Monotone Weak Distributive Laws over the Lifted Powerset Monad in Categories of Algebras

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Abstract

In both the category of sets and the category of compact Hausdorff spaces, there is a monotone weak distributive law that combines two layers of non-determinism. Noticing the similarity between these two laws, we study whether the latter can be obtained automatically as a weak lifting of the former. This holds partially, but does not generalize to other categories of algebras. We then characterize when exactly monotone weak distributive laws over powerset monads in categories of algebras exist, on the one hand exhibiting a law combining probabilities and non-determinism in compact Hausdorff spaces and showing on the other hand that such laws do not exist in a lot of other cases.

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1 Introduction

In the study of the semantics of programming languages, and since the seminal work of Moggi [26], effectful computations are usually modeled with monads: an effectful function of type $X \twoheadrightarrow Y$ is interpreted as a function of type $X \to \mathsf{T}Y$, where the monadic structure on T allows for having identities and compositions of such effectful functions. When considering several effects at the same time, a natural question arises: given monads corresponding to two effects, is it possible to construct a monad that corresponds to the combination of these two effects? In particular, combining probabilities and non-determinism has been a very popular subject of study in the topological and (dually) domain-theoretic settings: see for instance the introduction of [21] for an extensive bibliography on the topic.

The most straightforward way to combine two monads S and T would be to compose their underlying functors, but unfortunately in general the resulting endofunctor ST may not carry the structure of a monad. For this to hold, a sufficient condition is the existence of a *distributive law* of T over S, i.e. a natural transformation $\mathsf{TS} \Rightarrow \mathsf{ST}$ satisfying four axioms involving the units and multiplications of the two monads [2]. Such a distributive law makes ST into a monad, and its data is equivalent to the data of a *lifting* of S to the Eilenberg-Moore category $\mathbf{EM}(\mathsf{T})$ of T-algebras or to the data of an *extension* of T to the Kleisli category $\mathbf{Kl}(\mathsf{S})$ of free S-algebras.

Unfortunately, distributive laws turn out to be not so common: proving that some specific pairs of monads do not admit any distributive law between them has been the focus of several works [22, 29, 12], culminating in [31] where general techniques for proving the absence of distributive laws between monads on **Set**, so-called "no-go theorems", are exhibited. Among the culprits are the powerset monad $\mathbf{P}$ and the probability distributions monad $\mathbf{D}$: there is no distributive law $\mathbf{PP} \Rightarrow \mathbf{PP}$ nor $\mathbf{DP} \Rightarrow \mathbf{PD}$.

A next step in combining monads is thus to weaken the requirements asked for by distributive laws. In [3] a 2-categorical theory of such weakened distributive laws, where the axioms relating to the units of the monads are relaxed, is developed. This theory encompasses two orthogonal kinds of weakened laws, both called weak distributive laws: those of [28], extensively studied in [5, 6, 4], and those more recently put light on in [14] and given further instances of in [18, 19, 7]. This work focuses on the latter kind of laws: in the following a *weak distributive law* of T over S will thus be a natural transformation of type $\mathsf{TS} \Rightarrow \mathsf{ST}$ (again), where only three of the four axioms of distributive laws are required. Such a weak distributive law need not make ST into a monad, but does so of a retract of it when the category is well-behaved. Weak laws $\mathsf{TS} \Rightarrow \mathsf{ST}$ are equivalent to *weak extensions* of T to $\mathbf{Kl}(\mathsf{S})$ – extensions of the semi-monad underlying T, i.e. of its endofunctor and multiplication but not of its unit – or, when the category is well-behaved, *weak liftings* of S to $\mathbf{EM}(\mathsf{T})$ – liftings of S up to a retraction.

Weak extensions are a precious tool for building weak distributive laws, because *monotone* extensions to **Rel** – the category of sets and relations, which also happens to be the category of free algebras of the powerset monad **P** on **Set** – admit a nice characterization: this makes it possible to find weak distributive laws without having to guess their formulæ anymore, and the monotonicity of the extension is a strong indicator that the resulting law will be semantically interesting. In the context of weak distributive laws, this strategy was first used by Garner [14], who exhibited a law $\mathbf{PP} \Rightarrow \mathbf{PP}$, but also a law $\boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$ combining the ultrafilter monad $\boldsymbol{\beta}$ and the powerset monad and whose corresponding weak lifting turned out to be the Vietoris monad **V** of closed subsets on the category **KHaus** of compact Hausdorff spaces (the algebras of $\boldsymbol{\beta}$ [24]). The same strategy was then used for instance in [18], where a law $\mathbf{DP} \Rightarrow \mathbf{PD}$, combining probabilities and non-determinisim, is described.

In fact, most non-trivial weak distributive laws in the literature (trivial ones are described in [16, §2.2]) are laws of type $\mathsf{T}\mathbf{P} \Rightarrow \mathbf{P}\mathsf{T}$ built using this strategy. On the other hand, in [19] the goal was to find weak distributive laws in other categories than **Set**, and it was reached as a law $\mathbf{VV} \Rightarrow \mathbf{VV}$, combining two layers of non-determinism in a continuous setting, was described. This last law was also built by constructing a monotone weak extension, and its formula is thus very close to that of the law $\mathbf{PP} \Rightarrow \mathbf{PP}$. In topological settings, non-determinism can also be combined with probabilities: weak distributive laws for this purpose are constructed by hand in a very recent pre-print [15].

The goal of the present work is to take over this program of finding non-**Set**-based weak distributive laws: we focus here on categories of algebras, which fit in the general framework for monotone weak laws presented in [19]. In particular, we notice in Theorem 12 that the law $\mathbf{VV} \Rightarrow \mathbf{VV}$ is not only very similar to the law $\mathbf{PP} \Rightarrow \mathbf{PP}$, but is also actually some sort of weak lifting of it. We thus study whether there is a general framework for not only weakly lifting monads (as weak distributive laws do), but also weakly lifting weak distributive laws themselves: this framework should yield or simplify the construction of the law $\mathbf{VV} \Rightarrow \mathbf{VV}$, and hopefully generalize it to other categories of algebras, in particular $\mathbf{EM}(\mathbf{P})$ and $\mathbf{EM}(\mathbf{D})$ which have weakly lifted powerset monads thanks to the laws $\mathbf{PP} \Rightarrow \mathbf{PP}$ and $\mathbf{DP} \Rightarrow \mathbf{PD}$.

This question is largely related to the problem of composing weak distributive laws, which was investigated in [17] from the point of view of the Yang-Baxter equations – the usual tool for composing and lifting plain distributive laws [10]. We end up getting a general "no-go theorem" for monotone weak laws over weakly lifted powerset monads: in that sense this work is also close in spirit to [31], where general no-go theorems for (strict) distributive laws are given. While not restricted to monotone distributive laws, these theorems are unlikely to generalize to our setting because they are based on the correspondence between monads and algebraic theories, which is mostly restricted to **Set** and does not have any obvious generalization to semi-monads – to which weak distributive laws are deeply related.

This article is organized as follows. In Section 2, we recall definitions and notations for and give examples of monads and weak distributive laws, and also recall the framework of [19] for monotone weak distributive laws in regular categories. This culminates in Theorem 12, where we notice that the law $\mathbf{VV} \Rightarrow \mathbf{VV}$ is some sort of weak lifting of the law $\mathbf{PP} \Rightarrow \mathbf{PP}$. The next two sections focus on lifting weak distributive laws: in Section 3 we study the approach of the Yang-Baxter equation, showing it indeed allows for weakly lifting weak laws but does not apply to the examples we consider; while in Section 4 we focus on the monotonicity of the laws, giving a simple characterization for the existence of monotone weak laws in categories of algebras and applying it to several examples. We conclude in Section 5.

Our main contributions are the following:

- we show that, while the law $\mathbf{VV} \Rightarrow \mathbf{VV}$ is a kind of weak lifting of the law $\mathbf{PP} \Rightarrow \mathbf{PP}$ (Theorem 18), this lifting does not come from a Yang-Baxter equation (Theorem 16), the usual approch to lifting laws: it is an instance of a general no-go theorem for Yang-Baxter equations involving the law $\mathbf{PP} \Rightarrow \mathbf{PP}$ (Theorem 15);
- we characterize the Kleisli category of the weak lifting $\overline{\mathsf{S}}$ of a monad S to the algebras of a monad T as the category of T-algebras and $\mathbf{Kl}(\mathsf{S})$-morphisms between them (Theorem 22);
- we give a characterization of the Kleisli categories of weakly lifted powerset monads as subcategories of relations (Theorem 29), in which a certain class of *decomposable morphisms* play a central role: it follows that monads must preserve these decomposable morphisms to have monotone weak distributive laws over weakly lifted powerset monads, and this is in fact a sufficient condition for monads that are themselves weak liftings (Theorem 37);
- concrete instances of this result are then easily derived: we recover independently the law combining probabilities and non-determinism in compact Hausdorff spaces and recently exhibited in [15] (Theorem 38), but we observe otherwise that monotone weak distributive laws over weakly lifted powerset monads in categories of algebras seem very rare (Table 1).

2 Preliminaries

In this section we first recall the theory of weak distributive laws from [14] and the tools that come along, especially the ones developed in [19]. The reader is assumed to be familiar with the basics of category theory, an introduction to which appearing for instance in [8].

2.1 Monads and (Weak) Distributive Laws

▶ **Definition 1** (monad). *A* monad *on a category* C *is the data* $(\mathsf{T}, \eta^{\mathsf{T}}, \mu^{\mathsf{T}})$*, often abbreviated* T*, of an endofunctor* $\mathsf{T}\colon \mathsf{C} \to \mathsf{C}$ *and natural transformations* $\eta^{\mathsf{T}}\colon \mathsf{Id} \Rightarrow \mathsf{T}$ *and* $\mu^{\mathsf{T}}\colon \mathsf{TT} \Rightarrow \mathsf{T}$ *satisfying the axioms* $\mu^{\mathsf{T}} \circ \mathsf{T}\eta^{\mathsf{T}} = \mathrm{id} = \mu^{\mathsf{T}} \circ \eta^{\mathsf{T}}\mathsf{T}$ *and* $\mu^{\mathsf{T}} \circ \mathsf{T}\mu^{\mathsf{T}} = \mu^{\mathsf{T}} \circ \mu^{\mathsf{T}}\mathsf{T}$.

▶ **Example 2** (monads). In this work we will be particularly concerned with the following monads on **Set**. Thereafter X and Y are sets, x is an element of X and $f\colon X \to Y$ is a function.

The *powerset* monad P. $\mathbf{P}X = \{\text{subsets of } X\}$; for $e \subseteq X$, $(\mathbf{P}f)(e) = f[e] = \{f(x') \mid x' \in e\}$; $\eta^{\mathbf{P}}_X(x) = \{x\}$; for $E \subseteq \mathbf{P}X$, $\mu^{\mathbf{P}}_X(E) = \bigcup E$.

The *probability distributions* monad D. $\mathbf{D}X$ is the set of finitely-supported probability distributions on X, i.e. functions $\varphi\colon X \to [0,1]$ such that $\varphi^{-1}(0,1]$ is finite and $\sum_{x \in X} \varphi(x) = 1$; $\mathbf{D}f$ is the pushforward along f given by $(\mathbf{D}f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$ for $\varphi \in \mathbf{D}X$ and $y \in Y$; $\eta^{\mathbf{D}}_X(x)$ is the Dirac δ_x for $x \in X$, i.e. the probability distribution such that $\delta_x(x) = 1$; and $\mu^{\mathbf{D}}$ computes the mean of a distribution of distributions, so that for $\Phi \in \mathbf{DD}X$, $\mu^{\mathbf{D}}(\Phi)(x) = \sum_{\varphi \in \mathbf{D}X} \Phi(\varphi) \cdot \varphi(x)$.

The *ultrafilter monad* β. $\boldsymbol{\beta}X$ is the set of maximal filters on X, where filters are sets $E \in \mathbf{PP}X$ that are non-empty, up-closed (for inclusion), stable under finite intersections and do not contain the empty set; for an ultrafilter $E \in \boldsymbol{\beta}X$, $(\boldsymbol{\beta}f)(E) = \{e' \supseteq f[e] \mid e \in E\}$; $\eta^{\boldsymbol{\beta}}_X(x)$ is the principal filter $\{e \in \mathbf{P}X \mid x \in e\}$; and, for $\mathfrak{E} \in \boldsymbol{\beta\beta}X$, $\mu^{\boldsymbol{\beta}}_X(\mathfrak{E}) = \bigcup\{\bigcap \mathfrak{e} \mid \mathfrak{e} \in \mathfrak{E}\}$.

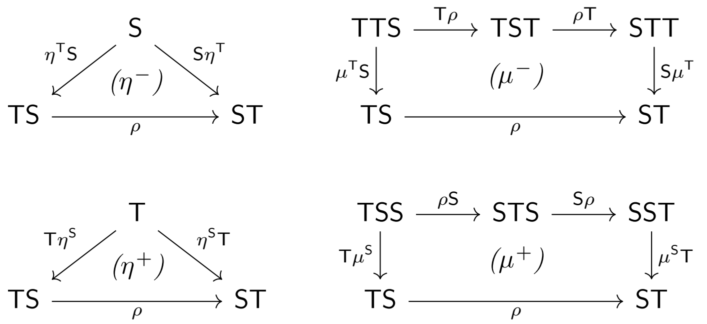
We will also be concerned with a topological analogue of the powerset monad, defined on the category **KHaus** of compact Hausdorff spaces and continuous functions. Thereafter X and Y are compact Hausdorff spaces, x is an element of X, and $f\colon X \to Y$ is a continuous function.

The *Vietoris monad* V. $\mathbf{V}X$ is the space of closed subsets of X equipped with the topology for which a subbase is given by the sets $\Box u = \{c \in \mathbf{V}X \mid c \subset u\}$ and $\Diamond u = \{c \in \mathbf{V}X \mid c \cap u \neq \varnothing\}$ where u ranges among all open sets of X; $(\mathbf{V}f)(c) = f[c] = \{f(x) \mid x \in c\}$ for $c \in \mathbf{V}X$; $\eta^{\mathbf{V}}_X(x) = \{x\}$; and $\mu^{\mathbf{V}}_X(C) = \bigcup C$ for $C \in \mathbf{VV}X$.

We also write $\mathbf{P}_*$ and $\mathbf{V}_*$ for the non-empty powerset and Vietoris monads, respectively obtained by removing the empty set from $\mathbf{P}X$ and $\mathbf{V}X$.

Weak distributive laws are a certain type of natural transformations involving monads.

▶ **Definition 3** ((weak) distributive law [14, Definition 9])**.** *A monad* T weakly distributes *over a monad* S *when there is a* weak distributive law *of* T *over* S*, i.e. a natural transformation* $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$ *such that the Diagrams* (η^+)*,* (μ^-)*, and* (μ^+) *below commute. A* (strict) distributive law *is a weak distributive law that moreover has the diagram* (η^-) *below commute.*



If $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$ is a distributive law, ST, $\eta^{\mathsf{ST}} = \eta^{\mathsf{S}}\eta^{\mathsf{T}}$ and $\mu^{\mathsf{ST}} = \mu^{\mathsf{S}}\mu^{\mathsf{T}} \circ \mathsf{S}\rho\mathsf{T}$ form a monad [2]. If it is only a weak distributive law this is not the case anymore, because $\mu^{\mathsf{ST}} \circ \eta^{\mathsf{ST}}\mathsf{ST}$ is not the identity anymore, only an idempotent (its composite with itself is itself). But suppose now it is a split idempotent, i.e. that there is a functor $\mathsf{S} \bullet \mathsf{T}$ and natural transformations $p\colon \mathsf{ST} \Rightarrow \mathsf{S} \bullet \mathsf{T}$ and $i\colon \mathsf{S} \bullet \mathsf{T} \Rightarrow \mathsf{ST}$ such that $p \circ i = \mathrm{id}$ and $i \circ p = \mu^{\mathsf{ST}} \circ \eta^{\mathsf{ST}}\mathsf{ST}$. Then $\mathsf{S} \bullet \mathsf{T}$ can be made into a monad [14] with unit $\eta^{\mathsf{S}\bullet\mathsf{T}} = p \circ \eta^{\mathsf{S}}\eta^{\mathsf{T}}$ and multiplication $\mu^{\mathsf{S}\bullet\mathsf{T}} = p \circ \mu^{\mathsf{S}}\mu^{\mathsf{T}} \circ \mathsf{S}\rho\mathsf{T} \circ ii$: the monad $\mathsf{S} \bullet \mathsf{T}$ is called the *weak composite* of S and T.

▶ **Example 4** (weak distributive laws)**.** In this work we will be particularly concerned with the following weak distributive laws:

- in **Set**, the weak distributive law $\lambda^{\mathbf{P}/\mathbf{P}}\colon \mathbf{PP} \Rightarrow \mathbf{PP}$ given by Equation (1) below has for weak composite the monad $\mathbf{P} \bullet \mathbf{P}$ of sets of subsets closed under non-empty unions [14];
- in **KHaus**, there is a weak distributive law $\lambda^{\mathbf{V}/\mathbf{V}}\colon \mathbf{VV} \Rightarrow \mathbf{VV}$ given by Equation (2) below [19];
- in **Set**, the weak distributive law $\lambda^{\mathbf{D}/\mathbf{P}}\colon \mathbf{DP} \Rightarrow \mathbf{PD}$ given by Equation (3) below has for weak composite the monad $\mathbf{P} \bullet \mathbf{D}$ of convex sets of finitely supported probability distributions [18].

$$\lambda_X^{\mathsf{P}/\mathsf{P}}(E) = \left\{e' \in \mathbf{P}X \;\middle|\; e' \subseteq \bigcup E \text{ and } \forall e \in E, e \cap e' \neq \varnothing\right\} \tag{1}$$

$$\lambda_X^{\mathsf{V}/\mathsf{V}}(C) = \left\{c' \in \mathbf{V}X \;\middle|\; c' \subseteq \bigcup C \text{ and } \forall c \in C, c \cap c' \neq \varnothing\right\} \tag{2}$$

$$\lambda_X^{\mathsf{D}/\mathsf{P}}(\Phi) = \left\{\left(\mu_X^{\mathsf{D}} \circ \mathbf{D}f\right)(\Phi) \;\middle|\; f\colon \mathbf{P}X \to \mathbf{D}X \text{ and } \forall e \in \mathbf{P}X, \Phi(e) \neq 0 \Rightarrow f(e) \in \mathbf{D}e\right\} \tag{3}$$

At this point two questions may come to the curious reader: how do we actually find these weak distributive laws – the formulas in Theorem 4 are not trivial – and how exactly is the monad structure on $\mathsf{S} \bullet \mathsf{T}$ constructed. These two questions will respectively be answered in the next two sections, which give two other equivalent presentations of weak distributive laws.

2.2 Regular Categories and Monotone (Weak) Extensions

Given a monad S, one may form its *Kleisli category* $\mathbf{Kl}(\mathsf{S})$, which is intuitively the category of S-effectful arrows of C: its objects are those of C, its arrows $X \multimap Y$ are those arrows $X \to \mathsf{S}Y$ in C, the identity arrow $X \multimap X$ is given by $\eta_X^{\mathsf{S}}\colon X \to \mathsf{S}X$, and the composite of two arrows $f\colon X \multimap Y$ and $g\colon Y \multimap Z$ is given by the composition $X \xrightarrow{f} \mathsf{S}Y \xrightarrow{\mathsf{S}g} \mathsf{SS}Z \xrightarrow{\mu_Z^{\mathsf{S}}} \mathsf{S}Z$ in C. $\mathbf{Kl}(\mathsf{S})$ comes with an adjunction $\mathbf{F}_{\mathsf{S}}\colon \mathsf{C} \leftrightarrows \mathbf{Kl}(\mathsf{S})\colon \mathbf{U}_{\mathsf{S}}$, such that $\mathbf{U}_{\mathsf{S}}\mathbf{F}_{\mathsf{S}} = \mathsf{S}$: the left adjoint $\mathbf{F}_{\mathsf{S}}$ sends objects on themselves and an arrow $f\colon X \to Y$ to the *pure* effectful arrow $X \xrightarrow{f} Y \xrightarrow{\eta_Y^{\mathsf{S}}} \mathsf{S}Y$, and the right adjoint $\mathbf{U}_{\mathsf{S}}$ sends an object X on $\mathsf{S}X$ and an effectful arrow $f\colon X \to \mathsf{S}Y$ to the arrow $\mathsf{S}X \xrightarrow{\mathsf{S}f} \mathsf{SS}Y \xrightarrow{\mu^{\mathsf{S}}} Y$. The unit of the adjunction is η^{S} and its counit is the natural transformation $\varepsilon^{\mathsf{S}}\colon \mathbf{F}_{\mathsf{S}}\mathbf{U}_{\mathsf{S}} \Rightarrow \mathsf{Id}$ (in $\mathbf{Kl}(\mathsf{S})$) with components the effectful arrows $\mathrm{id}_{\mathsf{S}X}\colon \mathsf{S}X \to \mathsf{S}X$.

An endofunctor $\mathsf{T}\colon \mathsf{C} \to \mathsf{C}$ extends to $\mathbf{Kl}(\mathsf{S})$ if there is an endofunctor $\underline{\mathsf{T}}\colon \mathbf{Kl}(\mathsf{S}) \to \mathbf{Kl}(\mathsf{S})$ such that $\underline{\mathsf{T}}\mathbf{F}_{\mathsf{S}} = \mathbf{F}_{\mathsf{S}}\mathsf{T}$. If T and T' have extensions $\underline{\mathsf{T}}$ and $\underline{\mathsf{T}}'$, a natural transformation $\alpha\colon \mathsf{T} \Rightarrow \mathsf{T}'$ extends to $\mathbf{Kl}(\mathsf{S})$ if $\underline{\alpha}$, given by $\underline{\alpha}\mathbf{F}_{\mathsf{S}} = \mathbf{F}_{\mathsf{S}}\alpha$, is a natural transformation $\underline{\mathsf{T}} \Rightarrow \underline{\mathsf{T}}'$.

▶ **Definition 5** (extensions of monads). *Let S be a monad on C. A monad $\left(\mathsf{T}, \eta^{\mathsf{T}}, \mu^{\mathsf{T}}\right)$ on C* weakly extends *to $\mathbf{Kl}(\mathsf{S})$ when T and μ^{T} extend to $\mathbf{Kl}(\mathsf{S})$. It* extends *to $\mathbf{Kl}(\mathsf{S})$ when η^{T} also extends to $\mathbf{Kl}(\mathsf{S})$.*

Giving an extension of a monad T to $\mathbf{Kl}(\mathsf{S})$ is equivalent to giving a distributive law $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$, just like giving a weak extension thereof is equivalent to giving a weak distributive law of the same type [14]: in both cases, the law ρ induces the extension which sends $f\colon X \to \mathsf{S}Y$ to $\mathsf{T}X \xrightarrow{\mathsf{T}f} \mathsf{TS}Y \xrightarrow{\rho} \mathsf{ST}Y$, and is computed from the extension as $\rho = \mathbf{U}_{\mathsf{S}}\underline{\mathsf{T}}\varepsilon^{\mathsf{S}}\mathbf{F}_{\mathsf{S}} \circ \eta^{\mathsf{S}}\mathsf{TS}$. In particular Theorem 4 also yields examples of weak extensions.

Monotone extensions

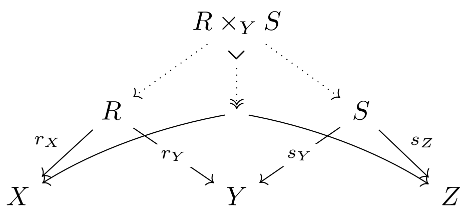
If $\mathbf{Kl}(\mathsf{S})$ carries more structure than just that of a category then it is natural to ask that extensions of functors preserve this additional structure: intuitively, the more structure preserved, the more semantically canonical the resulting law should be. For instance, recall that $\mathbf{Kl}(\mathbf{P})$ is the category $\mathbf{Rel}$ of sets and relations and that relations between two sets are ordered by inclusion. Laws arising from monotone extensions to $\mathbf{Rel}$ are thus of particular interest.

▶ **Definition 6** (monotone (weak) distributive laws and extensions). *A (weak) distributive law $\mathsf{TS} \Rightarrow \mathsf{ST}$ and its (weak) extension $\underline{\mathsf{T}}$ to $\mathbf{Kl}(\mathsf{S})$ are called* monotone *when the sets of morphisms of $\mathbf{Kl}(\mathsf{S})$ admit a canonical order that is preserved by $\underline{\mathsf{T}}$.*

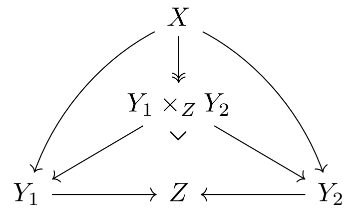
It turns out that such monotone extensions to relations can be defined and characterized in any *regular category*, as described in [19] and recalled now.

Let C be a finitely complete category. The *kernel pair* of an arrow $f\colon X \to Y$ is the pullback $p_1, p_2\colon X \times_Y X \rightrightarrows X$ of the cospan $X \xrightarrow{f} Y \xleftarrow{f} X$. Assume C has all the coequalizers of these kernel pairs: these coequalizers are called regular epimorphisms. C is then called *regular* if these conditions are satisfied and if the regular epimorphisms are stable under pullbacks[1]. Regular categories enjoy the fact that every arrow $f\colon X \to Y$ may be factored as a regular epimorphism (denoted with $\twoheadrightarrow$) followed by a monomorphism (denoted with $\hookrightarrow$), and this factorization is unique up to unique isomorphism – one should think of it as factoring an arrow through its image. **Set** is a canonical example of a regular category.

Regular categories are useful because they are categories where we can speak of relations: if C is a regular category, a C-relation between two objects X and Y is a subobject $r\colon R \hookrightarrow X \times Y$. Relations are preordered: $r\colon R \hookrightarrow X \times Y$ is smaller than $s\colon S \hookrightarrow X \times Y$, written $r \leq s$, when there is a monomorphism $m\colon R \hookrightarrow S$ such that $s \circ m = r$. One may form the category $\mathbf{Rel}(\mathsf{C})$ with objects those of C and arrows $X \rightsquigarrow Y$ the equivalence classes of relations between X and Y. The identity on X is the diagonal $\langle \mathrm{id}_X, \mathrm{id}_X\rangle\colon X \hookrightarrow X \times X$, and composition of relations $r = \langle r_X, r_Y\rangle\colon R \hookrightarrow X \times Y$ and $s = \langle s_Y, s_Z\rangle\colon S \hookrightarrow Y \times Z$, written $s \cdot r\colon X \rightsquigarrow Z$, is constructed as in Figure 1a: by considering the pullback $R \times_Y S \to R \times S$ of $R \xrightarrow{r_Y} Y \xleftarrow{s_Y} S$, and taking the monomorphism in the factorization of $R \times_Y S \to R \times S \xrightarrow{r_X \times s_Z} X \times Z$.



(a) Composition in $\mathbf{Rel}(\mathsf{C})$.



(b) A near pullback.

Figure 1 Diagrams involved in the definition of $\mathbf{Rel}(\mathsf{C})$.

The graph functor $\mathbf{Graph}_{\mathsf{C}}\colon \mathsf{C} \to \mathbf{Rel}(\mathsf{C})$ sends objects to themselves and an arrow $f\colon X \to Y$ to the relation $\langle \mathrm{id}_X, f\rangle\colon X \hookrightarrow X \times Y$. There is also a contravariant transpose functor $-^{\dagger}\colon \mathbf{Rel}(\mathsf{C}) \to \mathbf{Rel}(\mathsf{C})$ that sends objects on themselves and a relation $\langle r_X, r_Y\rangle\colon R \hookrightarrow X \times Y$ to $\langle r_Y, r_X\rangle\colon R \hookrightarrow Y \times X$. All in all, a relation $\langle f, g\rangle\colon R \hookrightarrow X \times Y$ can also be written as the composite $\mathbf{Graph}_{\mathsf{C}} g \cdot (\mathbf{Graph}_{\mathsf{C}} f)^{\dagger}$: we will often omit **Graph** and write directly $g \cdot f^{\dagger}$.

If $\mathbf{Kl}(\mathsf{S})$ is a wide[2] subcategory of $\mathbf{Rel}(\mathsf{C})$ (with $\mathbf{Graph}_{\mathsf{C}}$ being the left adjoint in the Kleisli adjunction), an extension to $\mathbf{Kl}(\mathsf{S})$ can be constructed by first finding a monotone extension to $\mathbf{Rel}(\mathsf{C})$ and then restricting this extension to $\mathbf{Kl}(\mathsf{S})$. This is a very useful technique because such monotone extensions to $\mathbf{Rel}(\mathsf{C})$, called *relational extensions*, have a very nice characterization in terms or near pullbacks: a square is a near pullback when its limiting morphism into the corresponding pullback is a regular epimorphism – as in Figure 1b. A functor between regular categories is said to be *nearly cartesian* when it sends pullbacks

[1] a class of arrows is stable under pullbacks when for every pullback square $a \circ b = c \circ d$, if a is in the class then so is d

[2] a wide subcategory is a subcategory that contains all objects of the bigger category

on near pullbacks and preserves[3] regular epimorphisms, or equivalently when it preserves near pullbacks, while a natural transformation $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ between nearly cartesian functors is *nearly cartesian* as well when its naturality squares $\alpha \circ \mathsf{F}f = \mathsf{G}f \circ \alpha$ are near pullbacks.

▶ **Theorem 7** ([19, Theorem 6])**.** *Let* $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ *be a functor between regular categories.* F *has a relational extension, i.e. an order-preserving functor* $\mathbf{Rel}(\mathsf{F})\colon \mathbf{Rel}(\mathsf{C}) \to \mathbf{Rel}(\mathsf{D})$ *such that* $\mathbf{Rel}(\mathsf{F})\,\mathbf{Graph}_{\mathsf{C}} = \mathbf{Graph}_{\mathsf{D}}\mathsf{F}$*, if and only if* F *is nearly cartesian. In that case there is only one possible such* $\mathbf{Rel}(\mathsf{F})$*, given by* $\mathbf{Rel}(\mathsf{F})\left(g \cdot f^\dagger\right) = (\mathsf{F}g) \cdot (\mathsf{F}f)^\dagger$*.*

Let $\mathsf{F}, \mathsf{G}\colon \mathsf{C} \rightrightarrows \mathsf{D}$ *be two such functors and* $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ *be a natural transformation between them. Then* $\mathbf{Rel}(\alpha)$*, given by* $\mathbf{Rel}(\alpha)\,\mathbf{Graph}_{\mathsf{D}} = \mathbf{Graph}_{\mathsf{D}}\alpha$*, is a natural transformation* $\mathbf{Rel}(\mathsf{F}) \Rightarrow \mathbf{Rel}(\mathsf{G})$*, called the relational extension of* α*, if and only if* α *is nearly cartesian.*

Just like we omit to write **Graph**, we will often omit to write the functor **Rel**, and instead write $\mathsf{F}(g \cdot f^\dagger)$ for $\mathbf{Rel}(\mathsf{F})\,(g \cdot f^\dagger) = \mathsf{F}g \cdot (\mathsf{F}f)^\dagger$ and α for $\mathbf{Rel}(\alpha)$.

▶ **Example 8** (monotone extensions)**. Set** is a regular category whose regular epimorphisms are the surjections and such that $\mathbf{Rel}(\mathbf{Set}) = \mathbf{Rel} \cong \mathbf{Kl}(\mathbf{P})$. The endofunctors and the multiplications of the monads **P** and **D** are nearly cartesian, hence **P** and **D** have monotone weak extensions to $\mathbf{Kl}(\mathbf{P})$: this is how the weak distributive laws $\mathbf{PP} \Rightarrow \mathbf{PP}$ and $\mathbf{DP} \Rightarrow \mathbf{PD}$ of Theorem 4 were constructed [14, 18]. **KHaus** is also a regular category: its regular epimorphisms are the surjective continuous functions and $\mathbf{Rel}(\mathbf{KHaus})$ is the category of compact Hausdorff spaces and closed relations, i.e. closed subsets of $X \times Y$, while $\mathbf{Kl}(\mathbf{V})$ is the category of compact Hausdorff spaces and continuous relations, i.e. closed relations $r\colon X \rightsquigarrow Y$ such that $r^{-1}[u]$ is open in X for every open u of Y. The endofunctor and multiplication of **V** are nearly cartesian hence they have a relational extension, which restricts to continuous relations: this yields a monotone weak extension of **V** to $\mathbf{Kl}(\mathbf{V})$, and this is how the weak distributive law $\mathbf{VV} \Rightarrow \mathbf{VV}$ of Theorem 4 was constructed [19].

2.3 Weak Liftings

Let T be a monad over a category C. Its category of algebras $\mathbf{EM}(\mathsf{T})$ is the category whose objects are pairs (A, a) of a C-object A and a C-arrow $a\colon \mathsf{T}A \to A$, and whose arrows $(A, a) \to (B, b)$ are those C-arrows $A \to B$ such that $f \circ a = b \circ \mathsf{T}f$ – the identity morphism is the one with the identity as its underlying C-arrow, and composition of morphisms is done by composing the underlying C-arrows. Just like for $\mathbf{Kl}(\mathsf{T})$, there is an adjunction $\mathbf{F}^\mathsf{T}\colon \mathsf{C} \leftrightarrows \mathbf{EM}(\mathsf{T})\colon \mathbf{U}^\mathsf{T}$ such that $\mathbf{U}^\mathsf{T}\mathbf{F}^\mathsf{T} = \mathsf{T}$: the left adjoint $\mathbf{F}^\mathsf{T}$ sends an algebra (A, a) to its carrier object A and a morphism $(A, a) \to (B, b)$ to the underlying arrow $A \to B$, while $\mathbf{U}^\mathsf{T}$ sends an object X to the free T-algebra on X, given by the pair $\left(\mathsf{T}X, \mu^\mathsf{T}_X\right)$, and an arrow $f\colon X \to Y$ to the morphism $\left(\mathsf{T}X, \mu^\mathsf{T}_X\right) \to \left(\mathsf{T}Y, \mu^\mathsf{T}_Y\right)$ with underlying C-arrow $\mathsf{T}f\colon \mathsf{T}X \to \mathsf{T}Y$. There is finally a natural transformation $\varepsilon^\mathsf{T}\colon \mathbf{F}^\mathsf{T}\mathbf{U}^\mathsf{T} \to \mathsf{Id}$ (in $\mathbf{EM}(\mathsf{T})$) given by $\mathbf{U}^\mathsf{T}\varepsilon^\mathsf{T}_{(A,a)} = a\colon \mathsf{T}A \to A$.

▶ **Definition 9** (weak liftings [3, Definitions 4.1 and 4.2])**.** *A* weak lifting *of an endofunctor* $\mathsf{S}\colon \mathsf{C} \to \mathsf{C}$ *to* $\mathbf{EM}(\mathsf{T})$ *is the data of an endofunctor* $\overline{\mathsf{S}}\colon \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T})$ *along with natural transformations* $\pi^\mathsf{T}_\mathsf{S}\colon \mathsf{S}\mathbf{U}^\mathsf{T} \Rightarrow \mathbf{U}^\mathsf{T}\overline{\mathsf{S}}$ *and* $\iota^\mathsf{T}_\mathsf{S}\colon \mathbf{U}^\mathsf{T}\overline{\mathsf{S}} \Rightarrow \mathsf{S}\mathbf{U}^\mathsf{T}$ *– also written* π_S *and* ι_S *when not ambiguous – such that* $\pi^\mathsf{T}_\mathsf{S} \circ \iota^\mathsf{T}_\mathsf{S} = \mathrm{id}$*. The composite* $\iota^\mathsf{T}_\mathsf{S} \circ \pi^\mathsf{T}_\mathsf{S}$ *is then written* $\kappa^\mathsf{T}_\mathsf{S}$*.*

[3] throughout this work we will say that a functor preserves a class of diagrams whenever it sends any diagram in that class to a diagram in that class

Let $\alpha\colon \mathsf{S} \Rightarrow \mathsf{R}$ *be a natural transformation between two* C*-endofunctors with weak liftings* $(\overline{\mathsf{S}}, \pi_{\mathsf{S}}, \iota_{\mathsf{S}})$ *and* $(\overline{\mathsf{R}}, \pi_{\mathsf{R}}, \iota_{\mathsf{R}})$*. If a natural transformation* $\overline{\alpha}\colon \overline{\mathsf{S}} \Rightarrow \overline{\mathsf{R}}$ *has Diagram* (π)*, Diagram* (ι) *or both Diagrams* (π) *and* (ι) *below commute, it is respectively a* weak π-*,* weak ι-*or a* weak lifting *thereof.*

$$
\begin{array}{ccc}
\mathsf{S}\mathbf{U}^{\mathsf{T}} & \xrightarrow{\pi^{\mathsf{T}}_{\mathsf{S}}} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} \\
{\scriptstyle \alpha\mathbf{U}^{\mathsf{T}}}\downarrow & (\pi) & \downarrow{\scriptstyle \mathbf{U}^{\mathsf{T}}\overline{\alpha}} \\
\mathsf{R}\mathbf{U}^{\mathsf{T}} & \xrightarrow[\pi^{\mathsf{T}}_{\mathsf{R}}]{} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow{\iota^{\mathsf{T}}_{\mathsf{S}}} & \mathsf{S}\mathbf{U}^{\mathsf{T}} \\
{\scriptstyle \mathbf{U}^{\mathsf{T}}\overline{\alpha}}\downarrow & (\iota) & \downarrow{\scriptstyle \alpha\mathbf{U}^{\mathsf{T}}} \\
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}} & \xrightarrow[\iota^{\mathsf{T}}_{\mathsf{R}}]{} & \mathsf{R}\mathbf{U}^{\mathsf{T}}
\end{array}
$$

As discussed in [3], weak π- and weak ι-liftings are necessarily unique and given by $\mathbf{U}^{\mathsf{T}}\overline{\alpha} = \pi^{\mathsf{T}}_{\mathsf{R}} \circ \alpha \circ \iota^{\mathsf{T}}_{\mathsf{S}}$ if they exist. Their existence is moreover fully characterized:

▶ **Theorem 10** ([3, Proposition 4.3 and Theorem 4.4])**.** *Suppose idempotents split in* C*. Then idempotents also split in* $\mathbf{EM}(\mathsf{T})$*, and having a weak lifting of* $\mathsf{S}\colon \mathsf{C} \to \mathsf{C}$ *to* $\mathbf{EM}(\mathsf{T})$ *is equivalent to having a law* $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$ *that has Diagram* (μ^+) *commute.*

Fix now liftings $\overline{\mathsf{S}}$ *and* $\overline{\mathsf{R}}$ *of* S *and* R *given by laws* $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$ *and* $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$*. Then,* $\alpha\colon \mathsf{S} \Rightarrow \mathsf{R}$ *respectively has a weak* π*-, weak* ι*- or weak lifting if and only if it makes Diagram (7), Diagram (8) or both Diagrams (7) and (8) below commute. The latter case holds equivalently if and only if* $\sigma \circ \mathsf{T}\alpha = \alpha\mathsf{T} \circ \rho$*.*

$$
\begin{array}{ccccccccc}
\mathsf{TS} & & & & \xrightarrow{\quad\rho\quad} & & & & \mathsf{ST} \\
{\scriptstyle \mathsf{T}\eta^{\mathsf{T}}\mathsf{S}}\downarrow & & & & (7) & & & & \downarrow{\scriptstyle \alpha\mathsf{T}} \\
\mathsf{TTS} & \xrightarrow{\mathsf{T}\rho} & \mathsf{TST} & \xrightarrow{\mathsf{T}\alpha\mathsf{T}} & \mathsf{TRT} & \xrightarrow{\sigma\mathsf{T}} & \mathsf{RTT} & \xrightarrow{\mathsf{R}\mu^{\mathsf{T}}} & \mathsf{RT} \\
{\scriptstyle \eta^{\mathsf{T}}\mathsf{TS}}\uparrow & & & & (8) & & & & \uparrow{\scriptstyle \sigma} \\
\mathsf{TS} & & & & \xrightarrow[\quad\mathsf{T}\alpha\quad]{} & & & & \mathsf{TR}
\end{array}
$$

This correspondence is moreover compositional: the composition of the weak (resp. weak π*-, weak* ι*-) liftings of two functors is a weak (resp. weak* π*-, weak* ι*-) lifting of their composite.*

In [14], Garner instantiates Theorem 10 to give another presentation of weak distributive laws: if idempotents split in C, a weak distributive law $\mathsf{TS} \Rightarrow \mathsf{ST}$ is equivalently given by a weak lifting of $(\mathsf{S}, \eta^{\mathsf{S}}, \mu^{\mathsf{S}})$ to $\mathbf{EM}(\mathsf{T})$, i.e. by weak liftings of S, η^{S} and μ^{S}, respectively coming from Diagrams (μ^-), (η^+), and (μ^+). That these two natural transformations weakly lift respectively means that Diagrams ($\pi \circ \eta$) and ($\iota \circ \eta$) and Diagrams ($\pi \circ \mu$) and ($\iota \circ \mu$) below commute. When Diagram (η^-) also commutes, ρ is a (strict) distributive law and this weak lifting is a (strict) lifting: π_{S} and ι_{S} are both the identity.

$$
\begin{array}{ccc}
 & \mathbf{U}^{\mathsf{T}} & \\
{\scriptstyle \eta^{\mathsf{S}}\mathbf{U}^{\mathsf{T}}}\swarrow & (\pi \circ \eta) & \searrow{\scriptstyle \mathbf{U}^{\mathsf{T}}\eta^{\overline{\mathsf{S}}}} \\
\mathsf{S}\mathbf{U}^{\mathsf{T}} & \xrightarrow[\pi^{\mathsf{T}}_{\mathsf{S}}]{} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{SS}\mathbf{U}^{\mathsf{T}} & \xrightarrow{\mathsf{S}\pi^{\mathsf{T}}_{\mathsf{S}}} & \mathsf{S}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow{\pi^{\mathsf{T}}_{\mathsf{S}}\overline{\mathsf{S}}} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\overline{\mathsf{S}} \\
{\scriptstyle \mu^{\mathsf{S}}\mathbf{U}^{\mathsf{T}}}\downarrow & & (\pi \circ \mu) & & \downarrow{\scriptstyle \mathbf{U}^{\mathsf{T}}\mu^{\overline{\mathsf{S}}}} \\
\mathsf{S}\mathbf{U}^{\mathsf{T}} & & \xrightarrow[\pi^{\mathsf{T}}_{\mathsf{S}}]{} & & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}
\end{array}
$$

$$
\begin{array}{ccc}
 & \mathbf{U}^{\mathsf{T}} & \\
{\scriptstyle \mathbf{U}^{\mathsf{T}}\eta^{\overline{\mathsf{S}}}}\swarrow & (\iota \circ \eta) & \searrow{\scriptstyle \eta^{\mathsf{S}}\mathbf{U}^{\mathsf{T}}} \\
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow[\iota^{\mathsf{T}}_{\mathsf{S}}]{} & \mathsf{S}\mathbf{U}^{\mathsf{T}}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\overline{\mathsf{S}} & \xrightarrow{\iota^{\mathsf{T}}_{\mathsf{S}}\overline{\mathsf{S}}} & \mathsf{S}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow{\mathsf{S}\iota^{\mathsf{T}}_{\mathsf{S}}} & \mathsf{SS}\mathbf{U}^{\mathsf{T}} \\
{\scriptstyle \mathbf{U}^{\mathsf{T}}\mu^{\overline{\mathsf{S}}}}\downarrow & & (\iota \circ \mu) & & \downarrow{\scriptstyle \mu^{\mathsf{S}}\mathbf{U}^{\mathsf{T}}} \\
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & & \xrightarrow[\iota^{\mathsf{T}}_{\mathsf{S}}]{} & & \mathsf{S}\mathbf{U}^{\mathsf{T}}
\end{array}
$$

The weak composite monad $\mathsf{S}\bullet\mathsf{T}$ corresponding to a weak distributive law $\mathsf{TS} \Rightarrow \mathsf{ST}$ can be retrieved from the weak lifting of S to $\mathbf{EM}(\mathsf{T})$ by setting $\mathsf{S} \bullet \mathsf{T} = \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\mathbf{F}^{\mathsf{T}}$, $\eta^{\mathsf{S}\bullet\mathsf{T}} = \mathbf{U}^{\mathsf{T}}\eta^{\overline{\mathsf{S}}}\mathbf{F}^{\mathsf{T}} \circ \eta^{\mathsf{T}}$ and $\mu^{\mathsf{S}\bullet\mathsf{T}} = \mathbf{U}^{\mathsf{T}}\mu^{\overline{\mathsf{S}}}\mathbf{F}^{\mathsf{T}} \circ \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\varepsilon^{\mathsf{T}}\overline{\mathsf{S}}\mathbf{F}^{\mathsf{T}}$. $\kappa^{\mathsf{T}}_{\mathsf{S}}\mathbf{F}^{\mathsf{T}}$ is moreover the idempotent $\mu^{\mathsf{ST}} \circ \eta^{\mathsf{ST}}\mathsf{ST}\colon \mathsf{ST} \Rightarrow \mathsf{ST}$, and its splitting is thus given by $\pi^{\mathsf{T}}_{\mathsf{S}}\mathbf{F}^{\mathsf{T}}\colon \mathsf{ST} \Rightarrow \mathsf{S} \bullet \mathsf{T}$ and $\iota^{\mathsf{T}}_{\mathsf{S}}\mathbf{F}^{\mathsf{T}}\colon \mathsf{S} \bullet \mathsf{T} \Rightarrow \mathsf{ST}$.

▶ **Example 11** (weak liftings). All idempotents split in **Set** (they factor through their image). The algebras of $\mathbf{P}$ are the *complete join-semilattices* (we write $\mathbf{EM}(\mathbf{P}) \cong \mathbf{JSL}$): the weak lifting corresponding to the law $\mathbf{PP} \Rightarrow \mathbf{PP}$ is the monad of *subsets closed under non-empty joins* [19]. The algebras of $\mathbf{D}$ are the *barycentric algebras*, also called *convex spaces* (we write $\mathbf{EM}(\mathbf{D}) \cong \mathbf{Conv}$): the weak lifting corresponding to the law $\mathbf{DP} \Rightarrow \mathbf{PD}$ is the monad of *convex-closed subsets* [18]. Finally, there is also a weak distributive law $\boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$. Assuming the axiom of choice, the algebras of $\boldsymbol{\beta}$ are the compact Hausdorff spaces [24] (we write $\mathbf{EM}(\boldsymbol{\beta}) \cong \mathbf{KHaus}$): the corresponding weak lifting is the Vietoris monad $\mathbf{V}$. In that case $\pi_{\mathbf{P}}$ computes the topological closure of a subset, while $\iota_{\mathbf{P}}$ embeds the set of closed sets into the set of all subsets.

3 Weakly Lifting Weak Distributive Laws

With Theorem 11 and the fact that $\mathbf{V}$ is a weak lifting of $\mathbf{P}$ to **KHaus** in mind, we may now notice that not only are the two laws $\mathbf{PP} \Rightarrow \mathbf{PP}$ (1) and $\mathbf{VV} \Rightarrow \mathbf{VV}$ (2) very similar, but the second one seems to be some kind of weak lifting to $\mathbf{KHaus} \cong \mathbf{EM}(\boldsymbol{\beta})$ of the first one:

▶ **Lemma 12.** $\mathbf{U}^{\boldsymbol{\beta}}\lambda^{\mathbf{V}/\mathbf{V}} = \pi_{\mathbf{P}}\mathbf{V} \circ \mathbf{P}\pi_{\mathbf{P}} \circ \lambda^{\mathbf{P}/\mathbf{P}}\mathbf{U}^{\boldsymbol{\beta}} \circ \mathbf{P}\iota_{\mathbf{P}} \circ \iota_{\mathbf{P}}\mathbf{V}$.

Proof. Recall from [14] that, if X is a compact Hausdorff space that we see as a $\boldsymbol{\beta}$-algebra, the X-component of $\pi_{\mathbf{P}}$ is the function $\mathbf{P}X \to \mathbf{V}X$ that takes a subset of a compact Hausdorff space and outputs its closure, while the X-component of $\iota_{\mathbf{P}}$ is the function $\mathbf{V}X \to \mathbf{P}X$ that embeds closed subsets in the set of all subsets. Hence for C a closed set of closed sets,

$$\begin{aligned}\left(\mathbf{P}\pi_{\mathbf{P}} \circ \lambda^{\mathbf{P}/\mathbf{P}}\mathbf{U}^{\boldsymbol{\beta}} \circ \mathbf{P}\iota_{\mathbf{P}} \circ \iota_{\mathbf{P}}\mathbf{V}\right)_X (C) &= \left\{\overline{e} \;\middle|\; e \subseteq \bigcup C \text{ and } \forall c \in C, c \cap e \neq \varnothing\right\} \\ &= \left\{c \in \mathbf{V}X \;\middle|\; c \subseteq \bigcup C \text{ and } \forall c' \in C, c \cap c' \neq \varnothing\right\} \\ &= \left(\iota_{\mathbf{P}}\mathbf{V} \circ \mathbf{U}^{\boldsymbol{\beta}}\lambda^{\mathbf{V}/\mathbf{V}}\right)_X (C)\end{aligned}$$

(where $\overline{e}$ denotes the closure of e). Indeed if $e \subseteq \bigcup C$ then $\overline{e} \subseteq \bigcup C$ because $\bigcup C = \mu^{\mathbf{V}}(C)$ is closed, and if $e \cap c \neq \varnothing$ then $\overline{e} \cap c \neq \varnothing$.

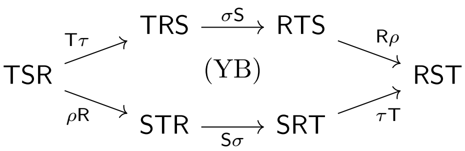
Because $\pi_{\mathbf{P}} \circ \iota_{\mathbf{P}} = \mathrm{id}$, $\mathbf{U}^{\boldsymbol{\beta}}\lambda^{\mathbf{V}/\mathbf{V}} = \pi_{\mathbf{P}}\mathbf{V} \circ \mathbf{P}\pi_{\mathbf{P}} \circ \lambda^{\mathbf{P}/\mathbf{P}}\mathbf{U}^{\boldsymbol{\beta}} \circ \mathbf{P}\iota_{\mathbf{P}} \circ \iota_{\mathbf{P}}\mathbf{V}$. ◀

Is this just a coincidence, or is this an instance of Theorem 10? And in the latter case, can the weak distributivity of $\lambda^{\mathbf{V}/\mathbf{V}}$ be automatically derived from that of $\lambda^{\mathbf{P}/\mathbf{P}}$, and does this law $\mathbf{PP} \Rightarrow \mathbf{PP}$ also weakly lift to laws on other categories of algebras where the powerset monad weakly lifts, say $\mathbf{EM}(\mathbf{P})$ and $\mathbf{EM}(\mathbf{D})$?

In this section we thus consider three monads $(\mathsf{T}, \eta^{\mathsf{T}}, \mu^{\mathsf{T}})$, $(\mathsf{S}, \eta^{\mathsf{S}}, \mu^{\mathsf{S}})$ and $(\mathsf{R}, \eta^{\mathsf{R}}, \mu^{\mathsf{R}})$ on a category C, and three weak distributive laws $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ (a mnemonic for which is which is that ρ does not involve R, σ does not involve S and τ does not involve T). We assume that idempotents split in C, so that the corresponding weak composite and weak liftings all exist.

3.1 The Yang-Baxter Equation for Weak Distributive Laws

The standard way to lift (strict) distributive laws is to use the so-called Yang-Baxter equation. The Yang-Baxter equation for the three laws ρ, σ and τ holds when diagram (YB) below commutes.



If these laws are strict distributive laws, then it is well-known since [10] that the Yang-Baxter equation is enough to show that $\mathsf{R}\rho \circ \sigma\mathsf{S} \colon \mathsf{TRS} \Rightarrow \mathsf{RST}$ is a distributive law of T over RS, and that the distributive law $\tau \colon \mathsf{SR} \Rightarrow \mathsf{RS}$ lifts to a distributive law $\overline{\tau} \colon \overline{\mathsf{S}}\overline{\mathsf{R}} \Rightarrow \overline{\mathsf{R}}\overline{\mathsf{S}}$ in $\mathbf{EM}(\mathsf{T})$. The composition of weak distributive laws using the Yang-Baxter equation was investigated in [17]. A notable result is the following:

▶ **Proposition 13** ([17, Theorem 4.3]). *If the weak distributive laws $\rho \colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma \colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau \colon \mathsf{SR} \Rightarrow \mathsf{RS}$ have Diagram (YB) commute, then $\pi^{\mathsf{S}}_{\mathsf{R}}\mathbf{F}^{\mathsf{S}}\mathsf{T} \circ \mathsf{R}\rho \circ \sigma\mathsf{S} \circ \mathsf{T}\iota^{\mathsf{S}}_{\mathsf{R}}\mathbf{F}^{\mathsf{S}}$ is a weak distributive law $\mathsf{T}(\mathsf{R} \bullet \mathsf{S}) \Rightarrow (\mathsf{R} \bullet \mathsf{S})\mathsf{T}$.*

The Yang-Baxter equation thus allows for weakly lifting $\mathsf{R} \bullet \mathsf{S}$ to $\mathbf{EM}(\mathsf{T})$. More importantly for our purpose, we show it also allows for weakly lifting the weak distributive law $\mathsf{SR} \Rightarrow \mathsf{RS}$:

▶ **Theorem 14.** *Weak distributive laws $\rho \colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma \colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau \colon \mathsf{SR} \Rightarrow \mathsf{RS}$ satisfy the Yang-Baxter equation if and only if $\tau \colon \mathsf{SR} \Rightarrow \mathsf{RS}$ weakly lifts to $\mathbf{EM}(\mathsf{T})$, i.e. if there is a natural transformation $\overline{\tau} \colon \overline{\mathsf{S}}\overline{\mathsf{R}} \Rightarrow \overline{\mathsf{R}}\overline{\mathsf{S}}$ such that Diagrams (14) and (15) commute.*

If this holds, $\overline{\tau}$ is a weak distributive law, and the weak composite $\overline{\mathsf{R}} \bullet \overline{\mathsf{S}}$ and the weak lifting $\overline{\mathsf{R} \bullet \mathsf{S}}$ (recall that $\mathsf{R} \bullet \mathsf{S}$ weakly lifts to $\mathbf{EM}(\mathsf{T})$ by Theorem 13) can be chosen to be equal (as monads).

$$
\begin{array}{ccccc}
\mathsf{SR}\mathbf{U}^{\mathsf{T}} & \xrightarrow{\mathsf{S}\pi_{\mathsf{R}}} & \mathsf{S}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}} & \xrightarrow{\pi_{\mathsf{S}}\overline{\mathsf{R}}} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\overline{\mathsf{R}} \\
{\scriptstyle \tau\mathbf{U}^{\mathsf{T}}}\downarrow & & (14) & & \downarrow{\scriptstyle \mathbf{U}^{\mathsf{T}}\overline{\tau}} \\
\mathsf{RS}\mathbf{U}^{\mathsf{T}} & \xrightarrow[\mathsf{R}\pi_{\mathsf{S}}]{} & \mathsf{R}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow[\pi_{\mathsf{R}}\overline{\mathsf{S}}]{} & \mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}}\overline{\mathsf{S}}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}}\overline{\mathsf{R}} & \xrightarrow{\iota_{\mathsf{S}}\overline{\mathsf{R}}} & \mathsf{S}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}} & \xrightarrow{\mathsf{S}\iota_{\mathsf{R}}} & \mathsf{SR}\mathbf{U}^{\mathsf{T}} \\
{\scriptstyle \mathbf{U}^{\mathsf{T}}\overline{\tau}}\downarrow & & (15) & & \downarrow{\scriptstyle \tau\mathbf{U}^{\mathsf{T}}} \\
\mathbf{U}^{\mathsf{T}}\overline{\mathsf{R}}\overline{\mathsf{S}} & \xrightarrow[\iota_{\mathsf{R}}\overline{\mathsf{S}}]{} & \mathsf{R}\mathbf{U}^{\mathsf{T}}\overline{\mathsf{S}} & \xrightarrow[\mathsf{R}\iota_{\mathsf{S}}]{} & \mathsf{RS}\mathbf{U}^{\mathsf{T}}
\end{array}
$$

When Theorem 14 holds it immediately follows that $\mathbf{U}\overline{\tau} = \pi_{\mathsf{R}}\overline{\mathsf{S}} \circ \mathsf{R}\pi_{\mathsf{S}} \circ \tau\mathbf{U}^{\mathsf{T}} \circ \mathsf{S}\iota_{\mathsf{R}} \circ \iota_{\mathsf{S}}\overline{\mathsf{R}}$, which is exactly the result we got in Theorem 12 for $\overline{\tau} = \lambda^{\mathbf{V}/\mathbf{V}} \colon \mathbf{VV} \Rightarrow \mathbf{VV}$ and $\tau = \lambda^{\mathbf{P}/\mathbf{P}} \colon \mathbf{PP} \Rightarrow \mathbf{PP}$. It would thus be a reasonable conjecture that $\lambda^{\boldsymbol{\beta}/\mathbf{P}} \colon \boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$, $\lambda^{\boldsymbol{\beta}/\mathbf{P}} \colon \boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$ and $\lambda^{\mathbf{P}/\mathbf{P}} \colon \mathbf{PP} \Rightarrow \mathbf{PP}$ satisfy the Yang-Baxter equation, from which we would immediately retrieve the weak distributivity of $\lambda^{\mathbf{V}/\mathbf{V}} \colon \mathbf{VV} \Rightarrow \mathbf{VV}$ but also learn that the weak composite $\mathbf{V} \bullet \mathbf{V}$ is a weak lifting of $\mathbf{P} \bullet \mathbf{P}$. Unfortunately, the Yang-Baxter equation does not hold in that case. A rather simple way to see this is to notice that $\lambda^{\mathbf{V}/\mathbf{V}}$ does not make Diagram (15) commute, i.e. it is not a ι-lifting of $\lambda^{\mathbf{P}/\mathbf{P}}$. More generally, we show the following no-go theorem for weak ι-liftings:

▶ **Proposition 15.** *Let $\lambda^{\mathsf{T}/\mathbf{P}} \colon \mathsf{T}\mathbf{P} \Rightarrow \mathbf{P}\mathsf{T}$ be a weak distributive law with corresponding weak lifting $\overline{\mathbf{P}}$, and write $\overline{\mathbf{P}}\overline{\mathbf{P}}A = \left(\mathbf{P}\iota_{\mathbf{P}} \circ \iota_{\mathbf{P}}\overline{\mathbf{P}}\right)\left[\mathbf{U}^{\mathsf{T}}\overline{\mathbf{P}}\overline{\mathbf{P}}(A,a)\right]$ when (A,a) is a T-algebra. If there is an (A,a) such that $\{A\} \in \overline{\mathbf{P}}\overline{\mathbf{P}}A$ and $\mathbf{P}_*A \notin \overline{\mathbf{P}}\overline{\mathbf{P}}A$, then $\lambda^{\mathbf{P}/\mathbf{P}} \colon \mathbf{PP} \Rightarrow \mathbf{PP}$ does not have a weak ι-lifting to $\mathbf{EM}(\mathsf{T})$.*

▶ **Corollary 16.** *There is no weak ι-lifting (let alone weak liftings) of $\lambda^{\mathbf{P}/\mathbf{P}}$ to* **KHaus**, **JSL** *or* **Conv**.

Proof sketch.

1. Given a compact Hausdorff space given as a $\boldsymbol{\beta}$-algebra (A,a), $\overline{\mathbf{P}}\overline{\mathbf{P}}A$ is the set of all closed sets (in the Vietoris topology) of closed sets of (A,a). $\{A\}$ is such a closed set of closed sets (all singletons and the whole set are always closed in a compact Hausdorff space) but $\mathbf{P}_*A$ is not in general because it contains all non-empty sets, in particular non-closed sets if there are any (which is the case for the unit interval, for instance). ◀

▶ **Remark 17.** It is not hard to show that the Yang-Baxter equation holding for $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ is also equivalent to $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$ having an extension $\underline{\rho}\colon \underline{\mathsf{TS}} \Rightarrow \underline{\mathsf{ST}}$ to $\mathbf{Kl}(\mathsf{R})$. In the case of $\lambda^{\boldsymbol{\beta}/\mathbf{P}}\colon \boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$, $\underline{\boldsymbol{\beta}}$ is in fact a lax monad on $\mathbf{Rel}$ whose algebras are the topological spaces [1]. Unfortunately, we have just shown that the Yang-Baxter equation does not hold for $\lambda^{\boldsymbol{\beta}/\mathbf{P}}\colon \boldsymbol{\beta}\mathbf{P} \Rightarrow \mathbf{P}\boldsymbol{\beta}$, $\lambda^{\mathbf{P}/\mathbf{P}}\colon \mathbf{PP} \Rightarrow \mathbf{PP}$ and $\lambda^{\mathbf{P}/\mathbf{P}}\colon \mathbf{PP} \Rightarrow \mathbf{PP}$, and so do not get a way to weakly lift $\underline{\mathbf{P}}$ to topological spaces for free.

Theorem 14 does have some concrete instances: in [17], Goy gives a substantial number of examples of triples of weak distributive laws for which the Yang-Baxter equation holds, although these examples all involve at least one strictly distributive law out of the three.

3.2 The π-Yang-Baxter Equation

We still do not have an explanation for why $\lambda^{\mathbf{V}/\mathbf{V}}$ looks so much like $\lambda^{\mathbf{P}/\mathbf{P}}$. But $\lambda^{\mathbf{V}/\mathbf{V}}$ being a weak lifting of $\lambda^{\mathbf{P}/\mathbf{P}}$ is not a necessary condition for retrieving Theorem 12: in fact, $\lambda^{\mathbf{V}/\mathbf{V}}$ being only a weak ι- or π-lifting of $\lambda^{\mathbf{P}/\mathbf{P}}$ would be enough. We saw in Theorem 16 that the weak ι-lifting hypothesis was a dead-end: how about $\lambda^{\mathbf{V}/\mathbf{V}}$ being a weak π-lifting of $\lambda^{\mathbf{P}/\mathbf{P}}$? This turns out to be true, although the proof is of course more involved than that of the weaker Theorem 12.

▶ **Lemma 18.** *$\lambda^{\mathbf{V}/\mathbf{V}}\colon \mathbf{VV} \Rightarrow \mathbf{VV}$ is a weak π-lifting of $\lambda^{\mathbf{P}/\mathbf{P}}\colon \mathbf{PP} \Rightarrow \mathbf{PP}$.*

Theorem 14 adapts to weak π-liftings, hence we immediately retrieve as a consequence of Theorem 18 that $\lambda^{\mathbf{V}/\mathbf{V}}\colon \mathbf{VV} \Rightarrow \mathbf{VV}$ is a weak distributive law.

▶ **Proposition 19.** *Weak distributive laws $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ satisfy the π-Yang-Baxter equation, given by Diagram (π-YB), if and only if $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ weakly π-lifts to $\mathbf{EM}(\mathsf{T})$, i.e. if there is a natural transformation $\overline{\tau}\colon \overline{\mathsf{S}}\overline{\mathsf{R}} \Rightarrow \overline{\mathsf{R}}\overline{\mathsf{S}}$ such that Diagram (14) commutes.*

If this holds, $\overline{\tau}$ is a weak distributive law.

$$
\begin{array}{ccccccc}
\mathsf{TSR} & \xrightarrow{\mathsf{T}\tau} & \mathsf{TRS} & \xrightarrow{\sigma\mathsf{S}} & \mathsf{RTS} & \xrightarrow{\mathsf{R}\rho} & \mathsf{RST} \\
\downarrow{\scriptstyle \rho\mathsf{R}} & & & & & & \uparrow{\scriptstyle \mathsf{RS}\mu^{\mathsf{T}}} \\
\mathsf{STR} & & & (\pi\text{-YB}) & & & \mathsf{RSTT} \\
\downarrow{\scriptstyle \mathsf{S}\sigma} & & & & & & \uparrow{\scriptstyle \mathsf{R}\rho\mathsf{T}} \\
\mathsf{SRT} & \xrightarrow[\tau\mathsf{T}]{} & \mathsf{RST} & \xrightarrow[\eta^{\mathsf{T}}\mathsf{RST}]{} & \mathsf{TRST} & \xrightarrow[\sigma\mathsf{ST}]{} & \mathsf{RTST}
\end{array}
$$

We also retrieve that $\lambda^{\mathbf{V}/\mathbf{V}}$ is a monotone weak distributive law:

▶ **Proposition 20.** *Consider a monotone weak distributive law $\lambda^{\mathsf{S}/\mathbf{P}}\colon \mathsf{S}\mathbf{P} \Rightarrow \mathbf{P}\mathsf{S}$ in $\mathbf{Set}$ that has a weak π-lifting to $\mathbf{EM}(\mathsf{T})$. If the components of $\kappa_{\mathbf{P}}\colon \mathbf{P}\mathsf{U}^{\mathsf{T}} \Rightarrow \mathbf{P}\mathsf{U}^{\mathsf{T}}$ are monotone functions (they preserve inclusion of subsets) then $\overline{\lambda^{\mathsf{S}/\mathbf{P}}}\colon \overline{\mathsf{S}}\overline{\mathbf{P}} \Rightarrow \overline{\mathbf{P}}\overline{\mathsf{S}}$ is also a monotone weak distributive law.*

Of course the point of Theorems 19 and 20 is that they make it easier to exhibit weak distributive laws in categories of algebras. Still, working with Diagram (π-YB) may be quite tedious, as it involves up to four composed layers of functors. In fact in Theorem 18 we did not use this π-Yang-Baxter equation at all, instead we directly proved that $\lambda^{\mathbf{V}/\mathbf{V}}$ was a weak π-lifting because we were already able to take for granted that its components were morphisms of $\boldsymbol{\beta}$-algebras, i.e. continuous functions. Another problem with Theorem 19 is that even if we manage to disprove its prerequisites for some examples, we only get that there is no weak π-lifting of the weak distributive law, but we do not learn anything about other possible meaningful weak distributive laws in the category of algebras.

For all of these reasons we do not try to apply Theorem 19 to weakly π-lift $\lambda^{\mathbf{P}/\mathbf{P}}\colon \mathbf{PP} \Rightarrow \mathbf{PP}$ to $\mathbf{EM}(\mathbf{P})$ and $\mathbf{EM}(\mathbf{D})$, and immediately turn towards another approach in Section 4 instead: we try to weakly lift the conditions for the existence of monotone weak distributive laws (described in Section 2.2). This is a reasonable strategy because monotone laws are easier to reason about (all non-trivial weak distributive laws described in the literature are monotone) and are closer to being fully characterized, meaning we should hopefully be able to prove no-go theorems for monotone weak distributive laws. In fact we will prove that there is no such law $\overline{\mathbf{P}}\overline{\mathbf{P}} \Rightarrow \overline{\mathbf{P}}\overline{\mathbf{P}}$ in $\mathbf{EM}(\mathbf{P})$ or $\mathbf{EM}(\mathbf{D})$, so that by Theorem 20 $\lambda^{\mathbf{P}/\mathbf{P}}$ cannot weakly π-lift to $\mathbf{EM}(\mathbf{P})$ nor to $\mathbf{EM}(\mathbf{D})$.

4 Weakly Lifting Monotone Weak Distributive Laws

Let T be a monad on a regular category C. It is folklore that, under mild conditions, $\mathbf{EM}(\mathsf{T})$ is regular as well. For instance on **Set**, all finitary monads, and even all monads if the axiom of choice is assumed to be true, have regular categories of algebras [9, Theorems 3.5.4 and 4.3.5]. Here we will assume that T is a nearly cartesian functor, but the following result also holds for monads that preserve reflexive coequalizers.

▶ **Theorem 21** (categories of algebras are regular)**.** *Let $(\mathsf{T}, \eta^{\mathsf{T}}, \mu^{\mathsf{T}})$ be a monad on a regular category C such that T is nearly cartesian. Then $\mathbf{EM}(\mathsf{T})$ is regular and $\mathbf{U}^{\mathsf{T}}$ creates finite limits and near pullbacks (a square is a near pullback in $\mathbf{EM}(\mathsf{T})$ if and only if its image by $\mathbf{U}^{\mathsf{T}}$ is so in C).*

Consider weak distributive laws $\rho\colon \mathsf{TS} \Rightarrow \mathsf{ST}$, $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$ and $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ on C. Note that because C is a regular category, idempotents split in C by way of the factorization into regular epimorphisms followed by monomorphisms: in particular, all the weak composites and weak liftings corresponding to these weak distributive laws exist. When $\tau\colon \mathsf{SR} \Rightarrow \mathsf{RS}$ is a monotone weak distributive law thanks to the framework of [19], it is now natural to ask when there is also a monotone weak distributive law $\overline{\mathsf{S}}\overline{\mathsf{R}} \Rightarrow \overline{\mathsf{R}}\overline{\mathsf{S}}$ in $\mathbf{EM}(\mathsf{T})$ arising in the same way: it is for instance the case for $\mathsf{T} = \boldsymbol{\beta}$ and $\mathsf{S} = \mathsf{R} = \mathbf{P}$.

To apply the framework for monotone weak distributive laws of [19] to monads $\overline{\mathsf{S}}$ and $\overline{\mathsf{R}}$, we need to characterize $\mathbf{Kl}(\overline{\mathsf{R}})$ as a subcategory of relations – we do this in Section 4.1 – and then prove that $\overline{\mathsf{S}}$ and $\mu^{\overline{\mathsf{S}}}$ are nearly cartesian and investigate when the extension of $\overline{\mathsf{S}}$ to $\mathbf{Rel}(\mathbf{EM}(\mathsf{T}))$ restricts to $\mathbf{Kl}(\overline{\mathsf{R}})$ – we do this in Section 4.2. We finally apply our results in Section 4.3.

Because we strive to be as general as possible, in the following the assumptions that we use vary from result to result. In the propositions and theorems we thus recall every time all the assumptions that are necessary.

4.1 Kleisli Categories of Weakly Lifted Monads

Let us forget about regular categories and internal relations for an instant and first describe the Kleisli categories of a weakly lifted monad $\overline{\mathsf{R}}$ in terms of the Kleisli category of R itself.

▶ **Proposition 22.** *Let $\sigma\colon \mathsf{TR} \Rightarrow \mathsf{RT}$ be a weak distributive law in a category C where idempotents split, so that R has a weak lifting $\overline{\mathsf{R}}$ to $\mathbf{EM}(\mathsf{T})$ and T a weak extension $\underline{\mathsf{T}}$ to $\mathbf{Kl}(\mathsf{R})$. Then $\mathbf{Kl}(\overline{\mathsf{R}})$-arrows $(A, a) \nrightarrow (B, b)$ are in one-to-one correspondence with $\mathbf{Kl}(\mathsf{R})$-arrows $f\colon A \nrightarrow B$ such that $f \circ \mathbf{F}_{\mathsf{R}} a = \mathbf{F}_{\mathsf{R}} b \circ \underline{\mathsf{T}} f$.*

Assume now that the framework of [19] applies: $\mathbf{Kl}(\mathsf{R})$ is a wide subcategory of $\mathbf{Rel}(\mathsf{C})$ (and the left adjoint coincides with the graph functor $\mathbf{Graph}_{\mathsf{C}}\colon \mathsf{C} \to \mathbf{Rel}(\mathsf{C})$), T and μ^{T} are nearly cartesian and the weak extension of $\left(\mathsf{T}, \eta^{\mathsf{T}}, \mu^{\mathsf{T}}\right)$ to $\mathbf{Kl}(\mathsf{R})$ is the restriction of the relational extension of T and μ^{T} to $\mathbf{Rel}(\mathsf{C})$.

By Theorems 7 and 21, $\mathbf{U}^{\mathsf{T}}$ has a relational extension $\mathbf{Rel}\left(\mathbf{U}^{\mathsf{T}}\right)\colon \mathbf{Rel}(\mathbf{EM}(\mathsf{T})) \to \mathbf{Rel}(\mathsf{C})$, and by Theorem 22 $\mathbf{Kl}(\overline{\mathsf{R}})$-morphisms $(A, a) \multimap\!\!\!\to (B, b)$ correspond to C-relations $\psi\colon A \leftrightsquigarrow B$ that are in $\mathbf{Kl}(\mathsf{R})$ and such that $\psi \cdot a = b \cdot \mathsf{T}\psi$. $\mathbf{Kl}(\overline{\mathsf{R}})$ is thus itself a wide subcategory of $\mathbf{Rel}(\mathbf{EM}(\mathsf{T}))$:

▶ **Lemma 23.** *When the endofunctor T is nearly cartesian, a C-relation $\psi\colon A \leftrightsquigarrow B$ is the image of an $\mathbf{EM}(\mathsf{T})$-relation $(A, a) \leftrightsquigarrow (B, b)$ by the faithful functor $\mathbf{Rel}\left(\mathbf{U}^{\mathsf{T}}\right)\colon \mathbf{Rel}(\mathbf{EM}(\mathsf{T})) \to \mathbf{Rel}(\mathsf{C})$ if and only if $\psi \cdot a \geq b \cdot \mathsf{T}\psi$.*

We now describe in more concrete terms which $\mathbf{EM}(\mathsf{T})$-relations are arrows in $\mathbf{Kl}(\overline{\mathbf{P}})$. *Decomposable* T-algebra morphisms play a central role in this description:

▶ **Definition 24** (decomposable morphisms of algebra)**.** *Let T be a monad on a category C such that $\mathbf{EM}(\mathsf{T})$ is regular. A T-algebra morphism $f\colon X \to Y$ is called* decomposable *when the square $f \circ \varepsilon^{\mathsf{T}}_X = \varepsilon^{\mathsf{T}}_Y \circ \mathbf{F}^{\mathsf{T}}\mathbf{U}^{\mathsf{T}} f$ is a near pullback.*

Given a jointly monic span $\langle \psi_X, \psi_Y \rangle$ in $\mathbf{EM}(\mathsf{T})$, the corresponding relation $\psi = \psi_Y \cdot \psi_X^{\dagger}$ is called decomposable *when ψ_X is so.*

Before looking at examples, let us give two lemmas that will make working with decomposability of morphisms and relations easier.

▶ **Lemma 25.** *If $\mathbf{U}^{\mathsf{T}}$ creates near pullbacks, a T-morphism $f\colon (A, a) \to (B, b)$ is decomposable if and only if $\mathbf{U}^{\mathsf{T}} f \circ a = b \circ \mathsf{T}\mathbf{U}^{\mathsf{T}} f$ is a near pullback in C. In particular if T is a monad on $\mathbf{Set}$, this holds if and only if for every $x \in A$ and $u \in \mathsf{T}B$ such that $f(x) = b(u)$, there is some $t \in \mathsf{T}A$ such that $(\mathsf{T}f)(t) = u$ and $a(t) = x$.*

▶ **Lemma 26.** *Suppose $\mathbf{U}^{\mathsf{T}}$ creates and T preserves regular epimorphisms. If $g = h \circ e$ is decomposable and e is a regular epimorphism, then h is decomposable as well. In particular, if $f\colon R \to X$ is decomposable then for any $g\colon R \to Y$, $g \cdot f^{\dagger}$ is a decomposable relation.*

Decomposable morphisms have been studied in [11, Definition 3.1.1] in the setting of *monoidal topology*. There, these morphisms are called *open* as they generalize open maps between compact Hausdorff spaces, as the next example shows. We prefer the term "decomposable" here as we focus on other examples that feel more algebraic than topological:

▶ **Example 27** (decomposable morphisms and relations in categories of algebras over **Set**)**.** In $\mathbf{EM}(\boldsymbol{\beta}) \cong \mathbf{KHaus}$, a continuous map is decomposable if and only if it is open (it preserves open sets), and decomposable relations are the continuous ones, i.e. those relations $\psi\colon X \leftrightsquigarrow Y$ such that $\psi^{-1}[u]$ is open in X for every open subset u of Y.

In $\mathbf{EM}(\mathbf{P}) \cong \mathbf{JSL}$, $\psi\colon X \leftrightsquigarrow Y$ is decomposable if and only if for every family $(x_i)_{i \in I}$ of elements of X and every $y \in Y$ such that $\left(\bigvee_{i \in I} x_i, y\right) \in \psi$, there is a family $(y_i)_{i \in I}$ of elements of Y such that $(x_i, y_i) \in \psi$ for all $i \in I$ and $\bigvee_{i \in I} y_i = y$.

In $\mathbf{EM}(\mathbf{D}) \cong \mathbf{Conv}$, $\psi\colon X \leftrightsquigarrow Y$ is decomposable if and only if for every $x \in X$, every *disintegration* of x as a barycenter $x = \sum_{i=1}^n \lambda_i x_i$ and every $y \in Y$ such that $(x, y) \in \psi$, y disintegrates as a barycenter $y = \sum_{i=1}^n \lambda_i y_i$ such that $(x_i, y_i) \in \psi$ for all $i \in I$.

A more general example of decomposable morphism is the following:

▶ **Lemma 28.** *When U^T creates near pullbacks, μ^T is nearly cartesian if and only if every free algebra morphism $\mathsf{F}^\mathsf{T} f\colon \left(\mathsf{T}X, \mu^\mathsf{T}_X\right) \to \left(\mathsf{T}Y, \mu^\mathsf{T}_Y\right)$ with $f\colon X \to Y$ is decomposable.*

We are now able to state the main result of this section. We first state it in full generality (Theorem 29), but in practice we will be especially concerned with the case $\mathsf{R} = \mathbf{P}$ on $\mathsf{C} = \mathbf{Set}$ (Theorem 30).

▶ **Theorem 29.** *Let $\left(\mathsf{T}, \eta^\mathsf{T}, \mu^\mathsf{T}\right)$ be a monad on a regular category C. When the endofunctor T is nearly cartesian, an $\mathbf{EM}(\mathsf{T})$-relation $\psi\colon (A, a) \leftrightsquigarrow (B, b)$ is decomposable if and only if $\mathsf{U}^\mathsf{T}\psi \cdot a = b \cdot \mathsf{T}\mathsf{U}^\mathsf{T}\psi$. If μ^T is also nearly cartesian and $\mathbf{Rel}(\mathsf{T})$ restricts to $\mathbf{Kl}(\mathsf{R}) \hookrightarrow \mathbf{Rel}(\mathsf{C})$ for some monad R on C, then the Kleisli category $\mathbf{Kl}\left(\overline{\mathsf{R}}\right)$ of the corresponding weakly lifted monad $\overline{\mathsf{R}}$ on $\mathbf{EM}(\mathsf{T})$ has for arrows $(A, a) \nrightarrow (B, b)$ the decomposable relations $\psi\colon (A, a) \leftrightsquigarrow (B, b)$ in $\mathbf{EM}(\mathsf{T})$ such that $\mathsf{U}^\mathsf{T}\psi$ is in $\mathbf{Kl}(\mathsf{R})$.*

▶ **Corollary 30.** *If T has a monotone weak distributive law over $\mathbf{P}$ in $\mathbf{Set}$, the Kleisli category of the lifted powerset monad $\overline{\mathbf{P}}$ on $\mathbf{EM}(\mathsf{T})$ is the category of T-algebras and decomposable relations between them.*

▶ **Remark 31** (subobject classifiers in categories of algebras)**.** Recall that an elementary topos is a regular category such that the **Graph** functor has a right adjoint [13, §1.911]; the corresponding monad is called the powerset monad. If C is an elementary topos with powerset monad $\mathbf{P}$, and if T is a monad on C such that the endofunctor T and the natural transformation μ^T are nearly cartesian, then by Theorem 29 $\mathbf{EM}(\mathsf{T})$ is an elementary topos as soon as every T-algebra morphism is decomposable. We retrieve for instance that the categories of group actions (algebras for monads $G \times -$ where G is a group) are toposes, because the corresponding morphisms of algebras are easily shown to all be decomposable.

This is not a necessary condition for a category of algebras to be an elementary topos: it is well known that categories of monoid actions (algebras for monads $M \times -$ where M is a monoid) are toposes, but there are equivariant morphisms that are not decomposable.

If $\mathbf{EM}(\mathsf{T})$ is an elementary topos as in Theorem 31, $\overline{\mathbf{P}}1$ classifies subobjects in the sense that subobjects $X \hookrightarrow Y$ are in one-to-one correspondence with morphisms $Y \to \overline{\mathbf{P}}1$, where 1 is the terminal object [13, §1.912].

This can be generalized when Theorem 29 holds as follows: $\overline{\mathsf{R}}1$ classifies decomposition-closed subobjects, in the sense that decomposable monomorphisms $X \hookrightarrow Y$ are in one-to-one correspondence with morphisms $Y \to \overline{\mathsf{R}}1$ (the correspondence comes from the adjunction $\mathbf{Kl}\left(\overline{\mathsf{R}}\right)(Y, 1) \cong \mathbf{EM}(\mathsf{T})\left(Y, \overline{\mathsf{R}}1\right)$). For instance, the Vietoris monad on $\mathbf{KHaus} \cong \mathbf{EM}(\boldsymbol{\beta})$ classifies clopen subsets of compact Hausdorff spaces, the non-empty-join-closed powerset monad on $\mathbf{JSL} \cong \mathbf{EM}(\mathbf{P})$ classifies downwards-closed subsets of join-semilattices, and the convex-closed powerset monad on $\mathbf{Conv} \cong \mathbf{EM}(\mathbf{D})$ classifies *walls*, i.e. subsets E such that if $x \in E$ and $\sum_{i=1}^n x_i = x$, $x_i \in E$ as well for all $1 \leq i \leq n$ (walls appear for instance in the structure theorem for convex algebras, which state that every convex algebra is a subalgebra of the *Płonka sum* of its walls [27, Theorem 4.5]).

4.2 Monotone Extensions to Kleisli Categories of Weakly Lifted Monads

In Section 4.1 we assumed T had a monotone weak distributive law over R coming from a relational extension of T and μ^T, and described the Kleisli category of the corresponding weakly lifted monad $\overline{\mathsf{R}}$. Suppose now there is another monad S that weakly lifts to $\mathbf{EM}(\mathsf{T})$ and that also has a monotone weak distributive law over R coming from a relational extension of S and μ^S. When do we also get a monotone weak distributive law of $\overline{\mathsf{S}}$ over $\overline{\mathsf{R}}$ coming from a relational extension of $\overline{\mathsf{S}}$ and $\mu^{\overline{\mathsf{S}}}$?

For the relational extension to exist, we need $\overline{\mathsf{S}}$ and $\mu^{\overline{\mathsf{S}}}$ to be nearly cartesian. This always holds:

▶ **Lemma 32.** *Let C be a regular category where idempotents split, and suppose T is nearly cartesian. If $\mathsf{F}\colon \mathsf{C} \to \mathsf{C}$ is nearly cartesian and weakly lifts to $\mathbf{EM}(\mathsf{T})$, then its weak lifting is nearly cartesian. If $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two such functors is nearly cartesian and weakly lifts to $\mathbf{EM}(\mathsf{T})$, then its weak lifting is nearly cartesian as well.*

$\overline{\mathsf{S}}$ thus has a relational extension $\mathbf{Rel}(\overline{\mathsf{S}})$. By adapting Theorem 7, we can not only characterize when relational extensions restrict to $\mathbf{Kl}(\overline{\mathsf{R}})$, but when any endofunctor or natural transformation has a monotone extension to $\mathbf{Kl}(\overline{\mathsf{R}})$. We can even state this characterization more generally, without necessarily speaking of decomposable morphisms: we do this now.

▶ **Definition 33.** *Let Γ be a wide subcategory of a regular category C such that, in C,*

- *Γ-arrows are stable under pullbacks (in C);*
- *if $f \circ e$ is a Γ-arrow and e is a regular epimorphism (in C), f is a Γ-arrow.*

Then we define $\mathsf{C} \cdot \Gamma^\dagger$ to be the wide subcategory of $\mathbf{Rel}(\mathsf{C})$ whose arrows are the C-relations $\psi\colon X \rightsquigarrow Y$ given by jointly monic spans $\langle \psi_X, \psi_Y\rangle$ such that ψ_X is a Γ-arrow, and we write $\mathbf{Graph}_\Gamma\colon \mathsf{C} \to \mathsf{C} \cdot \Gamma^\dagger$ for the restriction of $\mathbf{Graph}_\mathsf{C}\colon \mathsf{C} \to \mathbf{Rel}(\mathsf{C})$ to $\mathsf{C} \cdot \Gamma^\dagger$.

We also define a $\Gamma^\dagger \cdot \mathsf{C}$-square to be a square $a \circ b = c \circ d$ such that a or c is a Γ-arrow.

▶ **Definition 34.** *Let C and D be two regular categories with respective wide subcategories Γ and Δ as in Theorem 33. Let F be a functor $\mathsf{C} \to \mathsf{D}$. A (Γ, Δ)-*relational *extension of F is a functor $\underline{\mathsf{F}}_{\Gamma,\Delta}\colon \mathsf{C} \cdot \Gamma^\dagger \to \mathsf{D} \cdot \Delta^\dagger$ such that $\underline{\mathsf{F}}_{\Gamma,\Delta}\mathbf{Graph}_\Gamma = \mathbf{Graph}_\Delta \underline{\mathsf{F}}_{\Gamma,\Delta}$. If $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ is a natural transformation between functors $\mathsf{C} \to \mathsf{D}$ with (Γ, Δ)-relational extensions $\underline{\mathsf{F}}_{\Gamma,\Delta}$ and $\underline{\mathsf{G}}_{\Gamma,\Delta}$, a (Γ, Δ)-relational extension is a (necessarily unique) natural transformation $\underline{\alpha}_{\Gamma,\Delta}\colon \underline{\mathsf{F}}_{\Gamma,\Delta} \Rightarrow \underline{\mathsf{G}}_{\Gamma,\Delta}$ such that $\underline{\alpha}_{\Gamma,\Delta}\mathbf{Graph}_\Gamma = \mathbf{Graph}_\Delta \underline{\alpha}_{\Gamma,\Delta}$.*

▶ **Theorem 35.** *Let C, D, Γ, Δ and $\mathsf{F}\colon \mathsf{C} \to \mathsf{D}$ be as in Theorem 34. F has a monotone (Γ, Δ)-relational extension if and only if the following two conditions hold:*

- *F restricts to a functor $\Gamma \to \Delta$;*
- *F sends near pullback $\Gamma^\dagger \cdot \mathsf{C}$-squares on near pullback (necessarily $\Delta^\dagger \cdot \mathsf{D}$-) squares (this is always true when F is nearly cartesian).*

Such a monotone (Γ, Δ)-relational extension, if it exists, is necessarily unique and given by $\underline{\mathsf{F}}_{\Gamma,\Delta}(g \cdot f)^\dagger = \mathsf{F}g \cdot (\mathsf{F}f)^\dagger$.

Let $\alpha\colon \mathsf{F} \Rightarrow \mathsf{G}$ be a natural transformation between two functors $\mathsf{C} \to \mathsf{D}$ having such monotone (Γ, Δ)-relational extensions. α has a (necessarily unique) (Γ, Δ)-relational extension if and only if its has near pullbacks for its naturality squares along Γ-morphisms (this is always true when α is nearly cartesian).

A first corollary, while not especially ground-breaking, is the following:

▶ **Corollary 36.** *In $\mathbf{Set}$, a monad $(\mathsf{T}, \eta^\mathsf{T}, \mu^\mathsf{T})$ whose endofunctor and multiplication are nearly cartesian also has a (necessarily unique) monotone weak distributive law over the monad $\mathbf{P}_*$ of non-empty subsets, and has one over the monad $\mathbf{P}_f$ of finite subsets if and only if T preserves functions with finite pre-images of elements.*

We could more generally characterize the existence of monotone weak distributive laws over these powerset monads for any monad on $\mathbf{Set}$. Still, Theorem 36 is already enough to prove that $\mathbf{P}$ has monotone weak distributive laws over $\mathbf{P}_f$ and $\mathbf{P}_*$, that $\mathbf{D}$ has a monotone weak distributive law over $\mathbf{P}_*$ but not over $\mathbf{P}_f$, and that $\boldsymbol{\beta}$ has a monotone weak distributive law over $\mathbf{P}_*$. A more impactful corollary – we will apply it repeatedly in Section 4.3 – is the following:

▶ **Corollary 37.** *Let* T *be a monad on* **Set** *equipped with a monotone weak distributive law* $\mathsf{T}\mathbf{P} \Rightarrow \mathbf{P}\mathsf{T}$*, and let* S *be a monad on* $\mathbf{EM}(\mathsf{T})$*. If there is a monotone weak distributive law* $\mathsf{S}\overline{\mathbf{P}} \Rightarrow \overline{\mathbf{P}}\mathsf{S}$*,* S *preserves decomposable* T*-algebra morphisms. Moreover if* S *is itself the weak lifting of a monad on* **Set** *that has a monotone weak distributive law over* $\mathbf{P}$*, than the previous condition is not only necessary, but also sufficient.*

4.3 Monotone Weak Distributive Laws in Categories of Algebras

A first use of Theorem 37 is retrieving the monotone weak distributive law $\mathbf{VV} \Rightarrow \mathbf{VV}$: once $\mathbf{V}$ and $\mu^{\mathbf{V}}$ are shown to be nearly cartesian, we only need to prove that decomposable morphisms are the open maps (Theorem 27) instead of proving that $\mathbf{Kl}(\mathbf{V})$-arrows are the continuous relations, and that $\mathbf{V}$ preserves open maps, which is technically much simpler than proving that $\underline{\mathbf{V}}$ preserves continuous relations, as originally done in [19, Proposition 20]. We are also able to answer quite easily Goy's [16, Conjecture 7.31] on the weak distributivity of the Radon monad – the monad of Radon probability measures on a compact Hausdorff space – over the Vietoris monad:

▶ **Theorem 38.** *The Radon monad* $\mathbf{R}$ *does not have a monotone weak distributive law over the Vietoris monad* $\mathbf{V}$*, but it has (a unique) one over the non-empty Vietoris monad* $\mathbf{V}_*$*.*

Let us stress the importance of this new weak distributive law: the question of how to combine probability and non-determinism has been the topic of numerous works (again, see the introduction of [21]), and this law provides an answer in **KHaus** that is derived from a generic construction and thus comes with generic tools, e.g. generalized determinization and up-to techniques [18, 16]. In a very recent pre-print [15], Goubault-Larrecq also constructs this law $\mathbf{RV}_* \Rightarrow \mathbf{V}_*\mathbf{R}$ as an instance of weak distributive laws between monads of continuous valuations and non-deterministic choice in more general categories of topological spaces: our result is more restricted, but we derive the law from generic categorical principles instead of building it by hand, exhibit its canonicity (it comes from a relational extension), and show why the non-empty version of the Vietoris monad is needed.

A second use of Theorem 37 is in proving the absence of monotone weak distributive laws. In Section 3 we were able to prove that the law $\mathbf{PP} \Rightarrow \mathbf{PP}$ does not weakly lift to $\mathbf{EM}(\mathbf{P})$ nor $\mathbf{EM}(\mathbf{D})$, but we were not able to say anything about the existence of other monotone weak distributive laws $\overline{\mathbf{P}}\overline{\mathbf{P}} \Rightarrow \overline{\mathbf{P}}\overline{\mathbf{P}}$. Now, thanks to the framework developed above and in particular Theorem 37, we are able to prove that such laws cannot exist. We start with $\mathbf{EM}(\mathbf{P})$:

▶ **Example 39.** In $\mathbf{EM}(\mathbf{P}) \cong \mathbf{JSL}$, let $\overline{\mathbf{P}}$ be the monad of subsets closed under non-empty joins: the join of a family $(E_i)_{i\in I}$ of non-empty-joins-closed subsets of X is the non-empty-joins-closed subset $\left\{\bigvee_{i\in I} x_i \;\middle|\; x_i \in E_i\right\}$. Let $f\colon 4 \to 2$ (where $2 = \{0,1\}$ and $4 = \{0,1,2,3\}$) be the function given by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Then $\mathbf{F}^{\mathbf{P}}f$ is decomposable (by Theorem 28), but $\overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f$ is not. Indeed, let $A \in \overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}4$ and $B, B_1, B_2 \in \overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}2$ be as depicted in Figure 2a (page 17): $\left(\overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f\right)(A) = B = B_1 \vee B_2$ but there are no $A_1, A_2 \in \overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}4$ such that $A = A_1 \vee A_2$ and $\left(\overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f\right)(A_1) = B_1$ as well as $\left(\overline{\mathbf{P}}\mathbf{F}^{\mathbf{P}}f\right)(A_2) = B_2$.

Proof. Suppose indeed there are such A_1 and A_2. Then $A_1, A_2 \subseteq A$ and thus $\{0,1,2,3\} \in A_1 \cup A_2$ ($\{0,1,2,3\}$ is join-irreducible in A). This would imply $\{0,1\} \in B_1 \cup B_2$, which does not hold. ◀

$\overline{\mathbf{P}}$ does not preserve decomposable $\mathbf{P}$-algebra morphisms, and thus there is no monotone weak distributive law $\overline{\mathbf{P}\mathbf{P}} \Rightarrow \overline{\mathbf{P}\mathbf{P}}$. In fact because the counter-example decomposable morphism is surjective and has finite pre-images, this also proves that there are no monotone weak distributive laws $\overline{\mathbf{P}\mathbf{P}_*} \Rightarrow \overline{\mathbf{P}_*\mathbf{P}}$ or $\overline{\mathbf{P}\mathbf{P}_f} \Rightarrow \overline{\mathbf{P}_f\mathbf{P}}$ in $\mathbf{EM}(\mathbf{P})$.

Because there is a morphism of monads $\mathbf{D} \Rightarrow \mathbf{P}$ (that sends a probability distribution to its support), there is a functor $\mathbf{EM}(\mathbf{P}) \to \mathbf{EM}(\mathbf{D})$ which allows us to transfer Theorem 39 to $\mathbf{EM}(\mathbf{D})$: there are no monotone weak distributive laws $\overline{\mathbf{P}\mathbf{P}} \Rightarrow \overline{\mathbf{P}\mathbf{P}}$ or $\overline{\mathbf{P}\mathbf{P}_*} \Rightarrow \overline{\mathbf{P}_*\mathbf{P}}$ in $\mathbf{EM}(\mathbf{D})$. But this argument is not entirely satisfying, as the resulting example is that of a morphism of convex algebras with a very unnatural structure, namely that of complete join-semilattices: one could imagine restricting to a full subcategory of $\mathbf{EM}(\mathbf{D})$ that does not contain these semilattices, and perhaps $\overline{\mathbf{P}}$ would preserve decomposable morphisms there. As shown by the following example, due to Harald Woracek and Ana Sokolova (private communication), this cannot be the case as soon as free convex algebras come in the picture.

▶ **Example 40** ([30])**.** In $\mathbf{EM}(\mathbf{D}) \cong \mathbf{Conv}$, let $\overline{\mathbf{P}}$ be the monad of convex subsets: a convex combination of some convex subsets of X is the convex set of the corresponding convex combinations of their points (in X). Let $f\colon \{A, B, C\} \to \{B, C\}$ be the function given by $f(A) = f(B) = B$ and $f(C) = C$. Then $\mathbf{F}^{\mathbf{D}} f$ is decomposable (by Theorem 28) but $\overline{\mathbf{P}}\mathbf{F}^{\mathbf{D}} f$ is not. Indeed, depicting $\mathbf{F}^{\mathbf{D}}\{A, B, C\}$ as the triangle depicted in Figure 2b (page 17), $\mathbf{F}^{\mathbf{D}}\{B, C\}$ is the line segment $[BC]$ and $\mathbf{F}^{\mathbf{D}} f$ is the vertical projection. Now $\frac{1}{2}\{B\} + \frac{1}{2}[FC] = [DE] = \left(\overline{\mathbf{P}}\mathbf{F}^{\mathbf{D}} f\right)([GD])$, but $[GD]$ itself cannot be disintegrated as the mean of two convex subsets of ABC, one above B and the other above $[FC]$.

Proof. If such a disintegration existed, then the convex subset above B would contain both B (because $D \in [DG]$) and A (because $G \in [DG]$), hence would be $[AB]$. The subset above $[FC]$ would contain at least one point, and hence the mean of these two subsets would have to contain a non-trivial vertical line segment, which is not the case of $[DG]$. ◀

Using similar arguments, we are able to prove the existence or absence of monotone weak distributive laws over lifted powerset monads in several categories of algebras: these results are gathered in Table 1. All the negative results in this table come from the non-preservation of decomposable morphisms, and thus the absence of monotone extensions of the endofunctors themselves. The topmost row indicates in which category we work. A monad in the second topmost row has a monotone weak distributive law over a monad in the left column if the corresponding cell is filled with ✓, otherwise it is filled with ✗ . In $\mathbf{Set}$, $\overline{\mathbf{P}}$ and $\overline{\mathbf{P}_*}$ are the

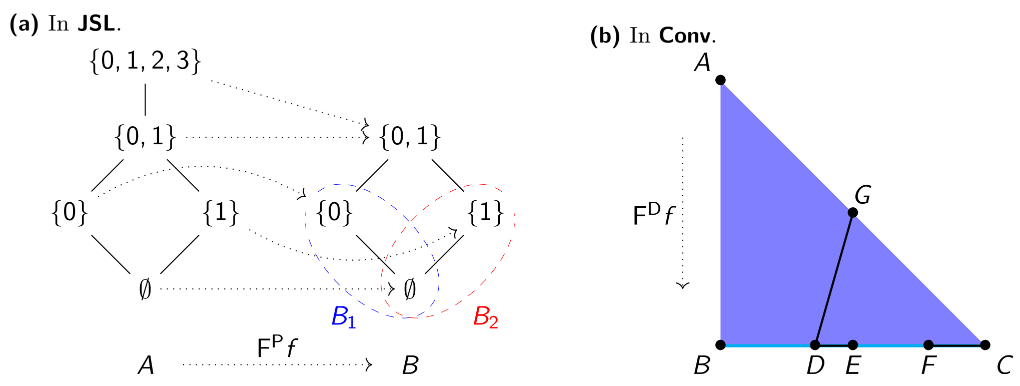


Figure 2 Counterexamples to preservation of decomposability.

Table 1 Existence or absence of monotone weak distributed laws over weakly lifted powerset monads in categories of algebras.

	Set						**KHaus**		**JSL**	**Conv**	**Mon**				**CMon**		
	L	M	D	P	β	M_S	V	R	$\overline{P}$	$\overline{P}$	$\overline{M}$	$\overline{D}$	$\overline{P}$	$\overline{M_S}$	$\overline{M}$	$\overline{D}$	$\overline{P}$
$\overline{P}$	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗
$\overline{P_*}$	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	✗

usual powerset monads **P** and $\mathbf{P}_*$. **L** is the monad of lists, **M** that of multisets and $\mathbf{M}_S$ that of modules for a semiring S satisfying the conditions of [7, Theorem 3.1]. **Mon** $\cong$ **EM**(**L**) is the category of monoids and **CMon** $\cong$ **EM**(**M**) that of commutative monoids. Linear theories distribute over commutative monads [25], hence **L** and **M** distribute over **M**, **P**, **D** and $\mathbf{M}_S$ when S is commutative, and so these four monads have liftings (in particular weak liftings) to **Mon** and **CMon**.

5 Conclusion

Noticing the similarity between the laws **VV** $\Rightarrow$ **VV** and **PP** $\Rightarrow$ **PP**, we developed the theory for weakly lifting weak distributive laws and showed that it only applied partially in this case. We then focused on the monotonicity of the laws, and gave full characterizations of monotone weak distributive laws over weakly lifted powerset monads in categories of algebras by characterizing the Kleisli categories of the latters, a key notion appearing then being that of decomposable morphisms. We finally applied this result to exhibit a new law $\mathbf{RV}_* \Rightarrow \mathbf{V}_*\mathbf{R}$ for combining probability and non-determinism in **KHaus**, but also to show that in general these monotone weak distributive laws seem to be quite rare.

We leave for further work the development of a full 2-categorical theory for iterating weak distributive laws (in the vein of [10, 4]), which would complete Section 3 but would likely be scarce in new examples. With monotone laws over powerset-like monads fully characterized, another natural question is now whether this can be done in other settings, e.g. in **Pos**-regular categories [23] or over other monads: the multiset monad for instance is a good candidate as its Kleisli category can also be described through spans. Finally, the author believes it would be interesting to tranpose the results of Section 4 in the setting of monoidal topology [20], where categories of algebras for nearly cartesian monads generalize the category of compact Hausdorff spaces in a formal sense: perhaps for instance the weakly lifted powerset monads we study are a generalization of the Vietoris monads of topological spaces.

References

1. Michael Barr. Relational algebras. In S. MacLane, H. Applegate, M. Barr, B. Day, E. Dubuc, Phreilambud, A. Pultr, R. Street, M. Tierney, and S. Swierczkowski, editors, *Reports of the Midwest Category Seminar IV*, pages 39–55, Berlin, Heidelberg, 1970. Springer Berlin Heidelberg.
2. Jon Beck. Distributive laws. In H. Appelgate, M. Barr, J. Beck, F. W. Lawvere, F. E. J. Linton, E. Manes, M. Tierney, F. Ulmer, and B. Eckmann, editors, *Seminar on Triples and Categorical Homology Theory*, Lecture Notes in Mathematics, pages 119–140, Berlin, Heidelberg, 1969. Springer. doi:10.1007/BFb0083084.

3 Gabriella Böhm. The weak theory of monads. *Advances in Mathematics*, 225(1):1–32, September 2010. doi:10.1016/j.aim.2010.02.015.

4 Gabriella Böhm. On the iteration of weak wreath products. *Theory and Applications of Categories*, 26(2):30–59, 2012. URL: http://www.tac.mta.ca/tac/volumes/26/2/26-02abs.html.

5 Gabriella Böhm, Stephen Lack, and Ross Street. On the 2-Categories of Weak Distributive Laws. *Communications in Algebra*, 39(12):4567–4583, December 2011. doi:10.1080/00927872.2011.616436.

6 Gabriella Böhm, Stephen Lack, and Ross Street. Idempotent splittings, colimit completion, and weak aspects of the theory of monads. *Journal of Pure and Applied Algebra*, 216(2):385–403, February 2012. doi:10.1016/j.jpaa.2011.07.003.

7 Filippo Bonchi and Alessio Santamaria. Convexity via Weak Distributive Laws. *Logical Methods in Computer Science*, Volume 18, Issue 4, November 2022. doi:10.46298/lmcs-18(4:8)2022.

8 Francis Borceux. *Handbook of Categorical Algebra: Volume 1: Basic Category Theory*, volume 1 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994. doi:10.1017/CBO9780511525858.

9 Francis Borceux. *Handbook of Categorical Algebra: Volume 2: Categories and Structures*, volume 2 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994. doi:10.1017/CBO9780511525865.

10 Eugenia Cheng. Iterated distributive laws. *Mathematical Proceedings of the Cambridge Philosophical Society*, 150(3):459–487, May 2011. doi:10.1017/S0305004110000599.

11 Maria Manuel Clementino, Eva Colebunders, and Walter Tholen. Lax algebras as spaces. In Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors, *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, Encyclopedia of Mathematics and Its Applications, pages 375–466. Cambridge University Press, Cambridge, 2014. doi:10.1017/CBO9781107517288.007.

12 Fredrik Dahlqvist and Renato Neves. Compositional semantics for new paradigms: Probabilistic, hybrid and beyond, April 2018. doi:10.48550/arXiv.1804.04145.

13 Peter John Freyd and Andre Scedrov. *Categories, Allegories*. North Holland, January 1990.

14 Richard Garner. The Vietoris Monad and Weak Distributive Laws. *Applied Categorical Structures*, 28(2):339–354, April 2020. doi:10.1007/s10485-019-09582-w.

15 Jean Goubault-Larrecq. Weak Distributive Laws between Monads of Continuous Valuations and of Non-Deterministic Choice, August 2024. doi:10.48550/arXiv.2408.15977.

16 Alexandre Goy. *On the Compositionality of Monads via Weak Distributive Laws*. PhD thesis, Université Paris-Saclay, October 2021. URL: https://theses.hal.science/tel-03426949.

17 Alexandre Goy. Weakening and Iterating Laws using String Diagrams. *Electronic Notes in Theoretical Informatics and Computer Science*, Volume 1 - Proceedings of MFPS XXXVIII, February 2023. doi:10.46298/entics.10482.

18 Alexandre Goy and Daniela Petrişan. Combining probabilistic and non-deterministic choice via weak distributive laws. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, pages 454–464, New York, NY, USA, July 2020. Association for Computing Machinery. doi:10.1145/3373718.3394795.

19 Alexandre Goy, Daniela Petrişan, and Marc Aiguier. Powerset-Like Monads Weakly Distribute over Themselves in Toposes and Compact Hausdorff Spaces. In *DROPS-IDN/v2/Document/10.4230/LIPIcs.ICALP.2021.132*. Schloss-Dagstuhl - Leibniz Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ICALP.2021.132.

20 Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors. *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge, 2014. doi:10.1017/CBO9781107517288.

21 Klaus Keimel and Gordon D. Plotkin. Mixed powerdomains for probability and nondeterminism. *Logical Methods in Computer Science*, Volume 13, Issue 1, January 2017. doi:10.23638/LMCS-13(1:2)2017.

22 Bartek Klin and Julian Salamanca. Iterated Covariant Powerset is not a Monad. *Electronic Notes in Theoretical Computer Science*, 341:261–276, December 2018. doi:10.1016/j.entcs.2018.11.013.

23 Alexander Kurz and Jiří Velebil. Quasivarieties and varieties of ordered algebras: Regularity and exactness. *Mathematical Structures in Computer Science*, 27(7):1153–1194, October 2017. doi:10.1017/S096012951500050X.

24 Ernest Manes. A triple theoretic construction of compact algebras. In H. Appelgate, M. Barr, J. Beck, F. W. Lawvere, F. E. J. Linton, E. Manes, M. Tierney, F. Ulmer, and B. Eckmann, editors, *Seminar on Triples and Categorical Homology Theory*, pages 91–118, Berlin, Heidelberg, 1969. Springer Berlin Heidelberg.

25 Ernie Manes and Philip Mulry. Monad compositions. I: General constructions and recursive distributive laws. *Theory and Applications of Categories*, 18:172–208, 2007. URL: https://eudml.org/doc/128434.

26 E. Moggi. Computational lambda-calculus and monads. In *[1989] Proceedings. Fourth Annual Symposium on Logic in Computer Science*, pages 14–23, June 1989. doi:10.1109/LICS.1989.39155.

27 Anna B. Romanowska and Jonathan D. H. Smith. On the Structure of Barycentric Algebras. *Houston Journal of Mathematics*, 16(3):431–448, 1990. URL: https://www.math.uh.edu/~hjm/restricted/archive/v016n3/0431ROMANOWSKA.pdf.

28 Ross Street. Weak distributive laws. *Theory and Applications of Categories*, 22(12):313–320, 2009. URL: http://www.tac.mta.ca/tac/volumes/22/12/22-12abs.html.

29 Daniele Varacca and Glynn Winskel. Distributing probability over non-determinism. *Mathematical Structures in Computer Science*, 16(1):87–113, February 2006. doi:10.1017/S0960129505005074.

30 Harald Woracek and Ana Sokolova. Re: Decomposing convex subsets of simplices along free morphisms, June 2024.

31 Maaike Zwart and Dan Marsden. No-Go Theorems for Distributive Laws. In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13, June 2019. doi:10.1109/LICS.2019.8785707.