


# Modal Automata: Analysing Modal Fixpoint Logics, One Step at a Time

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## Abstract

We present and investigate a general framework for studying modal fixpoint logics and some related versions of monadic second-order logic, by means of certain finite automata that operate on Kripke structures. Characteristic of these modal automata is that the co-domain of their transition function is a set of formulas of a so-called one-step logic. The motivation for taking this perspective is that if a logic is characterised by a class of modal automata, many of its properties are already determined at the level of the much simpler one-step logic.

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## 1 Modal automata

There is a long tradition in theoretical computer science connecting the research fields of automata theory and logic. This link becomes particularly strong when automata are used to classify (possibly) infinite objects like streams, trees or graphs. Interestingly, this research area has provided not only fundamental theoretical results, such as Rabin’s decidability theorem, but also quite concrete applications in computer science, such as tools for the automatic verification of reactive systems.

Building on this tradition, and in particular on the work by Janin & Walukiewicz [8, 9] and D’Agostino & Hollenberg [3] on the modal  $\mu$ -calculus, we present and investigate a general framework for studying modal fixpoint logics, and some versions of monadic second-order logic, by means of certain finite automata that operate on Kripke structures.

► **Definition 1.** Let  $\mathcal{Q}$  be a set of proposition letters. A (Kripke) structure over  $\mathcal{Q}$  is a triple  $\mathbb{S} = (S, R, \kappa)$  where  $S$  is a set of objects called points or worlds,  $R \subseteq S \times S$  is a binary relation and  $\kappa$  is a  $\mathcal{Q}$ -marking or colouring on  $S$ , that is, a map  $\kappa : S \rightarrow \wp(\mathcal{Q})$ . Given a state  $s$ , the set  $\{t \in S \mid (s, t) \in R\}$  of its successors is denoted as  $R[s]$ . A pointed structure is a pair  $(\mathbb{S}, s)$  where  $s$  is a point in  $\mathbb{S}$ , and a query is a class of pointed structures.

Characteristic of the modal automata that we are about to introduce is that the co-domain of their transition function is a set of formulas in a simple formalism that we call a one-step logic. The concept of a one-step logic was developed by several authors, including Cîrstea, Pattinson and Schröder, in the setting of coalgebraic modal logic.

► **Definition 2.** A one-step model over a set  $A$  of monadic predicates is a pair  $(D, m)$  where  $m : D \rightarrow \wp(A)$  is a  $A$ -marking on  $D$ .

A one-step logic is a pair  $(\mathbb{L}, \Vdash_1)$  where  $\mathbb{L}$  is a one-step language, that is, a map  $\mathbb{L}$  assigning to any set  $A$  a collection  $\mathbb{L}(A)$  of objects called one-step formulas; and  $\Vdash_1$  is a truth relation between one-step formulas and one-step models. If we have  $(D, m) \Vdash_1 \phi$  we will say that  $\phi$  is true of  $(D, m)$  or that  $(D, m)$  satisfies  $\phi$ .

We will require every one-step formula  $\phi$  to be monotone; that is:  $(D, m) \Vdash_1 \phi$  implies  $(D, m') \Vdash_1 \phi$  whenever  $m'$  is an extension of  $m$  (that is,  $m(d) \subseteq m'(d)$  for every  $d \in D$ ).



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In addition to the monotonicity requirement, we impose some natural coherence conditions on one-step logics; for instance, we require that  $L(A) \subseteq L(B)$  if  $A \subseteq B$ , that the truth relation is invariant under isomorphism, etc. We will usually blur the distinction between one-step logics and one-step languages, since the semantics of one-step formulas is generally fixed.

► **Example 3.** The one-step language  $\text{FOE}_1(A)$  of first-order logic with equality on a set of predicates  $A$  is given by the sentences (formulas without free variables) generated by the following grammar, where  $a \in A$  and  $x, y$  are individual variables:

$$\phi ::= a(x) \mid x = y \mid x \neq y \mid \exists x.\phi \mid \forall x.\phi \mid \phi \vee \phi \mid \phi \wedge \phi \quad (1)$$

We use  $\text{FO}_1$  for the equality-free fragment of  $\text{FOE}_1$ , where we omit the clauses  $x = y$  and  $x \neq y$ , and we write  $\text{FOE}_1^\infty$  for the extension of  $\text{FOE}_1$  with the infinity quantifiers  $\exists^\infty$  and  $\forall^\infty$ . The semantics of these languages is the obvious one.

► **Definition 4.** Let  $L$  and  $Q$  be, respectively, a one-step language and a set of proposition letters. An  $L$ -automaton over  $Q$  is a quadruple  $\mathbb{A} = (A, \Theta, \Omega, a_I)$  where  $A$  is a finite set of objects called states;  $\Theta : A \times \wp(Q) \rightarrow L(A)$  is its transition map;  $\Omega : A \rightarrow \omega$  is its priority function; and  $a_I$  is its initial state.

We will use the term “modal automaton” as a generic name for  $L$ -automata for some one-step logic  $L$ . The operational semantics of these automata is given in terms of parity games.

► **Definition 5.** Let  $L$  and  $Q$  be, respectively, a one-step language and a set of proposition letters. Furthermore, let  $\mathbb{S} = (S, R, \kappa)$  and  $\mathbb{A} = (A, \Theta, \Omega, a_I)$  be, respectively, a Kripke structure and an  $L$ -automaton over  $Q$ .

The acceptance game  $\mathcal{A}(\mathbb{A}, \mathbb{S})$  is given as the two-player game of which the set of positions, as well as their owners, admissible moves and priorities are given in the table below:

Position	Player	Admissible moves	Priority
$(a, s) \in A \times S$	$\exists$	$\{m : S \rightarrow \wp(A) \mid R[s], m \Vdash_1 \Theta(a, \kappa(s))\}$	$\Omega(a)$
$m : S \rightarrow \wp(A)$	$\forall$	$\{(b, t) \mid b \in m(t)\}$	$0$

Explained in words, the acceptance game  $\mathcal{A}(\mathbb{A}, \mathbb{S})$  proceeds in rounds, each round moving from one basic position  $(a, s) \in A \times S$  to the next. At such a basic position, it is  $\exists$ 's task to turn the set  $R(s)$  of successors of  $s$  into the domain of a one-step model for the formula  $\Theta(a, \kappa(s)) \in L(A)$ . That is, she needs to come up with a marking  $m : R[s] \rightarrow \wp(A)$  such that  $(R[s], m) \Vdash_1 \Theta(a, \kappa(s))$  (and if she cannot find such a valuation, she loses immediately). One may think of the set  $\{(b, t) \mid b \in m(t)\}$  as a collection of witnesses to her claim that, indeed, the one-step formula  $\Theta(a, \kappa(s))$  is true of  $(R[s], m)$ . The round ends with  $\forall$  picking one of these witnesses, which then becomes the basic position at the start of the next round. (Unless, of course,  $\exists$  managed to satisfy the formula  $\Theta(a, \kappa(s))$  with the empty set of witnesses, in which case  $\forall$  gets stuck and loses immediately.)

If  $\exists$  and  $\forall$  play the acceptance game  $\mathcal{A}(\mathbb{A}, \mathbb{S})$  and neither of them gets stuck, the resulting match  $\pi$  will take infinitely long. In this case we use the parity condition to assign a winner to  $\pi$ : we consider the set  $\text{Inf}(\pi)$  of states that occur infinitely often in  $\pi$ , and declare  $\exists$  to be the winner of  $\pi$  if the maximal priority of the states in  $\text{Inf}(\pi)$  is even, and  $\forall$  if this maximum priority is odd.

► **Definition 6.** Let  $\mathbb{A} = (A, \Theta, \Omega, a_I)$  be some modal automaton. In case the pair  $(a_I, s)$  is a winning position for  $\exists$  in the game  $\mathcal{A}(\mathbb{A}, \mathbb{S})$ , we say that  $\mathbb{A}$  accepts the pointed structure  $(\mathbb{S}, s)$  and we write  $\mathbb{S}, s \Vdash \mathbb{A}$ . A query is recognized by an automaton  $\mathbb{A}$  if it contains exactly those pointed structures that are accepted by  $\mathbb{A}$ .

We say that a logic is characterised by a class of modal automata if for every formula of the logic we can find an equivalent automaton in the class, and vice versa. Some well-known logics are of this kind.

► **Example 7.** Any one-step logic  $L$  naturally induces a fixpoint logic  $\mu L$  which is characterised by the class  $\text{Aut}(L)$ . The modal  $\mu$ -calculus  $\mu ML$  is characterised by the class  $\text{Aut}(\mathbf{FO}_1)$ , and various fragments of  $\mu ML$ , including the alternation-free fragment and computational tree logic (CTL), correspond to natural subclasses of  $\text{Aut}(\mathbf{FO}_1)$ . The same applies to propositional dynamic logic (PDL), if we consider a multi-sorted version of  $\mathbf{FO}_1$ .

Furthermore, if we restrict attention to tree-based structures, monadic second-order logic is characterised by the class  $\text{Aut}(\mathbf{FO}_1)$ , and weak monadic second-order logic is characterised by a natural fragment of  $\text{Aut}(\mathbf{FOE}_1^\infty)$ .

## 2 Some results

The motivation for studying logics from the perspective of modal automata is that much of their sometimes complex behaviour is already determined at the far simpler one-step level. This means that, in order to establish some property of a class of modal automata – or of the logic it characterises – it may suffice to prove a similar result for the one-step logic(s) underlying the automata. Here are some examples, where  $L$  and  $L'$  represent arbitrary one-step logics.

**Closure properties of recognisable queries** If  $L$  is closed under taking, respectively, disjunctions, conjunctions and boolean duals, then the class of  $\text{Aut}(L)$ -recognisable queries is closed under taking union, intersection, and complementation.

**Bisimulation invariance** We say that a one-step formula  $\phi$  is invariant under quotients if we have  $(D, m) \Vdash_1 \phi$  iff  $(D', m') \Vdash_1 \phi$ , whenever  $(D', m')$  is a quotient of  $(D, m)$ .

If  $L$  is the quotient-invariant fragment of  $L'$  (in some strong sense), then  $\text{Aut}(L)$  is the bisimulation-invariant fragment of  $\text{Aut}(L')$ . This result lies at the heart of the proof of the Janin-Walukiewicz Theorem, which identifies the modal  $\mu$ -calculus as the bisimulation-invariant fragment of monadic second-order logic.

**Nondeterminism** Modal automata are generally alternating in nature, but there is a natural notion of nondeterminism as well: we call a modal automaton  $\mathbb{A}$  nondeterministic if in any of its acceptance games, the role of  $\forall$  is essentially reducible to that of a pathfinder. At the level of one-step logic, we introduce the notion of a disjunctive formula; roughly the idea is that if a disjunctive formula holds of some model then we can also satisfy it in a related model where the marking assigns to each element of the domain either the empty set or some singleton. It is easy to see that a modal automaton  $\mathbb{A} = (A, \Theta, \Omega, a_I)$  is nondeterministic if the codomain of  $\Theta$  consists of disjunctive formulas.

**Disjunctive bases and simulation** Assume that  $L' \subseteq L$  and that  $L'$  consists of disjunctive formulas and is closed under disjunctions. If  $L$  and  $L'$  satisfy some natural distributive laws we call  $L$  a disjunctive basis for  $L'$ .

If  $L$  has a disjunctive basis  $L'$  then every (alternating)  $L$ -automaton can be simulated by an equivalent (nondeterministic)  $L'$ -automaton. Furthermore, the fixpoint logic characterised by  $L$  has several nice properties, such as the finite model property and uniform interpolation. This is how D'Agostino and Hollenberg established this property for the modal  $\mu$ -calculus.

### Coalgebraic generalisations

The concept of a modal automaton, and all applications in the theory of fixpoint logics and monadic second-order logics that we mentioned above, can be generalised to the setting of universal coalgebra. This means that the concept and the results become available for other coalgebraic modal (fixpoint) logics, such as graded, monotone, or probabilistic modal logic. A further result that we didn't mention above concerns the transfer of axiomatic completeness.

**Disjunctive bases and axiomatic completeness** Any one-step axiomatisation  $H$  for  $L$  naturally induces an axiom system  $\mu H$  for the  $\mu$ -calculus  $\mu L$  induced by  $L$ . If  $H$  is sound and complete for  $L$  and  $L$  has a disjunctive basis, then  $\mu H$  is sound and complete for the set of valid  $\mu L$ -formulas. By this result, which generalises Walukiewicz' completeness result for the modal  $\mu$ -calculus [12], one may for instance obtain complete axiomatisations for the graded and the monotone modal  $\mu$ -calculus.

Here are some pertinent publications in which the theory of modal automata has been developed: [2, 1, 4, 5, 6, 7, 10, 11]

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