

Minimality in Finite-Dimensional ZW-Calculi

Marc de Visme  

Université Paris-Saclay, Inria, CNRS, ENS Paris-Saclay,
Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Renaud Vilmart  

Université Paris-Saclay, Inria, CNRS, ENS Paris-Saclay,
Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Abstract

The ZW-calculus is a graphical language capable of representing 2-dimensional quantum systems (qubit) through its diagrams, and manipulating them through its equational theory. We extend the formalism to accommodate finite dimensional Hilbert spaces beyond qubit systems.

First we define a qudit version of the language, where all systems have the same arbitrary finite dimension d , and show that the provided equational theory is both complete – i.e. semantical equivalence is entirely captured by the equations – and minimal – i.e. none of the equations are consequences of the others. We then extend the graphical language further to allow for mixed-dimensional systems. We again show the completeness and minimality of the provided equational theory.

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1 Introduction

Graphical languages for quantum computations are a product of the categorical quantum mechanics program [1, 15] devoted to studying the foundations of quantum mechanics through the prism of category theory. These graphical languages come in different flavours, depending on which generators are used to build the diagrams (graphical representations of the quantum operators), and critically, displaying different kinds of interactions between said generators. The ZX-calculus describes the interaction between two complementary bases [13], the ZW-calculus, the interaction between the two “spiders” derived from the “GHZ” and “W” states, the only two fully entangled tripartite states up to SLOCC-equivalence [14], and the ZH-calculus the interaction between the same GHZ-state inferred spider and a spider obtained by generalising the Hadamard gate [3].

The equations that describe these interactions form “equational theories” that define syntactic equivalence classes of diagrams, that are also semantically equivalent. When the syntactic equivalence matches perfectly the semantic one (i.e. when two diagrams represent the same quantum operator iff they can be turned into one another), we say that the equational



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theory is *complete*. Complete equational theories have been found for the aforementioned graphical languages, betimes for restrictions of them [2, 4, 10, 11, 24, 26, 29, 31, 47].

As is customary for the computer science part of the quantum computation community, the focus was largely set onto *qubit* systems, i.e. systems where the base quantum system is 2-dimensional, yet this is enough to get applications in optimisation [5, 34], quantum error correction [18, 28, 46], verification [22, 27], simulation [35–37]... However, physics allows for *qudit* systems (where the base quantum system is d -dimensional with $d > 2$) and even infinite dimensional systems. Several attempts have hence been made to go beyond the qubit case [7, 25, 48], but it was only recently that a complete equational theory was found for d -dimensional (i.e. qudit) systems [42] and later for finite dimensional systems (so-called “qufinite”, i.e. for the category **FdHilb**) [50]. The results were obtained by generalising both the ZX and the ZW calculi and mixing them together. The W-node in particular allows for a neat intuitive (and unique) normal form for the diagrams. Satisfying the necessary conditions for every diagram to be normalisable then yields a complete equational theory. However, we believe that the ones obtained in [42, 50] are far from being *minimal*, due in particular to the presence of the generators from both the ZX and the ZW calculi.

From a foundational perspective, it can be enlightening to know if an equation is a defining property of quantum systems (and hence necessary), or on the contrary if it is derivable from more fundamental properties (see e.g. [16, 21]). The redundancy in the equational theory may also cause issues when trying to explore the space of equivalent diagrams, or to transport the completeness result to other diagrammatic languages for qudit systems (every equation has to be proven in the new language, hence the fewer the better); or when trying to generalise further, e.g. to the **FdHilb** setting.

We argue here that the ZW-calculus is enough to get a natural normal form (akin to that of [42]) even in the qudit version, and provide an elegant equational theory that we show to be *complete*, resorting to the normal form instead of transporting the completeness result from [42], for the reason described above. We also show that the equational theory is minimal, meaning that none of the equations can be derived from the others, hence avoiding the aforementioned redundancy in the presentation.

We then adapt diagrams and the equational theory of the graphical language to accommodate all finite dimensional Hilbert spaces (**FdHilb**), in a way that requires no additional generator and only one new equation. Here again we prove the completeness and the minimality of the equational theory, by leveraging that of the qudit setting.

The paper is split into two parts, Section 2 and Section 3, devoted respectively to the **Qudit** _{d} version, and to the **FdHilb** version. In the **Qudit** _{d} version, diagrams and their interpretation are introduced in Section 2.1 and the equational theory is introduced and discussed in Section 2.2. We then show its minimality in Section 2.3 and its completeness in Section 2.4. In the **FdHilb** version, diagrams and their interpretation are introduced in Section 3.1, and the equational theory is introduced and shown to be complete in Section 3.2. All missing proofs are provided in the full version.

The Dirac Notation

All the upcoming diagrams can be given an interpretation as a linear map, in the appropriate category. In quantum information, it is usual to express such linear maps using the so-called Dirac notation. The current section hence serves as a gentle introduction to this notation.

Let $d \geq 2$. In the d -dimensional Hilbert space \mathbb{C}^d , the canonical basis

$$\left\{ (1 \ 0 \ \cdots \ 0)^\top, (0 \ 1 \ \cdots \ 0)^\top, \dots, (0 \ 0 \ \cdots \ 1)^\top \right\}$$

is usually denoted $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ (with $(\cdot)^\top$ being the transpose). All 1-qudit systems have states that live in \mathbb{C}^d and that can hence be represented by linear combinations of the elements of this basis: $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + \dots + a_{d-1} |d-1\rangle$ (notice that the “ket” notation $|\cdot\rangle$ is used for states in general, not only basis elements).

To combine systems, we use the tensor product (Kronecker product): $(\cdot \otimes \cdot)$ which is a fairly standard operation on linear maps. In particular, the overall state obtained by composing two 1-qudit systems in respective states $|\psi\rangle$ and $|\varphi\rangle$ is simply $|\psi\rangle \otimes |\varphi\rangle$. Notice that $\{|i\rangle \otimes |j\rangle\}_{0 \leq i < d, 0 \leq j < d'}$ forms a basis of $\mathbb{C}^d \otimes \mathbb{C}^{d'} \simeq \mathbb{C}^{d \times d'}$. It is customary to write $|\psi, \varphi\rangle$ to abbreviate $|\psi\rangle \otimes |\varphi\rangle$.

The “bra” notation $\langle \cdot |$ is used to represent the dagger (the conjugate transpose) of a state, i.e. $\langle \psi | = |\psi\rangle^\dagger = |\psi\rangle^\top$. The choice of the “bra-ket” notation is such that composing a bra with ket forms the bracket, the usual inner product in \mathbb{C}^d : $\langle \psi | \circ |\varphi\rangle = \langle \psi | \varphi \rangle$. Linear combinations of “ket-bras” $|i\rangle \langle j|$ of the canonical basis can be used to represent any linear map of the correct dimensions, e.g. the 1-qudit identity: $id = \sum_{k=0}^{d-1} |k\rangle \langle k|$.

2 ZW-Calculus for Qudit Systems

In this section, we introduce a graphical language for quantum systems that all have the same fixed dimension d ($d \geq 2$): qudit systems.

2.1 Diagrams of ZW_d and their Interpretation

First, we need to introduce the mathematical objects at the heart of the graphical language – the diagrams – and what they represent.

As is traditional for graphical languages for finite-dimensional quantum systems, we work with a \dagger -compact prop [38,45,51]. Categorically speaking, this is a symmetric, compact closed monoidal category generated by a single object, endowed with a contravariant endofunctor that behaves well with the symmetry and the compact structure. The following explains some of these concepts in more detail.

Let us denote ZW_d the \dagger -compact prop generated by:

- the Z-spiders $\begin{matrix} \dots \\ \vdots \\ \textcircled{r} \\ \vdots \\ \dots \end{matrix} : n \rightarrow m$, for $r \in \mathbb{C}$ and $n, m \geq 0$
- the W-nodes $\begin{matrix} \dots \\ \vdots \\ \textcircled{} \\ \vdots \\ \dots \end{matrix} : 1 \rightarrow n$, for $n \geq 0$
- the state $|1\rangle \begin{matrix} \bullet \\ | \end{matrix} : 0 \rightarrow 1$
- the global scalars $r : 0 \rightarrow 0$, for $r \in \mathbb{C}$
- the swap $\begin{matrix} \diagup & \diagdown \\ & \end{matrix} : 2 \rightarrow 2$ representing the symmetry of the prop
- the cup $\begin{matrix} \cup \\ \end{matrix} : 2 \rightarrow 0$ and cap $\begin{matrix} \cap \\ \end{matrix} : 0 \rightarrow 2$ representing the compact structure
- and the identity $| : 1 \rightarrow 1$.

All these generators can be composed sequentially and in parallel, as follows:

$$\begin{matrix} \dots \\ | \\ \boxed{D_1} \\ \dots \\ \boxed{D_2} \\ \dots \end{matrix} := \begin{matrix} \dots \\ | \\ \boxed{D_2} \\ \dots \\ | \\ \dots \end{matrix} \circ \begin{matrix} \dots \\ | \\ \boxed{D_1} \\ \dots \\ | \\ \dots \end{matrix} \quad \text{and} \quad \begin{matrix} \dots \\ | \\ \boxed{D_1} \\ \dots \\ | \\ \dots \end{matrix} \begin{matrix} \dots \\ | \\ \boxed{D_2} \\ \dots \\ | \\ \dots \end{matrix} := \begin{matrix} \dots \\ | \\ \boxed{D_1} \\ \dots \\ | \\ \dots \end{matrix} \otimes \begin{matrix} \dots \\ | \\ \boxed{D_2} \\ \dots \\ | \\ \dots \end{matrix}$$

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The symmetry and the compact structure satisfy the following identities:

$$\begin{array}{ccccccc} \text{X} & = & \text{||} & & \text{D} & = & \text{D} \\ \text{Y} & = & \text{O} & & \text{Z} & = & \text{U} \end{array}$$

This compact structure in particular allows us to define the “upside-down” version of the generators, for instance:

$$\text{D}^\dagger := \text{D} \quad \text{and} \quad \text{Z}^\dagger := \text{Z}$$

The \dagger functor is defined inductively as:

$$\begin{array}{lll} (D_2 \circ D_1)^\dagger & = & D_1^\dagger \circ D_2^\dagger \\ (D_1 \otimes D_2)^\dagger & = & D_1^\dagger \otimes D_2^\dagger \\ (r)^\dagger & = & \bar{r} \end{array} \quad \left(\begin{array}{c} n \\ \text{O} \\ m \end{array} \right)^\dagger = \left(\begin{array}{c} m \\ \text{O} \\ n \end{array} \right) \quad \text{with the other generators being mapped to their upside-down version.}$$

Notice that thank to the identities satisfied by the \dagger -compact prop, the \dagger -functor is involutive.

As will be made clearer in what follows, in \mathbf{ZW}_d , $d \geq 2$ represents the dimension of the “base” quantum system, called qudit. As this d will be fixed in the following, we may forget to specify it. For convenience, we define an empty white node as a parameter-1 Z-spider: $\text{O} := \text{O}$ and give the $0 \rightarrow 1$ W-node a special symbol, akin to that of $|1\rangle$ (as its interpretation, as we will see later, is merely $|0\rangle$): $\text{Z} := \text{Z}$. We generalise the ket symbol inductively as follows (for $2 \leq k < d$): $\text{Z}^k := \text{Z}^k$. These last symbols can be given an upside-down definition using the compact structure as was done for $|1\rangle$ and the W-node.

2.1.1 The Interpretation

The point of the diagrams of the \mathbf{ZW}_d is to represent quantum operators on multipartite d -dimensional systems. The way those are usually specified is thanks to the category \mathbf{Qudit}_d . This forms again a symmetric \dagger -compact prop, where the base object is $1 := \mathbb{C}^d$, and morphisms $n \rightarrow m$ are linear maps $\mathbb{C}^{d^n} \rightarrow \mathbb{C}^{d^m}$. The symmetry and the compact structure correspond to their counterparts in \mathbf{ZW}_d , they will be stated out in the following, as part of the interpretation of \mathbf{ZW}_d diagrams. The \dagger functor is the usual \dagger of linear maps over \mathbb{C} .

We may hence interpret diagrams of the \mathbf{ZW}_d -calculus thanks to the functor $[\cdot] : \mathbf{ZW}_d \rightarrow \mathbf{Qudit}_d$ inductively defined as follows:

$$\begin{array}{ll} [D_2 \circ D_1] = [D_2] \circ [D_1] & \left[\begin{array}{c} n \\ \text{O} \\ m \end{array} \right] = \sum_{k=0}^{d-1} r^k \sqrt{k!}^{n+m-2} |k^m\rangle\langle k^n| \\ [D_1 \otimes D_2] = [D_1] \otimes [D_2] & \left[\begin{array}{c} \text{Z} \\ n \end{array} \right] = \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} \sqrt{\binom{k}{i_1, \dots, i_n}} |i_1, \dots, i_n\rangle\langle k| \\ \left[\text{||} \right] = \sum_k |k\rangle\langle k| & \left[\text{Z}^\dagger \right] = |1\rangle \\ \left[\text{X} \right] = \sum_{k, \ell} |\ell, k\rangle\langle k, \ell| & [r] = r \\ \left[\text{O} \right] = \left[\text{O}^\dagger \right]^\dagger = \sum_k |k, k\rangle & \end{array}$$

where $\binom{k}{i_1, \dots, i_n} = \frac{k!}{i_1! \dots i_n!}$ is a multinomial coefficient. Notice that the interpretation of the $0 \rightarrow 1$ W-node is simply: $\left[\text{Z} \right] = \sum_{k \in \{0, \dots, d-1\}} \sqrt{\binom{k}{0}} |k\rangle = |0\rangle$, and that of the black node symbol k for $k < d$ is $|k\rangle$ up to renormalisation: $\left[\text{Z}^k \right] = \sqrt{\binom{k}{1, \dots, 1}} |k\rangle = \sqrt{k!} |k\rangle$. The

presence of $\sqrt{\cdots}$ on the coefficients is not particularly relevant, and is simply an artefact of us maintaining some symmetry between generators and their dagger. Indeed, we want $\left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \right] = k!$, and for that the coefficient $k!$ needs to be split between both nodes, resulting in either an asymmetric presentation or a square root (see Section A for an equivalent semantics, without any $\sqrt{\cdots}$ but asymmetric instead).

Notice also that the interpretation of the Z-spider differs from more usual generalisations of its qubit counterpart, because of the $\sqrt{k!}^{n+m-2}$ which depends on the degree of the spider. While it makes the interpretation of the diagrams slightly more complicated, it allows us – as will be stated later – to quite conveniently generalise equations from the qubit ZW-calculus, and hence have a simpler equational theory. It will be shown in the following (Corollary 10), that the above set of generators makes for a universal calculus, i.e. any linear map of \mathbf{Qudit}_d can be represented by a \mathbf{ZW}_d -diagram.

To gain intuition about the upcoming equations between diagrams, it can be useful to *semantically* decompose a diagram into sums of simpler ones¹. To do so, it can be convenient to understand $|k\rangle$ as a bunch of k indistinguishable particles:

$$\left[\begin{array}{c} \bullet \\ \bullet \\ \vdots \\ \bullet \end{array} \right] = \sum_{i_1 + \dots + i_n = k} \binom{k}{i_1, \dots, i_n} \left[\begin{array}{c} \bullet \\ \bullet \\ \vdots \\ \bullet \end{array} \right] \quad (1)$$

$$\left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = \begin{cases} \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] & \text{if } k + \ell < d \\ \vec{0} & \text{if } k + \ell \geq d \end{cases} \quad (2)$$

$$\left[\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right] = r^k \left[\begin{array}{c} \bullet \\ \bullet \\ \vdots \\ \bullet \end{array} \right] \quad (3)$$

$$\left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = k! \langle k | \ell \rangle \quad (4)$$

$$\left[\begin{array}{c} | \\ | \end{array} \right] = \sum_{k=0}^{d-1} \frac{1}{k!} \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] \quad (5)$$

Equation (1) explains how the W-node spreads the k “particles” that enter it following a multinomial distribution. Equation (2) shows that the $2 \rightarrow 1$ W-node takes two bunches of particles k and ℓ and regroups them into one, and yields the null state if $k + \ell$ exceeds the “capacity” (i.e. the dimension) of a single wire. This will be proven graphically (Lemma 30) from the upcoming equational theory (Figure 1). When $k + \ell < d$, the fact that there is no additional scalar is due to the rescaling of the k -dots to represent $\sqrt{k!} |k\rangle$. This rescaling also makes the “copy” more natural: The Z-spider $1 \rightarrow n$ copies any bunch of k particles entering it, yielding global scalar r^k in the process, as is shown by Equation (3). This will again be proven graphically (Lemma 27) from the equational theory. The rescaling, however, forces Equation (4). Finally, it can be useful to decompose the identity as a linear combination of products of kets and bras as is done in Equation (5).

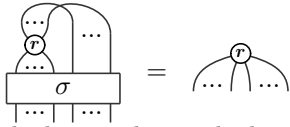
2.2 Equational Theory

With the above interpretation of the \mathbf{ZW}_d , different diagrams may yield the same linear map. All axioms of symmetric \dagger -compact props in particular preserve the interpretation. More generally, we may want to relate together *all* diagrams that have the same semantics. This is done through an equational theory, i.e. a set of equations that can be applied locally in a diagram without changing the semantics of the whole.

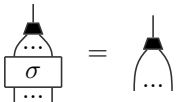
¹ Notice that here, such decompositions are merely semantical. The upcoming completeness is only interested in equivalence between single diagrams.

2.2.1 Equations of the ZW_d -Calculus

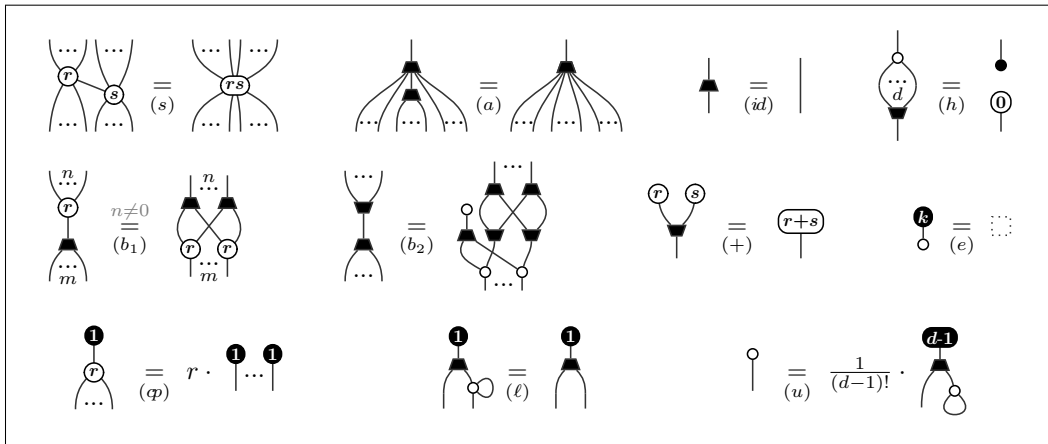
On top of the axioms of symmetric \dagger -compact props, we assume some conventional equations about the topology of the generators, which should align with the symmetries of the symbols used to depict them. The Z-spider does not distinguish between any of its connections: it is “flexsymmetric” [9], meaning that we can interchange any of its legs without changing the

semantics. Graphically, for any permutation of wires σ : . On the

other hand, the binary W -node is only co-commutative, which, together with the upcoming Equation (a), means that all the outputs of the n -ary W -node can be exchanged, i.e. for any

permutation of wires σ : . With all this in place, we can give the core of the

equational theory, in Figure 1. When diagram D_1 can be turned into diagram D_2 using the rules of ZW_d , we write $ZW_d \vdash D_1 = D_2$.



■ **Figure 1** Equational theory ZW_d for the qudit ZW -calculus.

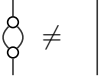
► **Remark 1.** In this framework, we can tensor global scalars together $r \otimes s$, which graphically could be confused with their product rs . This is actually unambiguous in the equational theory, as, using Equation (φ):

$$ZW_d \vdash r \otimes s \stackrel{(\varphi)}{=} r \cdot \begin{array}{c} \mathbf{1} \\ \circlearrowleft \\ s \end{array} \stackrel{(\varphi)}{=} \begin{array}{c} \mathbf{1} \\ \circlearrowleft \\ r \end{array} \stackrel{(s)}{=} \begin{array}{c} \mathbf{1} \\ \circlearrowleft \\ rs \end{array} \stackrel{(\varphi)}{=} rs$$

Moreover, using Equation (e), one can easily show that global scalar 1 is the empty diagram (Lemma 21). Scalar multiplication is assumed to be automatically applied, and scalar 1 is assumed to be automatically removed in the following after Lemma 21 is proven.

All rules up to (e) are fairly standard generalisations of rules of the qubit ZW -calculus (with (b_2) being inspired from [42] to avoid using a *fermionic swap*²). The non-conventional

² The fermionic swap, introduced in [24], is a generator that in many situations behave like the actual swap. The qubit version of ZW uses the fermionic swap, but this generator loses many properties (for instance its involution, or the fact that it maps $|k, \ell\rangle$ to $|\ell, k\rangle$ up to some coefficient) when going in larger dimensions. By not using it here, we avoid having to axiomatise it.

\sqrt{k}^{n+m-2} coefficients in the interpretation of the Z-spider seem to be necessary for Equations (s), (b₁) and (+) to all work. Notice that this makes the Z-spider *non-special*, meaning that: . Equation (ℓ) however gives a context in which that inequality becomes an equality. Finally, Equation (u) shows how a 0 → 1 Z-spider can be obtained by distributing d − 1 “particles” over two paths, and erasing (or post-selecting) adequately one of the two paths.

► **Remark 2.** Thanks to the compact structure and its interaction with the generators of the language, all upside-down version of the equations of Figure 1 are derivable, by simply deforming the diagrams to get the actual axiom. For instance, the upside-down version of (+) can be derived as follows:

$$ZW_d \vdash \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \stackrel{(+)}{=} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

► **Proposition 3.** All equations in ZW_d are sound, i.e.:

$$ZW_d \vdash D_1 = D_2 \implies \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$$

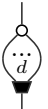
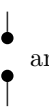
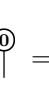
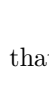
Proof. This is a straightforward verification for most of the equations. They can all be proven using the aforementioned identities (Equations 1 to 5) in the semantics of the diagrams, especially the decomposition of the identity (Equation 5). Equations (b₂) and (+) require respectively the Vandermonde identity and the binomial formula:

$$\sum_{k_1 + \dots + k_p = m} \binom{n_1}{k_1} \dots \binom{n_p}{k_p} = \binom{n_1 + \dots + n_p}{m} \quad \text{and} \quad (r + s)^n = \sum_{k=0}^n \binom{n}{k} r^k s^{n-k}. \quad \blacktriangleleft$$

2.3 Minimality

Minimality of an equational theory states that every single equation is necessary: none can be derived from the others. Said otherwise, as soon as we remove one of the equations, some equalities (that were previously provable) become unprovable. Minimality is fundamental, as it allows us to pinpoint properties that are necessary to our model, and a contrario those that are consequences of the necessary ones. Notice however that there is usually not a single minimal equational theory, for two reasons: 1) it often happens that one equation can be replaced by an equivalent one, and 2) one could start with a totally different set of equations. Most of the equations we chose are generalisations or adaptations of equations that were already used by other graphical languages, and are usually motivated by either categorical or physical considerations. We only “filled in the blanks” with equations (ℓ) and (u).

In trying to prove minimality, it often happens that two equations fail to be proven necessary individually, but that the pair (i.e. at least one of the two) can be proven necessary. Such cases underline some sort of proximity between the two equations, and the obstacle it poses to minimality can sometimes be circumvented (somewhat artificially) by merging them into a single, potentially slightly less intuitive, equation. This happened once here: we merged

equations  =  and  = , that we initially had as axioms, into Equation (h). This

finally provides us with a minimal equational theory for qudit ZW-calculus.

To prove that an equation is necessary, we define a non-standard interpretation which is preserved by all the equations (including the axioms of †-compact props), except the

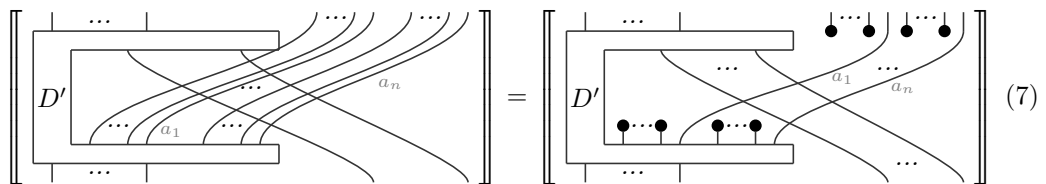
equation of interest. When such an interpretation is exhibited, we can safely conclude that the equation is necessary, since if it were a consequence of the others, it would also preserve this interpretation. Interpretations like these sometimes simply take the form of a quantity that turns out to be invariant for all equations except the one that is considered. In the realm of quantum graphical theories, such arguments were used for single equations in [23, 30, 32, 39, 44], partial minimality results were obtained for Clifford ZX-calculus [6], unrestricted ZX-calculus [47], and quite recently, full minimality (with completeness) was obtained for quantum circuits [12].

In the following, we show that the equational theory ZW_d from Figure 1 is minimal, i.e. that none of the equations can be derived from the others. It is to be noted that most of the equations in Figure 1 are schemas, that is they are parametrised, and the equation is assumed for all possible values of the parameters. Our minimality result is “weak” in the sense that for each equation schema, we show that *at least* one of the occurrences cannot be derived from the other equations, but we do not pinpoint for which parameters the equation is necessary or not. Nevertheless:

► **Theorem 4.** *All equations in ZW_d are necessary, hence ZW_d is minimal.*

Several arguments in the proof are graphical, meaning that they rely on the understanding that a diagram can also be seen as a special kind of graph with vertices (inputs, outputs, Z-spiders, W-nodes, states, global scalars) and edges (wires, cup, cap, with swaps representing crossings). As such, when we talk about “a W-node in a diagram D ”, we actually refer to a vertex (of the Z type) of said diagram. One argument in the proof require the notion of “effective Z-path”, that is a path that goes only through Z-spiders and trivial W-nodes – which W-nodes that could be replaced by identity wires without it making any difference – and that can be used in non-trivial computations.

► **Definition 5** (Sole effective output of a W-node). *Let D be a diagram with a collections of n distinguished W-node. Let’s call “ a_1 ”, ..., “ a_n ” one output of each, as shown in Equation (6) below. We say that wires “ a_1 ”, ..., “ a_n ” are jointly the sole effective outputs of the W-nodes if Equation (8) is satisfied:*



In particular, if we only consider one W-node that has a sole effective output – we call such a node a *trivial W-node* – it follows that:



► **Definition 6** (Effective Z-path). *An effective Z-path of a diagram D is a path going from a boundary to another (inputs and/or outputs), such that:*

- *For each W-node it goes through, it cannot use two outputs of the W-node (so it must use its input). Considering all those W-nodes at once, those outputs must be jointly the sole effective outputs of those W-nodes.*
- *There exists a state $|\phi\rangle$ that is not of the form $\lambda|0\rangle$ or $\lambda|1\rangle$ for some $\lambda \in \mathbb{C}$, that inhabits the path. That is, if we feed to $\llbracket D \rrbracket$ the state $|\phi\rangle$ and/or the effect $\langle\phi|$ to the two inputs and/or outputs of the path, then the result is not $\vec{0}$.*

We can now prove Theorem 4:

Proof. We consider each of these equations individually, and give more details in the full version:

- (s) When applying the transformation that turns all Z-spider parameters and global scalars to their real part ($r \mapsto \text{Re}(r)$), Equation (s) is the only one that is not preserved.
- (a) It is the only equation permitting to create non-trivial W-nodes with arity $> d$ from a diagram that only has non-trivial W-nodes with arity $\leq d$.
- (id) It is the only equation that can create nodes connected to boundaries from a node-free diagram.
- (h) To each wire in a diagram D , we associate a number $0 \leq k < d$ (or more graphically we annotate each wire by some number k)³. The procedure to do so is as follows:

1. annotate all wires with $d - 1$

2. rewrite the annotations using the following rules, until a fixed point is reached:

$$\begin{aligned}
 &= a \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \xrightarrow{a \neq 0} 0 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \xrightarrow{a \geq 1} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} a \\ \bigwedge \\ \dots \\ \bigvee \\ b_n \end{array} \rightarrow \begin{array}{c} \min(a, \sum b_i) \\ \bigwedge \\ \dots \\ \bigvee \\ \min(a, b_n) \end{array} \\
 &= \begin{array}{c} \dots \\ \bigwedge \\ a_0 \dots a_k \\ \bigvee \\ \dots \\ a_{k+1} \dots a_q \end{array} \rightarrow \begin{array}{c} \dots \\ \bigwedge \\ a \\ \bigvee \\ \dots \\ a \end{array} \quad \text{with } a := \min(a_i)
 \end{aligned}$$

This simple procedure obviously terminates, as a step is only applied if at least one of the annotations is decreased. By considering inputs and outputs of D (which are the only wires that can be guaranteed to remain during rewrites with ZW_d), we can check that Equation (h) is the only one that can modify the outcome of the procedure.

- (b₁) Consider diagrams as graphs, and use the above definition of an “effective Z-path”. All equations except (b₁) preserve the existence of effective Z-paths, although proving it for (b₂) is quite involved and relies on the following lemma:

► **Lemma 7.**

Let D be a diagram of the form shown in Equation (9), such that we have Equation (10).

$$\boxed{D} = \text{Diagram } D' \tag{9}$$

³ The annotations produced here are upper bounds on the value of the kets that can go through the wires. As such, they are very closely related to the *capacities* from Section 3.

Then Equation (11) is satisfied:

(b_2) Consider diagrams as graphs, and define a “W-path” in the diagram as a path 1) that goes from a boundary to another boundary, 2) which cannot use two outputs of a W-node (if it goes through a W-node, it has to use the input edge) and 3) that does not go through a Z-spider. All equations, except (b_2), preserve the existence of a W-path. (b_2) is the only equation that can bring the number of W-paths from non-zero to zero (which is done by adding a Z-spider on the path).

(+) Take the interpretation that maps all the Z-spider parameters (and the global scalars) to their absolute value ($r \mapsto |r|$). This interpretation preserves all equations except (+).

(e) It is the only equation that can create generators out of empty diagrams.

(φ) It is the only equation that can create a non-real scalar from a diagram with only real scalars.

(ℓ) Let $\varpi := e^{i\frac{\pi}{d-1}}$ be the first $(2d - 2)$ -th root of unity. Consider the \dagger -compact monoidal functor (i.e. functor that preserves compositions, symmetry and compact structure) that maps the generators as follows:

$$= \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mapsto \varpi \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad = \begin{array}{c} \binom{n}{\dots} \\ \left(\begin{array}{c} r \\ \dots \\ m \end{array} \right) \end{array} \mapsto \begin{array}{c} \binom{n}{\dots} \\ r\varpi^{n+m-2} \\ \left(\begin{array}{c} \dots \\ m \end{array} \right) \end{array} \quad = \begin{array}{c} | \\ \cap \\ \dots \\ n \end{array} \mapsto \begin{array}{c} | \\ \cap \\ \dots \\ n \end{array}$$

When $d > 2$, all equations but (ℓ) are preserved by this functor. We can make the argument work when $d = 2$ by choosing any ϖ such that $\varpi^2 \neq 1$ and by working up to a scalar factor.

(u) Take the interpretation that maps $\begin{array}{c} \bullet \\ | \\ \bullet \end{array}$ (and subsequently all $\begin{array}{c} \bullet \\ | \\ \bullet \end{array}$) to $\begin{array}{c} \bullet \\ | \\ \bullet \end{array}$. All rules hold (up to a the scalar factor) except (u). \blacktriangleleft

2.4 Completeness

Completeness of an equational theory with respect to a semantics is the fundamental property that ensures that semantical equivalence of diagrams is entirely captured by the equational theory. Minimality is worthless without some form of completeness, as it is extremely simple to design minimal, but not complete, equational theories. For instance, the empty equational theory (that contains no axiom) is minimal but clearly not complete for the qudit ZW-diagrams. We hence show in this section that we indeed have completeness.

2.4.1 Normal Form and Universality

The usual way to prove completeness is to show that any diagram can be put in a normal form, and that this normal form is unique and similar for all equivalent diagrams. As is customary in a category that is compact-closed, we can focus on states, as there is an isomorphism between operators and states [1]:

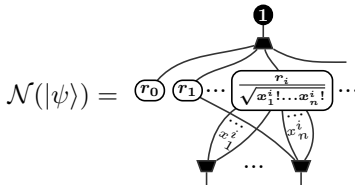
$$\begin{array}{|c|} \hline \dots \\ \hline D \\ \hline \dots \\ \hline \end{array} = \begin{array}{|c|} \hline D \\ \hline \dots \\ \hline \end{array} \begin{array}{|c|} \hline \dots \\ \hline \dots \\ \hline \end{array} =: \begin{array}{|c|} \hline \lceil D \rceil \\ \hline \dots \\ \hline \end{array}$$

Proving completeness requires a fair amount of diagrammatic derivations, especially when starting from a minimal equational theory, to get enough material to define a normalisation strategy.

► **Definition 8.** We define $\mathcal{N} : \mathbf{Qudit}_d \rightarrow \mathbf{ZW}_d$ as the functor that maps any n -qudit state

$$|\psi\rangle = r_0 |0\dots 0\rangle + r_1 |0\dots 01\rangle + \dots + r_i |x_1^i \dots x_n^i\rangle + \dots$$

to the diagram below. We say of any diagram in the image of \mathcal{N} that it is in normal form.



This construction is a direct generalisation of the normal form of the qubit ZW-diagrams [24], which is also considered in [25] in the context of q -arithmetic. It creates a diagram whose interpretation is the starting state:

► **Proposition 9.** $\forall |\psi\rangle \in \mathbf{Qudit}_d[0, n], \llbracket \mathcal{N}(|\psi\rangle) \rrbracket = |\psi\rangle.$

Where for any category \mathcal{C} , we write $\mathcal{C}[n, m]$ for the set all the morphisms from n to m . As a simple consequence of this proposition, any qudit operator can be represented by a diagram of \mathbf{ZW}_d :

► **Corollary 10 (Universality).** $\forall f \in \mathbf{Qudit}_d[n, m], \exists D_f \in \mathbf{ZW}_d[n, m], \llbracket D_f \rrbracket = f.$

Since we defined the normal form as the image of a map from the semantics, any diagram can only be associated to a unique normal form.

2.4.2 Completeness

Our goal now is to show that any diagram can be put in normal form. To do so, we show that all generators can be put in normal form, and that all compositions of diagrams in normal form can be put in normal form.

We start by showing the latter for the tensor product:

► **Proposition 11.** The spatial composition of diagrams in normal form can be put in normal form, i.e. $\mathbf{ZW}_d \vdash \mathcal{N}(v_1) \otimes \mathcal{N}(v_2) = \mathcal{N}(v_1 \otimes v_2).$

When turning arbitrary operators into states, the sequential composition turns into the application of cups \cup onto pairs of outputs of the state, as:

$$\begin{array}{|c|} \hline \dots \\ \hline D_1 \\ \hline \dots \\ \hline D_2 \\ \hline \dots \\ \hline \end{array} = \begin{array}{|c|} \hline \lceil D_2 \rceil \\ \hline \dots \\ \hline \lceil D_1 \rceil \\ \hline \dots \\ \hline \end{array} \begin{array}{|c|} \hline \dots \\ \hline \dots \\ \hline \end{array} \stackrel{\text{Prop. 11}}{=} \begin{array}{|c|} \hline \lceil D_2 \rceil \otimes \lceil D_1 \rceil \\ \hline \dots \\ \hline \end{array}$$

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► **Proposition 12.** *The diagram obtained by applying a cup \cup to two outputs of a diagram in normal form can be put in normal form.*

Then we move on to showing that all the generators can be put in normal form. To do so, the following lemma will prove useful:

► **Lemma 13.** *The diagram obtained by applying a cap \cap to two outputs of a normal form can be put in normal form.*

► **Proposition 14.** *All generators of the \mathbf{ZW}_d -calculus can be put in normal form.*

Putting all the latter results together, we can show the completeness of the language:

► **Theorem 15 (Completeness for Qudit Systems).** *The language is complete: for any two diagrams D_1 and D_2 of the \mathbf{ZW}_d -calculus:*

$$\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \mathbf{ZW}_d \vdash D_1 = D_2$$

Proof. By Proposition 14, any generator of the language can be put in normal form. Thanks to Propositions 11 and 12, compositions of diagrams in normal form can be put in normal form. As a consequence, any diagram can be put in normal form. By uniqueness of this normal form, if two diagrams share the same semantics, they can be rewritten into the same diagram. This proves completeness. ◀

3 Finite Dimensional Hilbert Spaces

In the previous setting, all systems are required to be d -dimensional for some fixed d . Here we relax that constraint, which allows us to go “mixed-dimensional” and to represent morphisms of \mathbf{FdHilb}^4 .

\mathbf{FdHilb} is the strict symmetric monoidal \dagger -compact category of finite dimensional Hilbert spaces [1]. Its objects are tensor products of finite dimensional Hilbert spaces $\mathbb{C}^d (d \in \mathbb{N} \setminus \{0\})$, and its morphisms are linear maps between them. The symmetry and the compact structure are naturally extended from that of \mathbf{Qudit}_d .

In this new setting, we will be able to represent *all* morphisms of \mathbf{FdHilb} , at the cost of annotating the wires of the diagrams to keep track of their dimensions. Instead of the dimension itself, we rather annotate the wire with its dimension -1 , i.e. with the largest k such that $|k\rangle$ is allowed on the wire. We call such k the *capacity* of the wire. This makes the bookkeeping a little bit less tedious.

3.1 Diagrams and Interpretation

We also require the following constraints for the capacities around each generator:

- All capacities around a Z-spider are the same
 - The input capacity of the W-node must be larger than (or equal to) each of its outputs
- The first constraint follows from the fact that Z-spiders in ZW can be seen as a generalisation of graph edges – more precisely they can be seen as hyperedges. Hence the whole hyperedge should have a single capacity. The second constraint simply comes from the fact that a larger capacity on the outputs of a W-node will never be used, so we might as well prevent

⁴ Technically, the skeleton of \mathbf{FdHilb} , i.e. where all d -dimensional Hilbert spaces are identified with the canonical representative \mathbb{C}^d . We take the liberty in this paper to name \mathbf{FdHilb} this skeleton.

it. When considering $1 \rightarrow 1$ W-nodes, which represent projections, this restriction allows us to see at a glance which side has the largest dimension.

The first restriction further allows us to put the capacity annotation on the Z-spider rather than on all its legs, making annotating diagrams less cumbersome.

We now work with a \dagger -compact symmetric monoidal category, which is not a prop anymore. Our base objects are \mathbb{C}^d for $d \in \mathbb{N} \setminus \{0\}$. Every pair of objects can be composed with \otimes to form a third object, with \otimes being associative, and with the tensor unit I being $I := \mathbb{C}^1$. We work with a strict monoidal category, so we consider $I \otimes \mathbb{C}^d = \mathbb{C}^d = \mathbb{C}^d \otimes I$. To simplify notations, we represent objects $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_n}$ by a list of capacities $\langle d_1 - 1, d_2 - 1, \dots, d_n - 1 \rangle$ (the tensor product simply becomes the concatenation of lists). The tensor unit is represented by $\langle \rangle$. Since $\langle 0 \rangle = \langle \rangle$, we forbid 0 capacities on the wires.

In this new setting, the generators are generalised as follows:

- Z-spiders $\begin{array}{c} \langle n \rangle \\ \circlearrowleft \\ \langle a \rangle \\ \circlearrowright \\ \langle m \rangle \end{array} : \langle \langle a, \dots, a \rangle \rangle \rightarrow \langle \langle a, \dots, a \rangle \rangle$ with $r \in \mathbb{C}$ and $a \geq 1$
- W-nodes $\begin{array}{c} \langle a \rangle \\ \blacktriangledown \\ \langle b_1 \rangle \dots \langle b_n \rangle \end{array} : \langle \langle a \rangle \rangle \rightarrow \langle \langle b_1, \dots, b_n \rangle \rangle$ with $a \geq \max_{1 \leq i \leq n} (b_i)$ and $b_i \geq 1$
- state $|1\rangle \begin{array}{c} \bullet \\ \langle a \rangle \end{array} : \langle \rangle \rightarrow \langle \langle a \rangle \rangle$ with $a \geq 1$
- global scalars $r : \langle \rangle \rightarrow \langle \rangle$ with $r \in \mathbb{C}$

with the symmetry and the compact structure being generalised to

- $\begin{array}{c} \langle a \rangle \quad \langle b \rangle \\ \diagdown \quad \diagup \\ \langle b, a \rangle \end{array} : \langle \langle a, b \rangle \rangle \rightarrow \langle \langle b, a \rangle \rangle$ ■ $\begin{array}{c} \langle a \rangle \\ \cup \\ \langle a, a \rangle \end{array} : \langle \rangle \rightarrow \langle \langle a, a \rangle \rangle$ and $\begin{array}{c} \langle a \rangle \\ \cup \\ \langle \rangle \end{array} : \langle \langle a, a \rangle \rangle \rightarrow \langle \rangle$,
- (again with $a, b \geq 1$) and the identity to $\begin{array}{c} \langle a \rangle \\ | \\ \langle a \rangle \end{array} : \langle \langle a \rangle \rangle \rightarrow \langle \langle a \rangle \rangle$ ($a \geq 1$).

Diagrams can still be composed together both sequentially and in parallel. The sequential composition prevents us from composing diagrams with unmatched objects (e.g. two Z-spiders with different capacities in sequence). Diagrams with capacities are called **ZW_f**-diagrams, and are graphical representations of the morphisms of the \dagger -compact symmetric monoidal category **FdHilb** (the dagger functor can be given in **ZW_f** in a similar way as **ZW_d**).

The compact structure still allows us to define upside-down versions of the W-node and the kets. Again, we give the $\langle \rangle \rightarrow \langle \langle a \rangle \rangle$ W-node a special symbol (for $a \geq 1$): $\begin{array}{c} \bullet \\ \langle a \rangle \end{array} := \begin{array}{c} \blacktriangledown \\ \langle a \rangle \end{array}$, and we generalise the ket symbol inductively as follows (for $k \geq 1$ and $a > k$): $\begin{array}{c} \langle k+1 \rangle \\ \langle a \rangle \end{array} := \begin{array}{c} \langle 1 \rangle \quad \langle k \rangle \\ \cup \\ \langle a \rangle \end{array}$.

The interpretation of these diagrams is now a monoidal functor $\llbracket \cdot \rrbracket : \mathbf{ZW}_f \rightarrow \mathbf{FdHilb}$ inductively defined as:

$$\begin{aligned}
 \llbracket D_2 \circ D_1 \rrbracket &= \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket & \left[\begin{array}{c} a \\ \diagdown \\ \diagup \\ b \end{array} \right] &= \sum_{k=0}^a \sum_{\ell=0}^b |k, \ell\rangle \langle k, \ell| \\
 \llbracket D_1 \otimes D_2 \rrbracket &= \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket & \left[\begin{array}{c} \dots \\ a \\ \diagdown \\ r \\ \diagup \\ m \\ \dots \end{array} \right] &= \sum_{k=0}^a r^k \sqrt{k!}^{n+m-2} |k^m\rangle \langle k^n| \\
 \left[\begin{array}{c} a \\ | \\ a \end{array} \right] &= \sum_{k=0}^a |k\rangle \langle k| & \left[\begin{array}{c} \bullet \\ | \\ a \end{array} \right] &= |1\rangle \\
 \left[\begin{array}{c} \curvearrowright \\ a \end{array} \right] &= \left[\begin{array}{c} \cup \\ a \end{array} \right]^\dagger = \sum_{k=0}^a |k, k\rangle & & \\
 \llbracket r \rrbracket &= r & & \\
 \left[\begin{array}{c} a \\ \bullet \\ \dots \\ b_1 \dots b_n \end{array} \right] &= \sum_{\substack{0 \leq k_i \leq b_i \\ k_1 + \dots + k_n \leq a}} \sqrt{\binom{k_1 + \dots + k_n}{k_1, \dots, k_n}} |k_1, \dots, k_n\rangle \langle k_1 + \dots + k_n|
 \end{aligned}$$

By composition, one can check that, for $k > 1$ and $a \geq k$: $\left[\begin{array}{c} k \\ | \\ a \end{array} \right] = \sqrt{k!} |k\rangle$.

Notice that we use the same notation for the interpretation of \mathbf{ZW}_d -diagrams, and for the interpretation of the \mathbf{ZW}_f -diagrams. Which interpretation we are referring to should be clear from the context.

3.2 Complete Equational Theory

We once again equip the language with an equational theory \mathbf{ZW}_f , defined in Figure 2. This equational theory only slightly differs from the one for qudit systems in Figure 1. It is interesting to notice that 1) the associativity of the W-node is broken down into two equations (a) and (o), whose choice depends on the capacities involved, 2) the W-bialgebra equation (b₂) does not need a context anymore, but instead side conditions on the capacities, 3) we managed to remove Equation (e), 4) we now need an equation (i) that states that a $|1\rangle$, when “injected” into a larger dimensional Hilbert space, is still a $|1\rangle$.

We also notice the existence of an interesting equation, that we did not include in Figure 2 as it turns out to be derivable; and which states that a $0 \rightarrow 1$ Z-state can be “copied” by the

W-node, as follows: $\left[\begin{array}{c} a+b \\ \bullet \\ a \quad b \end{array} \right] = \left[\begin{array}{c} a \\ \bullet \\ a \end{array} \right] \left[\begin{array}{c} b \\ \bullet \\ b \end{array} \right]$.

The category \mathbf{Qudit}_d is a full subcategory of \mathbf{FdHilb} , and as such there is an obvious inclusion functor $\mathbf{Qudit}_d \xrightarrow{i_d} \mathbf{FdHilb}$. This inclusion transports to the ZW-calculi: we can turn any \mathbf{ZW}_d -diagram into a \mathbf{ZW}_f -diagram through ι_d in such a way that the following

$$\begin{array}{ccc}
 \mathbf{ZW}_d & \xrightarrow{\quad} & \mathbf{ZW}_f \\
 \downarrow \llbracket \cdot \rrbracket & \iota_d & \downarrow \llbracket \cdot \rrbracket \\
 \mathbf{Qudit}_d & \xrightarrow{i_d} & \mathbf{FdHilb}
 \end{array}$$

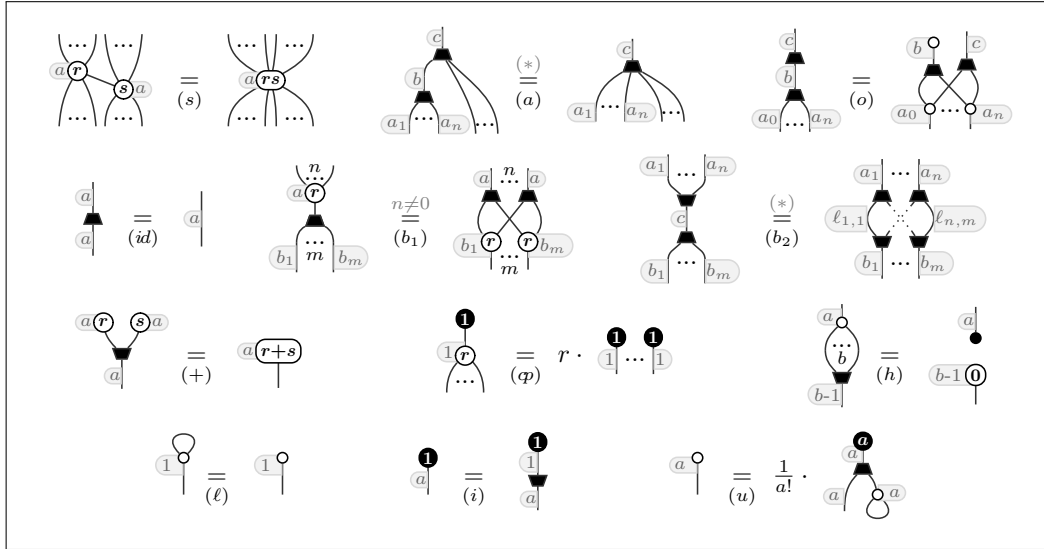
diagram commutes:

The functor ι_d simply takes a \mathbf{ZW}_d -diagram and annotates all its wires with $d - 1$.

We show that the present equational theory is complete. To do so, we need to adapt the notion of normal form from qudit systems (and we again use the map/state duality to focus on states rather than arbitrary morphisms):

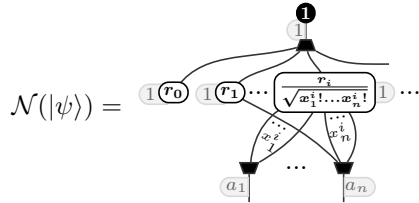
► **Definition 16.** We define $\mathcal{N} : \mathbf{FdHilb} \rightarrow \mathbf{ZW}_f$ as the functor that maps any n -ary state $|\psi\rangle \in \mathbf{FdHilb}(\emptyset, \langle a_1, \dots, a_n \rangle)$:

$$|\psi\rangle = r_0 |0\dots 0\rangle + r_1 |0\dots 01\rangle + \dots + r_i |x_1^i \dots x_n^i\rangle + \dots$$



■ **Figure 2** Equational theory ZW_f for the finite-dimensional ZW -calculus. In (a), we require that $b = c$ or $b \geq \sum_i a_i$; and in (b₂) that $c \geq \min(\sum a_i, \sum b_i)$ on the lhs, and that $\ell_{ij} = \min(a_i, b_j)$ on the rhs.

to the diagram below. We say of any diagram in the image of \mathcal{N} that it is in normal form.



We can once again show that \mathcal{N} builds a diagram that represents $|\psi\rangle$:

► **Lemma 17.** $\forall |\psi\rangle \in \mathbf{FdHilb}[\{\}, \langle a_1, \dots, a_n \rangle], \llbracket \mathcal{N}(|\psi\rangle) \rrbracket = |\psi\rangle$

We hence get universality of the language as a direct consequence:

► **Corollary 18** (Universality of ZW_f).

$$\forall f \in \mathbf{FdHilb}[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \exists D_f \in ZW_f[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \llbracket D_f \rrbracket = f$$

Most of the arguments given for the minimality of ZW_d can be adapted to arguments for the necessity of the equations of ZW_f , and the few remaining equations can be given a new argument. We hence have:

► **Theorem 19.** *The equational theory ZW_f is minimal.*

Proof. The argument given for the necessity of Equation (b₂) now works for the necessity of Equation (o), and that of Equation (e) now works for Equation (b₂). Namely, Equation (b₂) is the only that can create a non-empty diagram from an empty diagram. A less artificial argument for Equation (b₂) is that it is the only equation that can create a capacity $> k$ from a diagram whose capacities are all $\leq k$ for $k \geq 1$. Moreover:

Equation (i) is the only equation that can create a $|1\rangle$ with capacity $a \neq 1$, from a diagram whose $|1\rangle$ s are all on capacity 1.

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We can reformulate the argument of Equation (a) as: It is the only equation permitting to create non-trivial W-nodes with arity ≥ 3 from a diagram where all non-trivial W-nodes have arity ≤ 2 .

In the argument of Equation (ℓ), one can take for ϖ any complex number such that $\varpi^2 \neq 1$, and by working up to a scalar factor, as is done initially for $d = 2$.

In the argument of Equation (h), we can instantiate the protocol by annotating the wires with their capacities, then continue with the protocol as explained in the initial argument.

All the other arguments work right off the bat for their mixed-dimensional counterpart, hence the result of minimality. ◀

Using the normal form, we can then leverage the completeness from the qudit ZW-calculus to get the similar result in the current setting:

► **Theorem 20** (Completeness for Finite Dimensional Systems). *The language is complete: for any two diagrams D_1 and D_2 of the \mathbf{ZW}_f -calculus:*

$$\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \mathbf{ZW}_f \vdash D_1 = D_2$$

Proof. The right-to-left implication (soundness) is again a straightforward verification. The other is proven in its entirety in the full version, and uses the completeness of \mathbf{ZW}_d for qudit systems. The idea is to show that i) we can turn both D_1 and D_2 into a diagram with a high enough capacity d everywhere (except boundaries), and that ii) all the equations of \mathbf{ZW}_d can be proven in \mathbf{ZW}_f (through ι_d). ◀

4 Related Work

4.1 Qudit Framework

The first and only result to date of a complete equational theory for a graphical language describing qudit systems comes from the “ZXW-calculus” [42]. There, the authors start from a qudit version of the ZX-calculus and most probably end up requiring a W-node in the definition of a normal form, and hence in the equational theory leading to completeness. We argue here that we can get a complete equational theory purely inside the ZW-calculus. By keeping the number of generators as low as possible, we also end up with few, intuitive equations in the equational theory.

The W-node we used is a different generalisation of the qubit W-node than the one used in [42]. The version we used offers two advantages with respect to the aims of the paper. First, it allows to use a single parameter in the Z-spiders (which aligns with the spirit of keeping things as minimal as possible) and to sum such parameters together, while the other version requires to have $(d - 1)$ -sized lists of coefficients as parameters in order to get a (+)-like rule to sum coefficients together. Second, it allows us to define $|k\rangle$ (up to a scalar) as a composition using only $|1\rangle$ and the W-node. Again, this lowers the number of generators, as all the $|k\rangle$ (for $k > 1$) become syntactic sugar.

Focussing on ZW-calculus is not a new idea. The first ever completeness proof for qubit graphical languages was in the (qubit) ZW-calculus, introduced in [14] and tweaked and made complete in [24, 26]. The ZW-calculus noticeably has very nice combinatorial properties different from those of its counterparts, which in particular allows for a very natural notion of normal form. It is hence not surprising that some attempts were made to get a complete equational theory of qudit systems purely in ZW. There have then been tentative generalisations for qudit systems, in particular in [25] where q -arithmetic is used, and in [49]

where the W -node is generalised in a different way (and that we encounter in [42]). It is to be noted that our two main generators are essentially the same as in [25], except with usual arithmetic instead of q -arithmetic. While some equations are sound with respect to the q -arithmetic semantics, others are truly specific to the standard arithmetic. Adapting the results of the present paper to q -arithmetic semantics hence seems non-trivial. Other presentations for qudit systems have also been proposed (without proof of completeness) in [17, 43]. Finally, complete presentations for fragments of qudit quantum mechanics can be found e.g. in [8, 40].

Another system we are close to is **QPath** [19]. Our W -node is merely the “triangle” node of **QPath** that we truncated to a finite dimension⁵, and we generalised their “line weight” to an n -ary Z -spider. The degree-2 Z -spider furthermore has exactly the same interpretation as the line weight. While in **QPath** the triangle nodes satisfy a bialgebra, this is not the case when truncating to finite dimension. Here we could either resort to define a “fermionic swap” that would replace the usual swap in the bialgebra (as in [25] and [49]), or give a context in which the bialgebra works (as is done in [42]). While such a “fermionic swap” exists in our setting, it does not have all the nice properties of the qubit fermionic swap, that in particular allow us to see it as a quasi-proper swap. Instead we went with the latter solution, which as it turns out works in our setting, despite the W -node having a different interpretation from that of [42], and we end up with Equation (b_2).

4.2 Finite Dimensional Framework

Another complete presentation of a graphical language for **FdHilb** was announced recently before the first version of the current paper [50]. This one builds upon the aforementioned ZXW -calculus, and introduces a new generator that takes two systems, of dimensions a and b , and builds a system of dimension $a \times b$. Our approach builds upon **ZW_d**, the qudit version of the ZW -calculus from Section 2 and hence starts with fewer generators and equations. As a consequence, the graphical language for **FdHilb** we end up with has fewer equations as well. Moreover, we did not require a new generator, and simply promoted the qudit W -node to work with any mix of dimensions in a natural manner, which was enough to provide us with universality.

A version of the ZX -calculus for **FdHilb** was recently provided and shown to be complete [41]. The proof of completeness for their graphical language was obtained by transporting the property from the **ZW_F**-calculus of the first version of the current paper to the ZX -calculus, through a system of translations between the two languages.

5 Conclusion

In this paper, we explored the potential for a minimal yet complete diagrammatic language for quantum mechanics beyond qubit systems. This starts with a well-chosen generalisation of the generators of the ZW -calculus, allowing us to have few and intuitive equations. For both qudit systems and finite dimensional systems, we showed that the diagrams are universal, and that the equational theories are both minimal and complete for their respective interpretation.

⁵ The idea of truncating this tensor has also been considered in [20] during a translation between graphical languages.

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A Asymmetric Presentation for Qudit Systems

In this section, we give an alternative semantics for the ZW_d -diagrams, which breaks the up/down symmetry of the generators. On the one hand, the dagger-functor becomes less natural; on the other hand, the combinatorics associated to the diagrams becomes simpler (except for the cap):

$$\begin{aligned}
 \llbracket D_2 \circ D_1 \rrbracket_{\zeta} &= \llbracket D_2 \rrbracket_{\zeta} \circ \llbracket D_1 \rrbracket_{\zeta} & \llbracket \begin{array}{c} \dots \\ \text{cap} \\ \dots \end{array} \rrbracket_{\zeta} &= \sum_{k=0}^{d-1} r^k k!^{n-1} |k^m\rangle\langle k^n| \\
 \llbracket D_1 \otimes D_2 \rrbracket_{\zeta} &= \llbracket D_1 \rrbracket_{\zeta} \otimes \llbracket D_2 \rrbracket_{\zeta} & \llbracket \begin{array}{c} \text{cup} \\ \dots \\ n \end{array} \rrbracket_{\zeta} &= \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} \binom{k}{i_1, \dots, i_n} |i_1, \dots, i_n\rangle\langle k| \\
 \llbracket | \rrbracket_{\zeta} &= \sum_k |k\rangle\langle k| & \llbracket \begin{array}{c} \text{cup} \\ \dots \\ n \end{array} \rrbracket_{\zeta} &= \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} |k\rangle\langle i_1, \dots, i_n| \\
 \llbracket \text{cap} \rrbracket_{\zeta} &= \sum_{k, \ell} |\ell, k\rangle\langle k, \ell| & \llbracket \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rrbracket_{\zeta} &= |1\rangle \quad \text{and} \quad \llbracket \begin{array}{c} | \\ \bullet \end{array} \rrbracket_{\zeta} = \langle 1| \\
 \llbracket \text{cup} \rrbracket_{\zeta} &= \sum_k k! \langle k, k| & \llbracket r \rrbracket_{\zeta} &= r \\
 \llbracket \text{cap} \rrbracket_{\zeta} &= \sum_k \frac{1}{k!} |k, k\rangle
 \end{aligned}$$

As a direct consequence, we have:

$$\llbracket \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rrbracket_{\zeta} = |k\rangle \quad \text{and} \quad \llbracket \begin{array}{c} | \\ \bullet \end{array} \rrbracket_{\zeta} = k! \langle k|$$

This semantics being equivalent to $\llbracket - \rrbracket_{\zeta}$, the equational theory of ZW_d -diagrams remains universal, sound and complete in this setting.

B Lemmas for Completeness of Qudit

The full version on Arxiv details the proof of completeness of ZW_d 's equational theory, including the proof of the following derivable equations:

► **Lemma 21.**

$$ZW_d \vdash 1 = \text{cap}$$

► **Lemma 22.**

$$ZW_d \vdash \begin{array}{c} \circ \\ | \\ \circ \end{array} = \begin{array}{c} | \\ \circ \end{array}$$

► **Lemma 23.**

$$ZW_d \vdash \begin{array}{c} \text{cap} \\ | \\ \text{cup} \end{array} = \begin{array}{c} \text{cap} \\ | \\ \text{cup} \end{array}$$

► Lemma 24.

$$ZW_d \vdash \text{[Diagram: a spider with } \dots \text{ legs and one bottom leg]} = \text{[Diagram: two separate dots]}$$

► Lemma 25.

$$ZW_d \vdash \text{[Diagram: a spider with two legs, each labeled 1]} = \text{[Diagram: a dotted circle]}$$

► Lemma 26.

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 0]} = \text{[Diagram: a single dot]}$$

► Lemma 27.

$$ZW_d \vdash \text{[Diagram: a spider with } r \text{ legs, one labeled } k \text{, and } n \text{ other legs]} = r^k \cdot \text{[Diagram: a spider with } k \text{ legs, one labeled } k \text{, and } n \text{ other legs]}$$

► Lemma 28.

$$ZW_d \vdash \text{[Diagram: two spiders with } r \text{ and } s \text{ legs respectively]} = \text{[Diagram: a spider with } r+s \text{ legs]}$$

► Lemma 29.

$$ZW_d \vdash \text{[Diagram: a spider with one leg and a loop]} = \text{[Diagram: a spider with one leg and a loop]} = \text{[Diagram: a single dot]}$$

► Lemma 30.

$$ZW_d \vdash \text{[Diagram: a spider with } k \text{ and } \ell \text{ legs]} = \begin{cases} \text{[Diagram: a spider with } k+\ell \text{ legs]} & \text{if } k + \ell < d \\ 0 \cdot \text{[Diagram: a single dot]} & \text{if } k + \ell \geq d \end{cases}$$

► Lemma 31.

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and a loop]} = \text{[Diagram: two separate dots]}$$

► Lemma 32.

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and two loops]} = \text{[Diagram: a spider with one leg labeled 1 and two loops]}$$

► Lemma 33.

$$ZW_d \vdash \text{[Diagram: a spider with } n \text{ legs and } m \text{ loops]} = \text{[Diagram: a spider with } n \text{ legs and } m \text{ loops, decomposed into } m \text{ spiders with } i_1, \dots, i_m \text{ legs} \text{ where } n = i_1 + \dots + i_m \text{.]}$$

where on the right-hand-side, there are as many Z-spiders as there are ways to decompose n as a sum of m natural numbers: $n = i_1 + \dots + i_m$.

► Lemma 34.

$$ZW_d \vdash \text{[Diagram: a spider with two legs labeled 1 and a loop]} = \text{[Diagram: a spider with one leg labeled 1 and a loop]}$$

► Lemma 35. With $0 < k < d$:

$$ZW_d \vdash \text{[Diagram: a spider with } k \text{ legs and a loop]} = \text{[Diagram: a spider with } k \text{ legs labeled 1 and a loop labeled } k! \text{.]}$$

► Lemma 36. If $0 < k < d$:

$$ZW_d \vdash k! \text{ [Diagram: a spider with one leg labeled 1]} = \text{[Diagram: a spider with } k \text{ legs labeled } k \text{ and a loop]} = \text{[Diagram: a spider with } k \text{ legs labeled } k \text{ and a loop]}$$

► Lemma 37. If $k \neq \ell$:

$$ZW_d \vdash \text{[Diagram: a spider with } k \text{ legs labeled } k \text{]} = k! \quad \text{and} \quad ZW_d \vdash \text{[Diagram: a spider with } k \text{ legs labeled } \ell \text{]} = 0$$

► Lemma 38.

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and } r \text{ legs]} = r \cdot \text{[Diagram: a spider with one leg labeled } 1/r \text{ and } r \text{ legs]}$$

► Lemma 39. If $0 \leq k < d$:

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and a loop]} = \text{[Diagram: a spider with one leg labeled 1 and a loop]}$$

► Lemma 40. If $0 \leq k < d$, we have:

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and a loop labeled } k \text{]} = \text{[Diagram: a spider with one leg labeled 1 and a loop labeled } k! \text{.]}$$

► Lemma 41. The bialgebra between W -nodes can be used in the following context, if $0 \leq k_i < d$ for all i :

$$ZW_d \vdash \text{[Diagram: a spider with one leg labeled 1 and } k_1, \dots, k_n \text{ legs]} = \text{[Diagram: a spider with one leg labeled 1 and } k_1, \dots, k_n \text{ legs, decomposed into } n \text{ spiders with } k_1, \dots, k_n \text{ legs} \text{.]}$$