) is a game state (H, b) such that all $S \in H$ are initial sequents of $\mathcal{C}(\mathbb{R}^+)$ and $b \geq 0$.*

▶ **Definition 2** (Plays and strategies). *A* play *of $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ on a game state (H, b) is a sequence $(H_1, b_1), (H_2, b_2), \ldots, (H_n, b_n)$ of game states, where $(H_1, b_1) = (H, b)$ and each (H_{i+1}, b_{i+1}) arises by playing one round on (H_i, b_i). A* strategy *(for* **P***) on a game state (H, b) is defined as a function telling* **P** *how to move in any given state. A strategy on (H, b) is a* winning strategy (w.s.) *if all plays following it eventually reach a winning state. We write $\models_{\mathcal{G}_\mathcal{C}(\mathbb{R}^+)} (H, b)$ if* **P** *has a w.s. in the $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$-game starting on (H, b).*

The intuitive reading of $\models_{\mathcal{G}_\mathcal{C}(\mathbb{R}^+)} (H, b)$ is: The budget b suffices to win the game H.

▶ **Example 3.** Consider the following well-known riddle:

> You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

We can model the matching pair by the disjunction $(w \otimes w) \oplus (b \otimes b)$, and the act of drawing a random sock by the labelled formula $!^1(w \oplus b)$. The above question then becomes:

> What is the least budget n such that $\models_{\mathcal{G}_\mathcal{C}(\mathbb{R}^+)} (!^1(w \oplus b) \Rightarrow (w \otimes w) \oplus (b \otimes b), n)$?

The following play illustrates that $n = 3$ suffices, where $F = (w \otimes w) \oplus (b \otimes b)$ and $G = !^1(w \oplus b)$:
1. $(\{G \Rightarrow F\}, 3)$
2. $(\{G, w \oplus b, w \oplus b, w \oplus b \Rightarrow F\}, 0)$ (**P** plays $!^1{}_L$ $3\times$, budget decrease)
3. $(\{G, w, w \oplus b, w \oplus b \Rightarrow F\}, 0)$ (**O** chooses w)
4. $(\{G, w, b, w \oplus b \Rightarrow F\}, 0)$ (**O** chooses b)
5. $(\{G, w, b, b \Rightarrow F\}, 0)$ (**O** chooses b)
6. $(\{G, w, b, b \Rightarrow b \otimes b\}, 0)$ (**P** plays $\oplus_{R_2}$)
7. $(\{G, w, b \Rightarrow b\} \cup \{G, b \Rightarrow b\}, 0)$ (**P** plays $\otimes_R$, parallelism)

The other possible choices for **O** are similar or simpler, and show that $n = 2$ is not enough for winning the game.

We note that it is not necessary to consider all possible strategies in $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$: For example, **P** never needs to take the budget into account when deciding the next move. Also, it is easy to see that a $\mathcal{C}(\mathbb{R}^+)$-proof Ξ of a sequent S translates to a w.s. in $(\{S\}, b)$ for some *sufficiently large* budget b. Taking these observations together, one can prove the following.

▶ **Theorem 4** (Weak adequacy for $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ [28]). *Let S be a sequent. Then*

$$\exists b \left(\models_{\mathcal{G}_\mathcal{C}(\mathbb{R}^+)} (\{S\}, b) \right) \quad \textit{iff} \quad \vdash_{\mathcal{C}(\mathbb{R}^+)} S$$

This is a *weak* adequacy since information about the budget b is lost in the proof theoretic representation. In other words, the game $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ is more expressive than the calculus $\mathcal{C}(\mathbb{R}^+)$.

To overcome this discrepancy, we introduce a labelled extension of $\mathcal{C}(\mathbb{R}^+)$ that we call $\mathcal{C}^\ell(\mathbb{R}^+)$. A $\mathcal{C}^\ell(\mathbb{R}^+)$-proof is build from labelled sequents $b : \Gamma \Rightarrow A$ where $\Gamma \Rightarrow A$ is a sequent and $b \in \mathbb{R}^+$. The complete system is given in Figure 2. Now we can prove the desired correspondence.

labelled sequent system for $\mathcal{C}^\ell(\mathbb{R}^+)$

$$\frac{b : \Gamma, A, B \Rightarrow C}{b : \Gamma, A \otimes B \Rightarrow C}\otimes_L \qquad \frac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2 \Rightarrow B}{a + b : !\Gamma, \Delta_1, \Delta_2 \Rightarrow A \otimes B}\otimes_R$$

$$\frac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, B \Rightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2, A \multimap B \Rightarrow C}\multimap_L \qquad \frac{b : \Gamma, A \Rightarrow B}{b : \Gamma \Rightarrow A \multimap B}\multimap_R$$

$$\frac{b : \Gamma, A_i \Rightarrow B}{b : \Gamma, A_1 \& A_2 \Rightarrow B}\&_{L_i} \qquad \frac{a : \Gamma \Rightarrow A \quad b : \Gamma \Rightarrow B}{\max\{a, b\} : \Gamma \Rightarrow A \& B}\&_R$$

$$\frac{a : \Gamma, A \Rightarrow C \quad b : \Gamma, B \Rightarrow C}{\max\{a, b\} : \Gamma, A \oplus B \Rightarrow C}\oplus_L \qquad \frac{b : \Gamma \Rightarrow A_i}{b : \Gamma \Rightarrow A_1 \oplus A_2}\oplus_{R_i}$$

$$\frac{c : \Gamma, !^a A, A \Rightarrow C}{c + a : \Gamma, !^a A \Rightarrow C}\,!^a{}_L$$

$$\frac{}{0 : \Gamma, p \Rightarrow p}\,init \qquad \frac{}{0 : \Gamma \Rightarrow \mathbf{1}}\,\mathbf{1}_R \qquad \frac{}{0 : \Gamma, \mathbf{0} \Rightarrow A}\,\mathbf{0}_L \qquad \frac{a : \Gamma \Rightarrow A}{b : \Gamma \Rightarrow A}\,w_\ell(b \geq a)$$

Figure 2 The labelled sequent system $\mathcal{C}^\ell(\mathbb{R}^+)$.

▶ **Theorem 5** (Strong adequacy for $\mathcal{G}_\mathcal{C}(\mathbb{R}^+)$ [28]). $\models_{\mathcal{G}_\mathcal{C}(\mathbb{R}^+)} (\{\Gamma \Rightarrow A\}, b)$ *iff* $\vdash_{\mathcal{C}^\ell(\mathbb{R}^+)} b : \Gamma \Rightarrow A$.

This result can be further strengthened. In fact, proofs (and games) can be assigned a minimal budget, referred to as *the cost*: given a proof Ξ of a sequent, one can assign the label 0 to all initial sequents of Ξ and propagate the labels downward according to the rules of $\mathcal{C}^\ell(\mathbb{R}^+)$. However, the broader implications are even more interesting, as illustrated in the following example.

▶ **Example 6.** Suppose that a printer costs \$500 and it produces copies for \$0.1. Which is the budget needed for making 2 copies?

Since buying a printer and making a copy can be modelled as $!^{500}(!^{0.1}C)$, the goal is to find possible budgets for

$$b : \ !^{500}(!^{0.1}C) \Rightarrow C \otimes C$$

Now, there are many ways of proving this sequent in $\mathcal{C}^\ell(\mathbb{R}^+)$. For example, the proof below has a cost \$500.20:

$$\dfrac{\dfrac{\overline{\overline{0 : \ C, C \Rightarrow C \otimes C}}\ \otimes, init}{0.20 : \ !^{0.1}C \Rightarrow C \otimes C}\ !^{0.10} \times 2}{500.20 : \ !^{500}(!^{0.1}C) \Rightarrow C \otimes C}\ !^{500}$$

This proof corresponds to purchasing one printer and producing two copies from it.

Alternatively, one could overprice the scenario by purchasing two printers and making one copy with each, incurring a cost of \$1,000.20.

$$\dfrac{\dfrac{\overline{\overline{0:\ C, C \Rightarrow C \otimes C}}\ \otimes, init}{0.20:\ !^{0.1}C, !^{0.1}C \Rightarrow C \otimes C}\ !^{0.10}}{1,000.20:\ !^{500}(!^{0.1}C) \Rightarrow C \otimes C}\ !^{500} \times 2$$

Hence, different proofs of the same sequent can lead to different costs. Nevertheless, cost-optimal strategies exist for all provable sequents, as the following result shows.[3]

▶ **Theorem 7** (Cost-optimal proofs [28]). *If* $\vdash_{\mathcal{C}(\mathbb{R}^+)} \Gamma \Rightarrow A$*, then there exists a smallest* b *such that* $\vdash_{\mathcal{C}^\ell(\mathbb{R}^+)} b : \Gamma \Rightarrow A$*.*

2.2 About cut-admissibility

We begin by noting that establishing cut-admissibility in $\mathcal{C}^\ell(\mathbb{R}^+)$ critically relies on the ability to define a computable function f that relates the cost of the end-sequent to the labels of the premises in the cut rule. Given that exponentials only occur negatively in $\mathcal{C}^\ell(\mathbb{R}^+)$, no cut steps involve banged formulas. This allows us to demonstrate that $f(a, b) = a + b$ is the *minimal* such function.

▶ **Theorem 8** (Negative-cut [28]). *For* $f(a, b) = a + b$*, the following cut rule is admissible in* $\mathcal{C}^\ell(\mathbb{R}^+)$*:*

$$\dfrac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, A \Rightarrow C}{f(a, b) : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}\ cut_\ell$$

Moreover, whenever cut_ℓ *is admissible w.r.t. a given* f'*, then* $a + b \leq f'(a, b)$*.*

It turns out that extending cost-conscious reasoning to modalities occurring *positively* in sequents is far from trivial. While an intuitive game-theoretic interpretation of promotion could be provided in the style of [16], this *does not* align with a proof-theoretic notion of cut-admissibility. This is due to the inherent difficulty in defining a functional notion of the cut-label, as demonstrated below.

Let $\mathcal{CP}^\ell(\mathbb{R}^+)$ be the system resulting from $\mathcal{C}^\ell(\mathbb{R}^+)$ by adding the following *labelled promotion rule*

$$\dfrac{b : \Gamma^{\leq !^a} \Rightarrow A}{b : \Gamma \Rightarrow !^a A}\ !^a{}_R$$

where $\Gamma^{\leq !^a}$ denotes all formulas in Γ which are of the form $!^c B$ and $a \geq c$.

The question that arises is whether the cut-admissibility result can be extended to $\mathcal{CP}^\ell(\mathbb{R}^+)$. To address this, consider the following derivation:

$$\dfrac{\dfrac{b_1 : \Rightarrow A}{b_1 : \Rightarrow !^a A}\ !^a{}_R \quad \dfrac{b_2 : \Delta, !^a A, A \Rightarrow C}{b_2 + a : \Delta, !^a A \Rightarrow C}\ !^a{}_L}{b_1 + b_2 + a :\ \Delta \Rightarrow C}\ cut$$

[3] We note that the proof of this result is non-constructive!

This is usually reduced to

$$\dfrac{b_1 : \Rightarrow A \qquad \dfrac{b_1 : \Rightarrow !^a A \quad b_2 : \Delta, !^a A, A \Rightarrow C}{b_1 + b_2 : \Delta, A \Rightarrow C}\ cut}{2b_1 + b_2 : \Delta \Rightarrow C}\ cut$$

where the upper cut has a smaller rank, and the lower cut has a smaller degree than the original cut. However, this approach fails in the labelled setting because, whenever $a < b_1$, the label increases.

Although alternative reduction methods could be explored, the following result shows that it is impossible to define a labelled cut rule for $\mathcal{CP}^\ell(\mathbb{R}^+)$ where the label of the conclusion depends solely on the labels of the premises. We include the proof, as it is highly insightful.

▶ **Theorem 9** (Impossible-cut [28]). *There is no function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the rule*

$$\dfrac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, A \Rightarrow C}{f(a,b)!\Gamma, \Delta_1, \Delta_2 \Rightarrow C}\ cut$$

is admissible in $\mathcal{CP}^\ell(\mathbb{R}^+)$.

Proof. Let p, q be different propositional variables, and let $A^{\otimes n}$ denote the n-fold multiplicative conjunction of a formula A. The sequents

$$a : !^{1/k} p \Rightarrow !^{1/k} p^{\otimes(k\cdot a)} \qquad \text{and} \qquad b : !^{1/k} p^{\otimes(k\cdot a)} \Rightarrow p^{\otimes(k\cdot k\cdot a\cdot b)}$$

are provable in $\mathcal{CP}^\ell(\mathbb{R}^+)$ for all natural numbers a, b, k. The smallest label f which makes their cut conclusion $f : !^{1/k} p \Rightarrow p^{\otimes(k\cdot k\cdot a\cdot b)}$ provable in $\mathcal{CP}^\ell(\mathbb{R}^+)$ is $k \cdot a \cdot b$, which is not a function on the premise labels a, b. ◀

The theorem above indicates that, to find an admissible labelled cut rule, we must either:

1. restrict the form of the cut formula;
2. allow the labelling function f to incorporate more information from the premises than just their labels;
3. keep track of the use of contraction in the cut-elimination process.

We shall explore next different fragments and (admissible) cut-like rules that can be proposed for $\mathcal{CP}^\ell(\mathbb{R}^+)$.

2.2.1 Infinite costs

We start by observing that the inclusion of "worse costs" entails a trivial labelling that makes cut admissible. Let $\mathbb{R}^+_\infty$ be the completion of $\mathbb{R}^+$ with ∞ and $\mathcal{CP}^\ell(\mathbb{R}^+_\infty)$ the corresponding labeled proof system with *decreasing* for $b \leq a$ being defined as follows:

- If $a, b \neq \infty$, $a - b$ is defined as usual;
- If $a = \infty$, then $a - b = \infty$.

In the following theorem, the cut formula A is an arbitrary formula (containing, possibly, positive and/or negative occurrences of the modality $!^a$).

▶ **Theorem 10** (Infinite-cut). *The following rule is admissible in $\mathcal{CP}^\ell(\mathbb{R}^+_\infty)$*

$$\dfrac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, A \Rightarrow C}{\infty : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}\ cut_\infty$$

The proof follows the same steps of the cut-elimination proof for SELL [14, 33], using natural extensions of invertibility and permutability of rules to the labelled case.

But this still does not define a computable function relating the labels of the premises and the conclusion of the cut rule.

2.2.2 Linearity

Now we show cases where the cut formula is restricted, starting with the case where the cut formula is !-free.

▶ **Theorem 11** (Linear-cut). *Let A be a formula with no occurrences of $!^a$. Then, the following rule is admissible in $\mathcal{CP}^\ell(\mathbb{R}^+)$*

$$\frac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, A \Rightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}\ cut_L$$

Moreover, if $a : \Gamma \Rightarrow C$ is provable using cut_L, then there is a cut-free proof of $a' : \Gamma \Rightarrow C$ with $a \geq a'$.

The proof uses a standard cut-reduction strategy for SELL, observing in each case that the reduction of the label is possible.

Still, forcing cut formulas to be linear seems to be a very severe restriction to impose. We will now consider another, and less limiting, syntactic restriction on the cut formula.

▶ **Definition 12.** *A formula of the form $!^a A$ is* simply exp-labelled *if $a \neq 0$ and A is bang-free.*

Since the formulas used in the proof of Theorem 9 can be simply exp-labelled, it is clear that we cannot expect to find an admissible cut rule for all simply exp-labelled cut formulas where the labelling depends solely on the labels of the premises. However, we can also incorporate the information from the label a in the simply exp-labelled formula $!^a A$, as follows.

▶ **Theorem 13** (Exp-labelled-cut [27]). *For any simply exp-labelled formula $!^a A$, the following cut rule is admissible in $\mathcal{CP}^\ell(\mathbb{R}^+)$:*

$$\frac{b_1 : !\Gamma, \Delta_1 \Rightarrow !^a A \quad b_2 : !\Gamma, \Delta_2, !^a A \Rightarrow C}{f(b_1, b_2, a) : !\Gamma, \Delta_1, \Delta_1 \Rightarrow C}\ cut_{el}$$

where $f(b_1, b_2, a) = b_2 + \lfloor b_2/a \rfloor \cdot b_1$.

The intuition behind this labelling is as follows: if the right subproof R of the cut_{el} ends with the label b_2, then the formula $!^a A$ can be unpacked at most $\lfloor b_2/a \rfloor$ times within a multiplicative subtree of R. Therefore, we can assume that the rule $!^a{}_L$ is applied only $\lfloor b_2/a \rfloor$ times on such a subtree.

2.2.3 Accumulated costs

We will end the part of substructural modalities with a new approach towards cut-admissibility, where we keep track of the use of contraction in the cut-elimination process. The idea is that, if proving A costs b, then any use of A must pay this "extra cost". For that, we introduce the following notation.

▶ **Definition 14.** *Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that*
1. $a_b +_{\mathcal{E}} c_d = a + b + c + d$.
2. $a_b \geq_{\mathcal{E}} a_c$ *(i.e., the ordering $\geq_{\mathcal{E}}$ ignores the subindices).*
3. $a_b >_{\mathcal{E}} c_d$ *iff* $a > c$.

For any formula $A \in \mathcal{CP}^\ell(\mathbb{R}^+)$, we define $[A]_c$ as the formula that substitutes any modality $!^{a_b}$ with $!^{a_{b+c}}$.

Hence $\mathcal{CP}^\ell(\mathbb{R}^+)$ can be slightly modified so that sequent labels belong to $\mathbb{R}^+$, while modal labels belong to $\mathcal{E}$. Due to the ordering above, the promotion of $!^{a_0}$ has the same effect/constraints that the promotion of $!^{a_b}$. However, the dereliction of the latter requires a greater budget ($a + b$ instead of a). Moreover, the equivalence $!^{a_b}A \equiv\, !^{a_c}A$ can be proven, each direction requiring a different budget. Finally, note that $\mathcal{E}_0 = \{a_0 \mid a \in \mathbb{R}^+\} \simeq \mathbb{R}^+$, that is, each element $a \in \mathbb{R}^+$ can be seen as the equivalence class of a_0 in $\mathbb{R}^+ \times \mathbb{R}^+$ modulo $\mathbb{R}^+$. We will abuse of the notation and continue representing the resulting system by $\mathcal{CP}^\ell(\mathbb{R}^+)$, also unchanging the representation of sequents.

The following lemma has a straightforward proof.

▶ **Lemma 15.** *If $b : \Gamma, [A]_c \Rightarrow C$ then $b' : \Gamma, A \Rightarrow C$ with $b \geq b'$. More generally, if $b : \Gamma, [A]_c \Rightarrow C$ and $c \geq c'$ then $b' : \Gamma, [A]_{c'} \Rightarrow C$ with $b \geq b'$.*

The next definition restricts the occurrence of unbounded modalities only under linear implication.

▶ **Definition 16.** *We say that A is $\multimap$-linear if for all subformulas of the form $B \multimap C$ in A, B is bang-free.*

The following result presents the admissibility of an extended form of the cut rule, where the budget information from the left premise is passed to the cut-formula in the right premise. Observe that the label of the conclusion is now a function of the labels of the premises. Moreover, the cut-reduction is *label preserving*, meaning that the budget monotonically decreases in the cut-elimination process.

▶ **Theorem 17** ($\multimap$-linear-cut)**.** *The following rule is admissible*

$$\frac{a : !\Gamma, \Delta_1 \Rightarrow A \quad b : !\Gamma, \Delta_2, [A]_a \Rightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}\ cut_{LL} \quad A \text{ is a } \multimap\text{-linear formula}$$

Moreover, if $b : \Gamma \Rightarrow C$ is provable using cut_{LL}, then there is a cut-free proof of $b' : \Gamma \Rightarrow C$ with $b \geq b'$.

Proof. We will illustrate some cases.

- Note that: $[!^{a_b}A]_c = !^{a_{b+c}}[A]_c$; the promotion of $!^{a_b}A$, bottom-up, results in a context of ! formulas (that can be contracted at will); and the dereliction of $!^{a_b}[A]_c$ decreases the budget in $a + b$. Hence,

$$\frac{\dfrac{c : (!\Gamma)^{\leq !^{a_b}} \Rightarrow A}{c : !\Gamma, \Delta_1 \Rightarrow !^{a_b}A} \quad \dfrac{d : !\Gamma, \Delta_2, [A]_c, !^{a_{b+c}}[A]_c \Rightarrow C}{a + b + c + d : !\Gamma, \Delta_2, !^{a_{b+c}}[A]_c \Rightarrow C}}{a + b + 2c + d : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

reduces to

$$\frac{c : (!\Gamma)^{\leq !^{a_b}} \Rightarrow A \quad \dfrac{c : !\Gamma \Rightarrow !^{a_b}A \quad d : !\Gamma, !^{a_{b+c}}[A]_c, \Delta_2, [A]_c \Rightarrow C}{c + d : !\Gamma, \Delta_2, [A]_c \Rightarrow C}}{2c + d : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

where the "extra cost" a_b disappears after the reduction.

- Note that $[A \otimes B]_c = [A]_c \otimes [B]_c$. Here, let $c = c_1 + c_2$:

$$\dfrac{\dfrac{c_1 : !\Gamma, \Delta_1' \Rightarrow A \quad c_2 : !\Gamma, \Delta_1'' \Rightarrow B}{c : !\Gamma, \Delta_1 \Rightarrow A \otimes B} \quad \dfrac{b : !\Gamma, \Delta_2, [A]_c, [B]_c \Rightarrow C}{b : !\Gamma, \Delta_2, [A \otimes B]_c \Rightarrow C}}{b + c : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

reduces to

$$\dfrac{c_1 : !\Gamma, \Delta_1' \Rightarrow A \quad \dfrac{c_2 : !\Gamma, \Delta_1'' \Rightarrow B \quad b : !\Gamma, \Delta_2, [A]_{c_1}, [B]_{c_2} \Rightarrow C}{b + c_2 : !\Gamma, \Delta_1'', \Delta_2, [A]_{c_1} \Rightarrow C}}{b + c : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

It is worth noticing that in the first derivation, the cost $c = c_1 + c_2$ is "charged" to $A \otimes B$ (in the formula $[A \otimes B]_c$) while in the second one, in a finer way, the cost c_1 is charged to A and c_2 to B.

- The case of implication explains the restriction we impose. Here $b = b_1 + b_2$:

$$\dfrac{\dfrac{c : !\Gamma, \Delta_1, A \Rightarrow B}{c : !\Gamma, \Delta_1 \Rightarrow A \multimap B} \quad \dfrac{b_1 : !\Gamma, \Delta_2' \Rightarrow [A]_c \quad b_2 : !\Gamma, \Delta_2'', [B]_c \Rightarrow C}{b : !\Gamma, \Delta_2, [A \multimap B]_c \Rightarrow C}}{c + b : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

reduces to

$$\dfrac{b_1 : !\Gamma, \Delta_2' \Rightarrow A \quad \dfrac{c : !\Gamma, \Delta_1, [A]_{b_1} \Rightarrow B \quad b_2 : !\Gamma, \Delta_2'', [B]_c \Rightarrow C}{c + b_2 : !\Gamma, \Delta_1, \Delta_2'', [A]_{b_1} \Rightarrow C}}{c + b : !\Gamma, \Delta_1, \Delta_2 \Rightarrow C}$$

Note that the reduction above is correct since A does not have occurrences of $!^a$ and then $[A]_c = [A]_{b_1} = A$. ◀

2.3 Discussion – part I

This research line offers at least three promising directions for future exploration.

First, the work initiated in [28] highlights that our games and systems provide more precise control over resources appearing negatively in sequents, unlocking new opportunities for analysing the problem of comparing proofs. For instance, studying proof costs in labelled calculi could reveal deeper links between labels and computational bounds [2]. Similarly, examining the interplay between resource budgets and the complexity of the cut-elimination process, particularly within the multiplicative-(sub)exponential fragment, presents considerable opportunities [40, 41].

Second, there is substantial value in investigating how the dialogue games we have developed align with the framework of concurrent games [1, 15, 13]. Understanding these connections could enrich our framework and provide new perspectives on resource management in proof theory.

Lastly, an essential direction involves addressing compositionality in dialogue games governed by the cut rule. Regardless of the specific approach taken to achieve cut-admissibility, ensuring compositionality remains a critical and promising challenge [34].

3 A game model for polarisation

We now turn to the study of modalities in the classical setting, focusing on the positive-negative modal logic **PNL** with nominals [47, 35]. This logic is based on Kripke frames with two disjoint and symmetric reachability relations. Here we will outline the construction of an adequate semantic game for **PNL**, its transformation into a provability game, and the derivation of a corresponding sequent system. This opens a discussion on how to generalise this method to other modal systems.

We begin with a brief discussion of games for modal logics and the motivation for hybrid extensions. As studied in [9] and further developed in [19], extending Hintikka games [25] dialogue game to modal logic is conceptually straightforward: in addition to the current roles of the players and the current formula F, one only has to keep track of the current world w in the model. However, this extension introduces an unfortunate drawback: the *game trees*, *i.e.*, labelled trees whose nodes are game states, are no longer determined solely by the syntax of the formula, but instead depend on the relational structure of the model. This is in stark contrast to semantic games for propositional logic, where semantic information is required only at the final stage to determine the winner. The loss of uniformity in game trees across all models is a significant limitation of this approach.

As in [9, 18], we address this problem by turning to hybrid logic [10, 12, 11], allowing explicit references to worlds and the accessibility relation within the object language.

Let $\mathsf{A} = \{\mathsf{a}, \mathsf{b}, \ldots\}$ be a non-empty set of agents, $\mathsf{At} = \{p, q, \ldots\}$ be a countable set of propositional variables, and $N = \{i, j, \ldots\}$ be a countable set of *nominals*. The language of **PNL** is generated by the following grammar

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid R^+(i,j) \mid R^-(i,j) \mid \Diamond^{+} F \mid \Diamond^{-} F \mid [A]F$$

where $p \in \mathsf{At}$, and $i, j \in N$. Formulas of the form p, $R^+(i,j)$, or $R^-(i,j)$ are called *elementary*. We shall use F, G, H to range over formulas. The propositional connectives $\top$, $\bot$, $\to$, and the (dual) modalities $\boxplus$ and $\boxminus$ can be obtained in the usual way.

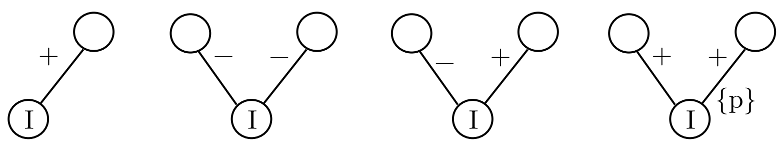
Intuitively, nominals are used as names for worlds of the model, while the propositions $R^{\pm}(i,j)$ state that agent i is a *friend*/*enemy* (or, more generally, *agrees*/*disagrees*) with j. The formula $\Diamond^{+} F$ (resp. $\Diamond^{-} F$) states that F holds for a friend (resp. an enemy). The global modality $[A]F$ states that F holds for all the agents. We use $R^{\pm}$ to denote either R^+ or R^-, and $\Diamond^{\pm}$ to denote either $\Diamond^{+}$ or $\Diamond^{-}$.

A model $\mathbb{M}$ is a tuple $\langle \mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g} \rangle$ where A is a set (of agents), $\mathsf{g} : N \to \mathsf{A}$ is called *denotation function*, $\mathsf{R}^+, \mathsf{R}^- \subseteq \mathsf{A} \times \mathsf{A}$, and $\mathsf{V} : \mathsf{At} \to \mathcal{P}(\mathsf{A})$. A model is a **PNL**-model if:

- g is surjective, *i.e.*, every agent has a name;
- R^+ is reflexive; and
- R^+ and R^- are both symmetric and non-overlapping, *i.e.*, for all $\mathsf{a}, \mathsf{b} \in \mathsf{A}$, $(\mathsf{a}, \mathsf{b}) \notin \mathsf{R}^+$ or $(\mathsf{a}, \mathsf{b}) \notin \mathsf{R}^-$.

The Kripke semantics of **PNL** is in Figure 3. A formula F is true over $\mathbb{M}$, written $\mathbb{M} \Vdash F$ iff $\mathbb{M}, \mathsf{a} \Vdash F$, for all agent $\mathsf{a} \in \mathsf{A}$. For a set of formulas Δ, we write $\mathbb{M} \models \Delta$ iff $\mathbb{M} \Vdash \Delta$ for all $F \in \Delta$. A formula F is valid iff $\mathbb{M} \Vdash F$ for every **PNL**-model $\mathbb{M}$. For a class of models $\mathfrak{M}$, we write $\Delta \models_{\mathfrak{M}} F$ iff $\mathbb{M} \Vdash F$ for every model $\mathbb{M} \in \mathfrak{M}$ with $\mathbb{M} \models \Delta$.

▶ **Example 18.** Consider the following models (omitting self loops for R^+):



$\mathbb{M}_1$ $\mathbb{M}_2$ $\mathbb{M}_3$ $\mathbb{M}_4$

$\mathbb{M}, \mathsf{a} \Vdash p$	iff $\mathsf{a} \in \mathsf{V}(p)$	$\mathbb{M}, \mathsf{a} \Vdash \neg F$	iff $\mathbb{M}, \mathsf{a} \nVdash F$
$\mathbb{M}, \mathsf{a} \Vdash F \wedge G$	iff $\mathbb{M}, \mathsf{a} \Vdash F$ and $\mathbb{M}, \mathsf{a} \Vdash G$	$\mathbb{M}, \mathsf{a} \Vdash F \vee G$	iff $\mathbb{M}, \mathsf{a} \Vdash F$ or $\mathbb{M}, \mathsf{a} \Vdash G$
$\mathbb{M}, \mathsf{a} \Vdash R^{\pm}(i, j)$	iff $(\mathsf{g}(i), \mathsf{g}(j)) \in \mathsf{R}^{\pm}$		
$\mathbb{M}, \mathsf{a} \Vdash \Diamond^{\pm} F$	iff there is $j \in N$ such that $\mathbb{M}, \mathsf{g}(j) \Vdash R^{\pm}(i, j)$ and $\mathbb{M}, \mathsf{g}(j) \Vdash F$		
$\mathbb{M}, \mathsf{a} \Vdash [A]F$	iff $\mathbb{M}, \mathsf{g}(j) \Vdash F$, for all $j \in N$.		

Figure 3 Kripke semantics for **PNL**.

The following holds:

- $\mathbb{M}_1$: I have a friend where $\neg p$: $\mathbb{M}_1, I \Vdash \Diamond^{+} \neg p$;
- $\mathbb{M}_2$: All *my* enemies do not believe in p: $\mathbb{M}_2, I \Vdash \boxminus \neg p$;
- $\mathbb{M}_3$: I have an enemy: $\mathbb{M}_3, I \Vdash \Diamond^{-} \top$;
- $\mathbb{M}_4$: Everybody has a friend where p: $\mathbb{M}_4, \mathsf{a} \Vdash [A]\Diamond^{+} p$ for any agent a.

3.1 Playing with models

Before starting playing, remember that in a **PNL**-model $\mathbb{M}$, every agent a has a name i, *i.e.*, there exists $i \in N$ s.t. $\mathsf{g}(i) = \mathsf{a}$. Hence, from now on, we will internalise the nominals, identifying an agent a with its respective nominal i.

The *semantic game* is played over a **PNL**-model $\mathbb{M} = (\mathsf{A}, \mathsf{R}^{+}, \mathsf{R}^{-}, \mathsf{V}, \mathsf{g})$ by two players, *Me* (or I) and *You*, who argue about the truth of a formula F at an agent i. At each stage of the game, one player acts as *proponent*, while the other acts as *opponent* of the claim that F is true at i.

We represent the situation where I am the proponent (and *You* are the opponent) by the *game state* $\mathbf{P}, i : F$, and the situation where I am the opponent (and *You* are the proponent) by $\mathbf{O}, i : F$.

We call a game state *elementary* if its involved formula is elementary. For a game state g, we denote the game starting at g over the model $\mathbb{M}$ by $\mathbf{G}_{\mathbb{M}}(g)$.

The game over a **PNL**-model $\mathbb{M}$ proceeds by reducing the involved formula F to an elementary formula by following the rules described in Figure 4.[4]

In general, every two-person, zero-sum, win-lose game is usually represented by a game tree. In our case, the root of the game tree representing the game $\mathbf{G}_{\mathbb{M}}(g)$ is g. The children of each node in the game tree are exactly the possible choices of the corresponding player. For instance, if $h = \mathbf{P}, i : F_1 \wedge F_2$ appears in the game tree, then its children are $\mathbf{P}, i : F_1$ and $\mathbf{P}, i : F_2$. Each node in the tree is labelled either "I", or "Y", depending on which player is to move in the corresponding game state, and we label the nodes $\mathbf{P}, i : \neg F$ and $\mathbf{O}, i : \neg F$ with "I" (even though there is no choice involved in these game states). For instance, the node corresponding to the game state h above is "Y", since it is *Your* choice in $\mathbf{P} : F_1 \wedge F_2$. The leaves of the tree receive the label of the winning player. A *run* of the game is a maximal path through the game tree.

Now we are ready to define winning strategies and state the main result of this section: the adequacy of the proposed game semantics with respect to the Kripke semantics for **PNL**.

▶ **Definition 19.** *A* strategy *for* Me *in the game* $\mathbf{G}_{\mathbb{M}}(g)$ *is a subtree* σ *of the associated game tree such that:* ***(1)*** $g \in \sigma$*,* ***(2)*** *if* $h \in \sigma$ *is a node labelled "Y", then all children of* h *are in* σ*,* ***(3)*** *if* $h \in \sigma$ *is a node labelled "I", then exactly one child of* h *is in* σ*. The strategy* σ *is called* winning *if all leaves in the tree* σ *are labelled "I". (Winning) strategies for* You *are defined dually.*

[4] The outcome of the game state $\mathbf{Q}, k : R^{\pm}(i, j)$ is independent of k (it only depends on the underlying model $\mathbb{M}$). Hence, we write $\mathbf{Q}, \_ : R^{\pm}(i, j)$ instead of $\mathbf{Q}, k : R^{\pm}(i, j)$.

($\mathbf{P}_\wedge$) At $\mathbf{P}, i : F_1 \wedge F_2$, *You* choose between $\mathbf{P}, i : F_1$ and $\mathbf{P}, i : F_2$ to continue the game.

($\mathbf{O}_\wedge$) At $\mathbf{O}, i : F_1 \wedge F_2$, *I* choose between $\mathbf{O}, i : F_1$ and $\mathbf{O}, i : F_2$ to continue the game.

($\mathbf{P}_\vee$) At $\mathbf{P}, i : F_1 \vee F_2$, *I* choose between $\mathbf{P}, i : F_1$ and $\mathbf{P}, i : F_2$ to continue the game.

($\mathbf{O}_\vee$) At $\mathbf{O}, i : F_1 \vee F_2$, *You* choose between $\mathbf{O}, i : F_1$ and $\mathbf{O}, i : F_2$ to continue the game.

($\mathbf{P}_\neg$) At $\mathbf{P}, i : \neg F$, the game continues with $\mathbf{O}, i : F$.

($\mathbf{O}_\neg$) At $\mathbf{O}, i : \neg F$, the game continues with $\mathbf{P}, i : F$.

($\mathbf{P}_{\Diamond^\pm}$) At $\mathbf{P}, i : \Diamond^\pm F$, *I* choose a nominal j, and *You* decide whether the game ends in the state $\mathbf{P}, \_ : R^\pm(i, j)$ or continues with $\mathbf{P}, j : F$.

($\mathbf{O}_{\Diamond^\pm}$) At $\mathbf{O}, i : \Diamond^\pm F$, *You* choose j, and *I* choose between $\mathbf{O}, \_ : R^\pm(i, j)$ and $\mathbf{O}, j : F$.

($\mathbf{P}_{[A]}$) At $\mathbf{P}, i : [A]F$, *You* choose a nominal j and the game continues with $\mathbf{P}, j : F$.

($\mathbf{O}_{[A]}$) At $\mathbf{O}, i : [A]F$, *I* choose a nominal j, and the game continues with $\mathbf{O}, j : F$.

($\mathbf{P}_{el}$) Let F_e be an elementary formula. *I* win and *You* lose at $\mathbf{P}, i : F_e$ iff $\mathbb{M}, i \models F_e$. Otherwise, *You* win and *I* lose.

($\mathbf{O}_{el}$) At $\mathbf{O}, i : F_e$, *I* win and *You* lose iff $\mathbb{M}, i \not\models F_e$. Otherwise, *You* win and *I* lose.

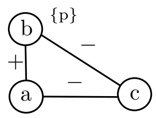
Figure 4 Semantic game given a **PNL**-model $\mathbb{M}$.

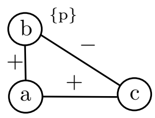
▶ **Theorem 20** (Adequacy - semantic games [22]). *Let* $\mathbb{M}$ *be a* ***PNL****-model,* a *an agent with nominal* i*, and* F *a formula.*

(1) I *have a winning strategy for* $\mathbf{G}_\mathbb{M}(\mathbf{P}, i : F)$ *iff* $\mathbb{M}, \mathsf{a} \models F$.

(2) You *have a winning strategy for* $\mathbf{G}_\mathbb{M}(\mathbf{P}, i : F)$ *iff* $\mathbb{M}, \mathsf{a} \not\models F$.

▶ **Example 21** ([22]). Let $(\mathbf{4B}) = ((\Diamond^+\Diamond^+ p \vee \Diamond^-\Diamond^- p) \to \Diamond^+ p) \wedge ((\Diamond^+\Diamond^- p \vee \Diamond^-\Diamond^+ p) \to \Diamond^- p)$. This formulas specifies *local balance* [35] and captures the idea that "the enemy of my enemy is my friend", "the friend of my enemy is my enemy", and "the friend of my friend is my friend". *I* have a winning strategy for the game $\mathbf{P}, \mathsf{a} : 4B$ on $\mathbb{M}_1$ while *You* have a winning strategy for the same game on $\mathbb{M}_2$ where (omitting self-loops for R^+):

$\mathbb{M}_1 =$ 

$\mathbb{M}_2 =$ 

For $\mathbb{M}_1$, in the first conjunct, *I* pick ($\mathbf{P}_\vee$) $\Diamond^+ p$ and then b in ($\mathbf{P}_{\Diamond^+}$); for the second conjunct, *I* pick the first disjunction in $F = (\Diamond^+\Diamond^- p \vee \Diamond^-\Diamond^+ p) \to \Diamond^- p)$ where, in any of *Your* choices ($\mathbf{P}_\neg$ followed by $\mathbf{O}_\vee$ and $\mathbf{O}_{\Diamond^\pm}$), *I* win all the elementary states. For $\mathbb{M}_2$, *I* do not have a winning strategy for the second conjunct: *I* can neither win $\Diamond^- p$ (no R^- successor), nor the first disjunct in F above since, after $\mathbf{P}_\neg$, *You* choose ($\mathbf{O}_\vee$) $\Diamond^-\Diamond^+ p$ and select c and then b ($\mathbf{O}_{\Diamond^\pm}$) where p holds and *You* win. See the complete game in our tool [23].

3.2 Playing all models

We now leverage semantic games to **PNL**-provability games. The key observation is that the rules of the semantic game remain independent of the underlying model, except at the level of elementary game states.

(Dupl) If no state in D is underlined, I can choose a non-elementary $g \in D$ and the game continues with $D \bigvee g$.
(Sched) If no state in $D = D' \bigvee g$ is underlined, and g is non-elementary, I can choose to continue the game with $D' \bigvee \underline{g}$.
(Move) If $D = D' \bigvee \underline{g}$ then the player who is to move in the semantic game $\mathbf{G}(g)$ at g makes a legal move to the game state g' and the game continues with $D' \bigvee g'$.
(End) The game ends if there are no non-elementary game states left in D, or if no game state is underlined and I win according to Definition 22. Otherwise, I must move according to **(Dupl)** or **(Sched)**.

Figure 5 Rules for the provability game.

The *provability game* $\mathbf{DG}(\mathbf{P}, i : F)$ can be thought of as *Me* and *You* playing all semantic games $\mathbf{G}(\mathbf{P}, i : F)$ over all **PNL**-models $\mathbb{M}$ simultaneously. We point out that the rules of the semantic game do not depend on the structure of $\mathbb{M}$ but merely on F. Truth degrees are only needed at the atomic level to determine who wins the particular run of the game. This allows us to require players to play "blindly", *i.e.*, without explicitly referencing a model $\mathbb{M}$. Clearly, if I have a winning strategy in such a game, then I can win in $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, i : F)$, for every $\mathbb{M}$, making this strategy an adequate witness of logical validity.

Provability game states are finite multisets of the game states defined in Section 3.1. We denote by $g_1 \bigvee ... \bigvee g_n$ the provability game state $\{g_1, ..., g_n\}$. We write $D_1 \bigvee D_2$ for the multiset sum $D_1 + D_2$ and $D \bigvee g$ for $D + \{g\}$. A provability state is called *elementary* if all its game states are elementary. We use $\mathbf{DG}(D)$ to denote the provability game starting at D.

▶ **Definition 22.** *Let D^{el} denote the provability state consisting of the elementary game states of D.* I *win and* You *lose at D if for every* ***PNL****-model there is a game state in D^{el} where* I *win the corresponding semantic game.*

In the provability game, I additionally take the role of a *scheduler*, deciding which game is to be played next. We signal the chosen game state by underlining it as in $\underline{g}$.

▶ **Definition 23.** *The rules of the provability game are in Figure 5. Infinite runs, and runs that end in elementary provability states where* I *do not win according to Definition 22, are winning for* You *and losing for* Me*.* ***(Dupl)*** *is referred to as the* duplication rule *and* ***(Sched)*** *as the* scheduling*, or* underlining rule*.*

▶ **Theorem 24** (Adequacy - provability games [22])**.** I *have a winning strategy in* $\mathbf{DG}(D)$ *iff for every* ***PNL****-model* $\mathbb{M}$*, there is some $g \in D$ such that* I *have a winning strategy in* $\mathbf{G}_{\mathbb{M}}(g)$*.*

▶ **Corollary 25.** *The formula F is* ***PNL****-valid iff* I *have a winning strategy in* $\mathbf{DG}(\mathbf{P}, i : [A]F)$*.*

▶ **Example 26.** Consider the game $\mathbf{P}, i : p \vee \neg p$. I duplicate the game state in the first round and the game continues with the provability state $\mathbf{P}, i : p \vee \neg p \bigvee \mathbf{P}, i : p \vee \neg p$.

Now I move to $\mathbf{P}, i : p$ in the first subgame and to $\mathbf{P}, i : \neg p$ in the second. After a role switch in the second subgame, the final state is $\mathbf{P}, i : p \bigvee \mathbf{O}, i : p$, where I win regardless of the underlying model.

3.3 From games to proofs

Theorems 20 and 24 establish that winning strategies for *Me* in the provability game correspond to the validity of formulas. In this section, we extend this result to proof systems by introducing a sequent calculus, **DS**, where proofs correspond to *My*'s winning strategies in the provability game.

Labelled nominal formulas are either *labelled formulas* of the form $i : F$ or *relational atoms* of the form $R(i, j)$, where i and j are nominals and F is a **PNL** formula.[5] *Labelled sequents* have the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets containing labelled nominal formulas.

Starting with sequents, every provability state of the form

$$\mathbf{O}, i_1 : F_1 \bigvee \ldots \bigvee \mathbf{O}, i_n : F_n \bigvee \mathbf{P}, j_1 : G_1 \bigvee \ldots \bigvee \mathbf{P}, j_m : G_m$$

can be rewritten as the labelled sequent $\Gamma \Rightarrow \Delta$ where $\Gamma = \{i_1 : F_1, \ldots, i_n : F_n\}$ and $\Delta = \{j_1 : G_i, \ldots, j_m : G_m\}$. In what follows, we will not distinguish between provability states and their corresponding labelled sequent. For example, the provability game state $\mathbf{O}, i : (\Diamond^{+}\Diamond^{+}p \vee \Diamond^{-}\Diamond^{-}p) \bigvee \mathbf{P}, i : \Diamond^{+}p$ will be identified with the sequent $i : (\Diamond^{+}\Diamond^{+}p \vee \Diamond^{-}\Diamond^{-}p) \Rightarrow i : \Diamond^{+}p$.

The inference rules must be tailored in such a way that *proofs* in the sequent system match exactly *My winning strategies* in the provability game. This means that the user of the proof system takes the role of *Me*, scheduling game states and choosing moves in **P**-states. Moreover, *provability* in the proof system should correspond to *validity* in the game. For that, it is crucial to establish the formal relationship between elementary game states and logical axioms.

▶ **Lemma 27** ([22]). *Let $\Gamma \Rightarrow \Delta$ be composed of elementary game states only.* I *win the provability game in $\Gamma \Rightarrow \Delta$ iff one of the following holds*[6]

- **i.** $R^-(i, i) \in \Gamma$ *or* $R^+(i, i) \in \Delta$ *for some* i*;*
- **ii.** $\{R^+(i, j), R^-(i, j)\} \subseteq \Gamma$ *for some* $i \neq j$*;*
- **iii.** $\Gamma \cap \Delta \neq \varnothing$*.*

Figure 6 presents the labelled sequent systems **DS** with the standard initial axiom and structural/propositional rules. The modal rules and the relational rules *sym* and $ref\pm$ coincides with the modal rules originally presented by Viganò in [45], adapted to multi-relational modal logics. It is routine to show that the rule *no* in Figure 6 correspond to the non-overlapping axiom $\forall i, j.\neg(R^+(i, j) \wedge R^-(i, j))$.

The following result immediately implies that the provability game **DG** is adequate with respect to the calculus **DS**.

▶ **Theorem 28** (Adequacy - sequent system [22]). I *have a winning strategy in the provability game* $\mathbf{DG}(\Gamma \Rightarrow \Delta)$ *iff* $\Gamma \Rightarrow \Delta$ *is provable in* **DS**.

Let us write $\models_{\mathbf{PNL}} \Gamma \Rightarrow \Delta$ iff for every **PNL**-model there is some $i : F \in \Gamma$ such that $\mathbb{M}, \mathsf{g}(i) \not\models F$, or there is some $i : G \in \Delta$ such that $\mathbb{M}, \mathsf{g}(i) \models G$. We have the following consequence of Theorems 20, 24, and 28:

▶ **Corollary 29.** *Let Γ, Δ be multisets of labelled formulas. Then $\models_{\mathbf{PNL}} \Gamma \Rightarrow \Delta$ iff there is a proof of $\Gamma \Rightarrow \Delta$ in* **DS**. *In particular, F is **PNL**-valid iff there is a proof of $\Rightarrow F$ in* **DS**.

[5] Observe that here we are abusing the notation, identifying $k : R(i, j)$ with $R(i, j)$ – see Footnote 4.
[6] Since relations are symmetric, we will identify $R^{\pm}(i, j)$ with $R^{\pm}(j, i)$.

Axiom and Structural Rules

$$\frac{}{\Gamma, i : F_{el} \Rightarrow \Delta, i : F_{el}}\ init \qquad \frac{\Gamma, i : F, i : F \Rightarrow \Delta}{\Gamma, i : F \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow i : F, i : F, \Delta}{\Gamma \Rightarrow i : F, \Delta}\ (R_c)$$

Propositional Rules

$$\frac{\Gamma \Rightarrow i : F, \Delta}{\Gamma, i : \neg F \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma \Rightarrow i : \neg F, \Delta}\ (R_\neg)$$

$$\frac{\Gamma, i : F \Rightarrow \Delta \quad \Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \vee G \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow i : F, \Delta}{\Gamma \Rightarrow i : F \vee G, \Delta}\ (R^1_\vee) \qquad \frac{\Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \vee G, \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, i : F \Rightarrow \Delta}{\Gamma, i : F \wedge G \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma, i : G \Rightarrow \Delta}{\Gamma, i : F \wedge G \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow i : F, \Delta \quad \Gamma \Rightarrow i : G, \Delta}{\Gamma \Rightarrow i : F \wedge G, \Delta}\ (R_\wedge)$$

Modal Rules

$$\frac{\Gamma, R^\pm(i, j) \Rightarrow \Delta}{\Gamma, i : \Diamond^\pm F \Rightarrow \Delta}\ (L_{\Diamond^\pm})_1 \qquad \frac{\Gamma, j : F \Rightarrow \Delta}{\Gamma, i : \Diamond^\pm F \Rightarrow \Delta}\ (L_{\Diamond^\pm})_2$$

$$\frac{\Gamma \Rightarrow R^\pm(i, j), \Delta \quad \Gamma \Rightarrow j : F, \Delta}{\Gamma \Rightarrow i : \Diamond^\pm F, \Delta}\ (R_{\Diamond^\pm}) \qquad \frac{\Gamma, j : F \Rightarrow \Delta}{\Gamma, i : [A]F \Rightarrow \Delta}\ (L_{[A]}) \qquad \frac{\Gamma \Rightarrow j : F, \Delta}{\Gamma \Rightarrow i : [A]F, \Delta}\ (R_{[A]})$$

Relational Rules

$$\frac{\Gamma \Rightarrow \Delta, R^\pm(j, i)}{\Gamma \Rightarrow \Delta, R^\pm(i, j)}\ sym \qquad \frac{}{\Gamma \Rightarrow \Delta, R^+(i, i)}\ ref+$$

$$\frac{}{\Gamma, R^-(i, i) \Rightarrow \Delta}\ ref- \qquad \frac{\Gamma \Rightarrow \Delta, R^+(i, j) \quad \Gamma \Rightarrow \Delta, R^-(i, j)}{\Gamma \Rightarrow \Delta}\ no$$

Figure 6 The proof system **DS**. In the rule init, F_{el} denotes an elementary formula. In the rules $(L_{\Diamond^\pm})_1$, $(L_{\Diamond^\pm})_2$, and $(R_{[A]})$, the nominal j is fresh. The rule $R_{\Diamond}$ has the proviso that $i \neq j$.

Proving cut-admissibility of labelled systems can be cumbersome due to the presence of relational rules. In [30], a systematic procedure for transforming axioms into rules was presented, based on *focusing* and *polarities* [5]. This procedure not only allows for generalizing different approaches for transforming axioms into sequent rules present in the literature [39, 45, 31], but it also provides a uniform way of proving cut-admissibility for the resulting systems.

The cut-admissibility result for **DS** is a particular instance of the general result in [30].

▶ **Theorem 30 (PNL-cut).** *The following cut rule is admissible in* **DS**

$$\frac{\Gamma \Rightarrow \Delta, i : F \quad i : F, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ cut$$

As a consequence, **DS** is consistent, since the only rule that can be applied in an empty sequent is *no*, and it is routine to show that it does not trivialise derivations.

3.4 Discussion – part II

This work opens up several promising directions for future exploration.

It would be interesting to explore extensions of **PNL** that relax symmetry assumptions, enabling the representation of scenarios where an agent a can influence the opinion of agent b, but not vice versa. Another potential direction involves incorporating the concept of a “budget,” as introduced in the game discussed in the first part of this paper, to model situations where proponents and opponents operate under a limited amount of *political capital*. In such scenarios, adding or modifying relations (*i.e.*, making new friends, making enemies to reconcile, etc) could reduce this capital. Preferences on how to "expend" the political capital could be expressed through a combination of **PNL** with a suitable choice logic – a framework where preferences are explicitly definable at the object level. Semantic games for choice logics have been explored in [20], and the extension of game-induced choice logic (**GCL**) to a provability game and proof system was proposed in [21]. Exploring these dynamics within our framework offers a compelling direction for future research.

Another particularly interesting avenue is extending the semantic-provability-proof system approach to other logics characterised by Kripke semantics. For instance, it would be worthwhile to investigate games for logics that involve model-change modalities [44, 36] or dynamic modalities [42]. Initial progress in this direction was made in [22], where we showed how the global link-adding and local link-changing modalities from [35] (inspired by sabotage modal logic [6, 7, 43]) can be incorporated into our framework.

We are also interested in exploring the application of this framework to develop games for constructive and intuitionistic modal logics [17, 38, 39, 8]. The constructive logic **CK** stands out as a promising candidate due to its intuitive semantics and straightforward sequent system. The main challenge lies in adapting the classical approach presented here to an intuitionistic setting.

Finally, building on ideas from [4, 3], we aim to establish a correspondence between winning innocent strategies in games played on Hyland-Ong arenas [26] and proofs in these constructive logics. This correspondence would deepen the connection between game semantics and constructive modal reasoning, opening new avenues for further study.

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