

# Local Equivalence of Stabilizer States: A Graphical Characterisation

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## Abstract

Stabilizer states form a ubiquitous family of quantum states that can be graphically represented through the graph state formalism. A fundamental property of graph states is that applying a local complementation – a well-known and extensively studied graph transformation – results in a graph that represents the same entanglement as the original. In other words, the corresponding graph states are LU-equivalent. This property served as the cornerstone for capturing non-trivial quantum properties in a simple graphical manner, in the study of quantum entanglement but also for developing protocols and models based on graph states and stabilizer states, such as measurement-based quantum computing, secret sharing, error correction, entanglement distribution... However, local complementation fails short to fully characterise entanglement: there exist pairs of graph states that are LU-equivalent but cannot be transformed one into the other using local complementations. Only few is known about the equivalence of graph states beyond local complementation. We introduce a generalisation of local complementation which graphically characterises the LU-equivalence of graph states. We use this characterisation to show the existence of a strict infinite hierarchy of equivalences of graph states. Our approach is based on minimal local sets, which are subsets of vertices that are known to cover any graph, and to be invariant under local complementation and even LU-equivalence. We use these structures to provide a type to each vertex of a graph, leading to a natural standard form in which the LU-equivalence can be exhibited and captured by means of generalised local complementation.

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## 1 Introduction

Stabilizer states, and in particular graph states, form a versatile family of entangled quantum states, which allow for easy and compact representations. They are used as entangled resource states in various quantum information applications, like measurement-based computation [29, 30, 5], error corrections [35, 34, 10, 33], quantum communication network routing [19, 26, 4, 7], and quantum secret sharing [25, 17], to cite a few. In all these applications, stabilizer states are used as multi-partite entangled resources, it is thus crucial to understand when two such states have the same entanglement. According to standard quantum information theory, two quantum states have the same entanglement if they can be transformed into each



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other using only local operations, where “local” refers to operations that can be applied to at most one qubit at a time. Indeed, intuitively, such local operations can only decrease the strength of the entanglement of a quantum state, thus if a state can be transformed into another and back to the original one using only local operations, the two states do have the same entanglement. Concretely, the most general case is the so-called SLOCC-equivalence (stochastic local operations and classical communications) that encompass the use of local unitaries and measurements. In the particular case of stabilizer states, it is enough to consider *LU-equivalence* (local unitaries), as two stabilizer states are SLOCC-equivalent iff there exists  $U = U_1 \otimes \dots \otimes U_n$  that transforms one state into the other, where each  $U_i$  is a 1-qubit unitary transformation [21]. Therefore, in this paper, we refer to LU-equivalence when stating that two stabilizer states possess the same entanglement.

Stabilizer states can be graphically represented through the graph state formalism, which consists in representing quantum states using graphs where each vertex corresponds to a qubit. Graph states form actually a sub-family of stabilizer states but any stabilizer state can be transformed into a graph state using local Clifford unitaries in a straightforward way [21], as a consequence it is sufficient to consider graph states when dealing with entanglement equivalence of stabilizer states. Notice that understanding the entanglement structure of graph states and how it can be characterised in a graphical way is the most fascinating fundamental question of the graph state formalism [21, 13, 14, 18, 23, 36, 37, 11].

An interesting combinatorial property of entanglement has been proved in [21]: LU-equivalent graph states have the same cut-rank function<sup>1</sup>. However, there exist pairs of graph states that have the same cut-rank function but which are not LU-equivalent: a counterexample involves two isomorphic Petersen graphs [16, 21].

A fundamental property of the graph state formalism is that a *local complementation* – a graph transformation that consists in complementing the neighbourhood of a given vertex – preserves entanglement: if a graph can be transformed into another by means of local complementations then the two corresponding graph states are LU-equivalent. More precisely the corresponding graph states are LC-equivalent (local Clifford). Van den Nest [13] proved that two graph states are LC-equivalent if and only if the corresponding graphs are related by a sequence of local complementations. LC-equivalence and LU-equivalence of graph states were conjectured to coincide [24], as they coincide for several families of graph states [14, 38]. However, a counterexample of order 27 has been discovered using computer assisted methods [23]. Since then, the existence of a graphical characterisation of LU-equivalence has remained an open question. Furthermore, families of graph states for which LC-equivalence and LU-equivalence coincide have been a subject of interest [33, 37].

**Contributions.** We introduce the *r-local complementation*, a graphical transformation parametrised by a positive integer  $r$ , which coincides with the usual local complementation when  $r = 1$ . We show that an  $r$ -local complementation can be performed on the corresponding graph states by means of local unitaries in  $LC_r = \langle H, Z(\pi/2^r) \rangle$ , the group generated by the local Clifford group augmented with the  $Z(\pi/2^r)$  gate. We additionally show that, conversely, for any fixed  $r$ , the  $LC_r$ -equivalence of graph states is captured by  $r$ -local complementations: if two graph states are  $LC_r$ -equivalent, then there is a sequence made of (usual) local complementations and at most one  $r$ -local complementation, that transforms one graph into the other. In particular, the known examples of graph states that are LU-equivalent but not

<sup>1</sup> The cut-rank function over a graph  $G = (V, E)$  is the function that maps a set  $A \subseteq V$  to the rank (in  $\mathbb{F}_2$ ) of the cut-matrix  $\Gamma_A = ((\Gamma_A)_{ab} : a \in A, b \in V \setminus A)$  where  $\Gamma_{ab} = 1$  if and only if  $(a, b) \in E$ .

LC-equivalent [23, 36], are actually  $LC_2$ -equivalent, thus they are captured graphically by the 2-local complementation. This leads to the natural question of the existence of pairs of graph states that are  $LC_3$ -equivalent but not  $LC_2$ -equivalent, and so on. To answer this question, we show that the  $r$ -local complementations form a strict hierarchy: for any positive  $r$  there exist pairs of  $LC_{r+1}$ -equivalent graph states that are not  $LC_r$ -equivalent. Finally, we show that any LU-equivalent graph states are  $LC_r$ -equivalent for some  $r$ , even if there exist some local unitaries, like  $R_Z(\pi/3)$ , that are not contained in  $LC_r$ , for any  $r$ . In other words, two graph states are LU-equivalent if and only if the corresponding graphs can be transformed into each other by a sequence of generalised local complementations, leading to a full graphical description of LU-equivalence for graph states. As an application, we prove a conjecture on a family of graph states for which LU-equivalence and LC-equivalence coincide.

**Related work.** Other graphical extensions of local complementation have been introduced in particular for weighted hypergraph states [36], a wider family of quantum states. The connection between generalised local complementation and local complementation of weighted hypergraph states is discussed in section 3. In [18, 39], the authors proved that when considering LU-equivalence of stabilizer states it suffices to consider semi-Clifford unitaries<sup>2</sup>. Symmetries of stabilizers and graph states have been explored in [15], where a characterisation of local operations that have graph states as fixed points, is provided; the characterisation of the LU-equivalence of graph states was however left as an open question.

**Structure of the paper.** In Section 2, we recall some definitions and notations and provide a handful of tools to manipulate graph states. In Section 3, we define the generalisation of the local complementation and we show that  $r$ -local complementations can be implemented on the corresponding graph states by means of local unitaries in  $LC_r$ . Section 4 is dedicated to the converse result: the graphs corresponding to any pair of  $LC_r$ -equivalent graph states can be transformed into each other by means of  $r$ -local complementations. To do so, we show that any graph can be put in a standard form by means of (usual) local complementations. This standard form is based on the so-called minimal local sets, which are subsets of vertices known to be invariant under LU-equivalence. As any graph can be covered by minimal local sets, we associate to any vertex of a graph a type on which is based the standard form. We then prove that if two graph states in standard form are  $LC_r$ -equivalent, there exists an  $r$ -local complementation transforming one graph into the other. We also prove that if two graph states on  $n$  qubits are LU-equivalent they are  $LC_{n/2}$ -equivalent. Finally, in Section 5, we introduce a family of variants of Kneser graphs, and show, using our graphical characterisation of  $LC_r$ -equivalence, that for any  $r > 1$  there exists pairs of graph states that are  $LC_r$ -equivalent but not  $LC_{r-1}$ -equivalent.

## 2 Preliminaries

Let us first give some notations and basic definitions. A graph<sup>3</sup>  $G$  is a pair  $(V, E)$ , where  $V$  is a finite set of vertices, and  $E$  is a set of unordered pairs of distinct vertices called edges. We assume the set of vertices to be totally ordered and use the notation  $V = \{u_1, \dots, u_n\}$  s.t.  $u_i \prec u_j$  iff  $i < j$ . The number  $n = |V|$  of vertices is called the order of the graph and

<sup>2</sup> A unitary  $U$  is semi-Clifford if there exists a Pauli operator  $P \in X, Y, Z$  such that  $UPU^\dagger$  is also a Pauli operator.

<sup>3</sup> We only consider simple (no selfloop) undirected graphs.

$|G| := |E|$  its size. We use the notation  $u \sim_G v$  when the vertices  $u$  and  $v$  are connected in  $G$ , i.e.  $(u, v) \in E$ . Given a vertex  $u$ ,  $N_G(u) = \{v \in V \mid (u, v) \in E\}$  is the neighbourhood of  $u$ . The odd and the common neighbourhoods are two natural generalisations of the notion of neighbourhoods to sets of vertices: for any  $D \subseteq V$ ,  $Odd_G(D) = \Delta_{u \in D} N_G(u) = \{v \in V \mid |N_G(v) \cap D| = 1 \pmod{2}\}$  is the odd neighbourhood of  $D$ , where  $\Delta$  denotes the symmetric difference on vertices. Informally,  $Odd_G(D)$  is the set of vertices that are the neighbours of an odd number of vertices in  $D$ . The common neighbourhood of  $D$  is denoted  $\Lambda_G^D = \bigcap_{u \in D} N_G(u) = \{v \in V \mid \forall u \in D, v \in N_G(u)\}$ . A local complementation with respect to a given vertex  $u$  consists in complementing the subgraph induced by the neighbourhood of  $u$ , leading to the graph  $G \star u = G \Delta K_{N_G(u)}$  where  $\Delta$  denotes the symmetric difference on edges and  $K_A$  is the complete graph on the vertices of  $A$ . Local complementation is an involution, i.e.  $G \star u \star u = G$ . A pivoting with respect to two connected vertices  $u$  and  $v$  is the operation that maps  $G$  to  $G \wedge uv := G \star u \star v \star u$ . Notice that pivoting is symmetric ( $G \wedge uv = G \wedge vu$ ) and is an involution ( $G \wedge uv \wedge uv = G$ ). With a slight abuse of notation we identify multisets of vertices with their multiplicity function  $V \rightarrow \mathbb{N}$ .<sup>4</sup> A (multi)set  $S$  of vertices is independent if there is no two vertices of  $S$  that are connected.

Graph states form a standard family of quantum states that can be represented using graphs (Ref. [20] is an excellent introduction to graph states). Given a graph  $G$  of order  $n$ , the corresponding **graph state**  $|G\rangle$  is the  $n$ -qubit state:

$$|G\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{|G[x]|} |x\rangle$$

where  $G[x]$  denotes the subgraph of  $G$  induced by  $\{u_i \mid x_i = 1\} \subseteq V$ .

The  $i^{\text{th}}$  qubit of  $|G\rangle$  can be associated with the vertex  $u_i$  of  $G$ , so, with a slight abuse of notation, we refer to  $V$  as the set of qubits of  $|G\rangle$ . Given a 1-qubit operation  $R$ , like  $X : |a\rangle \mapsto |1-a\rangle$  or  $Z : |a\rangle \mapsto (-1)^a |a\rangle$ , and a subset  $D \subseteq V$  of qubits,  $R_D := \bigotimes_{u \in D} R_u$  is the local operation which consists in applying  $R$  on each qubit of  $D$ .

The graph state  $|G\rangle$  is the unique quantum state (up to a global phase) that, for every vertex  $u \in V$ , is a fixed point of the Pauli operator  $X_u Z_{N_G(u)}$ . More generally,  $|G\rangle$  is an eigenvector of any Pauli operator  $X_D Z_{Odd(D)}$  with  $D \subseteq V$ :

**► Proposition 1.** *Let  $G = (V, E)$  be a graph. The corresponding graph state  $|G\rangle$  is a fixed point of  $\mathcal{P}_D^G = (-1)^{|G[D]|} X_D Z_{Odd(D)}$  for any  $D \subseteq V$ , where  $|G[D]|$  is the number of edges of the subgraph of  $G$  induced by  $D$ .*

The proof of Proposition 1 can be found in the full version of this paper [9]. The fact that a graph state can be defined as the common eigenvector of Pauli operators witnesses that graph states form a subfamily of stabilizer states<sup>5</sup>. Conversely, any stabilizer state can be turned into a graph state by means of local Clifford unitaries<sup>6</sup> in a straightforward way (see for instance [13]). Thus we will focus in the rest of this paper on graph states.

The action of measuring, in the standard basis, a qubit of a graph state leads, roughly speaking, to the graph state where the corresponding vertex has been removed. A diagonal measurement can also be used to remove a vertex when this vertex is isolated:

<sup>4</sup> Hence we also identify sets of vertices with their indicator functions  $V \rightarrow \{0, 1\}$ .

<sup>5</sup> A  $n$ -qubit state is a stabilizer state if it is the fixpoint of  $n$  independent commuting Pauli operators (or equivalently  $2^n$  distinct commuting Pauli operators).

<sup>6</sup> Local Clifford unitaries are tensor products of single-qubit Clifford gates, which map the Pauli group to itself under conjugation. Formally, a single-qubit gate  $U$  is Clifford if for any  $P \in \mathcal{P} := \{\pm 1, \pm i\} \times \{I, X, Y, Z\}$ ,  $UPU^\dagger \in \mathcal{P}$ .

► **Proposition 2.** For any graph  $G$  and any vertex  $u$ ,  $\langle 0|_u |G\rangle = \frac{1}{\sqrt{2}}|G \setminus u\rangle$ . Moreover, if  $u$  is an isolated vertex (i.e.  $N_G(u) = \emptyset$ ), then  $\langle +|_u |G\rangle = |G \setminus u\rangle$  where  $\langle +| = \frac{\langle 0| + \langle 1|}{\sqrt{2}}$ .

► **Remark 3.** A standard basis measurement consists in applying either  $\langle 0|$  or  $\langle 1|$  which correspond the classical outcomes 0 and 1 respectively. Whereas the action in the 0 case is directly described in Proposition 2, the action in the 1 case can be recovered thanks to Proposition 1:  $\langle 1|_u |G\rangle = \langle 0|_u X_u |G\rangle = \langle 0|_u Z_{N_G(u)} |G\rangle = \frac{1}{\sqrt{2}} Z_{N_G(u)} |G \setminus u\rangle$ . Thus it also corresponds to a vertex deletion up to some  $Z$  corrections on the neighbourhood of the measured qubit.

We are interested in the action of 1-qubit unitaries on graph states, in particular Hadamard  $H : |a\rangle \mapsto \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}}$ , and  $Z$ - and  $X$ -rotations respectively defined as follows:

$$Z(\alpha) := e^{i\frac{\alpha}{2}} \left( \cos\left(\frac{\alpha}{2}\right) I - i \sin\left(\frac{\alpha}{2}\right) Z \right) \quad X(\alpha) := H Z(\alpha) H = e^{i\frac{\alpha}{2}} \left( \cos\left(\frac{\alpha}{2}\right) I - i \sin\left(\frac{\alpha}{2}\right) X \right)$$

In particular, local complementations can be implemented by  $\pi/2$   $X$ - and  $Z$ -rotations:

► **Proposition 4** ([13]). For any graph  $G = (V, E)$  and any  $u \in V$ ,  $|G \star u\rangle = L_u^G |G\rangle$  where  $L_u^G := X_u(\frac{\pi}{2}) \otimes_{v \in N_G(u)} Z_v(-\frac{\pi}{2})$ .

Similarly, pivoting can be implemented by means of Hadamard transformations (up to Pauli transformations):

► **Proposition 5** ([27, 12]). For any graph  $G = (V, E)$  and any  $(u, v) \in E$ ,  $|G \wedge uv\rangle = H_u H_v Z_{\Lambda_G^{u,v}} |G\rangle$

Beyond  $\frac{\pi}{2}$  rotations, notice that any local unitary can be implemented using Hadamard and  $Z$ -rotations with arbitrary angles. We consider for any  $r \in \mathbb{N}$  the set  $LC_r$  of local unitaries generated by  $H$  and  $Z(\frac{\pi}{2^r})$ . In particular  $LC_1$  is the set of local Clifford operators, moreover, for any  $r$ ,  $LC_r \subseteq LC_{r+1}$ . Notice that there exist local unitaries, e.g.  $Z(\frac{\pi}{3})$ , that are not contained in  $LC_r$  for any  $r$ . We say that two states are LU-equivalent, denoted  $|\psi\rangle =_{LU} |\psi'\rangle$  when there exists a local unitary transformation  $U$  such that  $|\psi\rangle = U|\psi'\rangle$ . Similarly two states are  $LC_r$ -equivalent, denoted  $|\psi\rangle =_{LC_r} |\psi'\rangle$  when there exists a local unitary  $U \in LC_r$  s.t.  $|\psi\rangle = U|\psi'\rangle$ .

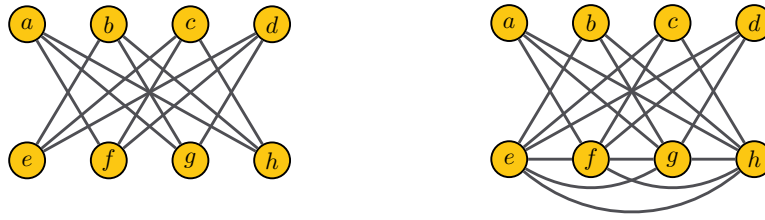
It is well-known that two graph states  $|G\rangle$  and  $|G'\rangle$  are  $LC_1$ -equivalent if and only if there exists a sequence of local complementations that transforms  $G$  into  $G'$  [13]. We slightly refine this result showing that there exists a reasonably small sequence of local complementation transforming  $G$  into  $G'$ , that corresponds to a particular Clifford operator:

► **Proposition 6.** If  $|G_2\rangle = C|G_1\rangle$  where  $C$  is a local Clifford operator up to a global phase, then there exists a sequence of (possibly repeating) vertices  $a_1, \dots, a_m$  such that  $G_2 = G_1 \star a_1 \star a_2 \star \dots \star a_m$  with  $m \leq \lfloor 3n/2 \rfloor$ , and the local Clifford operator that implements these local complementations is  $C$  up to a Pauli operator.

The proof of Proposition 6 can be found in the full version of this paper [9]

► **Remark 7.** The induced Clifford operator is of the form  $\mathcal{P}_D^G C$  for some subset  $D$  of vertices. As  $X_u Z_{N_G(u)} = (L_u^G)^2$ , there exists a sequence at most  $\lfloor 7n/2 \rfloor$  local complementations that transforms  $G_1$  into  $G_2$  and induces exactly the local Clifford operator  $C$ .

Whereas it has been originally conjectured that if two graphs states are LU-equivalent then there exists a sequence of local complementations transforming one in another [24], a couple of counter examples of pairs of  $LC_2$ - but not  $LC_1$ -equivalent graph states have been pointed out [23, 36]. To further understand the action of local unitaries on graph states, we thus introduce in the next section a generalisation of local complementation to graphically capture the equivalence of graph states that are LU- but not  $LC_1$ -equivalent.



■ **Figure 1** Example of non LU-equivalent graph states. With the notations of Section 5, the graph on the left is a  $C_{4,3}$  graph, and the one on the right is a  $C'_{4,3}$  graph. Notice that applying a local complementation on the upper part of the bipartition ( $S = \{a, b, c, d\}$ ) leaves the graphs invariant as each pair of vertices on the other part (e.g.  $(e, f)$ ), has two common neighbours in  $S$ . However a 2-local complementation over  $S$  is not valid in both graphs as  $S$  is not 2-incident. See Section 4.1 for a proof that these graph states are not LU-equivalent.

### 3 Generalising Local Complementation

#### 3.1 Local complementation over independent sets

As a first step towards a generalisation of local complementation, we introduce a natural shortcut  $G \star S := G \star a_1 \dots \star a_k$  to denote the graph obtained after a series of local complementations when  $S = \{a_1, \dots, a_k\}$  is an independent set in  $G$ . The independence condition is important as local complementations do not commute in general when applied on connected vertices.

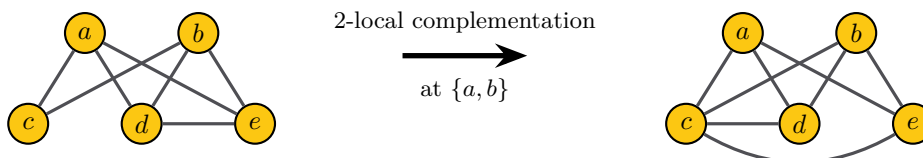
Notice that the action of a local complementation over  $S$  can be described directly as toggling edges which have an odd number of common neighbours in  $S$ :

$$u \sim_{G \star S} v \Leftrightarrow (u \sim_G v \oplus |S \cap N_G(u) \cap N_G(v)| = 1 \pmod 2) \tag{1}$$

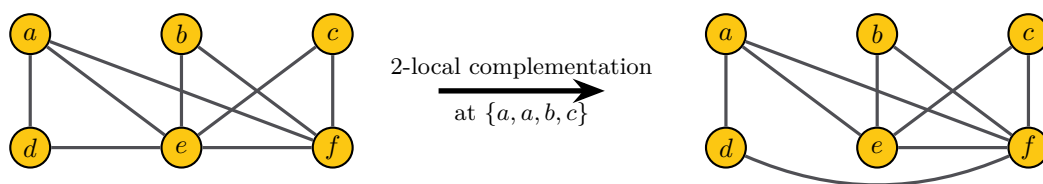
#### 3.2 Towards a 2-local complementation

We introduce 2-local complementations as a refinement of *idempotent* local complementations, i.e. when  $G \star S = G$ . According to Equation (1), an idempotent local complementation occurs when each pair  $(u, v)$  of vertices has an even number of common neighbours in  $S$ , one can then consider a new graph transformation which consists in toggling an edge  $(u, v)$  when its number of common neighbours in  $S$  is equal to 2 modulo 4. However, such an action may not be implementable using local operations in the graph state formalism (see Figure 1). To guarantee an implementation by means of local unitary transformations on the corresponding graph states, we add the condition, called 2-incident, that any set of at most three vertices has an even number of common neighbours in  $S$ .

For instance in the following example, a 2-local complementation over  $\{a, b\}$  performs the following transformation:







■ **Figure 2** Illustration on a 2-local complementation on the multiset  $S = \{a, a, b, c\}$ .  $S$  is 2-incident: indeed  $S \bullet \Lambda_G^{\{d,e,f\}} = 2$ , which is a multiple of  $2^{2-1-0} = 2$ . Similarly,  $S \bullet \Lambda_G^{\{d,e\}} = S \bullet \Lambda_G^{\{d,f\}} = 2$  and  $S \bullet \Lambda_G^{\{e,f\}} = 4$ . Edges  $de$  and  $df$  are toggled as  $S \bullet \Lambda_G^{\{d,e\}} = S \bullet \Lambda_G^{\{d,f\}} = 2 \pmod 4$ , but not edge  $ef$  as  $S \bullet \Lambda_G^{\{e,f\}} = 0 \pmod 4$ .

In the LHS graph  $G$ ,  $S = \{a, b\}$  is an independent set. Moreover, a 1-local complementation over  $S$  is idempotent:  $G \star S = G$ , as  $a$  and  $b$  are twins.  $S$  is also 2-incident as any pairs and triplets of vertices have an even number of neighbours in  $S$ . Thus, a 2-local complementation over  $S$  is valid and consists in toggling the edges  $(c, d)$ ,  $(c, e)$ , and  $(d, e)$  as each of them has a number of common neighbours in  $S$  equal to  $2 \pmod 4$ .

We also consider the case where  $S$  is actually a multiset, like in Figure 2 where a 2-local complementation over the multiset  $\{a, a, b, c\}$  is performed. When  $S$  is a multiset, the number of common neighbours in  $S$  should be counted with their multiplicity.

Notice that when a 2-local complementation is invariant one can similarly refine the 2-local complementation into a 3-local complementation, leading to the general definition of generalised local complementation provided in the next section.

### 3.3 Defining generalised local complementation

We introduce a generalisation of local complementation, that we call  $r$ -local complementation for any positive integer  $r$ . This generalised local complementation denoted  $G \star^r S$  is parametrised by a (multi)set  $S$ , that has to be independent and also  $r$ -incident, which is the following counting condition on the number of common neighbours in  $S$  of any set that does not intersect  $\text{supp}(S)$ , the support of  $S$ :

► **Definition 8** ( $r$ -Incidence). *Given a graph  $G$ , a multiset  $S$  of vertices is  $r$ -incident, if for any  $k \in [0, r)$ , and any  $K \subseteq V \setminus \text{supp}(S)$  of size  $k + 2$ ,  $S \bullet \Lambda_G^K$  is a multiple of  $2^{r-k-\delta(k)}$ , where  $\delta$  is the Kronecker delta<sup>7</sup> and  $S \bullet \Lambda_G^K$  is the number of vertices of  $S$ , counted with their multiplicity, that are neighbours to all vertices of  $K$ .<sup>8</sup>*

► **Definition 9** ( $r$ -Local Complementation). *Given a graph  $G$  and an  $r$ -incident independent multiset  $S$ , let  $G \star^r S$  be the graph defined as*

$$u \sim_{G \star^r S} v \Leftrightarrow (u \sim_G v \oplus S \bullet \Lambda_G^{u,v} = 2^{r-1} \pmod{2^r})$$

A detailed example of a 2-local complementation is given in Figure 2.

We will also use  $r$ -local complementation parametrised with a set (rather than a multiset), in this case  $S \bullet \Lambda_G^K = |S \cap \Lambda_G^K|$ , and  $S \bullet \Lambda_G^{u,v} = |S \cap N_G(u) \cap N_G(v)|$ . Note that we recover the vanilla local complementation when  $r = 1$ .

<sup>7</sup>  $\delta(x) \in \{0, 1\}$  and  $\delta(x) = 1 \Leftrightarrow x = 0$ .

<sup>8</sup>  $\bullet$  is the scalar product:  $A \bullet B = \sum_{u \in V} A(u).B(u)$ , so  $S \bullet \Lambda_G^K = \sum_{u \in \Lambda_G^K} S(u)$ .

We say that  $G \star^r S$  is valid when  $S$  is an  $r$ -incident independent multiset in  $G$ . Moreover, generalising the notion of local equivalence on graphs, we say that two graphs  $G, G'$  are  $r$ -locally equivalent when there exists a sequence of  $r$ -local complementations transforming  $G$  into  $G'$ . In the case of the usual local complementation, we simply say that the graphs are locally equivalent.

Generalised local complementations satisfy a few basic properties (proven in the full version of this paper [9]), in particular, it is easy to double check that they are self inverse:  $(G \star^r S) \star^r S = G$ . Moreover,  $r$ -local complementations can be related to  $(r + 1)$  and  $(r - 1)$ -local complementations:

- **Proposition 10.** *If  $G \star^r S$  is valid then:*
  - $G \star^{r+1} (2S)$  is valid and induces the same transformation:  $G \star^{r+1} (2S) = G \star^r S$ , where  $2S$  is the multiset obtained from  $S$  by doubling the multiplicity of each vertex,
  - $G \star^{r-1} S$  is valid (when  $r > 1$ ) and  $G \star^{r-1} S = G$ .

It implies in particular that two  $r$ -locally equivalent graphs are  $(r + 1)$ -locally equivalent. An  $r$ -local complementation over  $S$  preserves the neighbourhood of the vertices in  $\text{supp}(S)$ :

- **Proposition 11.** *If  $G \star^r S$  is valid, then for any  $u \in \text{supp}(S)$ ,  $N_{G \star^r S}(u) = N_G(u)$ .*

Notice that  $r$ -local complementations can be composed:

- **Proposition 12.** *If  $G \star^r S_1$  and  $G \star^r S_2$  are valid and the disjoint union<sup>9</sup>  $S_1 \sqcup S_2$  is independent in  $G$ , then  $G \star^r (S_1 \sqcup S_2) = (G \star^r S_1) \star^r S_2$ .*

Finally, it is easy to see that the multiplicity in  $S$  can be upperbounded by  $2^r$ : if  $G \star^r S$  is valid, then  $G \star^r S = G \star^r S'$ , where, for any vertex  $u$ ,  $S'(u) = S(u) \bmod 2^r$ .

### 3.4 Local Clifford operators and local complementation

It is well known that local complementations can be implemented on graph states by means of local Clifford operations [21, 20], we extend this result to  $r$ -local complementations that can be implemented using  $\frac{\pi}{2^r}$  X- and Z-rotations:

- **Proposition 13.** *Given a graph  $G = (V, E)$  and a  $r$ -incident independent multiset  $S$  of vertices,*

$$|G \star^r S\rangle = \bigotimes_{u \in V} X \left( \frac{S(u)\pi}{2^r} \right) \bigotimes_{v \in V} Z \left( -\frac{\pi}{2^r} \sum_{u \in N_G(v)} S(u) \right) |G\rangle$$

The proof of Proposition 13 can be found in the full version of this paper [9].

- **Remark 14.** The graph state formalism has been extended in various ways, notably through the introduction of *hypergraph states* [31] and even *weighted hypergraph states* [36]. Notice that the formalism of weighted hypergraph states provides a way to decompose a generalised local complementation over a multiset  $S$  of size  $k$  into  $k$  elementary transformations on weighted hypergraph states. Indeed, as shown in Proposition 13, a generalised local complementation can be implemented by means of  $k$  X-rotations (along with some Z-rotations), moreover the action of each X-rotation on weighted hypergraph states has been described in [36]. However, in such a decomposition, the intermediate weighted hypergraph states are not necessarily graph states.

<sup>9</sup> for any vertex  $u$ ,  $(S_1 \sqcup S_2)(u) = S_1(u) + S_2(u)$ .



In the next section, we prove the converse result: if a local unitary operation in  $\text{LC}_r$  transforms a graph state  $|G\rangle$  into a graph state  $|G'\rangle$  then there exists a sequence of  $r$ -local complementations transforming  $G$  into  $G'$ . More so, we show that  $r$ -local complementation actually captures completely the LU-equivalence of graph states. To prove these results, we make use of graphical tools, namely the so-called minimal local sets and a standard form for graphs.

## 4 Graphical characterisation of the action of local unitaries

### 4.1 Minimal local sets and Types

To characterise the action of local unitaries on graph states, we rely on properties that are invariant under local unitaries. A local set [22, 8] is one of them.

► **Definition 15.** *Given  $G = (V, E)$ , a local set  $L$  is a non-empty subset of  $V$  of the form  $L = D \cup \text{Odd}_G(D)$  for some  $D \subseteq V$  called a generator.*

Local sets of  $G$  are precisely the supports of the Pauli operators  $\mathcal{P}_D^G = (-1)^{|G[D]|} X_D Z_{\text{Odd}(D)}$ , with  $D$  a non-empty subset of qubits, that stabilises  $|G\rangle$ . Local sets are invariant under local complementation - hence their name: if  $L$  is a local set in a graph  $G$ , so is in  $G \star u$ , but possibly with a distinct generator [22]. Local sets are the same for graphs that have the same cut-rank function [8], thus graphs corresponding to LU-equivalent graph states have the same local sets.

A *minimal local set* is a local set that is minimal by inclusion. Minimal local sets have either 1 or 3 generators, and in the latter case, the minimal local sets are of even size. This was proved originally in the stabilizer formalism in [14], but an alternative graph-theoretic proof can be found in [8].

► **Proposition 16.** *Given a minimal local set  $L$ , only two cases can occur:*

- $L$  has exactly one generator,
- $L$  has exactly three (distinct) generators, of the form  $D_0$ ,  $D_1$ , and  $D_0 \Delta D_1$ . This can only occur when  $|L|$  is even.

In the first case, the minimal local set  $L$  is said to be of dimension 1; in the second, of dimension 2. The dimension of a minimal local set depends only on the cut-rank function, thus it is invariant by LU-equivalence. Just like (minimal) local sets, the dimension can be a useful tool to prove that two graph states are not LU-equivalent. For example, the two graphs of Figure 1 have the same minimal local sets, but they do not have the same dimension. Indeed,  $\{a, b, c, h\}$  is a minimal local set of dimension 2 in the first graph, while it is a minimal local set of dimension 1 in the second one, proving that the two graph states are not LU-equivalent.

Minimal local sets provide crucial informations on local unitaries acting on graph states: it is known that if a local unitary  $U$  transforms  $|G\rangle$  into  $|G'\rangle$  and  $L$  is a minimal local set of dimension 2, then for any  $v \in L$ ,  $U_v$  must be a Clifford operator [14]. Minimal local sets of dimension 1 are more permissive but also provide some constraints on  $U$ :

► **Lemma 17.** *Given two graphs  $G$ ,  $G'$  and a local unitary  $U$  such that  $|G\rangle = U|G'\rangle$ , if  $L$  is a 1-dimensional minimal local set with generators respectively  $D$  in  $G$  and  $D'$  in  $G'$ , then*

$$\mathcal{P}_D^G U = U \mathcal{P}_{D'}^{G'}$$

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**Proof.** Following [14, 39], we consider the density matrix  $\rho_G^L$  (resp.  $\rho_{G'}^L$ ) obtained by tracing out the qubits outside  $L$ . According to [20, 21]:  $\rho_G^L = \frac{1}{2}(I + \mathcal{P}_D^G)$  and  $\rho_{G'}^L = \frac{1}{2}(I + \mathcal{P}_{D'}^{G'})$ . As  $\rho_G^L = U\rho_{G'}^L U^\dagger$ , we get  $I + \mathcal{P}_D^G = I + U\mathcal{P}_{D'}^{G'}U^\dagger$ , hence  $\mathcal{P}_D^G U = U\mathcal{P}_{D'}^{G'}$ . ◀

So, intuitively, every minimal local set  $L$  induces some constraints on a local unitary transformation  $U$  acting on the corresponding graph state, more precisely on the qubits of  $U$  that are in  $L$ . It has been recently shown that minimal local sets cover every vertex of a graph [8], which unlocks the use of minimal local sets in characterizing local unitaries acting on graph states, as it guarantees that minimal local sets impose constraints on every qubit of the local unitary:

► **Theorem 18** ([8]). *Each vertex of any graph is contained in at least one minimal local set.*

We abstract away the set of minimal local sets, which can be exponentially many in a graph (see [8]), as a simple labelling of the vertices that we call a *type*.

► **Definition 19.** *Given a graph  $G$ , a vertex  $u$  is of type*

- *X if for any generator  $D$  of a minimal local set containing  $u$ ,  $u \in D \setminus \text{Odd}(D)$ ,*
- *Y if for any generator  $D$  of a minimal local set containing  $u$ ,  $u \in D \cap \text{Odd}(D)$ ,*
- *Z if for any generator  $D$  of a minimal local set containing  $u$ ,  $u \in \text{Odd}(D) \setminus D$ ,*
- $\perp$  *otherwise.*

The names X, Y and Z are chosen to match the Pauli operator at vertex  $u$  in  $\mathcal{P}_D^G$ . Notice that a vertex involved in a minimal local set of dimension 2 is necessarily of type  $\perp$ <sup>10</sup>. We define  $V_X^G \subseteq V$  (resp.  $V_Y^G, V_Z^G, V_\perp^G$ ) as the set of vertices of type X (resp. Y, Z,  $\perp$ ) in  $G$ .  $V_X^G, V_Y^G, V_Z^G$  and  $V_\perp^G$  form a partition of  $V$ : indeed, thanks to Theorem 18, every vertex has a type.

We show in the following a few properties of the vertex types. First notice that vertices of type Z represent at most half the vertices of a graph:

► **Lemma 20.** *For any graph  $G$  of order  $n$ ,  $|V_Z^G| \leq \lfloor n/2 \rfloor$ .*

**Proof.**  $V_Z^G$  contains no minimal local set. Indeed, as the generator of a local set is non-empty, at least one vertex of a given minimal local set  $L$  is not of type Z. Conversely, any subset of the vertices of size  $\lfloor n/2 \rfloor + 1$  contains at least one minimal local set [8]. ◀

Two LU-equivalent graph states share the same vertices of type  $\perp$ .

► **Lemma 21.** *Given two LU-equivalent graph states  $|G_1\rangle$  and  $|G_2\rangle$ ,  $V_\perp^{G_1} = V_\perp^{G_2}$ .*

**Proof.** Let  $U$  be a local unitary s.t.  $|G_1\rangle = U|G_2\rangle$ , and let  $v \in V$  a vertex of type  $\perp$  in  $G_1$ . If  $v$  is in a minimal local set of dimension 2 in  $G_1$ , so is in  $G_2$  as the dimension of minimal local sets is invariant under LU-equivalence. Otherwise, there exist two distinct minimal local sets  $L, L'$ , generated respectively by  $D_1$  and by  $D'_1$  in  $G_1$  such that  $\mathcal{P}_{D_1}^{G_1}$  and  $\mathcal{P}_{D'_1}^{G_1}$  do not commute on qubit  $v$ .<sup>11</sup> Let  $D_2$  (resp.  $D'_2$ ) be the generator of  $L$  (resp.  $L'$ ) in  $G_2$ . According to Lemma 17,  $\mathcal{P}_{D_1}^{G_1} = U\mathcal{P}_{D_2}^{G_2}U^\dagger$  and  $\mathcal{P}_{D'_1}^{G_1} = U\mathcal{P}_{D'_2}^{G_2}U^\dagger$ , as a consequence,  $\mathcal{P}_{D_2}^{G_2}$  and  $\mathcal{P}_{D'_2}^{G_2}$  do not commute on qubit  $v$ , so  $v$  must be of type  $\perp$  in  $G_2$ . ◀

<sup>10</sup> If a vertex  $v$  has the same type with respect to two distinct generators  $D, D'$  of a minimal local set  $L$ , then  $v$  would not be in the local set generated by  $D\Delta D'$  which contradicts the minimality of  $L$ .

<sup>11</sup>  $V = \bigotimes_u V_u$  commutes with  $W = \bigotimes_u W_u$  on qubit  $u_0$  if  $V_{u_0}$  and  $W_{u_0}$  commute.

A vertex having the same type in two LU-equivalent graph states implies strong constraints on the local unitaries that relate the two corresponding graph states.

► **Lemma 22.** *If  $|G_1\rangle =_{LU} |G_2\rangle$  i.e.  $|G_2\rangle = U|G_1\rangle$ , then  $U = e^{i\phi} \bigotimes_{u \in V} U_u$  where:*

- $U_u$  is a Clifford operator if  $u$  is of type  $\perp$  in both  $G_1$  and  $G_2$ ,
- $U_u = X(\theta_u)Z^{b_u}$  if  $u$  is of type  $X$  in both  $G_1$  and  $G_2$ ,
- $U_u = Z(\theta_u)X^{b_u}$  if  $u$  is of type  $Z$  in both  $G_1$  and  $G_2$ .

with  $b_u \in \{0, 1\}$ . Additionally, if  $|G_1\rangle$  and  $|G_2\rangle$  are  $LC_r$ -equivalent, there exists such a unitary  $U$  where every angle satisfies  $\theta_u = 0 \pmod{\pi/2^r}$ .

**Proof.** The first case, when  $u$  is of type  $\perp$ , is a variant of the minimal support condition [13] and was proved in [28] for the particular case of minimal local sets of dimension 2. If  $u$  is of type  $X$ , let  $L$  be a minimal local set such that  $u \in L$ . According to Lemma 17,  $U_u X U_u^\dagger = e^{i\phi} X$ .  $U_u|+\rangle$  and  $U_u|-\rangle$  are eigenvectors of  $U_u X U_u^\dagger$  of eigenvalues respectively 1 and -1, implying  $U_u X U_u^\dagger = \pm X$ . If  $U_u X U_u^\dagger = X$ , then  $U_u = e^{i\phi} X(\theta)$ . If  $U_u X U_u^\dagger = -X$ , define  $U' := U_u Z$ .  $U' X U'^\dagger = X$ , so  $U' = e^{i\phi} X(\theta)$ , thus  $U_u = e^{i\phi} X(\theta) Z$ . The proof is similar when  $u$  is of type  $Z$ . If  $|G_1\rangle$  and  $|G_2\rangle$  are  $LC_r$ -equivalent, there exists  $|G_2\rangle = U|G_1\rangle$  where  $U = e^{i\phi} \bigotimes_{u \in V} U_u$ , and each  $U_u$  is in  $LC_r$ . ◀

To fully characterise the unitaries that relate two graph states using Lemma 22, the type of every vertex needs to be the same. This is not the case in general, even for locally equivalent graphs<sup>12</sup>. Thus, we introduce a standard form on graphs, such that two graphs in standard form corresponding to LU-equivalent graph states have the same types.

## 4.2 Standard Form

We define a standard form of graphs up to local complementation. A graph in standard form satisfies the following properties: all vertices are of type  $X$ ,  $Z$  or  $\perp$ ; all the neighbours of a vertex of type  $X$  are of type  $Z$  (so in particular the vertices of type  $X$  form an independent set); and any vertex of type  $X$  is smaller than its neighbours according to the underlying total order  $\prec$  of the vertices. In other words:

► **Definition 23.** *A graph  $G$  is in said in standard form if*

- $V_Y^G = \emptyset$ ,
- $\forall u \in V_X^G$ , any neighbour  $v$  of  $u$  is of type  $Z$  and satisfies  $u \prec v$ .

Note that standard form is not unique in general, in the sense that a class of local equivalence of graphs may contain several graphs in standard form. For example, any graph with only vertices of type  $\perp$  is in standard form, along with its entire orbit generated by local complementation. Conversely, each class of local equivalence contains at least a graph in standard form.

► **Proposition 24.** *For any graph  $G$ , there exists a locally equivalent  $G'$  in standard form.*

The proof of Proposition 24 is an algorithm that puts a graph in standard form by means of local complementation and can be found in the full version of this paper [9]. Standard form implies a similar structure in terms of types assuming LU-equivalence.

<sup>12</sup>A simple example involves the complete graph  $K_3$  on 3 vertices and the line graph  $L_3$  on three vertices.  $K_3$  and  $L_3$  are related by a single local complementation, however every vertex of  $K_3$  is of type  $Y$ , while  $L_3$  contains one vertex of type  $Z$  and two vertices of type  $X$ .

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► **Proposition 25.** *If  $G_1$  and  $G_2$  are both in standard form and  $|G_1\rangle =_{LU} |G_2\rangle$ , each vertex has the same type in  $G_1$  and  $G_2$ , and any vertex  $u$  of type  $X$  satisfies  $N_{G_1}(u) = N_{G_2}(u)$ .*

The proof of Proposition 25 can be found in the full version of this paper [9].

### 4.3 Graphical characterisation of local equivalence

Thanks to the standard form, one can accommodate the types of two LU-equivalent graph states, to simplify the local unitaries mapping one to the other:

► **Lemma 26.** *If  $G_1$  and  $G_2$  are both in standard form and  $|G_1\rangle =_{LU} |G_2\rangle$ , there exists  $G'_1$  locally equivalent to  $G_1$  in standard form such that  $|G_2\rangle = \bigotimes_{u \in V_X^{G_1}} X(\alpha_u) \bigotimes_{v \in V_Z^{G_1}} Z(\beta_v) |G'_1\rangle$ .*

The proof of Lemma 26 can be found in the full version of this paper [9]. Note that if  $|G_1\rangle$  and  $|G_2\rangle$  are  $LC_r$ -equivalent, we can choose the angles such that  $\alpha_u, \beta_v = 0 \pmod{\pi/2^r}$ . Additionally,  $G_1$  and  $G'_1$  are related by local complementation only on vertices of type  $\perp$ . They are strong constraints relating the angles of the X- and Z-rotations acting on different qubits:

- **Lemma 27.** *Given  $G_1, G_2$  in standard form, if  $|G_2\rangle = \bigotimes_{u \in V_X^{G_1}} X(\alpha_u) \bigotimes_{v \in V_Z^{G_1}} Z(\beta_v) |G_1\rangle$ ,*
- $\forall v \in V_Z^{G_1}, \beta_v = -\sum_{u \in N_{G_1}(v) \cap V_X^{G_1}} \alpha_u \pmod{2\pi}$ ,
  - $\forall k \in \mathbb{N}, \forall K \subseteq V_Z^{G_1}$  of size  $k+2$ ,  $\sum_{u \in \Lambda_{G_1}^K \cap V_X^{G_1}} \alpha_u = 0 \pmod{\frac{\pi}{2^{k+\delta(k)}}}$ .

The proof of Lemma 27 can be found in the full version of this paper [9]. The constraints coincide with  $r$ -incidence, so, when angles are multiples of  $\pi/2^r$ , this unitary transformation implements a  $r$ -local complementation.

► **Lemma 28.** *Given  $G_1, G_2$  in standard form, if  $|G_2\rangle = \bigotimes_{u \in V_X^{G_1}} X(\alpha_u) \bigotimes_{v \in V_Z^{G_1}} Z(\beta_v) |G_1\rangle$  and  $\forall u \in V_X^{G_1}, \alpha_u = 0 \pmod{\pi/2^r}$ , then  $\bigotimes_{u \in V_X^{G_1}} X(\alpha_u) \bigotimes_{v \in V_Z^{G_1}} Z(\beta_v)$  implements an  $r$ -local complementation over the vertices of  $V_X^{G_1}$ .*

**Proof.** Let us construct an  $r$ -incident independent multiset  $S$  such that  $G_2 = G_1 \star^r S$ . We define  $S$  on the vertices of  $V_X^{G_1}$  such that  $\forall u \in V_X^{G_1}, \alpha_u = \frac{S(u)\pi}{2^r} \pmod{2\pi}$  with  $S(u) \in [1, 2^r)$ . Note that  $\sum_{u \in \Lambda_{G_1}^K \cap V_X^{G_1}} \alpha_u = \frac{\pi}{2^r} S \bullet \Lambda_G^K$ . Hence, by Lemma 27, For any  $k \in [0, r)$ , and any  $K \subseteq V \setminus S$  of size  $k+2$ ,  $S \bullet \Lambda_G^K$  is a multiple of  $2^{r-k-\delta(k)}$  meaning that  $S$  is  $r$ -incident. Also, by Lemma 27,  $\beta_v = -\frac{\pi}{2^r} \sum_{u \in N_{G_1}(v)} S(u) \pmod{2\pi}$ . Thus,  $\bigotimes_{u \in V_X^{G_1}} X(\alpha_u) \bigotimes_{v \in V_Z^{G_1}} Z(\beta_v)$  implements an  $r$ -local complementation on  $S$ . ◀

We can now easily relate  $LC_r$ -equivalence to  $r$ -local complementations for graphs in standard form:

► **Lemma 29.** *If  $G_1$  and  $G_2$  are both in standard form and  $|G_1\rangle =_{LC_r} |G_2\rangle$ , then  $G_1$  and  $G_2$  are related by a sequence of local complementations on the vertices of type  $\perp$  along with a single  $r$ -local complementation over the vertices of type  $X$ .*

► **Remark 30.** The sequence of local complementations commutes with the  $r$ -local complementation, as vertices of type  $X$  and vertices of type  $\perp$  do not share edges by definition of the standard form.

**Proof.** By Lemma 26, there exists  $G'_1$  locally equivalent to  $G_1$  in standard form such that  $|G_2\rangle = \bigotimes_{u \in V_X} X(\alpha_u) \bigotimes_{v \in V_Z} Z(\beta_v) |G'_1\rangle$  where  $V_X$  (resp.  $V_Z$ ) denotes the set of vertices of type X (resp. Z) in  $G'_1$  and  $G_2$ , and  $\forall u \in V$  of type X or Z,  $\alpha_u, \beta_u = 0 \pmod{\pi/2^r}$ . By Lemma 28,  $\bigotimes_{u \in V_X} X(\alpha_u) \bigotimes_{v \in V_Z} Z(\beta_v)$  implements an  $r$ -local complementation over the vertices of type X. ◀

Notice in particular that when there is no vertex of type  $\perp$ , a single  $r$ -local complementation is required.

► **Corollary 31.** *If two  $\perp$ -free  $r$ -locally equivalent graphs  $G_1$  and  $G_2$  are both in standard form, they are related by a single  $r$ -local complementation on the vertices of type X.*

We are ready to prove that  $r$ -local equivalence coincides with  $LC_r$ -equivalence.

► **Theorem 32.** *The following properties are equivalent:*

1.  $|G_1\rangle$  and  $|G_2\rangle$  are  $LC_r$ -equivalent.
2.  $G_1$  and  $G_2$  are  $r$ -locally equivalent.
3.  $G_1$  and  $G_2$  are related by a sequence of local complementations along with a single  $r$ -local complementation.

**Proof.** We proceed by cyclic proof. (2  $\Rightarrow$  1): Follows from Proposition 13. (3  $\Rightarrow$  2): A local complementation is, in particular, an  $r$ -local complementation. (1  $\Rightarrow$  3): Follows from Proposition 24 along with Lemma 29. ◀

More than just  $LC_r$ -equivalence,  $r$ -local complementations can actually characterise the LU-equivalence of graph states.

► **Theorem 33.** *If  $|G_1\rangle$  and  $|G_2\rangle$  are LU-equivalent then  $G_1$  and  $G_2$  are  $(\lfloor n/2 \rfloor - 1)$ -locally equivalent, where  $n$  is the order of the graphs.*

The proof of Theorem 33 can be found in the full version of this paper [9]. Remarkably, this result implies that for graph states, LU-equivalence reduces to equivalence up to operators in  $\bigcup_{r \in \mathbb{N}} LC_r$ , and even – as it has been hinted in [18] – to operators of the form  $C_1 \bigotimes_{u \in V} Z(\alpha_u) C_2$  where  $C_1, C_2$  are local Clifford operators and the angles are multiples of  $\pi/2^r$  for some integer  $r$ . Notice that these local operators are actually those of the well-known Clifford hierarchy (see for example [1]).

#### 4.4 Graph states whose class of LC and LU-equivalence coincide

Since the LU-LC conjecture was disproven in [38], classes of graph states whose class of  $LC_1$ - and LU-equivalence coincide has been a subject of interest. We say that  $LU \Leftrightarrow LC$  holds for a graph  $G$  if  $|G\rangle =_{LU} |G'\rangle \Leftrightarrow |G\rangle =_{LC_1} |G'\rangle$  for any graph  $G'$ . It is known that  $LU \Leftrightarrow LC$  holds for a graph  $G$  if either one of the following is true: **(1)**  $G$  is of order at most 8 [20, 6]; **(2)**  $G$  is a complete graph [14]; **(3)** every vertex of  $G$  is of type  $\perp$  (this is referred as the *minimal support condition*) [14]; **(4)** the graph obtained by removing all leaves (i.e. vertices of degree 1) vertices of  $G$  has only vertices of type  $\perp$  [38]; **(5)**  $G$  has no cycle of length 3 or 4 [38]; **(6)** the stabilizer of  $|G\rangle$  has rank less than 6 [23]. A review of these necessary conditions, especially for **(6)**, can be found in [37]. Our results imply a new criterion for  $LU \Leftrightarrow LC$  based on the standard form introduced in the last section. This criterion is actually a sufficient and necessary condition, meaning it can be used to prove both that  $LU \Leftrightarrow LC$  holds for some graph, or the converse.

► **Proposition 34.** *Given a graph  $G$ , the following are equivalent:*

- $LU \Leftrightarrow LC$  holds for  $G$ .
- For **some** graph locally equivalent to  $G$  that is in standard form, any  $r$ -local complementation over the vertices of type  $X$  can be implemented by local complementations.
- For **any** graph locally equivalent to  $G$  that is in standard form, any  $r$ -local complementation over the vertices of type  $X$  can be implemented by local complementations.

The proof of Proposition 34 can be found in the full version of this paper [9]. Checking that  $r$ -local complementations can be implemented by local complementations is not easy in general, nonetheless it is convenient in many cases, for example when there is few vertices of type  $X$  or that they have low degree. Namely, this criterion is stronger than the minimal support condition, as graphs with only vertices of  $\perp$  have no vertex of type  $X$ . Furthermore, it has been left as an open question in [37] whether  $LU \Leftrightarrow LC$  holds for some instances of repeater graph states. We use our new criterion to easily prove that this is the case. For this purpose, we prove that  $LU \Leftrightarrow LC$  holds for a broader class of graph.

► **Proposition 35.**  *$LU \Leftrightarrow LC$  holds for graphs where each vertex is either a leaf i.e. a vertex of degree 1, or is connected to a leaf.*

**Proof.** It is easy to prove (see the full version of this paper [9]) that such a graph  $G$  is in standard form (for some particular ordering of the vertices) and that the vertices of type  $X$  are exactly the leaves. Thus, any  $r$ -local complementation over the vertices of type  $X$  has no effect on  $G$ . According to Proposition 34, this implies that  $LU \Leftrightarrow LC$  holds for  $G$ . ◀

Such graphs include some instances of *repeater graph states* [2]. A *complete-graph-based repeater graph state* is a graph state whose corresponding graph of order  $2n$  is composed of a complete graph of order  $n$ , along with  $n$  leaves appended to each vertex. A *biclique-graph-based repeater graph state* is a graph state whose corresponding graph of order  $4n$  is composed of a symmetric biclique (i.e. a symmetric bipartite complete graph) graph of order  $2n$ , along with  $2n$  leaves appended to each vertex. Complete-graph-based repeater graph states are the all-photonic repeaters introduced in [3]. Biclique-graph-based repeater graph states are a variant introduced in [32] which is more efficient in terms of number of edges. It was left as an open question in [37] whether  $LU \Leftrightarrow LC$  holds for complete-graph-based or biclique-graph-based repeater graph states (although it was proved for some variants). These graph states satisfy the condition of Proposition 35, hence we can answer this question by the positive:

► **Corollary 36.**  *$LU \Leftrightarrow LC$  holds for complete-graph-based repeater graph states and biclique-graph-based repeater graph states.*

## 5 A hierarchy of generalised local equivalences

Two  $r$ -locally equivalent graphs are also  $(r+1)$ -locally equivalent. This can be seen graphically as a direct consequence of Proposition 10, or using graph states through Theorem 32, as two  $LC_r$ -equivalent states are obviously  $LC_{r+1}$ -equivalent. The known counter examples to the  $LU$ - $LC$  conjecture introduced in [23, 36] are actually  $LC_2$ -equivalent graph states that are not  $LC_1$ -equivalent, thus the corresponding graphs are 2-locally equivalent but not 1-locally equivalent. There was however no known examples of graph states that are  $LC_3$ -equivalent but not  $LC_2$ -equivalent, and more generally no known examples of graph states that are  $LC_r$ -equivalent but not  $LC_{r-1}$ -equivalent for  $r > 2$ . We introduce such examples in this section, by showing that for any  $r$  there exist pairs of graphs that are  $(r+1)$ -locally equivalent



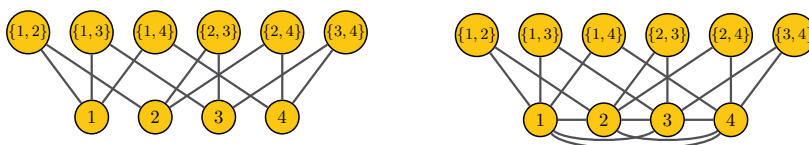
but not  $r$ -locally equivalent leading to an infinite hierarchy of generalised local equivalences. Our proof is constructive: for any  $r$  we exhibit pairs of graphs that are  $(r + 1)$ -locally equivalent but not  $r$ -locally equivalent.

We introduce a family of bipartite graphs  $C_{t,k}$  parametrized by two integers  $t$  and  $k$ . This family is a variant of the (bipartite) Kneser graphs and uniform subset graphs. The graph  $C_{t,k}$  is bipartite: the first independent set of vertices corresponds to the integers from 1 to  $t$ , and the second independent set is composed of all the subsets of  $[1, t]$  of size  $k$ . There is an edge between an integer  $u \in [1, t]$  and a subset  $A \subseteq [1, t]$  of size  $k$  if and only if  $u \in A$ :

► **Definition 37.** For any  $k \geq 1$  and  $t \geq k$  two integers, let  $C_{t,k} = (V, E)$  be a bipartite graph defined as:  $V = [1, t] \cup \binom{[1,t]}{k}$  and  $E = \{(u, A) \in [1, t] \times \binom{[1,t]}{k} \mid u \in A\}$ .

We introduce a second family of graphs  $C'_{t,k}$ , defined similarly to  $C_{t,k}$ , the independent set  $[1, t]$  being replaced by a clique:

► **Definition 38.** For any  $k \geq 1$  and  $t \geq k$  two integers, let  $C'_{t,k} = (V, E)$  be a graph defined as:  $V = [1, t] \cup \binom{[1,t]}{k}$  and  $E = \{(u, A) \in [1, t] \times \binom{[1,t]}{k} \mid u \in A\} \cup \{(u, v) \in [1, t] \times [1, t] \mid u \neq v\}$ .



■ **Figure 3** (Left) The graph  $C_{4,2}$ . (Right) The graph  $C'_{4,2}$ .

Examples of such graphs with parameters  $t = 4$  and  $k = 2$  are given in Figure 3. For any odd  $k \geq 3$ ,  $t \geq k + 2$ ,  $C_{t,k}$  and  $C'_{t,k}$  are in standard form for any ordering such that, for any  $u \in \binom{[1,t]}{k}$  and for any  $v \in [1, t]$ ,  $u \prec v$ . More precisely the vertices in  $\binom{[1,t]}{k}$  are of type X and the vertices in  $[1, t]$  are of type Z. The proof can be found in the full version of this paper [9].

The following proposition gives a sufficient condition on  $t$  and  $k$  for the  $r$ -local equivalence of  $C_{t,k}$  and  $C'_{t,k}$ .

► **Proposition 39.** For any  $t \geq k \geq 1$ ,  $C_{t,k}$  and  $C'_{t,k}$  are  $r$ -locally equivalent if

$$\binom{t-2}{k-2} = 2^{r-1} \pmod{2^r} \quad \text{and} \quad \forall i \in [1, r-1], \quad \binom{t-i-2}{k-i-2} = 0 \pmod{2^{r-i}}$$

**Proof.**  $C_{t,k}$  and  $C'_{t,k}$  are related by a  $r$ -local complementation on the set  $\binom{[1,t]}{k}$  (which can be seen as the multiset where each vertex of  $\binom{[1,t]}{k}$  appears once). Let  $K \subseteq [1, t]$  of size  $k' + 2$ .  $\binom{[1,t]}{k} \bullet \Lambda_G^K = \left| \left\{ x \in \binom{[1,t]}{k} \mid K \subseteq x \right\} \right| = \binom{t-k'-2}{k-k'-2}$  is a multiple of  $2^{r-k'-\delta(k')}$  by hypothesis. Thus,  $\binom{[1,t]}{k}$  is  $r$ -incident. Besides, for any  $u, v \in [1, t]$ ,  $\binom{[1,t]}{k} \bullet \Lambda_G^{u,v} = \binom{t-2}{k-2} = 2^{r-1} \pmod{2^r}$  by hypothesis. Thus,  $C_{t,k} \star^r \binom{[1,t]}{k} = C'_{t,k}$ . ◀

The following proposition provides a sufficient condition on  $t$  and  $k$  for the non  $r$ -local equivalence of  $C_{t,k}$  and  $C'_{t,k}$ .

► **Proposition 40.** For any odd  $k \geq 3$ ,  $t \geq k + 2$ ,  $C_{t,k}$  and  $C'_{t,k}$  are not  $r$ -locally equivalent if  $\binom{t}{2}$  is odd and  $\binom{k}{2} = 0 \pmod{2^r}$ .

**Proof.** Let us suppose by contradiction that  $C_{t,k}$  and  $C'_{t,k}$  are  $r$ -locally equivalent. By Corollary 31, they are related by a single  $r$ -local complementation on the vertices in  $\binom{[1,t]}{k}$ . Let  $S$  be the multiset in  $\binom{[1,t]}{k}$  such that  $C_{t,k} \star^r S = C'_{t,k}$ . For each  $u, v \in [1, t]$ ,  $S \bullet \Lambda_{C_{t,k}}^{u,v} = \sum_{x \in \Lambda_{C_{t,k}}^{u,v}} S(x) = 2^{r-1} \pmod{2^r}$ . Summing over all pairs  $u, v \in [1, t]$ , and, as by hypothesis  $\binom{t}{2}$  is odd,  $\sum_{u,v \in [1,t]} S \bullet \Lambda_{C_{t,k}}^{u,v} = \binom{k}{2} \sum_{x \in \binom{[1,t]}{k}} S(x) = \binom{t}{2} 2^{r-1} \pmod{2^r} = 2^{r-1} \pmod{2^r}$ . The first part of the equation is true because  $\sum_{u,v \in [1,t]} S \bullet \Lambda_{C_{t,k}}^{u,v} = \sum_{u,v \in [1,t]} \sum_{x \in \Lambda_{C_{t,k}}^{u,v}} S(x) = \sum_{x \in \binom{[1,t]}{k}} \sum_{u,v \in x} S(x)$ . This leads to a contradiction as  $\binom{k}{2} = 0 \pmod{2^r}$  by hypothesis. ◀

It remains to find, for any  $r$ , parameters  $t$  and  $k$  such that the corresponding graphs are  $r$ -locally equivalent but not  $(r-1)$ -locally equivalent. Fortunately, such parameters exist.

► **Theorem 41.** *For any  $r \geq 2$ ,  $C_{t,k}$  and  $C'_{t,k}$  are  $r$ -locally equivalent but not  $(r-1)$ -locally equivalent when  $k = 2^r + 1$  and  $t = 2^r + 2^{\lceil \log_2(r) \rceil + 1} - 1$ . Thus,  $|C_{t,k}\rangle$  and  $|C'_{t,k}\rangle$  are  $LC_r$ -equivalent but not  $LC_{r-1}$ -equivalent.*

The proof of Theorem 41 can be found in the full version of this paper [9].

► **Remark 42.** For the case  $r = 2$ , we obtain the pair  $(C_{7,5}, C'_{7,5})$ , which is the 28-vertex counter-example to the LU-LC conjecture from [36]. In particular, this translates to an example of a pair of graph states that are  $LC_2$ -equivalent but not  $LC_1$ -equivalent.

Thus, while Ji et al. proved that there exist pairs of graph states that are LU-equivalent but not  $LC_1$ -equivalent [23], we showed a finer result – the existence of a infinite strict hierarchy of graph states equivalence between  $LC_1$ - and LU-equivalence. There exist  $LC_2$ -equivalent graph states that are not  $LC_1$ -equivalent,  $LC_3$ -equivalent graph states that are not  $LC_2$ -equivalent... On the other end, for any integer  $r$ , there exist LU-equivalent graph states that are not  $LC_r$ -equivalent.

## 6 Conclusion

In this paper, we have introduced of a graphical characterisation of LU-equivalence for graph states, and consequently for stabilizer states. To achieve this, we have introduced a generalisation of the local complementation, and leveraged the structures of minimal local sets in graphs. A key outcome of this characterisation is the establishment of a strict hierarchy of equivalences for graph states, which significantly advances our understanding of the gap between the LC- and LU-equivalence. Thanks to this graphical characterisation, we have also proven a conjecture regarding families of graphs states where LC- and LU-equivalence coincide.

This graphical characterisation has additional potential applications, such as the search of minimal examples of graph states that are LU-equivalent but not LC-equivalent. The smallest known examples are of order 27, and it is known there is no counter example of order less than 8. More generally one can wonder whether there are smaller examples of graphs that are  $LC_r$  equivalent but not  $LC_{r-1}$  as the ones we have introduced are of order exponential en  $r$ . In terms of complexity, determining whether two graph states are LC-equivalent can be done in polynomial time. The graphical characterization of LU-equivalence also offers new avenues for exploring the complexity of the LU-equivalence problem, as allows one to study this computational problem from a purely graphical standpoint.

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