Faster Algorithms on Linear Delta-Matroids

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Abstract

We present new algorithms and constructions for linear delta-matroids. Delta-matroids are generalizations of matroids that also capture structures such as matchable vertex sets in graphs and path-packing problems. As with matroids, an important class of delta-matroids is given by linear delta-matroids, which generalize linear matroids and are represented via a "twist" of a skew-symmetric matrix. We observe an alternative representation, termed a *contraction representation* over a skew-symmetric matrix. This representation is equivalent to the more standard twist representation up to $O(n^\omega)$-time transformations (where n is the dimension of the delta-matroid and $\omega < 2.372$ the matrix multiplication exponent), but it is much more convenient for algorithmic tasks. For instance, the problem of finding a max-weight feasible set now reduces directly to finding a max-weight basis in a linear matroid. Supported by this representation, we provide new algorithms and constructions for linear delta-matroids. In particular, we show that the *union* and *delta-sum* of linear delta-matroids are again linear delta-matroids, and that a representation for the resulting delta-matroid can be constructed in randomized time $O(n^\omega)$ (or more precisely, in $O(n^\omega)$ field operations, over a field of size at least $\Omega(n \cdot (1/\varepsilon))$, where $\varepsilon > 0$ is an error parameter). Previously, it was only known that these operations define delta-matroids. We also note that every *projected* linear delta-matroid can be represented as an *elementary* projection. This implies that several optimization problems over (projected) linear delta-matroids, including the coverage, delta-coverage, and parity problems, reduce (in their decision versions) to a single $O(n^\omega)$-time matrix rank computation. Using the methods of Harvey, previously applied by Cheung, Lao and Leung for linear matroid parity, we furthermore show how to solve the search versions in the same time. This improves on the $O(n^4)$-time augmenting path algorithm of Geelen, Iwata and Murota, albeit with randomization. Finally, we consider the maximum-cardinality delta-matroid intersection problem (equivalently, the maximum-cardinality delta-matroid matching problem). Using Storjohann's algorithms for symbolic determinants, we show that such a solution can be found in $O(n^{\omega+1})$ time. This provides the first (randomized) polynomial-time solution for the problem, thereby solving an open question of Kakimura and Takamatsu.

1 Introduction

Matroids are important unifying structures in many parts of computer science and discrete mathematics, abstracting and generalizing notions from linear vector spaces and graph theory; see, e.g., Oxley [30] and Schrijver [32]. Formally, a matroid is a collection of *independent sets*, subject to particular axioms (see below). A maximum independent set is a *basis*. Among

other things, matroids are a very useful source of algorithmic meta-results, since there are many problems on matroids which admit efficient, general-purpose algorithms – such as the greedy algorithm for finding a max-weight basis, generalizing algorithms for max-weight spanning trees; or MATROID INTERSECTION, the problem of finding a maximum common feasible set in two given matroids, which generalizes bipartite matching.

An important class of matroids are *linear matroids*, where independence in the matroid is represented by the column space of a matrix. Linear matroids enjoy many properties not shared by generic matroids. For example, the famous MATROID PARITY problem, which generalizes the matching problem in general graphs, is known to be intractable in the general case but efficiently solvable over linear matroids [26]. In addition, linear representations, as a compact representation of combinatorial information, have seen many applications in parameterized complexity for purposes of *sparsification* and *kernelization* [24, 25], and algebraic algorithms over linear matroids have proven a very useful general-purpose tool in FPT algorithms [14, 12] (cf. [8, 15]).

Delta-matroids are a generalization of matroids, where, informally, the notion of bases is replaced by the notion of *feasible sets*, which satisfy an exchange axiom similar to matroid bases but need not all have the same cardinality. Delta-matroids were introduced by Bouchet (although similar structures were independently defined by others), and have connections to multiple areas of computer science such as structural and topological graph theory [28], constraint satisfaction problems [13, 22], matching and path-packing problems and more. Like with matroids, there is also a notion of *linear delta-matroids*, where the feasible sets are represented through a skew-symmetric matrix. These generalize linear matroids, although this fact (or indeed the fact that skew-symmetric matrices define delta-matroids) is not elementary [17]. Delta-matroids (linear or otherwise) are remarkably flexible structures, in that there are many ways to modify or combine given delta-matroids into new delta-matroids, including twisting (partial dualization), contraction and deletion, existential projection, and unions and delta-sums of delta-matroids (all described below).

Similarly to matroids, there is also a range of generic problems that have been considered over delta-matroids, including delta-matroid intersection, partition, and parity problems. Unfortunately, due to the generality of delta-matroids, these problems are all intractable in the general case, since they generalize matroid parity. However, they are tractable on linear delta-matroids, where Geelen et al. [17] gave an algorithm (and a corresponding min-max theorem) with a running time of $O(n^4)$, and $O(n^{\omega+1})$ using fast matrix multiplication. However, other variants remain open. Kakimura and Takamatsu [21] considered the maximum cardinality version of delta-matroid parity (as opposed to the result of Geelen et al. [17], which is more of a *feasibility* or *minimum error* version). They gave a solution for a restricted class of linear and projected linear delta-matroid, but left the general case open. Furthermore, the natural weighted optimization variants of the above appear completely open.

In this paper, we show new constructions of linear delta-matroids and new and faster randomized algorithms for the aforementioned problems on linear and projected linear delta-matroids. In particular, we show a new representation variant for linear delta-matroids – dubbed *contraction representation*, as opposed to the standard *twist representation* – which appears more amenable to efficient algorithms. Using this representation, we show for the first time that unions and delta-sums of linear delta-matroids (represented over a common field $\mathbb{F}$) define *linear* delta-matroids, and that a representation can be constructed in randomized polynomial time. We also show new algorithmic results, including solving the search version of LINEAR DELTA-MATROID PARITY (LINEAR DM PARITY for short) in $O(n^{\omega})$ field operations[1]

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[1] Throughout, we give our running times as field operations. If the field has size $n^{O(1)}$, as in most applications, then this is just polylogarithmic overhead.

and giving the first (randomized) polynomial-time algorithm for the maximum cardinality version of the problem in $O(n^{\omega+1})$ field operations, thereby settling an open question from Kakimura and Takamatsu [21].

1.1 Introduction to delta-matroids

Before we describe our results in detail, let us review some background on delta-matroids. For more material, we refer to the survey by Moffatt [28].

Like matroids, delta-matroids are formally defined as set systems satisfying particular axioms. Formally, a delta-matroid is a pair $D = (V, \mathcal{F})$ where V is a ground set and $\mathcal{F} \subseteq 2^V$ a non-empty collection of subsets of V, referred to as *feasible sets* in D, subject to the following *symmetric exchange axiom*:

For all $F_1, F_2 \in \mathcal{F}$ and $x \in F_1 \Delta F_2$ there exists $y \in F_1 \Delta F_2$ such that $F_1 \Delta \{x, y\} \in \mathcal{F}$,

where Δ denotes symmetric difference.

It should be enlightening to compare this to the definition of matroids. Formally, a matroid is most commonly defined as a collection of *independent sets*; i.e., a matroid is defined as a pair $M = (V, \mathcal{I})$ where V is the ground set and $\mathcal{I} \subseteq 2^V$ is a collection of sets, referred to as independent sets in M, subject to (1) $\emptyset \in \mathcal{I}$; (2) if $B \in \mathcal{I}$ and $A \subset B$ then $A \in \mathcal{I}$; and (3) if $A, B \in \mathcal{I}$ with $|A| < |B|$ then there exists an element $x \in B \setminus A$ such that $A + x \in \mathcal{I}$. The second condition encodes that $(V, \mathcal{I})$ is an *independence system*. However, matroids can also be equivalently defined from just the collection of *maximal* independent sets, known as *bases*. Under this definition, a matroid is a pair $M = (V, \mathcal{B})$ where $\mathcal{B} \subseteq 2^V$ is a non-empty collection of bases, subject to the *basis exchange property*:

For all $A, B \in \mathcal{B}$ and $x \in A \setminus B$ there exists $y \in B \setminus A$ such that $A\Delta\{x, y\} \in \mathcal{B}$.

In particular, all bases of a matroid have the same cardinality. Thus, delta-matroids can be seen as the relaxation of matroids where the feasible sets (analogous to the bases) need not all have the same cardinality. In fact, a delta-matroid where all feasible sets have the same cardinality is precisely a matroid (represented as a set of bases). A similar statement holds for independent sets – the independent sets of a matroid form the feasible sets of a delta-matroid, and a delta-matroid which is an independence system is precisely a matroid in this sense – but the formulation from the set of bases is standard and more convenient.

As a further illustration, consider the case of graph matchings. Let $G = (V, E)$ be a graph. The *matching matroid* of G is a matroid over ground set V where a set $B \subseteq V$ is a basis if and only if it is the set of endpoints of a maximum matching of G. Correspondingly, the independent sets $S \subseteq V$ of the matching matroid are vertex sets that can be covered by a matching. On the other hand, the *matching delta-matroid* over G is the delta-matroid where a set $S \subseteq V$ is feasible if and only if $G[S]$ has a perfect matching. Thus, the maximal feasible sets of the matching delta-matroid form the bases of the matching matroid, but clearly, the matching delta-matroid captures more of the structure of G than the matching matroid does.

Linear delta-matroids. As with matroids, an important class of delta-matroids are *linear delta-matroids*. A matrix A is *skew-symmetric* if $A^T = -A$. Let A be a skew-symmetric matrix with rows and columns indexed by a set V. Then A defines a delta-matroid $\mathbf{D}(A) = (V, \mathcal{F})$ where for $S \subseteq V$ we have $S \in \mathcal{F}$ if and only if the principal submatrix of A indexed by S, denoted by $A[S]$, is non-singular. We refer to $\mathbf{D}(A)$ as a *directly represented* linear delta-matroid. More generally, the *twist* of a delta-matroid $D = (V, \mathcal{F})$ by a set $S \subseteq V$, denoted $D\Delta S$, is the delta-matroid with feasible sets

$$\mathcal{F}\Delta S := \{F\Delta S \mid F \in \mathcal{F}\}.$$

It is easy to check that $D\Delta S$ is a delta-matroid. The twisting operation is also known as *partial dualization*, since the twist $D^* := D\Delta V$ corresponds to the dualization M^* of a matroid M. A general representation of a linear delta-matroid is given as $D = \mathbf{D}(A)\Delta S$ for some skew-symmetric matrix A and twisting set S. A delta-matroid D is *even* if all feasible sets have the same cardinality modulo 2; all linear delta-matroids are even. In addition, we consider *projected linear delta-matroids*, which is a delta-matroid $D = (V, \mathcal{F})$ defined via existential projection over a set X from a larger linear delta-matroid $D' = \mathbf{D}(A)\Delta S$ over ground set $V \cup X$. We denote this $D = D'|X$, where D has the feasible set

$$\mathcal{F} = \{F \setminus X \mid F \in \mathcal{F}(D')\}.$$

As a canonical example, the matching delta-matroid of a graph G is directly represented by the Tutte matrix over G. The set of bases of a linear matroid forms a linear delta-matroid, and the independent sets form a projected linear delta-matroid, under natural representations (see Section 3.1 of the full version of the paper).

Underpinning algorithms on linear delta-matroids are a number of fundamental operations on skew-symmetric matrices. For a skew-symmetric matrix A indexed by V and a set $S \subseteq V$ such that $A[S]$ is non-singular, there is a *pivoting* operation that constructs a new skew-symmetric matrix $A' = A * S$ such that for any $U \subseteq V$, $A'[U]$ is non-singular if and only if $A[S\Delta U]$ is. Via this operation, linear delta-matroids are closed under the *contraction* operation D/T as well as *deletion* $D \setminus T$ (see Section 2 for the definitions). Another fundamental property of skew-symmetric matrices is the *Pfaffian*, defined as follows. Let A be a skew-symmetric matrix with rows and columns indexed by V. The *support graph* of A is the graph $G = (V, E)$ where $uv \in E$ if and only if $A[u, v] \neq 0$. Then the Pfaffian of A is defined as

$$\operatorname{Pf} A = \sum_M \sigma(M) \prod_{e \in M} A[u, v],$$

where M ranges over all perfect matchings in G and $\sigma(M) \in \{1, -1\}$ is a sign term. It holds that $\det A = (\operatorname{Pf} A)^2$, thus A is non-singular if and only if $\operatorname{Pf} A \neq 0$. Via this connection to matchings, the Pfaffian forms a link between the combinatorial and algebraic aspects of linear delta-matroids, in a way that is often exploited in this paper. The Pfaffian also enjoys some useful algebraic properties, such as the Pfaffian sum formula and the Ishikawa-Wakayama formula, with clear combinatorial interpretations. See Section 2 for details.

1.2   Our results

We show a range of results regarding the representation and construction of linear delta-matroids, and new and faster algorithms for computational problems over them. We discuss these in turn.

Representations and constructions.   Our first result, which supports the others, is the introduction of a new representation for linear delta-matroids. Recall that a linear delta-matroid $D = (V, \mathcal{F})$ is represented as $D = \mathbf{D}(A)\Delta S$ for a skew-symmetric matrix A with rows and columns indexed by V. We refer to this as a *twist representation*. Although this representation is intimately connected to the structure of delta-matroids, it is less convenient for algorithmic purposes. For this, we introduce the *contraction representation*, representing a linear delta-matroid $D = (V, \mathcal{F})$ as $D = \mathbf{D}(A)/T$ for a skew-symmetric matrix A indexed by $V \cup T$. Thus, a set $F \subseteq V$ is feasible in D if and only if $A[F \cup T]$ is non-singular.

We show that the representations are equivalent, and given a representation in one form, we can efficiently and deterministically construct one in the other; see Section 3.1. Thus contraction representations do not change the class of representable delta-matroids; however, we find that the contraction representation is more compatible with the algorithmic methods of linear algebra.

Next, we consider two methods of composing linear delta-matroids. Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be given delta-matroids (padding the ground sets with dummy elements if necessary so that they are defined over the same ground set V). The *union* $D_1 \cup D_2$ is the delta-matroid $D = (V, \mathcal{F})$ where

$$\mathcal{F} = \{F_1 \cup F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, F_1 \cap F_2 = \emptyset\},$$

i.e., the feasible sets in D are the disjoint unions of feasible sets in D_1 and D_2. Additionally, the *delta-sum* $D_1 \Delta D_2$ is defined as the delta-matroid $D = (V, \mathcal{F})$ with feasible sets

$$\mathcal{F} = \{F_1 \Delta F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\},$$

where Δ denotes symmetric difference. Bouchet [2] and Bouchet and Schwärzler [4] showed that $D_1 \cup D_2$ and $D_1 \Delta D_2$ are delta-matroids. Using properties of Pfaffians and the contraction representation, we show that furthermore, the union and delta-sum of linear delta-matroids are linear delta-matroids.

The construction is randomized, and takes an error parameter $\varepsilon > 0$ which controls the size of the field that the output delta-matroid is represented over. For this purpose, we say that an algorithm constructs an *ε-approximate representation* of a delta-matroid $D = (V, \mathcal{F})$ if it constructs a representation of a delta-matroid $D' = (V, \mathcal{F}')$ where $\mathcal{F}' \subseteq \mathcal{F}$ and for every $F \in \mathcal{F}$ the probability that $F \in \mathcal{F}'$ is at least $1 - \varepsilon$. Setting $\varepsilon = O(1/2^n)$ where $n = |V|$ gives a representation that with good probability is correct for all subsets. However, this leads to a prohibitive field size, with significant overhead cost per field operation. Thus, for algorithmic applications, a smaller value of ε may be faster and sufficient.

▶ **Theorem 1.** *Let D_1 and D_2 be delta-matroids represented over a common field $\mathbb{F}$, and let $\varepsilon > 0$ be given. Let $\mathbb{F}'$ be an extension field of $\mathbb{F}$ with at least $n \cdot \lceil 1/\varepsilon \rceil$ elements. Then the delta-matroid union $D_1 \cup D_2$ and delta-sum $D_1 \Delta D_2$ are linear, and ε-approximate representations over $\mathbb{F}'$ can be constructed in $O(n^2)$ respectively $O(n^\omega)$ field operations.*

Algorithms. As a warm-up, we first consider the problem of finding a max-weight feasible set in a given delta-matroid $D = (V, \mathcal{F})$ with element weights $w \colon V \to \mathbb{R}$. Note that the weights may be negative, and since not all feasible sets have the same cardinality, unlike in matroids, they cannot simply be raised to be non-negative. Bouchet [3, 1] showed that there is a variant of greedy algorithm that solves this problem using only separation oracle calls. However, this requires $O(n)$ separation oracle calls, each of which requires $O(n^\omega)$ field operations in linear representation. We show that, using the contraction representation, the max-weight feasible set problem in a linear delta-matroid reduces to finding a max-weight column basis of an $O(n) \times O(n)$ matrix, which can be done significantly faster.

▶ **Theorem 2.** *Let $D = (V, \mathcal{F})$ be a linear or projected linear delta-matroid. In $O(n^\omega)$ field operations, we can find a max-weight feasible set in D.*

For more intricate questions, the literature contains a range of problems over delta-matroids [3, 17, 29]. The most important are arguably the following.

- DM INTERSECTION: Given delta-matroids $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$, find a common feasible set $F \in \mathcal{F}_1 \cap \mathcal{F}_2$.
- DM PARITY: Given a delta-matroid $D = (V, \mathcal{F})$ and a partition Π of V into pairs, is there a feasible set in D which is the union of pairs? More generally, find a feasible set $F \in \mathcal{F}$ to minimize the number of broken pairs, i.e., the number of pairs $p \in \Pi$ with $|p \cap F| = 1$.

Recall that the matroid versions differ in complexity; MATROID INTERSECTION is tractable in the oracle model while MATROID PARITY is intractable in general but tractable for linear matroids. However, due to the flexibility of delta-matroids, DM INTERSECTION and DM PARITY are equivalent under efficient transformations [17], hence both intractable in general. Given an instance (D_1, D_2) of DM INTERSECTION, a reduction to DM PARITY is immediate by constructing the disjoint union of D_1 and D_2 [17]. In the other direction, given a delta-matroid $D = (V, \mathcal{F})$ and a partition Π of V into pairs, let D_Π be the matching delta-matroid of the graph with edge set Π. Thus the feasible sets of D_Π are precisely the sets $S \subseteq V$ with no broken pairs, and the DM PARITY instance (D, Π) has a *perfect* solution – i.e., one with no broken pairs – if and only if D and D_Π have a common feasible set. For linear-delta-matroids the problems are solvable in $O(n^{\omega+1})$ time due to Geelen et al. [17].

Additionally, the following problems have been considered. Again, due to the flexibility of delta-matroids, these problems are related to one another; see [17, 29]. The previously fastest algorithm for all of them is by Geelen et al. [17] at $O(n^{\omega+1})$ time. Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be given delta-matroids.

- DM COVERING: Given D_1 and D_2, find $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ with $F_1 \cap F_2 = \emptyset$ to maximize $|F_1 \cup F_2|$
- DM DELTA-COVERING: Given D_1 and D_2, find $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ to maximize $|F_1 \Delta F_2|$
- DM PARTITION: Given D_1 and D_2, find a partition $V = P \cup Q$ such that $P \in \mathcal{F}_1$ and $Q \in \mathcal{F}_2$

The variant of DM COVERING where the disjointness constraint is dropped reduces to MATROID UNION, since the maximal feasible sets of a delta-matroid form a matroid, hence is of less interest for delta-matroids.

Using the methods of the previous subsection, the decision versions of the above for linear and projected linear delta-matroids all reduce to computing the rank of an $O(n) \times O(n)$ skew-symmetric matrix. Indeed, consider DM DELTA-COVERING. Assume that D_1 and D_2 are given in some linear representation and let $D = \mathbf{D}(A)/T = D_1 \Delta D_2$. Then a set $F \subseteq V$ is a solution $F = F_1 \Delta F_2$ to the delta-covering problem if and only if $F \cup T$ is a basis of A. Similarly, DM COVERING reduces to finding a maximum feasible set in $D_1 \cup D_2$. DM INTERSECTION and DM PARTITION are asking whether $\emptyset$ is feasible in $D_1 \Delta D_2$ and $D_1 \cup D_2$, respectively. For DM PARITY, as above let the input be (D, Π) and construct the delta-matroid D_Π. Then DM PARITY reduces to finding the rank of $D \Delta D_\Pi$ (or indeed $D \cup D_\Pi$). Thus the decision versions of all the above problems can be solved by a randomized algorithm using $O(n^\omega)$ field operations given linear representations of D resp. D_1 and D_2 thanks to Theorem 2.

For general delta-matroids, these problems are as hard as MATROID PARITY: Clearly, DM PARITY is as hard as MATROID PARITY. Given an instance (D, Π) of DM PARITY, let $D_1 = D$ and $D_2 = D_\Pi$. Then, the following is equivalent:

- The instance (D, Π) has a perfect solution for DM PARITY
- (D_1, D_2) has a solution of cardinality $V(D)$ for DM COVERING
- (D_1, D_2) has a solution of cardinality $V(D)$ for DM DELTA-COVERING
- (D_1, D_2) is a yes-instance of DM PARTITION

This shows that these problems are all intractable in the oracle model. For linear delta-matroids, achieving $O(n^\omega)$ time is arguably the best possible.

For the search versions, the situation changes slightly. Although these problems can be reduced to straightforward questions about non-singularity or rank of a skew-symmetric matrix A the resulting solution (e.g., a basis of A) provides only limited insight into the solution of the original problem. For example, if (D, Π) is an instance of DM PARITY, then the rank of $D\Delta D_\Pi$ tells us the number of broken pairs in an optimal solution (and a maximum feasible set tells us the set of broken pairs in such a solution), it does not give us the feasible set $F \in \mathcal{F}(D)$ that is the "actual" solution. Similarly, for DM COVERING and DM DELTA-COVERING the above reduction gives us the set $F = F_1 \cup F_2$ respectively $F = F_1\Delta F_2$ but not the pair (F_1, F_2). We refer to this (the set F for DM PARITY and the pair (F_1, F_2) for the other four problems) as the *witness*.

The algorithm of Geelen et al. [17] actually computes the witness for LINEAR DM PARITY, and can be implemented in $O(n^{\omega+1})$ field operations. Applying self-reducibility over the above-mentioned representation to find a witness would reproduce the same running time. Using methods of Harvey [18] we show the following improved result. These methods also underpin the currently fastest algorithm for linear matroid parity [6]. The condition of field size is for simplicity; given representations over a common field $\mathbb{F}$ we can easily move to a large enough extension field of $\mathbb{F}$.

▶ **Theorem 3.** *Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be (projected) linear delta-matroids over a common field with $\Omega(n^3)$ elements. For a feasible set $F \subseteq V$ in $D_1\Delta D_2$, we can, with high probability, find feasible sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ with $F_1\Delta F_2 = F$ in $O(n^\omega)$ field operations.*

Via the reductions given above, this lets us find a witness for all problems considered. For LINEAR DM COVERING, we would first find the optimal set $F = F_1 \cup F_2$ as above, then solve LINEAR DM PARTITION for the instance induced by F; for the remaining problems the solution is straightforward. See the full version of the paper for details.

▶ **Corollary 4.** *The following search problems can be solved in $O(n^\omega)$ field operations with high probability, given (projected) linear delta-matroids on ground sets of n elements represented over a common field with $\Omega(n^3)$ elements: DM COVERING; DM DELTA-COVERING; DM INTERSECTION; DM PARTITION; and DM PARITY.*

Finally, we consider the *weighted* versions of the above problems. Again, the picture becomes slightly different. For DM PARTITION, there is no sensible weighted version. For DM COVERING and DM DELTA-COVERING, the natural weight is the weight of the solution set F, in which case the problem is solved in strongly polynomial time of $O(n^\omega)$ field operations using Theorem 2 and 3. For DM PARITY, we would attach weights to the pairs, and consider two weighted versions. In the first version, we want to minimize the weight of the broken pairs; this problem is solved in $O(n^\omega)$ field operations using Theorem 2 and 3. For the more interesting version, we assume that there exists a perfect delta-matroid parity solution, and we wish to maximize the weight of such a set. This version, finally, is equivalent to the weighted version of DM INTERSECTION, which we focus on, and directly generalizes WEIGHTED MATROID PARITY. We use the matrix representation of the problem to construct a solution via algebraic methods.

As a special case, even the unit weight version, MAXIMUM LINEAR DM INTERSECTION, is an interesting problem which up to now has had no polynomial-time solution. Kakimura and Takamatsu [21] asked this as an open problem, and provided algorithms for some special cases of it. We solve the general case with a randomized algorithm.

▶ **Theorem 5.** *Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be (projected) linear delta-matroids over a common field with $\Omega(n^2)$ elements and let $w \colon V \to \{1, \ldots, W\}$ be element weights. In $O(Wn^\omega)$ field operations we can find with high probability the maximum weight of a common feasible set $F \in \mathcal{F}_1 \cap \mathcal{F}_2$. In particular, we can find the maximum cardinality of $|F|$ in $O(n^\omega)$ field operations with high probability.*

By self-reducibility, the search version can thus be solved by a factor $O(n)$ overhead.

▶ **Corollary 6.** MAXIMUM LINEAR DM INTERSECTION *can be solved with high probability in* $O(n^{\omega+1})$ *field operations.* WEIGHTED LINEAR DM INTERSECTION *and* WEIGHTED PERFECT DM PARITY *with maximum element weight W can be solved with high probability in* $O(Wn^{\omega+1})$ *field operations.*

Removing the overhead on this result appears significantly harder. Indeed, a similar overhead exists for algorithms for weighted linear matroid parity [6], and even removing the overhead for WEIGHTED PERFECT MATCHING was significantly non-trivial [9]. We leave this as a (challenging) open question.

Applications.   We now review some applications of the results. Not all of these results are new, but they serve to demonstrate the applicability of the setting.

One area of application is graph matching and factor problems. In the general factor problem, we are given a graph $G = (V, E)$, and a set of integers $f(v) \subseteq \mathbb{N}$ for each vertex $v \in V$. The task is to find a spanning subgraph $H = (V, F)$ such that $\deg_H(v) \in f(v)$ for every vertex $v \in V$. Cornuéjols [7] showed that this problem is polynomial-time solvable if each $f(v)$ has gaps of length at most 1, but becomes NP-hard otherwise.

For each $v \in V$, let $\delta(v)$ be the set of edges incident with v and define $\mathcal{F}_v = \{S \subseteq \delta(v) \mid |S| \in f(v)\}$. Under the gap-1 condition, $D_v = (\delta(v), \mathcal{F}_v)$ forms a delta-matroid for every $v \in V$. We refer to D_v as a *symmetric* delta-matroid.

The general factor problem can be reformulated as a DM INTERSECTION problem. We define two delta-matroids on the ground set $E_2 = \{(e, v) \mid e \in E, v \in e\}$, where each edge $uv \in E$ is represented by (uv, u) and (uv, v). Let D_E be the matching delta-matroid of G', where G' is the graph on E_2 with edges between the pairs (uv, u) and (uv, v) for each edge $uv \in E$. Furthermore, let D_f be the direct sum of symmetric delta-matroids $D_v = (\partial_v, \mathcal{F}_v)$ with $\partial_v = \{(uv, v) \mid uv \in E\}$ and $\mathcal{F}_v = \{S \subseteq \partial_v \mid |S| \in f(v)\}$. Then the intersection of D_E and D_f captures the general factor problem.[2]

When the symmetric delta-matroids correspond to cases where $f(v) = \{a, a + 2, \ldots, b\}$ and $f(v) = \{a, a + 1, \ldots, b\}$, they are linear and projected linear, respectively. The associated factor problems are known as *parity (a, b)-factors* and *(a, b)-factors*. Thus, these problems can be reduced to LINEAR DM INTERSECTION, and solved via Corollary 4 (although a naive formulation gives only time $O(m^\omega)$ here). The algebraic formulation of Gabow and Sankowski [16] for the f-factor problem, where $|f(v)| = 1$ for all $v \in V$, can essentially be derived from this method in a generic way using the Ishikawa-Wakayama formula (see Lemma 10), as the corresponding systems form matroids. See also recent work on *weighted general factors* [10, 23].

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[2] We find it interesting that in this formulation, the GENERAL FACTOR problem is tractable if and only if every set system D_v, $v \in V$ forms a delta-matroid. For more such occurrences, see the BOOLEAN EDGE CSP [13, 22] and BOOLEAN PLANAR CSP [11, 22] problems, both of which appear (roughly speaking) to be interestingly tractable if and only if the constraints form delta-matroids in natural ways.

For another example, let $G = (V, E)$ be a graph and $T \subseteq V$ a set of terminals. Let $\mathcal{S}$ be a partition of T. An $\mathcal{S}$-*path packing* is a vertex-disjoint packing of paths where every path has endpoints in distinct parts of $\mathcal{S}$ and internal vertices disjoint from T. A classical theorem of Mader shows a min-max theorem characterizing the maximum number of paths in an $\mathcal{S}$-path packing, generalizing Menger's theorem and the Tutte-Berge formula; see Schrijver [32]. Wahlström [36] recently showed the following. Let a set $F \subseteq T$ be *feasible* if there is an $\mathcal{S}$-path packing whose set of endpoints is precisely F. Then $D = (T, \mathcal{F})$ is a linear delta-matroid. Then, via LINEAR DM INTERSECTION as above, we can solve the following "$\mathcal{S}$-factor" problem: Given G, $\mathcal{S}$ and a prescribed set of degrees $f(T_i) \in \mathbb{N}$, $T_i \in \mathcal{S}$, is there an $\mathcal{S}$-path packing in $(G, \mathcal{S})$ where precisely $f(T_i)$ paths have an endpoint in T_i? The generalizations to (a, b)-factors and (a, b)-parity factors can be handled similarly.

Structure of the paper. Section 2 contains all definitions. Section 3 gives results about new representations, and Section 4 gives the algorithmic results. Section 5 concludes the paper. For missing proofs (for results marked with ★) and additional results, including a more extensive related work review, see the full version of the paper.

2 Preliminaries

For two sets A, B, we let $A \Delta B = (A \setminus B) \cup (B \setminus A)$ denote their symmetric difference.

For a matrix A and a set of rows S and columns T, we denote by $A[S, T]$ the submatrix containing rows S and columns T. If S contains all rows (T contains all columns), then we use the shorthand $A[\cdot, T]$ ($A[S, \cdot]$, respectively). The $n \times m$ zero matrix and the $n \times n$ identity matrix is denoted by $O_{n \times m}$ and I_n, respectively. We often drop the subscript when clear from context.

Delta-matroids. A *delta-matroid* is a pair $D = (V, \mathcal{F})$ where V is a ground set and $\mathcal{F} \subseteq 2^V$ a collection of *feasible sets*, subject to the rule

$$\forall A, B \in \mathcal{F}, \, x \in A \Delta B \, \exists y \in A \Delta B : A \Delta \{x, y\} \in \mathcal{F}.$$

This is known as the *symmetric exchange axiom*. For $D = (V, \mathcal{F})$ we let $V(D) = V$ and $\mathcal{F}(D) = \mathcal{F}$. A delta-matroid is *even* if all feasible sets have the same parity. Note that in this case we must have $x \neq y$ in the symmetric exchange axiom, although this does not necessarily hold in general.

A *separation oracle* for a delta-matroid $D = (V, \mathcal{F})$ is an oracle that, given a pair (S, T) of disjoint subsets of V, reports whether there is a set $F \in \mathcal{F}$ such that $S \subseteq F$ and $F \cap T = \emptyset$. If so, the pair (S, T) is *separable*. A delta-matroid is *tractable* if it has a polynomial-time separation oracle.

For a delta-matroid $D = (V, \mathcal{F})$ and $S \subseteq D$, the *twisting* of D by S is the delta-matroid $D \Delta S = (V, \mathcal{F} \Delta S)$ where $\mathcal{F} \Delta S = \{F \Delta S \mid F \in \mathcal{F}\}$. This generalizes some common operations from matroid theory. The *dual* delta-matroid of D is $D \Delta V(D)$. For a set $S \subseteq V(D)$, the *deletion* of S from D refers to the set system $D \setminus S = (V \setminus S, \{F \in \mathcal{F} \mid F \subseteq V \setminus S\})$. The *contraction* of S refers to $D/S = (D \Delta S) \setminus S = (V \setminus S, \{F \setminus S \mid F \in \mathcal{F}, S \subseteq F\})$.

Skew-symmetric matrices. A square matrix A is *skew-symmetric* if $A = -A^T$. In the case that A is over a field of characteristic 2, we will additionally assume that it has zero diagonal, unless stated otherwise. For a skew-symmetric matrix A with rows and columns indexed by a set $V = [n]$, the *support graph* of A is the graph $G = (V, E)$ where $E = \{uv \mid A[u, v] \neq 0\}$. A fundamental tool for working with skew-symmetric matrices is the *Pfaffian*, defined for a skew-symmetric matrix A as $\text{Pf } A = \sum_M \sigma(M) \prod_{e \in M} A[u, v]$, where M ranges over all perfect

matchings of the support graph of A and $\sigma(M) \in \{1, -1\}$ is the sign of the permutation:
$\begin{pmatrix} 1 & 2 & \cdots n-1 & n \\ v_1 & v_1' & \cdots v_{n/2} & v_{n/2}' \end{pmatrix}$, where $M = \{v_i v_i' \mid i \in [n/2]\}$ with $v_i < v_i'$ for all $i \in [n/2]$. It is
known that $\det A = (\operatorname{Pf} A)^2$, hence in particular $\operatorname{Pf} A \neq 0$ if and only if A is non-singular.
However, for many algorithms it will be more convenient to work directly with the Pfaffian.
In fact, Pfaffian generalizes the notion of determinants as follows.

▶ **Lemma 7.** *For an $n \times n$-matrix M, it holds that* $\det M = (-1)^{n(n-1)/2} \operatorname{Pf} \begin{pmatrix} O & M \\ -M^T & O \end{pmatrix}$.

An important operation on skew-symmetric matrices is *pivoting*. Let $A \in \mathbb{F}^{n \times n}$ be
skew-symmetric and let $S \subseteq [n]$ be such that $A[S]$ is non-singular. Order the rows and
columns of A so that $A = \begin{pmatrix} B & C \\ -C^T & D \end{pmatrix}$, where $A[S] = B$. Then the *pivoting* of A by S
is $A * S = \begin{pmatrix} B^{-1} & B^{-1}C \\ C^T B^{-1} & D + C^T B^{-1} C \end{pmatrix}$. Note that this is a well-defined, skew-symmetric
matrix.

▶ **Lemma 8** ([34]). *It then holds, for any $X \subseteq [n]$, that* $\det(A * S)[X] = \frac{\det A[X \Delta S]}{\det A[S]}$, *In
particular, $(A * S)[X]$ is non-singular if and only if $A[X \Delta S]$ is non-singular.*

Finally, let us note a formula on the Pfaffian of a sum of two skew-symmetric matrices:

▶ **Lemma 9** ([29, Lemma 7.3.20]). *For two skew-symmetric matrices A_1 and A_2 both indexed
by $V = [n]$, we have*

$$\operatorname{Pf}(A_1 + A_2) = \sum_{U \subseteq V} \sigma_U \operatorname{Pf} A_1[U] \cdot \operatorname{Pf} A_2[V \setminus U],$$

where $\operatorname{Pf} A_i[\emptyset] = 1$ for $i = 1, 2$ and $\sigma_U \in \{1, -1\}$ is a sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & |U| & |U|+1 & \cdots & |V|-1 & |V| \\ u_1 & u_2 & \cdots & u_{|U|} & v_1 & \cdots & v_{|V \setminus U|-1} & u_{|V \setminus U|} \end{pmatrix},$$

where u_i and v_i are the i-th largest elements of U and $V \setminus U$, respectively.

The following is a generalization of the Cauchy-Binet formula to skew-symmetric matrices.
The algebraic approach of Lovász [26] for matroid parity can be derived from this formula
(see [27]).

▶ **Lemma 10** (Ishikawa-Wakayama formula [19]). *For a skew-symmetric $2n \times 2n$-matrix A
and a $2k \times 2n$-matrix B with $k \leq n$, we have*

$$\operatorname{Pf} BAB^T = \sum_{U \in \binom{[2n]}{2k}} \det B[\cdot, U] \operatorname{Pf} A[U].$$

Linear representation. A skew-symmetric matrix defines a delta-matroid as follows. For a
skew-symmetric matrix $A \in \mathbb{F}^{V \times V}$ over a field $\mathbb{F}$, define $\mathcal{F} = \{X \subseteq V \mid A[X] \text{ is non-singular}\}$.
Then, $(V, \mathcal{F})$, which is denoted by $\mathbf{D}(A)$, is a delta-matroid. We say that a delta-matroid
$D = (V, \mathcal{F})$ is *representable* over $\mathbb{F}$ if there is a skew-symmetric matrix $A \in \mathbb{F}^{V \times V}$ and
a twisting set $S \subseteq V$ such that $D = \mathbf{D}(A) \Delta S$. If $A[X]$ is non-singular, or equivalently
$\emptyset \in \mathcal{F}(D)$, we say that D is *directly representable* over $\mathbb{F}$. Note that a directly representable
delta-matroid D can be represented without a twisting set X, as $\mathbf{D}(A) \Delta X = \mathbf{D}(A * X)$. We

will say that D is *directly represented* by A if $D = \mathbf{D}(A)$. A delta-matroid is called *normal* if $\emptyset$ is feasible. Note that every linear delta-matroid is even, and that a linear delta-matroid is directly representable if and only if it is normal. Linear delta-matroids are tractable [3].

In addition, we consider *projected* linear delta-matroids. Let $D = (V, \mathcal{F})$ be a delta-matroid and $X \subseteq V$. Then the projection $D|X$ is defined as $D|X = (V \setminus X, \mathcal{F}|X)$ where $\mathcal{F}|X = \{F \setminus X \mid F \in \mathcal{F}\}$. Then $D|X$ is a delta-matroid, although it is in general not even, hence not linear. When D is linear, then we refer to $D' = D|X$ as a *projected linear delta-matroid*. When $|X| = 1$ we refer to this as an *elementary projection*, following Geelen et al. [17].

As noted above, we assume that A has a zero diagonal (this naturally follows from the definition over all fields with a characteristic not 2). However, if $\mathbb{F}$ is a field of characteristic 2, then linear delta-matroids over $\mathbb{F}$ with a non-zero diagonal correspond to projected linear delta-matroids over $\mathbb{F}$; see Geelen et al. [17].

For a matroid $M = (V, \mathcal{I})$ with the basis family $\mathcal{B}$, $D = (V, \mathcal{B})$ is a delta-matroid. If M is represented by A, D can be represented as follows. Fix a basis $B \in \mathcal{B}$. We may assume w.l.o.g. that $A[\cdot, B] = I$. Define

$$A' = \begin{matrix} B \\ V \setminus B \end{matrix} \begin{pmatrix} \overset{B}{O} & \overset{V \setminus B}{A[\cdot, V \setminus B]} \\ -A^T[V \setminus B, \cdot] & O \end{pmatrix}.$$

Observe that for every $F \subseteq V$, $A'[F]$ is non-singular if and only if $A[F \cap B, F \setminus B]$ is non-singular. Since $A[\cdot, B] = I$, this is equivalent to $A[\cdot, (B \setminus F) \cup (F \setminus B)]$ being non-singular, and thus $D = \mathbf{D}(A') \Delta B$.

Conversely, let $D = (V, \mathcal{F})$ be a delta-matroid. Then the set of maximum-cardinality feasible sets in D forms the set of bases of a matroid $M = (V, \mathcal{I})$. Furthermore, if $D = \mathbf{D}(A)$ is directly represented, then A (as a column space) is also a representation of the matroid M. This is because if B is a column basis for a skew-symmetric matrix A, then $A[B]$ is non-singular (see e.g., [31] or [29, Proposition 7.3.6]).

To avoid intricate representation issues, we assume that every linear representation is given over some finite field. We note that a representation over the rationals can be efficiently transformed into an equivalent representation over a finite field.

Approximate linear representation. For a delta-matroid $D = (V, \mathcal{F})$, we say that a delta-matroid $D' = (V, \mathcal{F}')$ is an *ε-approximate representation* of D if $\mathcal{F}' \subseteq \mathcal{F}$ and for every $F \in \mathcal{F}$, the probability that $F \in \mathcal{F}'$ is at least $1 - \varepsilon$. For constructing an ε-approximate linear representation, the Schwartz-Zippel lemma [33, 37] (also referred to as the DeMillo-Lipton-Schwartz-Zippel lemma) comes in handy. It states that a polynomial $P(X)$ of total degree at most d over a field $\mathbb{F}$ becomes nonzero with probability at least $1 - d/|\mathbb{F}|$ when evaluated at uniformly chosen elements from $\mathbb{F}$, unless $P(X)$ is identically zero.

Let $G = (V, E)$ be an undirected graph and let $\mathcal{F} \subseteq 2^V$ contain all sets $F \subseteq V$ such that $G[F]$ has a perfect matching. Then $D(G) = (V, \mathcal{F})$ is a delta-matroid referred to as the *matching delta-matroid* of G. The Tutte matrix gives rise to an approximate linear representation. Note that setting $\varepsilon = O(2^{-|V|})$ (or lower) gives a matrix of polynomial size which with high probability is a correct representation of $D(G)$. However, this will inflate the time needed for field operations over $\mathbb{F}$ by at least $\Omega(n)$, so for efficiency reasons we work with ε-approximate representations where ε is a parameter.

▶ **Lemma 11 (★).** *Let $G = (V, E)$ be a graph on n vertices and $\mathbb{F}$ be a field with at least $n \cdot \lceil 1/\varepsilon \rceil$ elements. We can construct a ε-approximate linear representation of the matching delta-matroid of G over $\mathbb{F}$.*

Operations in matrix product time. Determinant, rank, basis, inverse can be found in $O(n^\omega)$ time. Given an $n \times 2n$-matrix, its row echelon form can be computed in $O(n^\omega)$. We can find a lexicographically smallest column basis in $O(n^\omega)$ time. See [35].

3   Contraction representation of linear delta-matroids

In this section, we introduce a novel linear representation for delta-matroids, called *contraction representation*. For the sake of clarity, we will say that the representation of a delta-matroid as $D = \mathbf{D}(A)\Delta S$ is a *twist representation*. As we will see in Section 4, the contraction representation is useful in the design of more efficient algorithms for linear delta-matroids. We also give further results, supported by the new representation. First, we show that the union and delta-sum of linear delta-matroids is linear. Previously, this was only known to define delta-matroids [2, 4]. We also use this to provide a compact representation of projected linear delta-matroids. All of these additional results will be useful in our algorithms.

3.1   Contraction representations

For a delta-matroid $D = (V, \mathcal{F})$, a *contraction representation* of D is a pair (A, T) where A is a skew-symmetric matrix over a field $\mathbb{F}$ whose rows and columns are labelled by $V \cup T$, such that $D = \mathbf{D}(A)/T$, i.e., for every $F \subseteq V$, F is feasible in D if and only if $F \cup T$ is feasible in $\mathbf{D}(A)$. This is closely related to *strong maps* of delta-matroids. For two delta-matroids D and D°, D° is a *strong map* of D if there exists a delta-matroid $D^+ = (V \cup Z, \mathcal{F})$ such that $D = D^+ \setminus Z$ and $D^\circ = D^+/Z$ (see Geelen et al. [17]). Hence, if $D = (V, \mathcal{F}) = \mathbf{D}(A)/T$ is a contraction representation of a delta-matroid D, then D is a strong map of the directly representable delta-matroid $\mathbf{D}(A[V])$. We show that the contraction and twist representations are equivalent.

▶ **Lemma 12.** *Given a delta-matroid D in twist representation, we can construct a contraction representation $D = \mathbf{D}(A)/T$ of D deterministically in $O(n^2)$ time.*

Proof. Let $D = (V, \mathcal{F})$ be given as $D = \mathbf{D}(A)\Delta S$, $S \subseteq V$. For a set T of size $|S|$, define a skew-symmetric matrix A' over $V \cup T$ by

$$
A' = \begin{array}{c} \\ T \\ S \\ V \setminus S \end{array}
\begin{array}{c} \overset{T}{} \\ \left(\begin{array}{ccc} O & I & O \\ -I & A[S] & A[S, V \setminus S] \\ O & A[V \setminus S, S] & A[V \setminus S] \end{array} \right) \end{array}
\begin{array}{c} \overset{S}{} \overset{V \setminus S}{} \end{array}
$$

where I is an identity matrix. Note that the support graph of $A'[T \cup S]$ has a unique perfect matching (namely, every vertex in T has degree one), thus $\operatorname{Pf} A'[T \cup S] = \pm 1$ and $A'[T \cup S]$ is non-singular. Thus, we can construct the matrix $A^* = A' * (T \cup S)$. Note that

$$
(A'[T \cup S])^{-1} = \begin{pmatrix} O & I \\ -I & A[S] \end{pmatrix}^{-1} = \begin{pmatrix} A[S] & -I \\ I & O \end{pmatrix},
$$

and consequently, the result of pivoting is

$$
A^* = \begin{array}{c} \\ T \\ S \\ V\setminus S \end{array} \overset{\begin{array}{ccc} T & S & V\setminus S \end{array}}{\left(\begin{array}{ccc} A[S] & -I & -A[S, V\setminus S] \\ I & O & O \\ -A[V\setminus S, S] & O & A[V\setminus S] \end{array} \right)}.
$$

Clearly, A^* can be constructed in $O(n^2)$ time. Now by Lemma 8, for any $F \subseteq V$, $A^*[F \cup T]$ is non-singular if and only if $A'[(F \cup T)\Delta(S \cup T)] = A'[F\Delta S]$ is. Since $F\Delta S \subseteq V$ and $A'[V] = A[V]$, this is equivalent to $F \in \mathcal{F}(D)$, thereby showing that $\mathbf{D}(A^*)/T$ is a contraction representation of D. ◄

▶ **Lemma 13.** *Given a contraction representation $D = \mathbf{D}(A)/T$, we can find a twist representation of D deterministically using $O(n^\omega)$ field operations.*

Proof. Let $S \subseteq V$ be a set such that $A[S\cup T]$ is non-singular; such a set exists since $\mathcal{F}(D) \neq \emptyset$ by assumption, and can be found efficiently over A. Then $A' = A * (S \cup T)$ is well-defined. Let $A^* = A' \setminus T$. Observe that $D = \mathbf{D}(A)\Delta S$, that is, for every $F \subseteq V$, $A^*[F\Delta S] = A'[F\Delta S]$ is non-singular if and only if $A[(F\Delta S)\Delta(S \cup T)] = A[F \cup T]$ is non-singular by Lemma 8. All operations above can be performed in matrix multiplication time. ◄

We also observe that the contracted set T in a representation of a delta-matroid $D = (V, \mathcal{F})$ never needs to be larger than $|V|$.

▶ **Lemma 14 (★).** *Given a contraction representation $D = \mathbf{D}(A)/T$ of a delta-matroid $D = (V, \mathcal{F})$, in $O(n^\omega)$ field operations, where $n = |V| + |T|$, we can find a contraction representation $D = \mathbf{D}(A')/T'$ where $|T'| \leq |V|$.*

3.2 Constructions

We next consider an immediate way to combine two delta-matroids into a new delta-matroid, the *delta-matroid union* (surveyed in the introduction). Let $D_1 = (V_1, \mathcal{F}_1)$ and $D_2 = (V_2, \mathcal{F}_2)$ be two delta-matroids on not necessarily disjoint ground sets and let $V = V_1 \cup V_2$. Define $\mathcal{F} = \mathcal{F}_1 \uplus \mathcal{F}_2 := \{F_1 \cup F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, F_1 \cap F_2 = \emptyset\}$ as the collection of sets that can be produced as disjoint unions from $\mathcal{F}_1$ and $\mathcal{F}_2$, and write $D = (V, \mathcal{F}) = D_1 \cup D_2$. Then Bouchet [2] showed that $D = (V, \mathcal{F})$ is a delta-matroid. We show that furthermore, if D_1 and D_2 are linear or projected linear then so is D, and an ε-approximate representation can be constructed in polynomial time. Note that we may as well assume that $V = V_1 = V_2$, by adding the missing elements to the respective delta-matroid as loops.

▶ **Lemma 15 (★).** *Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be linear (respectively projected linear) delta-matroids defined over a common field $\mathbb{F}$ and given in contraction representation. Then the delta-matroid union $D = D_1 \cup D_2$ is a linear (respectively projected linear) delta-matroid, and an ε-approximate representation of D can be constructed in $O(n^2)$ field operations over an extension field of $\mathbb{F}$ with at least $n \cdot \lceil 1/\varepsilon \rceil$ elements.*

Let $D_1 = (V_1, \mathcal{F}_1)$ and $D_2 = (V_2, \mathcal{F}_2)$ be delta-matroids on not necessarily disjoint ground sets. Let $V = V_1 \cup V_2$ and $\mathcal{F} = \{F_1\Delta F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. Then $D = (V, \mathcal{F})$ is called the *delta-sum* $D = D_1\Delta D_2$ of D_1 and D_2, and is itself a delta-matroid. Bouchet and Schwärzler [4] give a proof, citing unpublished work by Duchamp for the result. We show that the delta-sum of linear delta-matroids is linear when D_1 and D_2 are given as representations over a common field. By Lemma 12, we can work with contraction representations.

▶ **Lemma 16 (★).** *The delta-sum of (projected) linear delta-matroids over a common field is a (projected) linear delta-matroid, and a representation can be computed in randomized polynomial time. More precisely, let D_1 and D_2 be linear delta-matroids given in contraction representation over a common finite field $\mathbb{F}$. Let $\varepsilon > 0$ be given and let $\mathbb{F}'$ be a field extension of $\mathbb{F}$ with at least $n \cdot \lceil 1/\varepsilon \rceil$ elements. We can construct an ε-approximate contraction representation of $D_1 \Delta D_2$ in $O(n^\omega)$ field operations over $\mathbb{F}'$.*

Proof sketch. Let $D_1 = (V, \mathcal{F}_1) = \mathbf{D}(A_1)/T_1$ and $D_2 = (V, \mathcal{F}_2) = \mathbf{D}(A_2)/T_2$, w.l.o.g. represented over the same ground set and with $T_1 \cap T_2 = \emptyset$. Create three copies $V \uplus V_1 \uplus V_2$ of the ground set and define three sets of variables $Y_i = \{y_{v,i} \mid v \in V\}$, $i = 1, 2, 3$. Let

$$
A = \begin{array}{c} \\ V \\ V_1 \\ T_1 \\ V_2 \\ T_2 \end{array}
\begin{array}{c}
\begin{array}{ccccc} V & V_1 & T_1 & V_2 & T_2 \end{array} \\
\left(
\begin{array}{ccccc}
O & B_1 & O & B_2 & O \\
-B_1 & A_1[V_1] & A_1[V_1, T_1] & B_3 & O \\
O & A_1[T_1, V_1] & A_1[T_1] & O & O \\
-B_2 & -B_3 & O & A_2[V_2] & A_2[V_2, T_2] \\
O & O & O & A_2[T_2, V_2] & A_2[T_2]
\end{array}
\right)
\end{array}
$$

where B_i, $i = 1, 2, 3$ is a diagonal matrix whose v:th entry is $y_{v,i}$. Note that A can be seen as the sum of the representation of the disjoint union of D_1 and D_2 and the Tutte matrix of the graph consisting of a disjoint union of triangles on $\{v, v^1, v^2\}$ for every $v \in V$, where v^i is the copy of v in V_i, $i = 1, 2$. It now follows from the Pfaffian sum formula (Lemma 9) that $\mathbf{D}(A)/(V_1 \cup T_1 \cup V_2 \cup T_2)$ is an ε-approximate representation of $D_1 \Delta D_2$. ◀

Recall that a projected linear delta-matroid $D = (V, \mathcal{F})$ is a delta-matroid represented as $D = D'|X$ where $D' = (V \cup X, \mathcal{F}')$ is a linear delta-matroid. Projections of linear delta-matroids were studied by Geelen et al. [17] in the context of linear delta-matroids over fields of characteristic 2, and by Kakimura and Takamatsu [21] regarding generalizations of constrained matching problems. We observe that if D is linear, then the even (respectively odd) sets of $D|X$ form a linear delta-matroid, and that every projected linear delta-matroid $D|X$ can be represented via an elementary projection.

▶ **Lemma 17 (★).** *Let $D = (V \cup X, \mathcal{F})$ be a linear delta-matroid. Then the following delta-matroids are linear and approximate representations can be constructed efficiently.*
1. *A linear delta-matroid $D' = (V \cup X', \mathcal{F}')$ such that $D|X = D'|X'$ and $|X'| \leq 1$*
2. *The delta-matroid $D_e = (V, \mathcal{F}_e)$ where $\mathcal{F}_e$ contains the sets of $\mathcal{F}|X$ of even cardinality*
3. *The delta-matroid $D_o = (V, \mathcal{F}_o)$ where $\mathcal{F}_o$ contains the sets of $\mathcal{F}|X$ of odd cardinality*
More precisely, let $D = \mathbf{D}(A)/T$ in contraction representation over a finite field $\mathbb{F}$ and let $\varepsilon > 0$ be given. Let $\mathbb{F}'$ be a field extension of $\mathbb{F}$ with at least $n \cdot \lceil 1/\varepsilon \rceil$ elements. We can construct an ε-approximate contraction representation of each of the above delta-matroids in $O(n^2)$ operations over $\mathbb{F}'$.

4   Algorithms for fundamental delta-matroid problems

We now present the various algorithms over linear delta-matroids.

4.1   Max-weight feasible sets

We show an $O(n^\omega)$-time algorithm for finding a max-weight feasible set in a linear delta-matroid. More precisely, let $D = (V, \mathcal{F})$ be a delta-matroid and $w(v) \in \mathbb{Q}$, $v \in V$ a set of element weights. Let $n = |V|$. The goal is to find a feasible set $F \in \mathcal{F}$ to maximize the

weight $w(F) = \sum_{v \in F} w(v)$. Note that since the feasible sets of D do not necessarily all have the same cardinality, the negative element weights cannot easily be removed by any simple transformation (as, e.g., shifting by a constant would affect different feasible sets differently). This problem can be solved via the "signed greedy" algorithm, which extends the normal greedy algorithm (in fact, like for matroids, the success of signed greedy can be taken as a definition of delta-matroids) [3, 1]. However, this requires $O(n)$ calls to a separation oracle. If D is linear, this algorithm thus runs in $O(n^{\omega+1})$ time. We show an improvement using the contraction representation. We begin with the following observation.

▶ **Lemma 18.** *Let $D = (V, \mathcal{F})$ be a delta-matroid and $w\colon V \to \mathbb{Q}$ a set of weights. Let $N \subseteq V$ be the set of elements $v \in V$ such that $w(v) < 0$. Let $D' = D \Delta N$, and define a set of weights w' by $w'(v) = |w(v)|$ for every $v \in V$. For any feasible set $F \in \mathcal{F}$, we have $w(F) = w'(F \Delta N) + w(N)$.*

Proof. The following hold: $w(F) = w'(F \setminus N) - w'(F \cap N)$ and $w'(F \Delta N) = w'(F \setminus N) + w'(N \setminus F)$. Thereby $w(F) - w'(F \Delta N) = -w'(N) = w(N)$ as promised. ◀

The problem of finding a max-weight feasible set in a linear delta-matroid in contraction representation now reduces to the problem of finding a max-weight basis of a linear matroid.

▶ **Theorem 2.** *Let $D = (V, \mathcal{F})$ be a linear or projected linear delta-matroid. In $O(n^\omega)$ field operations, we can find a max-weight feasible set in D.*

Proof. By Lemma 17, it suffices to prove the statement for linear delta-matroids.

Let $D = (V, \mathcal{F})$ and $w\colon V \to \mathbb{Q}$ be given as input, and as above define $N = \{v \in V \mid w(v) < 0\}$, $D' = D \Delta N$ and $w'\colon V \to \mathbb{Q}$ where $w'(v) = |w(v)|$ for all $v \in V$. Let $\mathbf{D}(A)/T$ be a contraction representation of D', which can be constructed using Lemma 12. Finally, order the columns of A to begin with T and thereafter elements $v \in V$ in order of non-increasing weight $w'(v)$. Let B be a lex-min column basis for A with respect to this ordering. Then B can be computed in $O(n^\omega)$ field operations over A (see e.g., [5]), and B is a max-weight column basis of A with respect to the weights w'. We claim that $F = (B \setminus T) \Delta N$ is a max-weight feasible set in D. For this, let F^* be a max-weight feasible set in D with respect to the weights w. Then by Lemma 18, $F^* \Delta N$ is a max-weight feasible set in D' with respect to the weights w'. Hence $B' = (F^* \Delta N) \cup T$ is feasible in $\mathbf{D}(A)$ and $w'(B \setminus T) \geq w'(B' \setminus T) = w(F^*) - w(N)$. On the other hand, let B be a lex-min basis of A in the above ordering. Then $T \subseteq B$ by construction. By Lemma 18, $F = (B \setminus T) \Delta N$ is a feasible set in D with $w(F) = w'(B \setminus T) + w(N) \leq w(F^*)$. Hence $w'(B \setminus T) = w(F^*) - w(N)$ by sandwiching and $w((B \setminus T) \Delta N) = w'(B \setminus T) + w(N) = w(F^*)$. ◀

4.2 Intersection, Parity, and Delta-Covering

Recall that the DM PARITY problem is defined as follows. Let $D = (V, \mathcal{F})$ be a delta-matroid with V partitioned into n pairs Π. The problem is to find a feasible set $F \in \mathcal{F}$ minimizing the number of *broken* pairs $\delta_\Pi(F) = |\{P \in \Pi : |F \cap P| = 1\}|$. Let $\delta(D, \Pi) = \min_{F \in \mathcal{F}} \delta_\Pi(F)$. We consider the equivalent DM DELTA-COVERING problem defined as follows. Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be two given delta-matroids. The problem is to find $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ maximizing $|F_1 \Delta F_2|$. Let $\tau(D_1, D_2) = \max_{F_i \in \mathcal{F}_i} |F_1 \Delta F_2|$. As described in Section 1, DM PARITY and DM DELTA-COVERING reduce to each other. Moreover, DM COVERING and DM INTERSECTION are special cases of DM PARITY and DM DELTA-COVERING.

We can compute $\tau(D_1, D_2)$ in $O(n^\omega)$ field operations as follows. Observe that $\tau(D_1, D_2)$ is the maximum feasible set size in the delta-sum $D_1 \Delta D_2$. By Lemma 16, we can find a linear representation $\mathbf{D}(A)/T$ of $D_1 \Delta D_2$ in $O(n^\omega)$ field operations. We then have $\tau(D_1, D_2) =$

rank $A - |T|$. Thus the decision variants of DM PARITY and DM DELTA-COVERING can be solved in $O(n^\omega)$ field operations. Via self-reducibility, we obtain an algorithm that finds a witness for DM PARITY and DM DELTA-COVERING in $O(n^{\omega+1})$ field operations, matching the result of Geelen et al. [17]. We present an improvement to $O(n^\omega)$, using the method of Harvey [18]. Specifically, we prove the following:

▶ **Lemma 19** (★). *Let A be an $n \times n$ skew-symmetric, non-singular polynomial matrix with its row and column indexed by $V = \{v_1, \ldots, v_n\}$. Suppose that $A = B + Y$, where (i) B is a matrix defined over a field $\mathbb{F}$ containing $\Omega(n^3)$ elements, and (ii) Y is the Tutte matrix of a graph $G = (V, E)$ with variables $y_e, e \in E$. Suppose that G has connected components $C_1, \ldots, C_\gamma$, with $|C_i| \in O(1)$ for every $i \in [\gamma]$. It is possible to find an inclusion-wise maximal set $S \subseteq E$ for which A remains non-singular when setting y_e to zero for all $e \in S$. This can be done with probability $1 - 1/\Omega(n)$ using $O(n^\omega)$ field operations over $\mathbb{F}$.*

Using Lemma 19, we prove Theorem 3, which yields algorithms for DM DELTA-COVERING and DM PARITY as well as DM COVERING and DM INTERSECTION using $O(n^\omega)$ field operations (Corollary 4).

▶ **Theorem 3.** *Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be (projected) linear delta-matroids over a common field with $\Omega(n^3)$ elements. For a feasible set $F \subseteq V$ in $D_1 \Delta D_2$, we can, with high probability, find feasible sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ with $F_1 \Delta F_2 = F$ in $O(n^\omega)$ field operations.*

▶ **Corollary 4.** *The following search problems can be solved in $O(n^\omega)$ field operations with high probability, given (projected) linear delta-matroids on ground sets of n elements represented over a common field with $\Omega(n^3)$ elements: DM COVERING; DM DELTA-COVERING; DM INTERSECTION; DM PARTITION; and DM PARITY.*

Proof sketch. We treat the problems in turn. Throughout, by Lemma 17 (and by exhaustively guessing the parities of the feasible sets involved in the solution), we assume that every delta-matroid D in the input is linear, and by Lemma 12 that it is provided in contraction form $D = \mathbf{D}(A)/T$ for some skew-symmetric matrix A. We omit details of field size and approximate representations.

DM Parity. Let the input be (D, Π) with $D = \mathbf{D}(A)/T$. Construct the matching delta-matroid D_Π of the graph with edge set Π and construct a contraction representation of $D' = D \Delta D_\Pi$ using Lemma 16. Let F be a maximum feasible set in D' and apply Theorem 3.

DM Intersection. Given input (D_1, D_2), construct $D = D_1 \Delta D_2$. If $\emptyset$ is feasible in D, then apply Theorem 3 to $F = \emptyset$; otherwise report a no-instance.

DM Partition. Given input (D_1, D_2), construct $D = D_1 \Delta D_2$. If the full ground set V is feasible in D, apply Theorem 3; otherwise report a no-instance.

DM Covering. Given input (D_1, D_2), construct $D = D_1 \cup D_2$ and let F be a maximal feasible set. Delete all elements of $V \setminus F$ to create the induced delta-matroids D_1' and D_2' with ground set F, and apply Theorem 3 to $D' = D_1' \Delta D_2'$.

DM Delta-Covering. Given input (D_1, D_2), construct $D = D_1 \Delta D_2$ and let F be a maximal feasible set. Apply Theorem 3 to F.   ◀

Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be two linear delta-matroids with weights $w(v) \in \mathbb{N}, v \in V$. We consider the *intersection* problem, where the goal is to find a common feasible set $F \subseteq V$, i.e., $F \in \mathcal{F}_1$ and $F \in \mathcal{F}_2$. The decision problem, i.e., whether there exists $F \in \mathcal{F}_1 \cap \mathcal{F}_2$, can be solved in $O(n^\omega)$ by testing whether V is feasible in the delta-sum $D_1 \Delta D_2^*$. Moreover, we can find a common feasible set in $O(n^\omega)$ time using Theorem 3. We next give a (pseudo)polynomial-time randomized algorithm for the WEIGHTED DELTA-MATROID

INTERSECTION, where we are tasked with finding a common feasible set of maximum weight, answering an open question of Kakimura and Takamatsu [21]. Previously, there has been no polynomial-time algorithm even for the unweighted case.

▶ **Theorem 5.** *Let $D_1 = (V, \mathcal{F}_1)$ and $D_2 = (V, \mathcal{F}_2)$ be (projected) linear delta-matroids over a common field with $\Omega(n^2)$ elements and let $w\colon V \to \{1, \ldots, W\}$ be element weights. In $O(Wn^\omega)$ field operations we can find with high probability the maximum weight of a common feasible set $F \in \mathcal{F}_1 \cap \mathcal{F}_2$. In particular, we can find the maximum cardinality of $|F|$ in $O(n^\omega)$ field operations with high probability.*

5 Conclusions

We have introduced a novel representation for linear delta-matroids, the *contraction* representation, which enable us to derive a range of new results, including linear representations of delta-matroid union and delta-sum, and faster algorithms for various problems including LINEAR DELTA-MATROID PARITY. We also show the first (pseudo)polynomial-time, randomized algorithms for the maximum cardinality and weighted versions of LINEAR DELTA-MATROID INTERSECTION, solving an open question of Kakimura and Takamatsu [21].

We note a few open questions. First, all our running times are stated purely in terms of the number of elements n. It would be interesting to explore faster algorithms for linear delta-matroids of bounded "rank". But we also note two specific challenging questions.

1. Is there a strongly polynomial-time algorithm for WEIGHTED LINEAR DELTA-MATROID PARITY, extending the recent result for WEIGHTED LINEAR MATROID PARITY [20]?
2. Is there a $\tilde{O}(Wn^\omega)$-time algorithm for SHORTEST DISJOINT $\mathcal{S}$-PATHS? This would extend results for graph matching [9].

Additionally, does there exist a good characterization of, and/or a deterministic algorithm for the maximum cardinality version of LINEAR DELTA-MATROID INTERSECTION?

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