Insights from Univalent Foundations: A Case Study Using Double Categories

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Abstract

Category theory unifies mathematical concepts, aiding comparisons across structures by incorporating not just objects, but also morphisms capturing interactions between objects. Of particular importance in some applications are double categories, which are categories with two classes of morphisms, axiomatizing two different kinds of interactions between objects. These have found applications in many areas of mathematics and theoretical computer science, for instance, the study of lenses, open systems, and rewriting.

However, double categories come with a wide variety of equivalences, which makes it challenging to transport structure along equivalences. To deal with this challenge, we propose the *univalence maxim*: each notion of equivalence of categorical structures has a corresponding notion of univalent categorical structure which induces that notion of equivalence. We also prove corresponding univalence principles, which allow us to transport structure and properties along equivalences. In this way, the usually informal practice of reasoning modulo equivalence becomes grounded in an entirely formal logical principle.

We apply this perspective to various double categorical structures, such as (pseudo) double categories and double bicategories. Concretely, we characterize and formalize their definitions in Coq UniMath up to chosen equivalences, which we achieve by establishing their univalence principles.

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1 Introduction

The advancement of mathematics has resulted in ever more intricate structures, which come with various commonly used identifications, weakening set-theoretic equalities. For instance, groups have one common type of identification: isomorphisms. Two groups have the same group-theoretic properties if and only if they are isomorphic. Hence, structures on, and properties of, groups can be transported along isomorphisms. Already in this first example we can witness the challenge of dealing with equivalences in set-theoretic foundations. Indeed, one needs to prove every time that a suitable structure or property can be transported along isomorphisms of groups, as only transport along equalities would come for free.

If we generalize groups, we encounter one of the first examples of a structure with more than one important type of sameness: categories. Indeed, it comes with two major identifications, isomorphisms and categorical equivalences. Here, our choice depends on whether we study objects of a category up to equality, as is often done in the study of syntax [5, 12], in which case we use isomorphisms; or whether we study objects up to isomorphism, the more common choice, in which case we use categorical equivalences. Hence, we now need suitable results regarding transporting structures and properties both for isomorphisms and categorical equivalences [10, 17, 24], which is significantly more challenging given the amount of data an equivalence entails. At least we can still benefit from the fact that categorical equivalences generalize isomorphism, meaning it often suffices to restrict to the case of categorical equivalences.

The situation becomes untenable when we further generalize to higher categories, and particularly *double categories*, a structure that is a key ingredient in the study of lenses [11, 13], rewriting [9, 8], open systems [7, 6], and programming language theory [28, 14]. Double categories come with objects, two classes of morphisms, known as horizontal and vertical morphisms, and suitable data detailing how they interact, known as squares. This additional structure enables a wide range of configurations and identifications. In particular we can employ the squares to relax the associativity and unitality of the composition of horizontal or vertical morphisms. Also, we can use the intricate structure to define a wide range of equivalences, with the fascinating observation that none of them is more general than all the others.

As a result, a significant part of the double categorical literature has exclusively focused on one type of equivalences, called *horizontal equivalence* (Section 5), which prioritizes horizontal morphisms and hence ignores the natural symmetry between horizontal and vertical morphisms inherent to the definition of a double category. Examples include the work on limits [20], adjunctions [21], formal category theory [31], lenses [13] and open systems [6]. At the same time, symmetric notions of equivalences, such as *gregarious equivalences* (Section 9), have received far less attention, even though they are the natural context for other important double categorical constructions, such as the *square functor* [16]. The square functor is already conjecturally applied to modern aspects of algebraic geometry [18], with other anticipated applications currently hindered by insufficient theoretical advances in this direction. Addressing this situation requires the ability to adjust our definition of double categories based on the equivalence of interest, which can then be employed in the aforementioned examples.

Beyond mathematics, the fact that double categories do not come with a distinguished notion of equivalence also complicates any effort formalizing double category theory in intensional type theories via proof assistants. Indeed, we would like to have suitable principles that permit transporting results regarding double categories along equivalences. However, as there are several incomparable equivalences in the literature, it is not possible to only have one principle. We would rather need one such transport principle, and corresponding notion of double category, for each equivalence.

The Univalence Maxim. In this paper, we propose the *univalence maxim* to resolve the aforementioned challenges and to provide suitable transport principles, formally. Our univalence maxim takes place in univalent type theory, a variant of Martin-Löf type theory with the univalence axiom.

In type theory, Martin-Löf's identity type is a fundamental concept, capturing formally when two objects are considered "the same". The univalence axiom adds extensionality principles, postulating that the identity type of types coincides with the type of equivalences between these types. That is, two types are identified if we have an equivalence between them, and thus equivalent types have the same properties. This perspective also extends to structures. Indeed, in univalent foundations, each notion of structure comes with a notion of equivalence such that identity of structures corresponds to equivalence. For instance, one can prove that identity of algebraic structures, such as groups, corresponds to isomorphism. The correspondence between identity and equivalence for structures is known as the *structure identity principle*. Such principles allow us to prove directly that structure and properties can be transported along equivalences.

The *univalence maxim* we propose says the following: for every chosen equivalence of a given categorical structure, there exists a tailored definition for which identity precisely coincides with the chosen notion of equivalence. We can already witness manifestations of the maxim for categories, which, as we discussed, can be considered up to isomorphism or up to categorical equivalence. Indeed, in the univalent setting we have two notions of categories, namely *setcategories*, whose properties are automatically invariant under isomorphisms, and *univalent categories*, whose properties are automatically invariant under categorical equivalences. See Section 2 for further details.

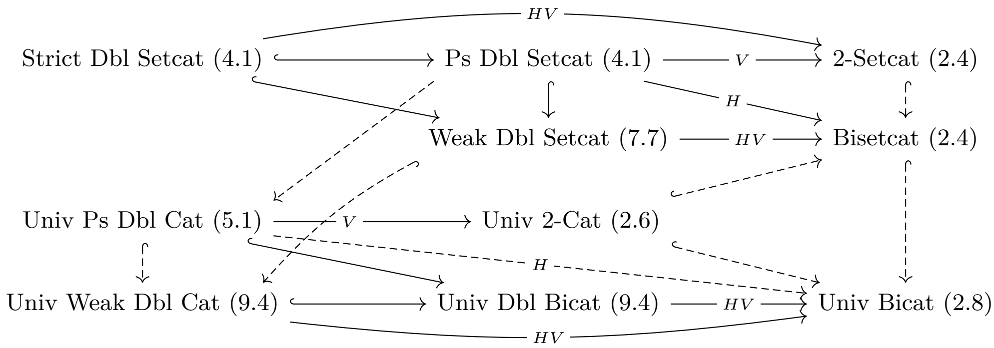
While we can already apply the univalence maxim to categories, its true power manifests in the more complicated framework of double categories. Indeed, beyond obtaining precise transport principles which assist formalization, by establishing a correspondence between various double categorical equivalences and suitably defined double categories, we are for the first time able to structure the existing zoo of double categorical notions that can be found in the literature, the results of which we summarized in the *Diagrammatic Summary* below. Having such principles available also helps with the formalization of double categories, because it allows us to view equivalences via the usual Martin-Löf identity type.

Contributions. In this paper we apply the univalence maxim to categories, 2-categories and double categories. More specifically,

1. in Section 4 we define *(pseudo) double setcategories* and prove that they are invariant under isomorphisms;
2. we provide a further generalization to *double bicategories* (Section 7) and *univalent double bicategories* invariant under gregarious equivalences (Section 9).
3. we also develop the theory of double bicategories, such as companions (Section 8) and computationally feasible methods to establish univalence (Section 10).

We review univalent categories and 2-categories following [3, 2] in Section 2 and *univalent pseudo double categories* in Section 5 following our previous work in [34].

Diagrammatic Summary. We can summarize our major double categorical notions and their relations diagrammatically. Here a dashed arrow represents an inclusion that only respects the categorical structure (i.e. only holds in classical setting), whereas a solid arrow indicates an inclusion that respects categorical structure and univalence conditions. Moreover, an arrow labeled V denotes the underlying vertical 2-category or bicategory (Definitions 3.7 and 7.1), whereas an arrow labeled H is the underlying horizontal 2-category or bicategory (Definitions 3.7 and 7.1).



Formalization. The main results have been formalized using the Coq proof assistant and the UniMath library. Links to the corresponding identifier in the code are in blue.

There are two differences between the formalization and the definitions presented in the paper. While in the paper, we present strict double categories as a special instance of pseudo double categories (Definition 3.1), we use an unfolded approach in the formalization. Second, here we define Verity double bicategories (Definition 7.1) using two bicategories whose types of objects are equal. This is not so in the formalization, where instead a more unfolded approach is used.

Related Work. The study of (higher) categories in univalent foundations has a rich history. Indeed, a study of univalent categories was commenced by Ahrens, Kapulkin, and Shulman in [3], and later extended by Ahrens, Frumin, Maggesi, Veltri, and Van der Weide to a study of univalence for 2-categories and bicategories in [2]. Both these works are reviewed more carefully in Section 2. This existing work on (2-)categories motivated us to pursue univalence principles for double categorical structures. This first commenced with [34], where we focused on univalence principles for horizontal equivalences, given their centrality in the current double categorical literature. In that paper we introduced univalent pseudo double categories and proved that their identities correspond to horizontal equivalences. Further details regarding our past work can be found in Section 5. The current paper is a natural continuation of that effort, generalizing from horizontal equivalences to gregarious equivalences and from double categories to double bicategories. This effort was also motivated by work in [4], where Ahrens, North, Shulman, and Tsementzis established a very broad univalence principle, which in particular applies to univalent double (bi)categories and hence establishes the basis for our work in Section 9.

Double categories acquired some attention from the formalization community, and several libraries on formalized mathematics contain double categories. Murray, Pronk, and Szyld [27] worked towards defining double categories in the Lean proof assistant[1]. In 1lab [32], internal

[1] https://github.com/leanprover-community/mathlib/pull/18204

categories are defined, and thus double categories are also defined as category objects in the category of setcategories. Finally, in the library by Hu and Carette [22] a definition of double category has been implemented[2]. Each of these formalization only considers strict double categories, whereas we also consider weaker notions. In addition, our formalization contains a study of various univalence principles.

2 (2-)Categories in Univalent Foundations

In this section we realize the vision of the univalence maxim for categories and bicategories, based on work done in [2, 3]. More precisely, we analyze two notions of equivalences for categories (isomorphisms, equivalences) and three notions of equivalences for 2-categories and bicategories (isomorphisms, equivalences, biequivalences). For each of these notions, we define a categorical structure whose identities correspond to that notion of equivalence.

We start with categories. In classical mathematics a category is defined as a class of objects and a set of morphisms, depending on two objects, with a unital and associative composition of morphisms. In univalent foundations, categories are defined to have a type of objects and a set of morphisms [3, Definition 3.1]. Here a type is called a set if any two identities between two terms are equal.

Note that categories have two important notions of equivalences: isomorphisms and equivalences. Categories that are invariant under isomorphism, are also known has *setcategories*. The other notion of interest, *univalent categories*, are categories that are invariant under equivalence. These notions are defined as follows.

▶ **Definition 2.1.** A category is said to be a **setcategory** if its type of objects is a set. A category C is said to be **univalent** if for all objects $x, y : \mathsf{C}$ the map $\mathsf{idtoiso}_{x,y}$, which sends identities $p : x = y$ to isomorphisms $\mathsf{idtoiso}_{x,y}(p) : x \cong y$, is an equivalence of types.

The univalence condition implies that equalities of categories are precisely categorical equivalences [3, Theorem 6.17], giving us the desired invariance property. Univalent categories provide us with an alternative way to characterize the fact that setcategories are invariant under isomorphisms:

▶ **Proposition 2.2.** *The category of setcategories and functors is univalent.*

Unfortunately, we cannot repeat the argument and incorporate the equivalence invariance of univalent categories into the construction of a univalent category, as the type of equivalences between two univalent categories is generally too complex and does not form a set. However, we can in fact construct a univalent bicategory of univalent categories [2, Proposition 3.19].

▶ **Proposition 2.3.** *The bicategory of univalent categories, functors, and natural transformations is univalent.*

Next we look at bicategories, and 2-categories, which are bicategories with identity associators and unitors. As bicategories not only have objects and 1-morphisms, but also 2-morphisms between 1-morphisms, the number of relevant equivalences increases significantly. Here we focus in particular on three equivalences: isomorphisms of bicategories, equivalences of bicategories (equivalences of the underlying 1-categories that are isomorphisms of 2-morphisms), and biequivalences. Our goal is to construct for each type of equivalence a bicategory (2-category) such that their identities correspond to the chosen equivalence.

[2] https://github.com/agda/agda-categories/blob/36abe6bff98be027bd4fcc3306d6dac8b2140079/src/Categories/Double/Core.agda

For the first kind of equivalence, taking Definition 2.1 as motivation we analogously impose appropriate set level restrictions to obtain isomorphism invariance.

▶ **Definition 2.4.** A bicategory (2-category) is said to be a **bisetcategory** (**2-setcategory**) if its type of objects and of 1-morphisms are sets.

▶ **Proposition 2.5.** *The two categories given by bisetcategories and functors, and by* 2-*setcategories and functors are univalent.*

For the second kind of equivalence, we combine Proposition 2.2 and Proposition 2.3.

▶ **Definition 2.6.** A 2-category is said to be **univalent** if the underlying 1-category is univalent and the 2-morphisms form a set.

▶ **Proposition 2.7.** *Identities of univalent* 2*-categories correspond to equivalences.*

Finally, we want a 2-categorical structure invariant under biequivalence. In light of Proposition 2.3, all hom categories need to be univalent, which we call the *local univalence* condition. Moreover, we also need a *global univalence* condition, stating that identities of objects correspond to equivalences in the 2-category. In general this means that objects form a 2-type and compositions of 1-morphisms is generally not strictly associative or unital. Hence, this univalence condition only applies to bicategories.

▶ **Definition 2.8.** A bicategory is **univalent** if it is globally and locally univalent.

See [2, Definition 3.1] for a more explicit description of its definition. Finally, univalent bicategories do exhibit the anticipated invariance property; see [4, Example 9.1].

▶ **Proposition 2.9.** *Identities of univalent bicategories correspond to biequivalences.*

3 Definition of Pseudo Double Categories

In the previous section, we reviewed (bi)categories and showed how imposing additional conditions in univalent foundations lead to (bi)categories up to a desired notion of sameness. For the rest of this paper, we conduct a similar analysis for double categorical structures. As mentioned in the introduction, defining a double category is more complex and involves more data, providing a wider range of examples and equivalences. Consequently, our analysis is more challenging and valuable, and more relevant to the broader literature.

In this transitional section we commence with a review of a general definition of pseudo double categories and explicate our examples. Then, in the next two sections we show how imposing additional conditions result in the desired equivalences.

▶ **Definition 3.1.** A **pseudo double category** consists of

1. a category C called the **vertical category**;
2. for all objects $x : \mathsf{C}$ and $y : \mathsf{C}$, a type $x \nrightarrow y$ of **horizontal morphisms**;
3. for every object $x : \mathsf{C}$ a **horizontal identity** $\mathsf{id}_x : x \nrightarrow x$;
4. for all horizontal morphisms $h : x \nrightarrow y$ and $k : y \nrightarrow z$, a **horizontal composition** $h \odot k : x \nrightarrow z$;
5. for all horizontal morphisms $h : x_1 \nrightarrow y_1$ and $k : x_2 \nrightarrow y_2$ and vertical morphisms $v_x : x_1 \nrightarrow x_2$ and $v_y : y_1 \nrightarrow y_2$, a set $\left(v_x \ {}^{h}_{k}\ v_y\right)$ of **squares**;
6. for all horizontal morphisms $h : x \nrightarrow y$, we have a **vertical identity** $\mathsf{id}^v_{\mathsf{sq}}(h) : \left(\mathsf{id}_x \ {}^{h}_{h}\ \mathsf{id}_y\right)$;
7. for all $\tau_1 : \left(v_1 \ {}^{h}_{k}\ w_1\right)$ and $\tau_2 : \left(v_2 \ {}^{k}_{l}\ w_2\right)$, a square $\tau_1 \cdot_{\mathsf{sq}} \tau_2 : \left(v_1 \cdot v_2 \ {}^{h}_{l}\ w_1 \cdot w_2\right)$;

8. for all $v : x \longrightarrow y$, we have a **horizontal identity** $\mathsf{id}^h_{\mathsf{sq}}(v) : \left(v \ \substack{\mathsf{id}_x \\ \mathsf{id}_y} \ v\right)$;

9. for all $\tau_1 : \left(v_1 \ \substack{h_1 \\ k_1} \ v_2\right)$ and $\tau_2 : \left(v_2 \ \substack{h_2 \\ k_2} \ v_3\right)$, a square $\tau_1 \odot_{\mathsf{sq}} \tau_2 : \left(v_1 \ \substack{h_1 \odot h_2 \\ k_1 \odot k_2} \ v_3\right)$;

10. for all $h : x \nrightarrow y$, we have a **left unitor** $\lambda_h : \left(\mathsf{id}_x \ \substack{\mathsf{id}_x \odot h \\ h} \ \mathsf{id}_y\right)$;

11. for all $h : x \nrightarrow y$, we have a **right unitor** $\rho_h : \left(\mathsf{id}_x \ \substack{h \odot \mathsf{id}_y \\ h} \ \mathsf{id}_y\right)$;

12. for all $h_1 : w \nrightarrow x$, $h_2 : x \nrightarrow y$, and $h_3 : y \nrightarrow z$, we have an **associator**

$$\alpha_{(h_1,h_2,h_3)} : \left(\mathsf{id}_w \ \substack{h_1 \odot (h_2 \odot h_3) \\ (h_1 \odot h_2) \odot h_3} \ \mathsf{id}_z\right).$$

This data is required to satisfy several laws, which are found in the literature [34].

A **strict double category** is a pseudo double category in which horizontal morphisms form a set and in which the unitors and associators are identities.

We now give precise definitions of the four major classes of examples that play a prominent role throughout this paper: squares, spans, structured cospans and profunctors.

▶ **Example 3.2.** Let C be a category. Then we define a pseudo double category $\mathsf{Sq}(\mathsf{C})$ **of squares** . The objects are objects in C and the horizontal and vertical morphisms are morphisms in C. The type of squares $\left(v_1 \ \substack{h_1 \\ h_2} \ v_2\right)$ is defined to be $h_1 \cdot v_2 = v_1 \cdot h_2$.

▶ **Example 3.3.** Let C be a category with pullbacks. Then we define a pseudo double category $\mathsf{Span}(\mathsf{C})$ **of spans** . The objects are objects in C and the vertical morphisms are morphisms in C. The horizontal morphisms are spans, which are diagrams of the form $x \xleftarrow{\varphi} z \xrightarrow{\psi} y$; and Given morphisms $f : x_1 \longrightarrow x_2$ and $g : y_1 \longrightarrow y_2$, then a square with f and g as vertical sides and spans $x_1 \xleftarrow{\varphi_1} z_1 \xrightarrow{\psi_1} y_1$ and $x_2 \xleftarrow{\varphi_2} z_2 \xrightarrow{\psi_2} y_2$ as horizontal sides is a morphism $h : z_1 \longrightarrow z_2$ such that the following diagram commutes.

$$\begin{array}{ccccc} x_1 & \xleftarrow{\varphi_1} & z_1 & \xrightarrow{\psi_1} & y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ x_2 & \xleftarrow[\varphi_2]{} & z_2 & \xrightarrow[\psi_2]{} & y_1 \end{array}$$

Even if C is a setcategory, $\mathsf{Span}(\mathsf{C})$ is generally only a pseudo double category. This is because composition of spans is given by pullbacks, which is only weakly unital and associative. Note that spans have been used in the study of rewriting systems [9, 8].

▶ **Example 3.4.** Suppose that we have a functor $L : \mathsf{C}_1 \longrightarrow \mathsf{C}_2$. We define the double category $\mathsf{StructCospan}(L)$ **of structured cospans** . The objects are objects in C_1 and the vertical morphisms are morphisms in C_1. The horizontal morphisms are **structured cospans**, which are diagrams of the form $L(x) \xrightarrow{\varphi} z \xleftarrow{\psi} L(y)$; Given two structured cospans $L(x_1) \xrightarrow{\varphi_1} z_1 \xleftarrow{\psi_1} L(y_1)$ and $L(x_2) \xrightarrow{\varphi_2} z_2 \xleftarrow{\psi_2} L(y_2)$, and two morphisms $f : x_1 \longrightarrow x_2$ and $g : y_1 \longrightarrow y_2$, a square consists of a morphism $h : z_1 \longrightarrow z_2$ such that the following diagram commutes

$$\begin{array}{ccccc} L(x_1) & \xrightarrow{\varphi_1} & z_1 & \xleftarrow{\psi_1} & L(y_1) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle L(g)} \\ L(x_2) & \xrightarrow[\varphi_2]{} & z_2 & \xleftarrow[\psi_2]{} & L(y_1) \end{array}$$

Note that structured cospans are used to study open systems [7, 6].

▶ **Example 3.5.** We define the pseudo double category Prof **of profunctors** . The objects are small setcategories and the vertical morphisms are functors. The horizontal morphisms are profunctors from a category C to D, meaning functors of the form $\mathsf{D}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{Set}$, and given profunctors $P : \mathsf{C}_1 \longrightarrow \mathsf{D}_1$ and $Q : \mathsf{C}_2 \longrightarrow \mathsf{D}_2$ and functors $F : \mathsf{C}_1 \longrightarrow \mathsf{C}_2$ and $G : \mathsf{D}_1 \longrightarrow \mathsf{D}_2$, we define squares $\left(F \; {}^{P}_{Q} \; G\right)$ to be natural transformations $P \Rightarrow (F \times G) \cdot Q$. Again, this pseudo double category will not be a strict double category, as profunctors do not compose strictly. Note that profunctors are important in the study of lenses [11, 13].

Note that the composition of profunctors is defined as a colimit in the category of sets. To guarantee that the desired colimit exists, we require that the involved categories are small.

▶ **Example 3.6** (Gradual type theory, [28, Definition 5.2])**.** Let C be a 2-category. We define a **pseudo double category** Coreflect(C) **of coreflections** . The objects are objects in C, the vertical morphisms are morphisms in C, and the horizontal morphisms are adjunctions in C whose unit is an equality (i.e., coreflections). The squares are given by 2-cells in C.

Double categories include the data of several (2-)categories. These are known as the *underlying 2-categories*, and they are defined as follows.

▶ **Definition 3.7.** Given a pseudo double category C, we define a strict 2-category $\mathsf{Ver}(\mathsf{C})$, which we call the **underlying vertical 2-category** as the 2-category whose objects are objects in C, whose 1-cells are vertical morphisms in C, and whose 2-cells are squares with horizontal identity sides.

In addition, we define a bicategory $\mathsf{Hor}(\mathsf{C})$, which we call the **underlying horizontal bicategory** , as the bicategory whose objects are objects in C, whose 1-cells are horizontal morphisms in C, and whose 2-cells are squares with vertical identity sides.

4 (Pseudo) Double Set-categories

Having defined strict and pseudo double categories, we now impose conditions to obtain isomorphism invariant definitions, following the vision of the univalence maxim. Concretely, we first construct isomorphism invariant notions of (pseudo) double categories and then study several classes of examples. Following Section 2 and Definition 2.1, we need to impose a set level condition to obtain isomorphism invariance, motivating the following definitions.

▶ **Definition 4.1.** A **strict double setcategory** is a strict double category whose objects form a set. In addition, a **pseudo double setcategory** is a double category whose objects and horizontal morphisms form a set.

Using similar ideas to the one used in Proposition 2.2, we confirm their desired invariance.

▶ **Theorem 4.2.** *The category of strict double setcategories, with objects being strict double setcategories and morphisms being strict double functors, is univalent.*

▶ **Theorem 4.3.** *The category of pseudo double setcategories, with objects being pseudo double setcategories and morphisms being strict double functors, is univalent.*

We now review our motivating examples in light of this invariance property.

▶ **Proposition 4.4.** *Let* C *be a setcategory. Then* $\mathsf{Sq}(\mathsf{C})$ *is a strict double setcategory.*

▶ **Proposition 4.5.** *For a setcategory* C *with pullbacks,* $\mathsf{Span}(\mathsf{C})$ *is a pseudo double setcategory.*

▶ **Proposition 4.6.** *Suppose that* C_1 *and* C_2 *are setcategories and that* C_2 *has pushouts. Then* $\mathsf{StructCospan}(L)$ *is a pseudo double setcategory.*

▶ **Proposition 4.7.** *Let* C *be a 2-setcategory. Then* $\mathsf{Coreflect}(\mathsf{C})$ *as defined in Example 3.6 is a pseudo double setcategory.*

The previous examples are very consistent with our intuition that an isomorphism invariant category should indeed result in isomorphism invariant double categories of squares, (co)spans, coreflections, or polynomials. Observe the class of examples of profunctors, Example 3.5, is in fact not a pseudo double setcategory. See Proposition 5.7 for a more detailed discussion. Let us instead present one further example coming from 2-category theory.

▶ **Example 4.8.** For a bisetcategory B we define the pseudo double setcategory $\widehat{\mathsf{B}}$, whose objects are objects in B. Vertical morphisms are morphisms in B, and horizontal morphisms $x \nrightarrow y$ are identities $x = y$. Squares $\left(f \ {}^{p}_{q} \ g\right)$ are 2-cells $f \cdot \mathsf{idtoiso}(q) \Rightarrow \mathsf{idtoiso}(p) \cdot g$.

We end this section by observing that the set condition is preserved by taking underlying bicategories. This results supports the intuitive fact that taking underlying 2-categories preserves isomorphism invariance.

▶ **Proposition 4.9.** *If* C *is a strict double setcategory, then* $\mathsf{Ver}(\mathsf{C})$ *is a strict* 2*-setcategory.*

▶ **Proposition 4.10.** *If* C *is a pseudo double setcategory, then* $\mathsf{Ver}(\mathsf{C})$ *is a strict* 2*-setcategory.*

▶ **Proposition 4.11.** *If* C *is a pseudo double setcategory, then* $\mathsf{Hor}(\mathsf{C})$ *is a bisetcategory.*

5 Univalent Pseudo Double Categories

In the previous section we focused on isomorphism invariance. In this section we continue realizing our univalence maxim, this time studying pseudo double categories invariant under *vertical equivalences*, which are characterized by inducing equivalences on the underlying vertical category and for any two objects x, y inducing equivalences on the category given by horizontal morphisms $x \nrightarrow y$ and squares with trivial vertical sides. Our analysis relies on our previous work done in [34]. We then end this section analyzing several examples.

Following Definition 2.1, we would expect a univalence condition for these two underlying categories. This, however, implies that the underlying horizontal category is neither univalent nor has a set of objects, meaning horizontal composition cannot be strict. Thus we have to work with pseudo double categories, resulting in the following definition.

▶ **Definition 5.1.** A pseudo double category C is said to be **univalent** if its underlying vertical category is univalent and if for all $x, y : \mathsf{C}$ the category whose objects are horizontal morphisms $x \nrightarrow y$ and whose morphisms are squares with vertical identity sides, is univalent.

Building on our insights in Proposition 2.3, we similarly construct a univalent bicategory of univalent pseudo double categories [34].

▶ **Theorem 5.2.** *The bicategory of univalent pseudo double categories with lax double functors is univalent.*

Note that we use lax double functors in Theorem 5.2 whereas we use strict double functors in Theorems 4.2 and 4.3. As the univalence condition is motivated by vertical equivalences, it is not symmetric. For examples identities of objects only correspond to vertical isomorphisms, and identities of horizontal morphisms correspond to isomorphisms of squares (composed vertically). However, some double categories satisfy a stronger univalence condition that is in fact symmetric.

▶ **Definition 5.3.** A pseudo double category C is said to be **symmetrically univalent** if the horizontal morphisms form a set, C is univalent, the category of objects and horizontal morphisms is univalent, and for all $x, y : \mathsf{C}$ the category of vertical morphisms $x \longrightarrow y$ and squares with horizontal identity sides, is univalent.

Let us present a variety of examples of univalent and symmetrically univalent pseudo double categories. Again we focus on our three classes of interest, namely squares, spans and profunctors [34], but we also provide additional examples.

▶ **Proposition 5.4.** *If* C *is a univalent category, then* $\mathsf{Sq}(\mathsf{C})$ *is a symmetrically univalent pseudo double category.*

▶ **Proposition 5.5.** *Let* C *be a univalent category with pullbacks. Then the pseudo double category* $\mathsf{Span}(\mathsf{C})$*, is univalent, but not symmetrically univalent.*

▶ **Proposition 5.6.** *If* C_1 *and* C_2 *are univalent categories such that* C_2 *has pushouts, then* $\mathsf{StructCospan}(L)$ *is a univalent pseudo double category.*

▶ **Proposition 5.7.** *The pseudo double category of profunctors is univalent.*

▶ **Proposition 5.8.** *If* C *is a univalent 2-category. Then* $\mathsf{Coreflect}(\mathsf{C})$ *is a univalent pseudo double category.*

In the later sections, we analyze enriched versions of double categories of profunctors benefiting from appropriately defined weak double categories (Example 7.5). However, if we focus on categories enriched over a poset, which includes quantales and has found applications in automata theory [1, 30] and fuzzy logic [15], we do get stricter double categories.

▶ **Proposition 5.9.** *Suppose that* V *is a complete and cocomplete symmetric monoidal category and suppose that* V *is a poset. Then we have a univalent pseudo double category whose objects are univalent categories enriched over* V*, vertical morphisms are enriched functors, horizontal morphisms are enriched profunctors, and whose squares are given by enriched natural transformations.*

▶ Remark 5.10. Note we cannot construct a univalent pseudo double category given by univalent categories, functors and profunctors. Indeed, the type of univalent categories is a 2-type as it includes all 1-types, by [33, Example 9.9.6] and [2, Example 2.18], and hence cannot be the objects of a univalent category, which is at most a 1-type ([3, Lemma 3.8]).

In the last section we observed that double setcategories (both pseudo and strict) preserve isomorphism invariance when taking underlying bicategories. We might hence anticipate similar results for univalent pseudo categories. Unfortunately we only have a partial result. Indeed, vertical equivalences are preserved by taking the underlying vertical 2-category, confirmed by the following result.

▶ **Proposition 5.11.** *If* D *is univalent, then so is* $\mathsf{Ver}(\mathsf{D})$*.*

However, it is generally untrue that univalent pseudo double categories will induce globally univalent underlying horizontal bicategories. Indeed, the underlying horizontal bicategory of the pseudo double category Prof is given by small setcategories, profunctors and 2-morphisms, and we already observed in Proposition 5.7 that profunctors can be an isomorphism without the underlying categories being isomorphic. However, not all hope is lost and we do recover a local univalence condition.

▶ **Proposition 5.12.** *If* D *is a univalent, then* $\mathsf{Hor}(\mathsf{D})$ *is locally univalent.*

Note that $\mathsf{Hor}(\mathsf{D})$ is not necessarily univalent. This necessitates a double categorical notion that accommodates biequivalences of bicategories.

6 Motivating Verity Double Bicategories

Pseudo-double categories only exhibit non-strict compositions in the horizontal direction and are hence unable to incorporate all relevant examples (Remark 5.10) or biequivalences as an invariant (Proposition 5.12). We hence require a notion of a doubly weak double category with non-strict compositions in both directions. However, providing a direct definition similar to Definition 3.1 results in a fully faithful embedding from pseudo double-categories that does not preserve vertical equivalences. In univalent foundations, in addition to the non-preservation of equivalences, this embedding also does not preserve univalence. Concretely, due to the strictness of vertical compositions, identities in the type of objects correspond to vertical **isomorphisms**, whereas the weak vertical composition and symmetric nature of weak double categories demands that identities correspond to a pair of horizontal and vertical **equivalences** which interact well with each other, i.e. form a companion pair (Section 8). Hence, a univalent pseudo double category results in a non-univalent weak double category.

Univalent foundations hence can propose a solution to non-preservation of equivalences, namely by introducing a *pre-double categorical structure*, whose notion of univalence can incorporate both univalent pseudo double categories and univalent weak double categories, which will in particular imply preservation of equivalences. Taking the previous paragraph as motivation what such a structure should entail is a notion of 2-morphisms divorced from the 2-morphisms induced by squares (as defined in Definition 3.7). By appropriately choosing the 2-cells we can then restrict to both cases of interest:

- If we choose the vertical 2-cells to be identities then equivalences in the vertical 2-category would be isomorphism, recovering the identities of univalent pseudo-categories;
- if we choose the vertical (and horizontal) 2-cells to coincide with 2-cells induced by squares, then identities of objects correspond to equivalences we expect to see in weak double categories.

Thus, in order to pursue our study of equivalences of double categories, we first develop pre-double category theory, whose structure involves objects, horizontal (vertical) morphisms, horizontal (vertical) 2-morphisms, and squares, along with appropriately coherent compositions. Fortunately, a suitable candidate for such a notion has already been proposed by Verity , where it is called a *double bicategory* [35], along with a univalence principle [4]. Hence, for the remainder of the paper the aim is to study and formalize double bicategories, study its univalence principle, and establish an embedding from univalent pseudo categories to univalent double bicategories.

7 Verity Double Bicategories

Following the discussion of Section 6 we commence with the definition and formalization of double bicategories due to Verity [35], and also describe various examples. In Section 9 we will then pursue its univalence principles.

▶ **Definition 7.1.** A **Verity double bicategory** B consists of

1. a bicategory H_B whose objects, 1-cells, and 2-cells are called **horizontal**;
2. a bicategory V_B with the same type of objects as H_B, and whose objects, 1-cells, and 2-cells are called **vertical**;
3. for all objects $x_1, x_2, y_1, y_2 : \mathsf{H}_\mathsf{B}$, horizontal 1-cells $h_1 : x_1 \longrightarrow x_2$ and $h_2 : y_1 \longrightarrow y_2$, and vertical 1-cells $v_1 : x_1 \longrightarrow y_1$ and $v_2 : x_2 \longrightarrow y_2$ a set $\left(v_1 \ {}^{h_1}_{h_2} \ v_2\right)$ of **squares**;
4. for all horizontal 1-cells $h : x \longrightarrow y$ a square $\mathsf{id}^h_{\mathsf{sq}}(h) : \left(\mathsf{id}_x \ {}^{h}_{h} \ \mathsf{id}_y\right)$;

5. for all vertical 1-cells $v : x \longrightarrow y$ a square $\mathsf{id}^v_{\mathsf{sq}}(v) : \left(v \;{}^{\mathsf{id}_x}_{\mathsf{id}_y}\; v\right)$;
6. for all $s_1 : (v_1 \;{}^{h}_{k}\; w_1)$ and $s_2 : (v_2 \;{}^{k}_{l}\; w_2)$, a square $s_1 \cdot_{\mathsf{sq}} s_2 : (v_1 \cdot v_2 \;{}^{h}_{l}\; w_1 \cdot w_2)$;
7. for all $s_1 : \left(v_1 \;{}^{h_1}_{k_1}\; v_2\right)$ and $s_2 : \left(v_2 \;{}^{h_2}_{k_2}\; v_3\right)$, a square $s_1 \odot_{\mathsf{sq}} s_2 : \left(v_1 \;{}^{h_1 \odot h_2}_{k_1 \odot k_2}\; v_3\right)$;
8. for all vertical 2-cells $\tau : v_1 \Rightarrow v_2$ and squares $s : (v_2 \;{}^{h}_{k}\; w)$, a square $\tau \lhd s : (v_1 \;{}^{h}_{k}\; w)$;
9. for all vertical 2-cells $\tau : w_1 \Rightarrow w_2$ and squares $s : (v \;{}^{h}_{k}\; w_1)$, a square $\tau \rhd s : (v \;{}^{h}_{k}\; w_2)$;
10. for all horizontal 2-cells $\tau : h_1 \Rightarrow h_2$ and squares $s : \left(v \;{}^{h_2}_{k}\; w\right)$, a square $\tau \vartriangle s : \left(v \;{}^{h_1}_{k}\; w\right)$;
11. for all horizontal 2-cells $\tau : k_1 \Rightarrow k_2$ and squares $s : (v \;{}^{h}_{k_1}\; w)$, a square $\tau \triangledown s : (v \;{}^{h}_{k_2}\; w)$.

We call $\mathsf{H_B}$ the **underlying horizontal bicategory** and $\mathsf{V_B}$ the **underlying vertical bicategory**. In addition to the data explicated here, we have various laws governing their behavior. See the formalization or [35, Definition 1.4.1] for further details

Let us note that this definition does in fact satisfy all the desired conditions outlined in Section 6. Indeed, we have independently defined horizontal and vertical 2-cells and compositions of all 1-morphisms are defined weakly, giving us a symmetric definition. Following our vision, we now define *weak double categories* by adding an appropriate saturation condition identifying 2-cells with certain squares.

▶ **Definition 7.2.** Suppose we have a Verity double bicategory B. For all horizontal 1-cells $h_1, h_2 : x \longrightarrow y$ we have a map $\mathsf{CellToSq_H}$ sending 2-cells $\tau : h_1 \Rightarrow h_2$ to the square $\tau \vartriangle \mathsf{id}^h_{\mathsf{sq}}(h_2) : \left(\mathsf{id}_x \;{}^{h_1}_{h_2}\; \mathsf{id}_y\right)$. We say that B is **horizontally saturated** if $\mathsf{CellToSq_H}$ is an equivalence. Similarly, the map $\mathsf{CellToSq_V}$ sends vertical 2-cells $\tau : v_1 \Rightarrow v_2$ to the square $\tau \lhd \mathsf{id}^v_{\mathsf{sq}}(v_2) : \left(v_1 \;{}^{\mathsf{id}_x}_{\mathsf{id}_y}\; v_2\right)$, and B is **vertically saturated** if $\mathsf{CellToSq_V}$ is an equivalence. A **weak double category** is a horizontally and vertically saturated Verity double bicategory.

Our definition of weak double categories by definition comes with an inclusion in Verity double bicategories. We now establish the second inclusion suggested in Section 6 and show that every pseudo double category gives rise to a Verity double bicategory.

▶ **Example 7.3.** Suppose that we have a pseudo double category C. We define a Verity double bicategory $\overline{\mathsf{C}}$. Its underlying horizontal bicategory is the discrete bicategory on the underlying vertical category of C, and the underlying vertical bicategory is the underlying horizontal bicategory of C. The squares are defined to be squares in C. Note that $\overline{\mathsf{C}}$ is vertically saturated, but not necessarily horizontally saturated.

Notice that the assignment of the horizontal 2-cells in Example 7.3 is not unique, and our choice is motivated by the desire to realize our programme, meaning to guarantee that univalent pseudo-double categories give us univalent Verity double bicategories. See Remark 10.10 for more details

We now proceed to look at several examples of Verity double bicategories. By Example 7.3, every pseudo double category results in a Verity double bicategory. So here we focus on examples giving us weak double categories, again motivated by our three classes of examples introduced in Section 3, squares, profunctors and spans.

▶ **Example 7.4.** Let B be a bicategory. We define a Verity double bicategory $\mathsf{Sq(B)}$ **of squares** . The horizontal bicategory $\mathsf{H_{Sq(B)}}$ is B, and the vertical bicategory $\mathsf{V_{Sq(B)}}$ is $\mathsf{B^{co}}$. The squares $(v \;{}^{h}_{k}\; w)$ are defined to be 2-cells $h \cdot w \Rightarrow v \cdot k$. Note that $\mathsf{Sq(B)}$ is both horizontally and vertically saturated, meaning it is a weak double category.

In Example 7.4, the 2-cells in the vertical bicategory $\mathsf{V_{Sq(B)}}$ are reversed. This is necessary to get the right whiskering operations.

▶ **Example 7.5.** We define a Verity double bicategory Prof **of profunctors** . The underlying horizontal bicategory $\mathsf{H}_{\mathsf{Prof}}$ is the bicategory $\mathsf{UnivCat}^{\mathsf{co}}$ of small univalent categories, and the objects of $\mathsf{V}_{\mathsf{Prof}}$ are small univalent categories, the 1-cells are profunctors, and the 2-cells are natural transformations. The squares are defined in the same way as in Example 3.5.

The identity and composition operations for profunctors are defined in the same way as in Example 3.5. For the details of the whiskering constructions, we refer the reader to the formalization. Note that Prof is both vertically and horizontally saturated, giving us a weak double category.

Similarly, we define the Verity double bicategory $\mathsf{Prof}_{\mathsf{V}}$ **of enriched profunctors** . Let V be a complete and cocomplete symmetric monoidal closed category. The underlying horizontal bicategory $\mathsf{H}_{\mathsf{Prof}_{\mathsf{V}}}$ is the bicategory $\mathsf{UnivCat}^{\mathsf{co}}_{\mathsf{V}}$ of small univalent enriched categories, and the objects of the underlying vertical $\mathsf{V}_{\mathsf{Prof}_{\mathsf{V}}}$ are small univalent categories, 1-cells are enriched profunctors, and 2-cells are enriched natural transformations. The squares are defined in a similar way as in Example 3.5.

Identity, composition, and whiskering operations are defined similarly to Example 7.5. Since $\mathsf{Prof}_{\mathsf{V}}$ is both vertically and horizontally saturated, it is a weak double category.

Note here we use small categories in Example 7.5 to guarantee that the desired coends exist, analogous to Example 3.5.

Unlike the previous sections we do not construct a weak double category of spans in a bicategory, as such a construction would require additional categorical layers; see also [25, Section 4]. However, we do have one additional example motivated by *mate calculus*.

▶ **Example 7.6** (Mate calculus, [23, Proposition 2.2])**.** Let B be a bicategory. We define a Verity double bicategory LAdj(B) . The horizontal bicategory $\mathsf{H}_{\mathsf{LAdj(B)}}$ is the bicategory whose objects are objects in B, 1-cells are left adjoints, and 2-cells are mate-pairs, and the vertical bicategory $\mathsf{V}_{\mathsf{LAdj(B)}}$ is B^{co}. Given adjunctions $h : x \dashv y$ and $k : x' \dashv y'$, and 1-cells $v : x \longrightarrow x'$ and $w; y \longrightarrow y'$, squares $\left(v \ {}^{h}_{k} \ w\right)$ are defined to be 2-cells $h \cdot w \Rightarrow v \cdot k$.

▶ Remark 7.7. Similar to Section 4, we can impose a set condition on the types of objects and morphisms to define *Verity double bisetcategories* and *weak double setcategories*, whose identities, analogous to Theorems 4.2 and 4.3, correspond to isomorphisms. However, similar to the classical setting (Section 6), pseudo double setcategories fully faithfully embeds in weak double setcategories, hence obviating the need to generalize to any pre-double categorical notion, such as Verity double bicategories.

Given these similarities, we proceed to the study of univalent Verity double bicategories and their relation to univalent pseudo categories and univalent weak double categories.

8 Companion Pairs

In the previous section we defined Verity double bicategories and showed that this general notion includes both pseudo double categories and weak double categories. Our major aim is to show that these assignments preserves univalence. Due to the symmetric nature of Verity double bicategories, equivalences of objects are symmetric as well, so they need to be given by a pair of horizontal and vertical equivalences that interact well with each other. In this section we provide a precise characterization of the interaction between horizontal and vertical morphisms, via *companion pairs*.

▶ **Definition 8.1.** Suppose that we have a Verity double bicategory B and a horizontal morphism $h : x \longrightarrow y$ and a vertical morphism $v : x \longrightarrow y$. Then we say that h and v form a **companion pair** if we have squares $\eta : \left(v \; {}^{h}_{\mathsf{id}_y} \; \mathsf{id}_y\right)$ and $\varepsilon : \left(\mathsf{id}_x \; {}^{\mathsf{id}_x}_{h} \; v\right)$ such that the squares $\rho \rhd \ell^{-1} \lhd (\varepsilon \odot_{\mathsf{sq}} \eta)$ and $\rho \triangledown \ell^{-1} \vartriangle (\varepsilon \cdot_{\mathsf{sq}} \eta)$ are identity squares. Here, ℓ is the left unitor. We call η the **unit** and ε the **counit**.

An alternative approach to the interaction between horizontal and vertical morphisms is given by *conjoints*, which are companion pairs in the horizontal dual of B. This notion is prevalent in *formal category theory* [29, 37, 38].

Beyond the definition we also need several key properties of companion pairs that we confirm here. Concretely, we want to know that companion pairs include identities and are closed under composition. Moreover, companions, if they exist, are unique up to isomorphism. Finally, companions of equivalences are also equivalences.

▶ **Proposition 8.2.** *Given an object x in a Verity double bicategory B, then the horizontal identity id_x and vertical identity id_x form a companion pair.*

▶ **Proposition 8.3.** *Given companion pairs $h_1 : x \longrightarrow y$ and $v_1 : x \longrightarrow y$, and $h_2 : y \longrightarrow z$ and $v_2 : y \longrightarrow z$, then $h_1 \cdot h_2$ and $v_1 \cdot v_2$ also form a companion pair.*

▶ **Proposition 8.4.** *Let B be a Verity double bicategory such that H_{B} and V_{B} are locally univalent and such that B is vertically saturated. For every horizontal 1-cell $h : x \longrightarrow y$, we have that all vertical 1-cells $v, v' : x \longrightarrow y$ that form a companion pair with h are equal.*

▶ **Proposition 8.5.** *Suppose that we have a Verity double bicategory B such that B is vertically saturated. Given a horizontal adjoint equivalence $l \dashv r$ such that l and r have companion pairs l' and r' respectively, then we have a vertical adjoint equivalence given by $l' \dashv r'$.*

In many important examples of Verity double bicategories, every horizontal 1-morphism has a companion, which is a key ingredient towards establishing its univalence condition. This holds for $\mathsf{Sq}(\mathsf{B})$, Prof, and $\mathsf{Prof}_{\mathsf{V}}$, whereas if we have a pseudo double category C, then $\overline{\mathsf{C}}$ has companion pairs if C has.

▶ **Proposition 8.6.** *Let B be a bicategory. Given a 1-cell $f : x \longrightarrow y$ in B, then f and f form a companion pair in $\mathsf{Sq}(\mathsf{B})$.*

▶ **Proposition 8.7.** *Suppose that we have a functor $F : \mathsf{C}_1 \longrightarrow \mathsf{C}_2$. Note that F gives rise to a profunctor $\mathsf{rep}_\ell(F) : \mathsf{C}_1 \longrightarrow \mathsf{C}_2$ that sends objects $x : \mathsf{C}_1$ and $y : \mathsf{C}_2$ to the set of morphisms $F(y) \longrightarrow x$. Then F and $\mathsf{rep}_\ell(F)$ form a companion pair.*

9 Univalent Double Bicategories

In this section we use companion pairs introduced in Section 8 to present a univalence principle for double bicategories (Section 7), further advancing our general maxim. Given the amount data a Verity double bicategory involves, we split up the univalence condition into two parts. The first one is a local conditions imposed on the hom-categories in the underlying horizontal and vertical bicategories.

▶ **Definition 9.1.** A Verity double bicategory B is said to be **locally univalent** if both H_{B} and V_{B} are locally univalent.

The second univalence condition is global and focuses on the type of objects in Verity double bicategories. Here we use companion pairs.

▶ **Definition 9.2.** A **gregarious equivalence** from x to y in a Verity double bicategory B consists of a horizontal adjoint equivalence $h : x \longrightarrow y$, a vertical adjoint equivalence $v : x \longrightarrow y$ such that h and v form a companion pair.

▶ **Proposition 9.3.** *Given an object x in a Verity double bicategory, then the horizontal identity id_x and the vertical identity id_x form a gregarious equivalence.*

Following [4], we define *gregarious univalence* using gregarious equivalences.

▶ **Definition 9.4.** Given a Verity double bicategory and objects x and y, we define the map $\mathsf{IdToGregEq}_{x,y}$ sending identities $x = y$ to gregarious equivalences using path induction and the fact that the identity is a gregarious equivalence (Proposition 9.3). A Verity double bicategory is said to be **gregarious univalent** if the map $\mathsf{IdToGregEq}_{x,y}$ is an equivalence of types for all x and y.

Following the univalence principle in [4, Example 9.3], identities of gregarious univalent Verity double bicategories are equivalent to the type of gregarious equivalences between them. Finally, by Definition 7.2, a weak double category is univalent if it is univalent as a double bicategory, confirming our intuition presented in Section 6.

10 Univalence and Weak Horizontal Invariance

In this final section we tie up our discussion of univalent double categorical structures, by establishing the following two facts. First, there is a large class of univalent double bicategories, such as $\mathsf{Sq}(\mathsf{B})$, Prof, and $\mathsf{Prof}_{\mathsf{V}}$. Second, every univalent pseudo double category gives rise to a univalent Verity double bicategory.

To deal with the complicated nature of gregarious univalence, we use a more conceptual approach. We define a notion of *weakly horizontally invariant double bicategory* and show that in this case (with some minor conditions) gregarious univalence reduces to horizontal univalence. We end this section with checking these two properties for our cases of interest.

Let us start with the definition of weak horizontal invariance.

▶ **Definition 10.1.** A Verity double bicategory is **weakly horizontally invariant** if every horizontal adjoint equivalence has a companion pair.

▶ **Proposition 10.2.** *Let B be a Verity double bicategory such that H_{B} is globally univalent. Then B is weakly horizontally invariant.*

This is because the horizontal identity has a companion, and to construct companions for arbitrary adjoint equivalence, we use induction on adjoint equivalences. Proving the main theorem requires the following proposition, which characterizes gregarious equivalences.

▶ **Proposition 10.3.** *Let B be a weakly horizontally invariant Verity double bicategory such that B is vertically saturated and such that H_{B} and V_{B} are locally univalent. Then a horizontal morphism $h : x \longrightarrow y$ is an adjoint equivalence if and only if we have a vertical 1-cell $v : x \longrightarrow y$ such that h and v are a gregarious equivalence.*

▶ **Theorem 10.4.** *Let B be a locally univalent, vertically saturated, and weakly horizontally invariant Verity double bicategory. Then B is gregarious univalent if and only if the bicategory H_{B} is globally univalent.*

We now apply Theorem 10.4 to our cases of interest.

▶ **Proposition 10.5.** *Let* B *be a univalent bicategory. Then* $\mathsf{Sq}(\mathsf{B})$ *is weakly horizontally invariant and univalent.*

▶ **Proposition 10.6.** Prof *is weakly horizontally invariant and univalent.*

▶ **Proposition 10.7.** $\mathsf{Prof}_{\mathsf{V}}$ *is weakly horizontally invariant and univalent.*

▶ **Proposition 10.8.** *If* B *be a univalent bicategory, then* $\mathsf{LAdj}(\mathsf{B})$ *is weakly horizontally invariant and univalent.*

▶ **Proposition 10.9.** *Suppose that we have a univalent pseudo double category* C*. Then the double bicategory* $\overline{\mathsf{C}}$ *is weakly horizontally invariant and univalent.*

▶ Remark 10.10. The choice of 2-cells in Example 7.3 was not unique, and our choice was motivated by Proposition 10.9. To obtain gregarious univalence, the horizontal 2-morphisms need to be trivial so that identities are given by isomorphisms. For instance, in the univalent pseudo-double category of setcategories, functors and profunctors, the resulting Verity double bicategory is univalent because the 2-cells are identities.

Let us end by observing that weak horizontal invariance has already been employed to study equivalences of double categories, however, from a categorical perspective [26].

11 Conclusion

In this paper we presented a connection between equivalences of categorical structures and formalizations thereof in `UniMath`. More specifically, we introduced the univalence maxim for categorical structures, which says that every notion of equivalence comes with a corresponding notion of univalent categorical structure, whose identity type corresponds to the given notion of equivalence. We studied the maxim for many different examples, namely categories, 2-categories, and double categories.

As a result of the maxim, we realize that the univalent setting is the appropriate framework to articulate and formalize (higher) categorical definitions and results, as it provides direct access to valuable transport principles along (higher) categorical equivalences of interest, by transforming categorical equivalences into identities (Proposition 2.9 and Theorem 5.2). Moreover, the univalence maxim empowers us to compare and contrast different double categorical notions, tying together disparate results in the double categorical literature and facilitating a holistic approach to all double categorical notions. Concretely, in the univalent setting, we can compare double categories that differ in their strictness properties, for example strict vs. pseudo double categories, and double categories that differ in their chosen notion of sameness, for example pseudo double categories up to isomorphism vs. pseudo double categories up to horizontal equivalence. Here we note that the second type of comparison is not possible in set theoretical foundations.

Finally, let us note that in several cases we obtained a univalence principle by establishing the univalence of an appropriately defined (bi)category. One future aim is to generalize this approach to bicategories and Verity double bicategories, which would require developing *tricategory theory* [19].

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