

Forbidden Patterns in Mixed Linear Layouts

Deborah Haun  

Karlsruhe Institute of Technology, Germany

Laura Merker  

Karlsruhe Institute of Technology, Germany

Sergey Pupyrev  

Menlo Park, CA, USA

Abstract

An ordered graph is a graph with a total order over its vertices. A linear layout of an ordered graph is a partition of the edges into sets of either non-crossing edges, called stacks, or non-nesting edges, called queues. The stack (queue) number of an ordered graph is the minimum number of required stacks (queues). Mixed linear layouts combine these layouts by allowing each set of edges to form either a stack or a queue. The minimum number of stacks plus queues is called the mixed page number. It is well known that ordered graphs with small stack number are characterized, up to a function, by the absence of large twists (that is, pairwise crossing edges). Similarly, ordered graphs with small queue number are characterized by the absence of large rainbows (that is, pairwise nesting edges). However, no such characterization via forbidden patterns is known for mixed linear layouts.

We address this gap by introducing patterns similar to twists and rainbows, which we call thick patterns; such patterns allow a characterization, again up to a function, of mixed linear layouts of bounded-degree graphs. That is, we show that a family of ordered graphs with bounded maximum degree has bounded mixed page number if and only if the size of the largest thick pattern is bounded. In addition, we investigate an exact characterization of ordered graphs whose mixed page number equals a fixed integer k via a finite set of forbidden patterns. We show that for $k = 2$, there is no such characterization, which supports the nature of our first result.

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1 Introduction

An *ordered graph* is a graph given with a fixed linear vertex order, \prec . A *linear layout* of an ordered graph is a partition of its edges such that each part satisfies certain requirements with respect to the order. In a *stack layout* each part, also called a *stack*, is required to be crossing-free with respect to \prec , that is, two edges in the same stack may not have alternating endpoints. A *queue layout* is a “dual” concept which forbids two edges to nest in the same part, called a *queue*; that is, if $u \prec x \prec y \prec v$, then edges uv and xy must be in different queues. The two concepts are generalized in *mixed linear layouts*, where each part (called a *page* then) may either be a stack or a queue. A linear layout using s stacks and/or q queues is called pure s -*stack*, pure q -*queue*, and mixed s -*stack* q -*queue*, respectively. In all three cases, the objective is to minimize the number of parts. The *stack number* $sn(G)$ (*queue number* $qn(G)$), *mixed page number* $mn(G)$) of an ordered graph G is the smallest k such that there is a stack (queue, mixed) layout with at most k stacks (queues, pages).



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Stack layouts and queue layouts are well understood and a rich collection of tools has been developed. Most notably, the product structure theory [22], layered path decompositions [7, 26], track layouts [23], and different kinds of H -partitions that were used successfully both for queue layouts [8, 22, 35, 45] and stack layouts [48]. A fundamental technique in this context is a characterization of stack and queue layouts via forbidden ordered patterns. Formally, a *pattern* is an ordered graph with at least one edge. The *size* of a pattern is the number of edges. An ordered graph (G, \prec_1) *contains* a pattern (H, \prec_2) if H is a subgraph of G and \prec_2 is a suborder of \prec_1 ; otherwise, the graph *avoids* the pattern. A k -*twist* denotes a set of k pairwise crossing edges with respect to some vertex order, and a k -*rainbow* is a set of k pairwise nesting edges, where symbol k can be omitted if not needed. We define a *graph parameter* to be a function assigning a non-negative integer to every graph. A parameter p is *bounded* for a family of graphs if there is a constant c such that $p(G) \leq c$ for every graph G of the family. Now, the characterization of stack and queue layouts can be formulated as follows.

► **Theorem 1** ([17, 40]). *A family of ordered graphs has bounded stack number if and only if the size of the largest twist is bounded.*

► **Theorem 2** ([44]). *A family of ordered graphs has bounded queue number if and only if the size of the largest rainbow is bounded.*

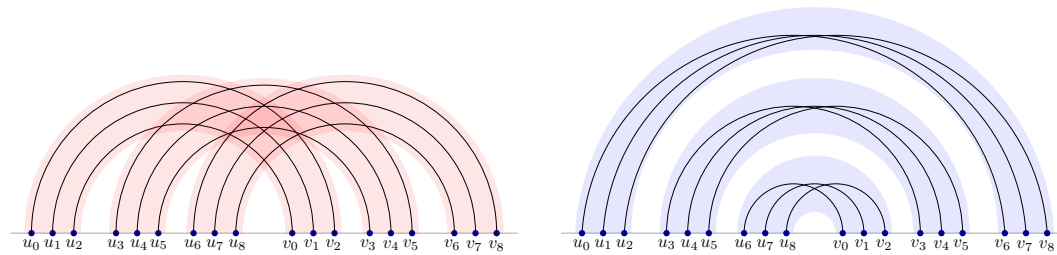
These theorems are useful both for upper bounds (as an explicit assignment of edges to stacks or queues is not needed) and for lower bounds (as a tedious case distinction which edge could go to which stack or queue gets superfluous). Unfortunately, no analogous characterization is known for mixed linear layouts. As a consequence, all known results on mixed linear layouts rely on an explicit assignment of edges to stacks and queues. In this paper, we aim to close this gap and find a set of patterns whose absence characterizes mixed linear layouts similarly to twists and rainbows in Theorems 1 and 2. The following open question asks whether there is a theorem similar to Theorems 1 and 2 for mixed linear layouts, where stack/queue is replaced by mixed page number and the largest twist/rainbow is replaced by the largest pattern in \mathcal{P} .

► **Open Problem 3.** *Do there exist finite sets, $\mathcal{P}_1, \mathcal{P}_2, \dots$, of patterns and a binding function, $f : \mathbb{N} \rightarrow \mathbb{N}$, such that for all $k \in \mathbb{N}$ and every ordered graph, G , the following holds:*

- $\text{mn}(G) < f(k)$ if G avoids all patterns in \mathcal{P}_k , and
- $\text{mn}(G) \geq k$ if G contains some pattern in \mathcal{P}_k ?

Indeed, an affirmative answer would give a theorem of the form of Theorems 1 and 2: There is a set \mathcal{P} of patterns, namely the union of $\mathcal{P}_1, \mathcal{P}_2, \dots$, such that a family of ordered graphs has bounded mixed page number if and only if the size of the largest pattern of \mathcal{P} that occurs in a graph of the family is bounded. To see the implication, note that there are only finitely many patterns of a certain size, so if the size of the largest pattern of \mathcal{P} is bounded, then the first property of Open Problem 3 gives that the mixed page number is bounded. And conversely, a family whose mixed page number is at most k contains no patterns of \mathcal{P} that are larger than the largest pattern in $\mathcal{P}_1, \dots, \mathcal{P}_k$ by the second property.

Note that for pure stack/queue layouts, the answer to the question is positive, since the corresponding sets \mathcal{P}_k of forbidden patterns (also called *obstruction sets*) contain a single element, a k -twist and a k -rainbow, respectively. Indeed, a more technical formulation of Theorem 1 would be that for every ordered graph G and every $k \in \mathbb{N}$, it holds that the stack number of G is less than $14k \log k$ if G avoids a k -twist, and is at least k if G contains a k -twist [17, 40]. We remark that in contrast to stack layouts, queue layouts even admit the



■ **Figure 1** The two 3-thick patterns: Three pairwise crossing 3-rainbows (left) and three pairwise nesting 3-twists (right).

identity as binding function [44]. To complement this, we answer the problem affirmatively for mixed linear layouts of bounded-degree graphs and provide negative results for an exact characterization with the identity binding function.

The remainder of this section is organized as follows. First, we present our main results and describe technical contributions with concrete bounds that guide through the subsequent sections. Then, we relate our findings to the state-of-the-art. Finally, we discuss linear layouts from various perspectives targeted to readers not familiar with the topic.

1.1 Main Results

For an explicit description of the set \mathcal{P} of patterns, we define a k -thick pattern to be obtained either from a k -twist by replacing each edge by a k -rainbow, or from a k -rainbow by replacing each edge by k -twist; see Figure 1 and Section 2.2 for a more elaborate introduction.

► **Theorem 4.** *For every family of ordered graphs with bounded maximum degree, the size of the largest thick pattern is bounded if and only if the mixed page number is bounded.*

Note that Open Problem 3 asks for a characterization up to a function bounding the mixed page number depending on the size of the largest pattern occurring in the graph. That is, even if we know the size of the largest pattern, we do not learn the exact mixed page number but only an upper bound. A more granular characterization would guarantee the exact mixed page number given that a set of forbidden patterns is excluded. As a negative result, we present an infinite family of patterns that needs to be included in an obstruction set of graphs with mixed page number 2. Thus, Open Problem 3 does not admit such a precise answer.

► **Theorem 5.** *Let \mathcal{P} be the set of patterns such that an ordered matching has mixed page number at most 2 if and only if it contains no pattern of \mathcal{P} . Then \mathcal{P} is infinite.*

1.2 Summary

When answering Open Problem 3, which we do positively for bounded-degree graphs, there is a trade-off between three objectives: First, the number of patterns should be small since each of them needs to be excluded to prove upper bounds. Second, the bound we obtain on the mixed page number should be small. And third, we want our results to hold for graph classes that are as large as possible, while smaller graph classes have the potential of better results in the first two objectives. In this subsection, we present explicit bounds on the mixed page number that are different trade-offs between these three objectives.

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As a base for subsequent results, we start with (ordered) matchings having a separated layout and a relatively large obstruction set. Here, a layout of a bipartite graph is called *separated* if we first have all vertices of one part of the bipartition, and then all vertices of the other part. We provide an explicit list of the patterns in Section 2.1.

► **Theorem 6** (Section 2.1). *There is a set \mathcal{P} of patterns such that for every matching M with a separated layout, if the largest pattern of \mathcal{P} in M is of size k^2 , then the mixed page number of M is at least k and at most $2k$.*

For stack layouts and queue layouts, it turned out to be a powerful tool to have only one pattern, namely twists, respectively rainbows. Close enough, we present two patterns, namely the two options for thick patterns, to play the same role for mixed linear layouts by combining twists and rainbows.

► **Theorem 7** (Section 2.2). *If the largest thick pattern in a matching M with a separated layout is a k -thick pattern, then the mixed page number of M is at least k and at most $2k^7$.*

We finish the matching case by generalizing the separated layout to an arbitrary vertex ordering. We remark that we expect the patterns to always be matchings, as is the case with twists and rainbows, and even conjecture that the same set of patterns also works for general ordered graphs.

► **Theorem 8** (Section 3). *If the largest thick pattern in an ordered matching M is a k -thick pattern, then the mixed page number of M is at least k and at most $k^{O(k)}$.*

To prove this, we work with so-called quotients of linear layouts that can capture the global structure of an (ordered) graph and show how to transfer linear layouts of a quotient to the initial graph (Lemma 17), which might be of independent interest.

With Vizing's theorem [76], all our results on matchings generalize to bounded-degree graphs, in particular we obtain the following bounds for Theorem 4.

► **Theorem 9** (Section 3). *If the largest thick pattern in an ordered graph G with maximum degree Δ is a k -thick pattern, then the mixed page number of G is at least k and at most $\Delta k^{O(k)}$.*

Note that this is indeed a specification of Theorem 4 since the upper bound shows that if there is no large thick pattern, then the mixed page number is bounded, and conversely, if the mixed page number is bounded, there is no large thick pattern due to the lower bound.

Finally, we aim for the second objective to be optimal, i.e., we aim for an obstruction set such that the mixed page number is at most k if and only if none of the forbidden patterns is contained. For $k = 2$, we show that such a finite obstruction set exists if the ordered graph is bipartite and has a separated layout, but not in the general case.

► **Theorem 10** (Section 4.1). *There is a finite set \mathcal{P} of patterns such that a bipartite graph with separated layout has mixed page number at most 2 if and only if it contains no pattern of \mathcal{P} .*

For the negative side, we remark that it is not surprising that such an exact characterization is infeasible as no such characterization is known for stack layouts. Here, we know that an ordered graph without k -twist has stack number at most $O(k \log k)$ [17] and that this bound is tight [55]. However, one might think that with a larger obstruction set, an exact characterization is possible. For $k \geq 4$ this is not true [74] but to the best of our knowledge no explicit infinite family of patterns was known to be in the obstruction set. We provide

constructions for the following theorem, which in particular proves Theorem 5 as all minimal patterns need to be included in the obstruction set. Here, minimal refers to edge-minimal for a given mixed page number (stack number), i.e., if any edge is removed, then the mixed page number (stack number) decreases.

► **Theorem 11** (Section 4.2). *There is an infinite family of minimal ordered matchings with mixed page number > 2 and an infinite family of minimal ordered matchings with stack number k for all $k > 2$.*

As some theorems have only a proof sketch, we refer to [41] for the full paper.

1.3 Related Work

Building on earlier notions [50, 64], the concepts of stack and queue number were first investigated by Bernhart and Kainen [12] in 1979 and Heath and Rosenberg [44] in 1992, respectively. Over the last three decades, there has been extensive research on these concepts leading to numerous results, primarily for planar graphs [2, 8, 9, 20, 22, 35, 65, 80, 81], but also for 1-planar graphs [10] and graphs with bounded genus [42, 57] or bounded treewidth [56]. In light of our results it is worth mentioning that there is a particular interest in bounded-degree graphs [11, 25, 78]. In this context, the vertex ordering is usually not given, so there are two steps to solve: First one needs to find a good vertex ordering and second, the resulting ordered graph is analyzed, often using the characterization with twists and rainbows. In addition, the stack and queue number for directed acyclic graphs [43, 48, 63], where the vertex ordering must respect the orientation of the edges, have been studied fruitfully, e.g., for (planar) posets [4, 32, 53, 62, 69], or upward planar graphs [34, 48, 49].

Although Heath and Rosenberg [44] already suggested a study of mixed linear layouts back in 1992, specifically conjecturing that every planar graph has a 1-stack 1-queue layout, significant progress began only quite recently when Pupyrev [68] disproved their conjecture. Subsequently, other papers have followed strengthening Pupyrev's result by showing that not even series-parallel [5] or planar bipartite graphs [35] admit a 1-stack 1-queue layout. Further investigations into more general mixed linear layouts have often been proposed [9, 11, 63]. Results on this include complete and complete bipartite graphs [3], subdivisions [24, 60, 68], and computational hardness results [19]. Very recently, Katheder, Kaufmann, Pupyrev, and Ueckerdt [51] related the queue, stack, and mixed page number to each other and asked for an improved understanding of graphs with small mixed page number both in the separated and in the general setting as, together with their results, this would fully reveal the connection between these three concepts. All these investigations have in common that it is quite difficult to analyze mixed linear layouts due to the lack of a simple characterization similar to rainbows and twists for stack and queue layouts, which is the gap we address in this paper.

An explicit investigation of rainbows and twists in linear layouts can be found from early on. In their seminal paper on queue layouts, Heath and Rosenberg [44] identified rainbows as a simple characterization for queue layouts. They showed that the queue number with respect to a fixed vertex order always equals the size of the largest rainbow, a result that has often been used to derive bounds on the queue number [2, 4, 8, 11, 22, 27, 32, 53, 69]. Similarly, twists offer a straight-forward characterization for stack layouts, although the stack number of an ordered graph is not exactly the size of the largest twist. While the size of the largest twist is clearly a lower bound on the stack number, Davies [17] recently showed an upper bound of $\mathcal{O}(k \log k)$ on the stack number, where k denotes the size of the largest twist. This very recent and asymptotically tight [55] bound, along with earlier larger upper bounds [15, 18, 40, 55] has proven useful for bounding the stack number of various graph

classes [34, 48, 49, 63]. For small k , it is known that an order without a 2-twist (that is, when $k = 1$) corresponds to an outerplanar drawing of a graph, which is a 1-stack layout. For $k = 2$ (that is, an order without a 3-twist), five stacks are sufficient and sometimes necessary [1, 54]; for $k = 3$, it is known that 19 stacks suffice [17].

Similarly, for the mixed page number we do not expect an exact characterization, but rather a characterization up to a function. For general graphs, previous results [19, 74] have shown that deciding whether an ordered matching has an s -stack q -queue layout is NP-complete for all $s \geq 4$ and $q \geq 0$, making the question for a finite set of patterns exactly characterizing s -stack q -queue layouts obsolete. However, in some special cases, such as for separated layouts of bipartite graphs or for smaller s and q , there is still potential for an exact characterization, while in the general case we aim to find a finite characterization up to a function that is as small as possible.

Our investigations start with bipartite graphs having a separated layout, which is a setting that has been widely studied (sometimes, implicitly) under different names, such as *2-track layouts*, *2-layer drawings*, or *partitioned drawings* [3, 6, 16, 21, 23, 24, 28, 29, 61, 67, 70, 79]. Initially, we further narrow down our focus to matchings with a separated layout. Separated matchings can also be viewed as permutations of the right endpoints of the edges, relative to the order of the left endpoints. In this context, the mixed page number of the ordered matching equals the minimum number of monotone subsequences into which the permutation can be partitioned. For this problem it is known that permutations that can be partitioned into a fixed number of monotone subsequences, and thus matchings with a fixed mixed page number, are characterized by a finite, though potentially large set of forbidden subsequences [31, 52, 77].

Another related topic is the extremal theory of ordered graphs [66, 72], which has also been studied from the perspective of 0-1-matrices in the case of bipartite graphs with a separated layout [36, 38, 47, 58, 71]. In this field, the focus is on determining how many edges (or 1-entries) an ordered graph (or a 0-1-matrix) can have while avoiding a specified family of patterns (or submatrices). A necessary condition for a set of patterns to characterize a fixed mixed page number is that these patterns must enforce the number of edges to be linear in the number of vertices. If the patterns are matchings, this necessary condition is met by the proof of the Füredi-Hajnal conjecture [36], which was proven by Marcus and Tardos [58].

1.4 Connections to Other Fields

Before proving our results in the subsequent sections, let us review linear layouts from different perspectives. We briefly state some connections here and refer to the full version [41] for a more detailed discussion.

Stack/Queue Perspective. Stack layouts and queue layouts capture how well an ordered graph can be processed by stacks and queues, which can be best seen in the case of an ordered matching (but also works for general ordered graphs). Here, the vertices are considered in the given order, and an edge is pushed into the data structure when its first endpoint is reached, and it is popped when its second endpoint is reached. With a stack, an ordered matching can be processed if and only if the edges obey a first-in-last-out order, i.e., if and only if no two edges have alternating endpoints. Similarly, an ordered matching can be processed with a queue if and only if the edges obey a first-in-first-out order, which is equivalent to having no two nesting edges. Thus, the minimum number of stacks or queues equals the stack number, respectively queue number, and allowing to use both as needed represents mixed linear layouts. That is, investigating linear layouts improves our understanding of how three very fundamental data structures, namely graphs, stacks, and queues, interact.

Coloring Perspective. Finding a stack layout for an ordered graph is equivalent to coloring a circle graph [17]. A *circle graph* is the intersection graph of chords of a circle, where two chords intersect if they cross but not if they share an endpoint. Hence, all findings on colorings of circle graphs [15, 18, 37, 40, 55, 74, 75] also apply to stack layouts and vice versa. Most importantly, Theorem 1 is proved in the language of circle graphs by showing that they are χ -bounded, that is their chromatic number is bounded by a function of their clique number [17, 40]. A twist in an ordered graph corresponds to pairwise crossing chords, and similarly the size of the largest rainbow is the maximum number of parallel chords, up to a factor of 2. Together, a mixed linear layout asks for a partition of the chords into two groups: The first group is supposed to have only few pairwise crossing chords and corresponds to the set of stacks in the mixed linear layout. And the second group should have only small sets of parallel chords and thereby corresponds to the set of queues.

Upward Stack Number Perspective. One of the most prominent open questions in the field of linear layouts is whether or not planar posets and upward planar graphs have bounded stack number [62]. A directed acyclic graph is called *upward planar* if it can be drawn in the plane such that the edges are crossing-free and y -monotone. Despite intensive research in this direction [13, 14, 34, 46, 48, 49, 59, 63], this problem is still widely open. Hoping for a positive answer, bounding the mixed page number is an intermediate step before answering the question for pure stack layouts. In light of this, our results provide tools in this direction.

2 Separated Layouts

We start with matchings having a separated layout, which is the base for proving Theorem 4 in Section 3. Recall that a linear layout of a bipartite graph $G = (V_1 \cup V_2, E)$ is *separated* if all vertices of V_1 precede all vertices of V_2 in the vertex ordering. First, we give a linear upper bound in Section 2.1 using a comparably large obstruction set of patterns with mixed page number k . We then reduce the number of patterns to 2 in Section 2.2 and still obtain a polynomial dependency on k .

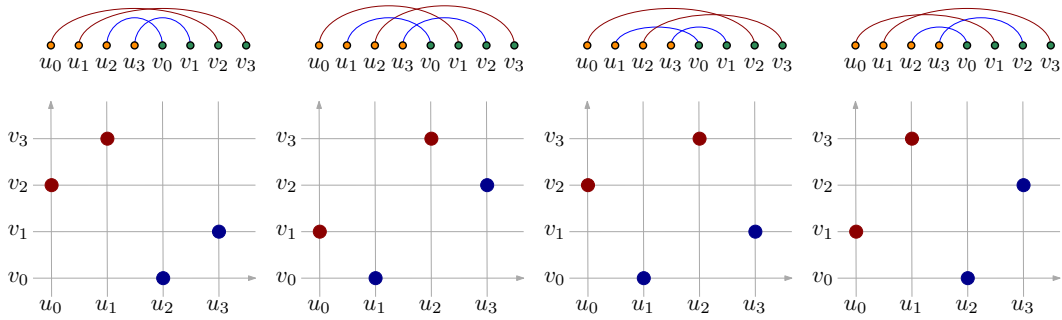
2.1 \diamond -Patterns

Consider the *grid representation* of a bipartite graph G with a fixed separated layout. That is, one part of the bipartition is represented by columns, the other by rows, and each edge is described by an x - and a y -coordinate indicating the column, respectively the row, of its endpoints. As we consider matchings, we have only one edge in each row and in each column. For two edges e_1, e_2 of G we write $e_1 \nearrow e_2$ if $x(e_1) < x(e_2)$ and $y(e_1) < y(e_2)$, i.e., if e_1 and e_2 cross. Similarly, we write $e_1 \searrow e_2$ if $x(e_1) < x(e_2)$ and $y(e_1) > y(e_2)$, i.e., if e_1 and e_2 nest. Consider a set of k^2 edges indexed by $e_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq k$. We say that the edges form a k - \diamond -*pattern* if the following holds:

- (i) $e_{i,j} \nearrow e_{i,j+1}$ for every $1 \leq i \leq k$ and $1 \leq j < k$, and
- (ii) $e_{i,j} \searrow e_{i+1,j}$ for every $1 \leq i < k$ and $1 \leq j \leq k$.

That is, in the grid representation we have k increasing sequences of length k , where the j -th elements in each chain together form a decreasing sequence for each $j = 1, \dots, k$. We denote the *size* of a k - \diamond -pattern by $k \times k$ to indicate that we have k^2 edges, where k is the parameter we are usually interested in. Notice that there are exactly four \diamond -patterns of size 2×2 ; see Figure 2.

As it turns out, \diamond -patterns are deeply connected to mixed linear layouts. Our main result on \diamond -patterns is summarized in the following theorem.



■ **Figure 2** 2- \diamond -patterns for Theorem 12.

► **Theorem 12.** *Let M be a matching with a separated layout. If the largest \diamond -pattern in M has size $k \times k$, then the separated mixed page number of G is at least k and at most $2k$.*

We translate Theorem 12 to the language of posets so we can use a result by Greene [39] to prove it in this regime. For this, we interpret a \diamond -pattern, or more generally any matching M with a separated vertex ordering, as a poset $P(M)$ whose elements are the edges of M . Since M is a matching, for every two elements, e_1, e_2 of $P(M)$ with $x(e_1) < x(e_2)$, it holds that either $e_1 \nearrow e_2$ or $e_1 \searrow e_2$. In the former case, we set $e_1 < e_2$ in $P(M)$ and in the latter we make the two elements incomparable, for which we write $e_1 \parallel e_2$. We call $P(M)$ the *poset of M* . Observe that a chain of $P(M)$ is increasing in the grid representation and corresponds to a twist in M , and an antichain is decreasing in the grid representation and corresponds to a rainbow. That is, to bound the mixed page number, we aim to cover each element of the poset by a chain or an antichain (or both) and thereby minimize the number of chains plus antichains.

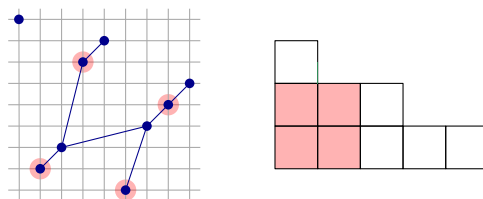
► **Observation 13.** *For the mixed page number $\text{mn}(M)$ of a matching M with separated vertex ordering we have $\text{mn}(M) \leq m$ if and only if its poset $P(M)$ can be covered by m chains and antichains.*

Now, the lower bound of Theorem 12 asks for a proof that $P(M)$ cannot be covered by less than k chains and antichains. The omitted proof of the following slightly stronger statement is straight-forward by counting the maximum number of elements in an (anti)chain.

► **Lemma 14.** *Let M be a k - \diamond -pattern and let $P(M)$ be its poset. Then every decomposition of $P(G)$ into a minimum number of chains and antichains consists either of k chains or of k antichains.*

That is, \diamond -patterns are particularly hard graphs for mixed linear layouts as they do not profit from the possibility to mix chains and antichains, respectively queues and stacks, which makes them a suitable candidate for a characterization. The remainder of this subsection is devoted to the upper bound of Theorem 12, which confirms that these candidates are, up to a factor of 2, the only reason for a large mixed page number.

The upper bound relies on insights into a Ferrer’s diagram, a standard combinatorial tool [30], that represents how many elements can be covered by a certain number of chains, respectively antichains. In general, a Ferrer’s diagram represents a partition of an integer by left-aligned rows of cells, where a summand s is represented by a row of length s and the rows are ordered by decreasing length from bottom to top. In our context, we partition the number $|P|$ of elements of a poset. Roughly speaking, the Ferrer’s diagram of a poset P consists of $|P|$ cells arranged in such a way that the number of cells in the first i rows is the



■ **Figure 3** Left: A grid representation of a matching M with edges indicating the comparabilities in the poset $P(M)$ of M . The 2- \diamond -pattern is highlighted red. Right: The Ferrer's diagram of $P(M)$ showing the partition $|P(M)| = 5 + 3 + 1$ by rows of length 5, 3 and 1. The diagram expresses that one chain can cover five elements (bottommost row), two chains can cover $5 + 3 = 8$ chains (two bottommost rows), and that three chains can cover all $5 + 3 + 1 = 9$ elements (all three rows). Note, however, that eight elements cannot be covered with two chains of length 5 and 3.

number of elements in P that can be covered by i chains, and the number of cells in the first i columns equals the number of elements that can be covered by i antichains. It is proved by Greene [39] that such diagrams exist. We refer readers not familiar with this kind of Ferrer's diagrams, sometimes also called Greene's diagrams, to the full version [41, Section 2.1] and also point to Figure 3 (right) for an example.

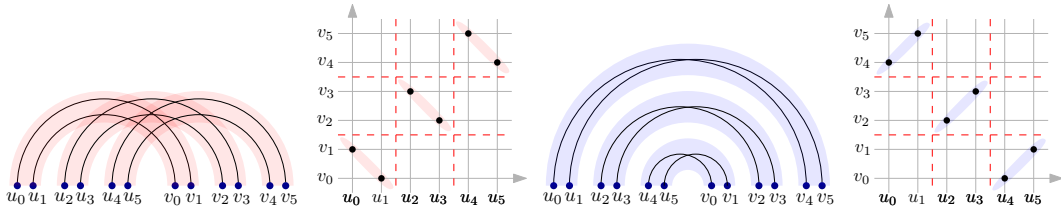
For Theorem 12, we want a bound depending on the size of the largest \diamond -pattern, so we first need to identify these patterns in the Ferrer's diagrams. Let us first discuss why this is not a trivial task. Recall that the Ferrer's diagram only indicates the number of elements that can be covered by a certain number of (anti)chains but does not associate the cells with elements of the poset as this is not always possible. For an example, consider Figure 3 where we have three rows of size 5, 3, and 1 but there is no chain decomposition with chains of lengths 5, 3, 1. That is, there is no way of writing elements into all cells of the diagram such that elements in the same row form a chain. Therefore, we cannot simply use the “elements in the $k \times k$ -square” to identify a k - \diamond -pattern.

Instead, besides the translation of the linear layout problem to posets and its Ferrer's diagrams, the core of the proof of Theorem 12 is a counting argument that allows to identify a \diamond -pattern of the same size as the largest square in the Ferrer's diagram. Having this, we may use $2k$ (anti)chains, where k is the size of the largest square instead of the \diamond -pattern, which can be done using the properties of a Ferrer's diagram (see full version [41, Section 2.1]). Finally, we remark that Theorem 12 is tight (see full version [41, Lemma 18]) and yields a 2-approximation in $O(n^{3/2} \log n)$ [33, 73].

2.2 Connection to Twists and Rainbows

Having Theorem 12, we can characterize which patterns cause a large mixed page number up to a factor of 2. In contrast to stack layouts and queue layouts, however, we have not only a single pattern like twists or rainbows, respectively, but a family of patterns. Note that all k - \diamond -patterns are necessary to forbid if we want an exact characterization as they all have mixed page number k . In this section, we give up the idea of an (almost) exact characterization in favor of reducing the number of patterns we forbid. More precisely, we point out two specific \diamond -patterns whose absence characterizes the mixed page number up to a polynomial factor.

As we may use both stacks and queues in mixed linear layouts, we combine twists and rainbows to obtain two new patterns for characterizing the mixed page number. A t -thick k -twist is obtained from a k -twist by replacing each edge by a t -rainbow. That is, we have



■ **Figure 4** From left to right: A 2-thick 3-twist with its grid representation, and a 2-thick 3-rainbow with its grid representation.

k pairwise vertex-disjoint t -rainbows, where each edge nests with the edges belonging to the same rainbow but crosses the edges of all other rainbows, see Figure 4 (left). Similarly, a t -thick k -rainbow is obtained from a k -rainbow by replacing each edge by a t -twist, see Figure 4 (right). In both cases, we call t the *thickness* and write the *size* as $k \times t$ to emphasize that we have kt edges organized as k smaller groups of size t . Throughout the paper, we often have $t = k$ and simply write k -thick rainbow, respectively k -thick twist. If we mean any of the two patterns, we say k -thick pattern.

In this section, we investigate the relation between thick patterns and \diamond -patterns. Specifically, we show that thick patterns occur if and only if \diamond -pattern occur, where the sizes are tied by a polynomial function. Together with Theorem 12 we obtain:

► **Theorem 15.** *Let M be a matching with a separated layout. If the largest thick pattern in M has size $k \times k$, then the mixed page number of M is at least k and at most $2k^7$.*

Note that the lower bound is already given by Theorem 12, so our task is to bound the mixed page number in terms of the largest thick pattern. Theorem 12 also gives an upper bound using \diamond -patterns instead of thick patterns, so we aim to show that if there is a large \diamond -pattern, than there also is a large thick pattern.

► **Proposition 16.** *Let M be a k^7 - \diamond -pattern, then M contains a k -thick pattern.*

The proof makes use of the grid representation, which is subdivided repeatedly. First, after each subdivision, we find a sufficiently large \diamond -pattern, and second these smaller patterns can be combined to a thick pattern. See the full version [41, Lemma 19] for the proof.

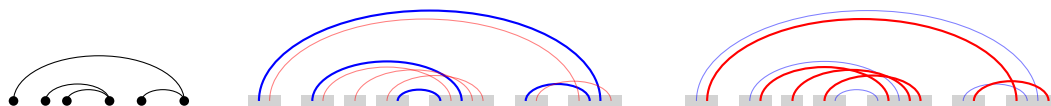
3 Non-separated Matchings and Bounded-Degree Graphs

To transfer our findings for separated layouts to the general matching case, we first need the notion of quotients. For this, consider a matching G with a fixed vertex ordering \prec together with a partition \mathcal{I} of the vertices into *intervals*, i.e., the vertices in each part are consecutive. The *quotient* G/\mathcal{I} of G and \mathcal{I} with respect to \prec is the graph obtained from contracting each interval together with the inherited vertex ordering. The following lemma shows that mixed linear layouts of quotients can be transferred to the original matching.

► **Lemma 17.** *Let $k \geq 1$, let G be a matching with a fixed vertex ordering without a $(k + 1)$ -thick pattern, let \mathcal{I} be a partition of the vertices into intervals, and let $H = G/\mathcal{I}$ be the quotient. Then we have*

$$\text{mn}(G) \leq 6\ell \cdot 2k^7(1 + 14(k + 1) \log(k + 1) + k) + 2m \in O(\ell k^8 \log(k) + m),$$

where $\ell = \text{mn}(H)$ is the mixed page number of the quotient and $m = \max_{I \in \mathcal{I}}(\text{mn}(G[I]))$ is the maximum mixed page number induced by some interval.



■ **Figure 5** A stack of the quotient H consisting of one-sided stars (left) and the matching induced in G (middle and right). Each star induces a separated pattern admitting a page assignment α with at most $2k^7$ stacks (blue) and queues (red). The set Q is formed by one queue per star (red). The stacks can be reused for distinct stars, while the edges in the queues of all stars together are partitioned into two groups – one having only small twists and one having only small rainbows. The intervals of \mathcal{I} are indicated by gray boxes.

Note that the lemma is not false for $k = 0$ but does not make sense to state as a 1-thick pattern is a 1-thick 1-rainbow or a 1-thick 1-twist, i.e., a single edge. Thus, forbidding a 1-thick pattern means that G has no edges.

Proof. First, observe that all edges with both endpoints in the same interval I can be covered by at most $\text{mn}(G[I])$ stacks plus the same number of queues. As the pages can be reused for all intervals, we have $2 \max_{I \in \mathcal{I}} (\text{mn}(G[I])) = 2m$ pages for all edges with both endpoints in the same interval.

Thus, our main task is to deal with edges in G between distinct intervals, that is with the edges of G induced by some edge of the quotient H . We only consider a single page of H , which we pay with a factor of ℓ . Now having a stack or a queue, it is well known that it can be partitioned into six one-sided star forests, i.e., for each part either all stars have their center to the left of their leaves or all to the right. That is, at the cost of a factor of 6, we may now consider a stack, respectively a queue, of H consisting of one-sided stars. Without loss of generality, we may assume that the stars have their center as their rightmost vertex. Note that a one-sided star in H induces a separated pattern in G consisting of a single interval (corresponding to the center of the star) on one side and possibly several intervals on the other. This is particularly convenient as we already know how to deal with separated matchings by Theorem 15. We fix a $2k^7$ -page assignment α for each subgraph of G induced by some one-sided star.

Stacks of H . We are now ready to construct a mixed linear layout for a subgraph of G induced by a one-sided star forest forming a stack in the layout of H , see Figure 5. First, edges in G that are assigned to a stack by α are easy to handle: As the stars do not cross, we may reuse the same $2k^7$ stacks. That is, we are left with the edges of G assigned to queues by α . Consider only one out of the up to $2k^7$ queues of each star and let Q denote the set of these edges. We pay this with a factor of $2k^7$. As the stars may nest, we cannot simply join the queues but need a more involved strategy. Observe that the edges of Q belonging to the same star in H are separated but do not nest, i.e., they form a twist. Next we aim to use these twists to either cover the edges with few pages or to find a large thick rainbow. For each twist, take the k leftmost edges (or all if there are less than k), and denote the union of all these edges of all twists by L . As the stars do not cross, the edges in L form twists of size at most k and can be covered by $14k \log(k)$ stacks [17].

For the remaining edges, we show that a rainbow implies a thick rainbow of the same size in G , and thus they can be covered by k queues. Indeed, consider a rainbow e_1, e_2, \dots in $Q - L$, where the edges nest above each other in this order. We refer to Figure 6 for an illustration of the upcoming argument. Recall that the edges of Q belonging to the same star form a twist and thus the edges in the rainbow belong to pairwise distinct stars. Let T_i



■ **Figure 6** Left: Non-crossing one-sided stars in H . Right: A rainbow $e_1, e_2, e_3 \in Q - L \subseteq E(G)$ together with the twist T_2 . Recall that e_2 is chosen such that it is the rightmost edge of T_2 . The right endpoints of T_2 are consecutive as they are in a common interval that is contracted to the center of the star in H . The left endpoints are consecutive as the respective edges in the stars do not cross and thus the edges of T_2 do not cross any edges of other stars like e_3 .

denote the $(k + 1)$ -twist consisting of e_i and the k edges in L induced by the same star. Also recall that the stars are in a stack of H and as their edges do not cross, the same holds for edges in G belonging to distinct stars. In particular, the edges of the twist T_i do not cross e_{i+1} nor e_{i-1} . Therefore, the left endpoints of T_i are between the left endpoints of e_i and e_{i+1} . For the right endpoints of T_i , recall that the stars are one-sided with the center to the very right. As such, the right endpoints of T_i belong to a common interval of \mathcal{I} , and thus are to the right of e_{i-1} . Since T_i is a twist, its right endpoints are between the right endpoints of e_{i-1} and e_i . This yields $(k + 1)$ -twists that are pairwise nesting, i.e., if there is a $(k + 1)$ -rainbow in $Q - L$, then we obtain a $(k + 1)$ -thick rainbow in G , a contradiction. Hence, there is no $(k + 1)$ -rainbow in $Q - L$ and k queues suffice for these edges.

To sum up, we have $2k^7$ stacks for the edges assigned to stacks by α , plus $14k \log(k)$ stacks for L and k queues for $Q - L$, which we do $2k^7$ times as this is the number of queues α may use. This yields $2k^7 + (14k \log(k) + k) \cdot 2k^7$ for all edges induced by one star forest in H that is in a stack in the layout of H .

Queues of H . Although not completely symmetric, the approach for a queue of H is fairly similar, which is why we refer to the full version [41, Lemma 20] here. ◀

Next we use Lemma 17 to prove our main result of this section, namely that it suffices to bound the size of the largest thick pattern to obtain bounded mixed page number. Recall that this is also necessary as k -thick patterns have mixed page number k .

► **Theorem 18.** *Let G be a matching with a fixed vertex ordering. If G does not contain a k -thick pattern, then the mixed page number of G is in $k^{O(k)}$.*

Proof. The idea is to start with G and apply Lemma 17 repeatedly, i.e., we define suitable intervals, contract them, and repeat the procedure on the quotient graph. After at most $k - 1$ steps, we show that the mixed page number of the obtained quotient graph is bounded. Applying Lemma 17 to the levels of contractions then gives that the mixed page number of G is bounded.

More detailed, we define a sequence of graphs H_1, H_2, \dots , starting with $H_1 = G$. If for some $i \geq 1$, there is no $(k + 1)$ -twist in H_i , the stack number, and thus the mixed page number of H_i is bounded by $\mathcal{O}(k \log k)$ [17] and we stop the procedure. Otherwise, we define H_{i+1} as the quotient of H_i and the partition \mathcal{I}_i into intervals I_1^i, I_2^i, \dots of H_i as follows. The first interval I_1^i starts with the first vertex of H_i in the given vertex ordering. From left to right, we add vertices to I_1^i until it contains a $(k + 1)$ -twist. In particular, the last vertex of I_1^i is the rightmost right endpoint of a $(k + 1)$ -twist and is followed by the first vertex of I_2^i . The next interval I_2^i is then defined in the same way, i.e., starting from its first vertex, it includes the minimum number of vertices such that it contains a $(k + 1)$ -twist (or all if there

is no $(k + 1)$ -twist in the remaining vertices). We continue until every vertex is in one of the intervals. Then, H_{i+1} is defined as the quotient H_i/\mathcal{I}_i . Note that every interval, except possibly the last one, contains a $(k + 1)$ -twist. On the other hand, by construction, there is no $(k + 2)$ -twist in the subgraph of H_i induced by any of the intervals, so the mixed page number within the intervals is bounded by $\mathcal{O}(k \log k)$ [17].

Next we show that the procedure stops with H_k (or before). Suppose to the contrary that there is always a $(k + 1)$ -twist in H_i for $i = 1, \dots, k$. We show that this implies a k -thick rainbow in G , which is a contradiction. That is, we now look for k -twists, one in each H_i , that nest above each other. For this, consider a $(k + 1)$ -twist with vertices $u_1, \dots, u_{k+1}, v_1, \dots, v_{k+1}$ in H_i , for $2 \leq i \leq k$, and note that the first k edges form a k -twist which nests above u_{k+1} . Now recall that every vertex of H_i (except for the last) is the result of contracting an interval of H_{i-1} containing a $(k + 1)$ -twist. Thus, a $(k + 1)$ -twist in H_i corresponds to a k -twist nesting above a $(k + 1)$ -twist of H_{i-1} . By transitivity of the nesting relation, this gives a k -thick rainbow, a contradiction.

It follows that we make at most $k - 1$ steps until H_k is the last quotient we obtain and has no further $(k + 1)$ -twist. Then, the mixed page number of H_k is bounded by $\mathcal{O}(k \log k)$. Also recall that the mixed page number within the intervals is $\mathcal{O}(k \log k)$ and thus is dominated by the quotient. Hence for H_{k-1} , Lemma 17 yields $\text{mn}(H_{k-1}) \leq \text{mn}(H_k) \cdot c \cdot k^8 \log k + \max_j \text{mn}(I_j^{k-1}) \in \mathcal{O}(k \log k \cdot k^8 \log k)$, where $c \leq 6 \cdot 2 \cdot 14 = 168$ is the constant from the big-O notation of Lemma 17. Similarly, applying Lemma 17 to H_{i-1} and $H_i = H_{i-1}/\mathcal{I}_{i-1}$ for $2 \leq i \leq k$ gives us a factor of $c \cdot k^8 \log k + \mathcal{O}(k \log k)$ each time. After $k - 1$ steps, we obtain an upper bound of $\mathcal{O}(c^{k-1} \cdot k^{8(k-1)+1} \log^k(k)) \in k^{\mathcal{O}(k)}$ on the mixed page number of $H_1 = G$. ◀

With Vizings's theorem [76], Theorem 18 generalizes to bounded-degree graphs. Note that for both theorems, the size of the largest thick pattern gives a trivial linear lower bound.

► **Theorem 19.** *Let G be an ordered graph with maximum degree Δ . If G does not contain a k -thick pattern, then the mixed page number of G is in $\Delta k^{\mathcal{O}(k)}$.*

4 Critical Graphs

In this section, we focus on the existence of a one-to-one characterization of mixed linear layouts. Note that up until now, we only identified patterns characterizing the mixed page number up to a function, i.e., we only get an upper and lower bound on the mixed page number even if we know exactly the size of the largest pattern. Here, we are attacking the question for an exact characterization, which would guarantee the exact mixed page number, given that all of the specified patterns are excluded. To this end, we introduce the concept of *critical graphs* that are minimal graphs that do not admit a mixed linear layout with a certain number of pages. Suppose $G = (V, E)$ is a graph with a fixed vertex order \prec . For $s, q \geq 0$, we call G (s, q) -critical if it does not admit an s -stack q -queue layout under \prec but every subgraph $G - e$ for all edges e does. Similarly, we define k -critical graphs as minimal graphs not admitting a layout on mixed $k = s + q$ (for some $s, q \geq 0$) pages.

We begin by investigating critical graphs for separated layouts. In the case of separated matchings, prior results [31, 52, 77] imply that the number of critical graphs is finite. However, for separated non-matchings, we can only bound the number of k -critical graphs when $k = 2$, and it remains open for larger k . In the non-separated case, prior hardness results [19, 74] imply that the number of (s, q) -critical graphs is infinite for $s \geq 4$ and $q \geq 0$, even for matchings. For $k = 2$ and for $s \geq 2$ we construct infinite sets of matchings that are k - and

	matching	non-matching
separated	finite	$k = 2$: finite (Theorem 21); $k > 2$: open
non-separated	$k = 2$: infinite (Theorem 26); $k \neq 2$: open; $0 < s < 4, q > 0$: open; $s = 0, q \geq 0$: finite; $s \geq 2, q = 0$: infinite (Theorem 25); $s \geq 4, q \geq 0$: infinite	

■ **Figure 7** An overview visualizing whether the number of k -critical and (s, q) -critical graphs is finite or infinite in the cases of separated/non-separated layouts of matchings/non-matchings. The blue cases follow immediately from previous results, which we discuss in Section 4.1 and Section 4.2.

$(s, 0)$ -critical, respectively. For all other cases, particularly for all $k \neq 2$, the question of whether there exists a finite set of patterns that exactly characterizes a mixed page number of k remains unresolved. Our results are summarized in Figure 7.

4.1 Critical Graphs for Separated Layouts

A separated matching G corresponds to a permutation $\pi(G)$. The separated mixed page number of a matching G equals the minimum number of monotone subsequences that $\pi(G)$ can be partitioned into. In this setting the following is known: permutations that can be partitioned into a fixed number of monotone subsequences are characterized by a finite set of forbidden subsequences [31, 52, 77]. Thus, for every pair (s, q) , there exists a finite number of (s, q) -critical matchings with a fixed separated layout. For separated non-matchings our main result is that there is only a finite number of 2-critical graphs.

► **Theorem 20.** *There exists a finite number of separated $(1, 1)$ -critical and 2-critical graphs.*

On the way to proving this, we additionally show that if the maximum degree of all (s, q) -critical graphs is bounded, then the total number of (s, q) -critical graphs must be finite. Further, we show that if the number of (s, q) -critical graphs is bounded for all pairs (s, q) with $s + q = k$, then the number of k -critical graphs is bounded. Therefore, to show that the number of k -critical graphs is finite, not only for $k = 2$ but for arbitrary k , it suffices to show that the maximum degree of all (s, q) -critical graphs is bounded. Then, for $(1, 1)$ -critical graphs, we show that the maximum degree is indeed bounded, so we can conclude that the number of $(1, 1)$ -critical graphs is finite.

Before proving Theorem 20, we remark that a similar characterization of graphs with a bounded separated pure stack (queue) number is straightforward. In fact, there exists a single $(s, 0)$ -critical graph and a single $(0, q)$ -critical graph for all $s, q \geq 1$. In contrast, characterizations for mixed linear layouts are significantly more complex. We computationally identified all 9 graphs that are 1-critical, all 20 graphs that are $(1, 1)$ -critical, and all 3128 graphs that are 2-critical.

Our proof uses essentially the same arguments as [77] to show that if the maximum degree in an (s, q) -critical graph G is bounded, then the total number of edges of G is bounded. The key in the arguments in [77] is that partial Boolean functions, interpreted as an assignment of the edges of a subgraph to the set of stacks or to the set of queues, can be combined to a total Boolean function corresponding to the given graph. See the full version [41, Section 4.1] for details.

The following lemma is the result of these arguments and potentially useful for proving that the number of (s, q) -critical graphs is finite for arbitrary s and q . For this, it now suffices to show that the maximum degree in every (s, q) -critical graph is bounded.

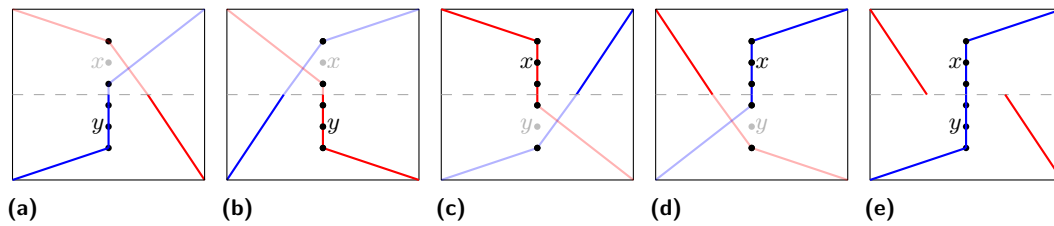


Figure 8 There are two cases each for the 1-stack 1-queue layouts of $G - x$ and $G - y$: Either the increasing sequence (blue) contains all points below x , resp. y , while the decreasing sequence (red) covers all points above (a and c), or the other way around (b and d). (e) visualizes the combination of (a) and (d) into a 1-stack 1-queue layout of G .

► **Lemma 21.** *There is a function $N^*(s, q, \Delta)$ such that for every s, q it holds that every separated (s, q) -critical graph with maximum degree Δ contains at most $N^*(s, q, \Delta)$ edges.*

Additionally, the following lemma allows us to extend results on (s, q) -critical graphs to more general k -critical graphs. Specifically, it shows that if the number of (s, q) -critical graphs is bounded for all pairs (s, q) with $s + q = k$, then the number of k -critical graphs is also bounded. The proof is not surprising and also inspired by [77] so we refer to the full version [41, Lemma 22]

► **Lemma 22.** *Suppose that the number of edges in an (s, q) -critical graph is bounded by some function $m(s, q)$ for every s, q . Then, for any k , the number of edges in a k -critical graph is bounded by $\sum_{s+q=k} m(s, q)$.*

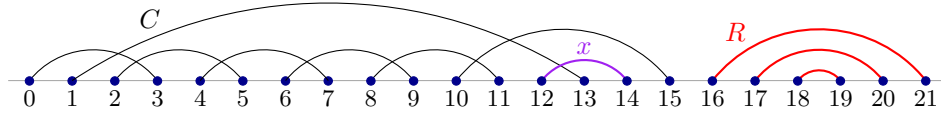
Note that in the separated case there is exactly one $(2, 0)$ -critical graph – a 3-twist – and exactly one $(0, 2)$ -critical graph – a 3-rainbow. Moreover, the following lemma, together with Lemma 21, shows that the number of $(1, 1)$ -critical graphs is also bounded. Applying Lemma 22, it then follows that the number of 2-critical graphs is finite. With Lemma 21, the only thing that is left to finish Theorem 20 is to prove that the maximum degree in any $(1, 1)$ -critical graph is bounded.

► **Lemma 23.** *Let G be a separated $(1, 1)$ -critical graph. Then $\Delta(G) < 6$.*

The following proof sketch is worked out more detailed in the full version [41, Lemma 24].

Proof Sketch. Consider the grid representation of G , i.e., vertices are represented by columns and rows, with a point in the intersection if the two vertices share an edge. Suppose that G has a vertex of degree 6, say represented by a column. Since G is critical, removing any point yields a 1-stack 1-queue graph, i.e., all remaining points can be covered by an increasing and a decreasing sequence together. We choose two such points x and y and obtain two pairs of sequences that we combine to one increasing and one decreasing sequence covering all points, i.e., all edges of G . This contradicts G being $(1, 1)$ -critical.

Figure 8 shows how two such layouts are recombined. To see that the situation indeed is as illustrated, observe that none of the sequences contains points above and below the removed point as then the point could simply be added. For the combination to work, it is crucial to choose x and y such that at least two vertices are between them and they are not the topmost or bottommost vertex. Together with the first observation, this guarantees that the order in which the sequences reach the cutting line fit together. ◀



■ **Figure 9** A forbidden edge pattern with $k = 6$ for $mn = 2$.

4.2 Critical Graphs for Non-Separated Layouts

Based on the literature, we first observe that the number of critical graphs characterizing non-separated s -stack q -queue layouts is expected to be infinite even for matchings, for $s \geq 4$. To this end, we use a hardness result stating that it is NP-complete to recognize 4-stack layout for matchings with a fixed layouts (from coloring circle graphs) [74]. Moreover, an already NP-complete mixed linear layout recognition problem with given vertex ordering remains NP-complete under addition of a stack or queue [19]. Thus, deciding whether a matching with a given vertex ordering admits an s -stack q -queue layout is NP-complete for all $s \geq 4$ and $q \geq 0$. On the other hand, a finite forbidden set of critical graphs implies a poly-time recognition, which would contradict the two mentioned hardness results under the assumption that $P \neq NP$. However, it is still interesting to construct an explicit infinite set of critical graphs, for fixed $k = s + q$, especially for $k < 4$ where the state of the art does not give whether or not the obstruction set is finite. In the following, we construct such an infinite set of $(s, 0)$ - and 2-critical graphs for $s \geq 2$.

As a first step, we give an infinite obstruction set for layouts with $s = 2, q = 0$. We use edges such that the conflicting edges (that cannot share a stack) form an odd-length cycle; clearly, the cycle requires three colors (stacks). In the full version [41, Lemma 25] this is generalized to an infinite obstruction set for all pure stack layouts with stack number at least 2.

► **Theorem 24.** *For every integer $s \geq 2$ and every $n \geq 3$, there exists an $(s, 0)$ -critical matching with at least n vertices.*

Observe that Theorem 24 does not rule out a possibility of a finite obstruction set for mixed linear layouts with $mn = 2$. The next theorem closes this gap.

► **Theorem 25.** *For every $n \geq 3$, there exists a 2-critical matching with at least n vertices.*

Proof. Let $k \geq 2$ be an even integer. We build a matching with $n = 2(k + 2) + 6$ vertices (starting with 0) having three types of edges (see Figure 9):

- $C = \{(2i, 2i + 3), 0 \leq i < k - 1\} \cup \{(2k - 2, 2k + 3)\} \cup \{(1, 2k + 1)\}$;
- a single edge $x = (2k, 2k + 2)$;
- $R = \{(n - 6, n - 1)\} \cup \{(n - 5, n - 2)\} \cup \{(n - 4, n - 3)\}$.

We claim that the graph $G_k = (\{0, \dots, n - 1\}, C \cup \{x\} \cup R)$ is 2-critical, that is, $mn(G_k) = 3$ but $mn(G_k - e) = 2$ for every edge e . First, we show that $G_k - e$ admits a mixed linear layout on two pages for every edge e , then that G_k requires at least 3 pages. We consider the three cases $e \in C$, $e = x$, and $e \in R$ separately. First, if we remove an edge from C , then the graph admits a 2-stack layout: The edges of $C \setminus \{e\}$ can be assigned to two stacks, while x and edges from R are assigned to one of the two stacks. Second, if we remove edge x , then the graph admits a 1-stack 1-queue layout: The “long” edge $(1, 2k + 1)$ of C together with edges from R are assigned to a stack, while the “short” edges $(2i, 2i + 3)$ of C are assigned to a queue. And third, if we remove an edge from R , then the graph admits a 2-queue layout: One queue contains an edge from R , edge x , and the “long” edge $(1, 2k + 1)$ from C . Another queue contains an edge from R along with the “short” edges $(2i, 2i + 3)$ from C .

Finally, graph G_k does not admit a layout on two (mixed) pages, since on one hand (i) it cannot be assigned to two queues (R forms a 3-rainbow), and (ii) it cannot be assigned to two stacks (C is an odd cycle). On the other hand, (iii) G_k cannot be assigned to a stack and a queue, since otherwise the “long” edge $(1, 2k + 1)$ of C is in the queue (it crosses two edges forming a 2-rainbow) and hence, all edges covered by the “long” edge would be in a stack, which implies a crossing between two such stack edges. ◀

To summarize, we know that the number of (s, q) -critical graphs is infinite if $s \geq 4$, even for matchings, due to previous hardness results [19, 74]. Furthermore, Theorem 24 and Theorem 25 show that for $s \geq 2$ the numbers of $(s, 0)$ - and 2-critical matchings are also infinite, providing a construction for an infinite, though not necessarily complete, set of such graphs. Recall that the $(0, q)$ -critical graphs are exactly the $(q + 1)$ -rainbows [44], i.e., there is exactly one such graph for every $q \in \mathbb{N}$. What remains open is the case $s < 4$ and $q \geq 1$, as well as the more general k -critical graphs for any $k \neq 2$. Notably, it is even unknown whether the set of 1-critical graphs is finite. For $s = q = 1$, we conjecture that there there is a finite number of critical graphs and present candidates in the full version [41, Conjecture 27].

5 Conclusions

In this paper we made the first steps towards characterizing mixed linear layouts of ordered graphs via forbidden patterns. The most prominent open question for fully resolving Open Problem 3 is to transfer Theorem 4 to general graphs with unbounded maximum degree. We remark that the proofs in Section 3 work similarly for general ordered graphs; hence, the challenge is to bound the mixed page number of bipartite graphs in the separated settings. We expect that thick patterns is the correct choice even for large-degree graphs.

Another interesting question is whether separated mixed linear layouts are characterized by a finite obstruction set; that is, whether the statement of Section 4.1 holds for all k . Again we expect a positive answer here, and observe that for a proof, it is sufficient to bound the maximum degree of separated k -critical graphs for $k \geq 3$, that is, extend Lemma 23 for the case.

Finally, we highlight a possible application of the studied characterizations. Is the mixed page number of upward planar (possibly, bounded-degree) graphs bounded by a constant? To answer the question affirmatively, it is sufficient for a graph to construct a topological ordering containing no k -thick pattern for some $k \in \mathbb{N}$.

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