Commutative ℕ-Rational Series of Polynomial Growth

Aliaume Lopez
University of Warsaw, Poland

Abstract

This paper studies which functions computed by ℤ-weighted automata can be realised by ℕ-weighted automata, under two extra assumptions: commutativity (the order of letters in the input does not matter) and polynomial growth (the output of the function is bounded by a polynomial in the size of the input). We leverage this effective characterization to decide whether a function computed by a commutative ℕ-weighted automaton of polynomial growth is star-free, a notion borrowed from the theory of regular languages that has been the subject of many investigations in the context of string-to-string functions during the last decade.

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1 Introduction

Given a semiring $\mathbb{S}$, and a finite alphabet Σ, the class of *(noncommutative)* $\mathbb{S}$*-rational series* is defined as functions from Σ^* to $\mathbb{S}$ that are computed by $\mathbb{S}$-weighted automata [1]. This computational model is a generalization of the classical notion of non-deterministic finite automata to the weighted setting, where transitions are labeled with elements of $\mathbb{S}$. The semantics of $\mathbb{S}$-weighted automata on a given word w is defined by the sum over all accepting runs reading w, of the product of the weights of the transitions taken along this run. In this paper, we are interested in the case where $\mathbb{S}$ equals $\mathbb{N}$ or $\mathbb{Z}$, hence, in ℕ-rational series (ℕSeries) and ℤ-rational series (ℤSeries). It is clear that ℕSeries is a proper subclass of ℤSeries, and a longstanding open problem is to provide an algorithm that decides whether a given ℤSeries is in ℕSeries [10].

▶ **Problem 1.** *Input: A* ℤSeries f*. Output: Is* f *in* ℕSeries*?*

Problem 1 recently received attention in the context of polyregular functions (Poly), a computational model that aims to generalize the theory of regular languages to the setting of string-to-string functions [2]. In the case of regular languages, *star-free languages* form a robust subclass of regular languages described equivalently in terms of first order logic [13], counter-free automata [13], or aperiodic monoids [16]. Analogously, there exists a *star-free* fragment of polyregular functions called star-free polyregular functions (SF) [2]. One open question in this area is to decide whether a given polyregular function is star-free.

▶ **Problem 2.** *Input: A polyregular function* f*. Output: Is* f *star-free?*

In order to approach decision problems on polyregular functions, restricting the output alphabet to a single letter has proven to be a fruitful method [6, 7]. Because words over a unary alphabet are canonically identified with natural numbers, unary output polyregular functions are often called ℕ-polyregular functions (ℕPoly), and their *star-free* counterpart star-free ℕ-polyregular functions (ℕSF). Coincidentally, polyregular functions with unary

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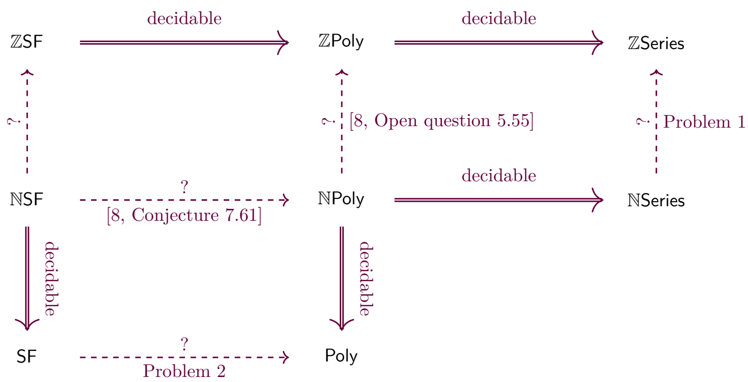

Figure 1 Decidability and inclusions of classes of functions, arranged along two axes. The first one is the complexity of the output alphabet ($\mathbb{Z}$, $\mathbb{N}$, Σ). The second one is the allowed computational power (star-free polyregular functions, polyregular functions, rational series). Arrows denote strict inclusions, and effectiveness (both in terms of decidability and of effective representation) is represented by thick double arrows. Inclusions that are suspected to be effective are represented using a dashed arrow together with a question mark.

output forms a subclass of $\mathbb{N}$-rational series, namely the class of $\mathbb{N}$-rational series of *polynomial growth*, i.e. the output of the function is bounded by a polynomial in the size of the input. In [5], the authors introduced the class of $\mathbb{Z}$-polyregular functions ($\mathbb{Z}$Poly) as a subclass of $\mathbb{Z}$-rational series that generalizes $\mathbb{N}$-polyregular functions by allowing negative outputs, and showed that membership in the *star-free* subclass $\mathbb{Z}$SF inside $\mathbb{Z}$Poly is decidable [5, Theorem V.8]. Although this could not be immediately leveraged to decide $\mathbb{N}$SF inside $\mathbb{N}$Poly, it was conjectured that $\mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF} = \mathbb{N}\mathsf{SF}$ [8, Conjecture 7.61]. It was believed that understanding the membership problem of $\mathbb{N}$Poly inside $\mathbb{Z}$Poly, that is, a restricted version of Problem 1, would be a key step towards proving $\mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF} = \mathbb{N}\mathsf{SF}$, which itself would give hope in designing an algorithm for Problem 2. We illustrate in Figure 1 the known inclusions and related open problems between the discussed classes of functions.

Contributions. In this paper, we work under the extra assumption of *commutativity*, that is, assuming that the function is invariant under the permutation of its input. In this setting, we prove that $\mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF} = \mathbb{N}\mathsf{SF}$ [8, Conjecture 7.61] and design an algorithm that decides whether a function in $\mathbb{Z}$Poly is in $\mathbb{N}$Poly [8, Open question 5.55]. As a consequence, the upper left square of Figure 1 has all of its arrows decidable and with effective conversion procedures under this extra assumption. Because $\mathbb{Z}$-rational series with *polynomial growth* are exactly $\mathbb{Z}$-polyregular functions [5], this can be seen as decision procedure for Problem 1 under the extra assumption of *commutativity* and *polynomial growth*. Similarly, our results provide an algorithm for Problem 2 under the extra assumption of *commutativity* and *unary output alphabet*.

As an intermediate step, we provide a complete and decidable characterization of polynomials in $\mathbb{Q}[\vec{X}]$ that can be computed using $\mathbb{N}$Series (resp. $\mathbb{Z}$Series). These characterizations uncover a fatal flaw in the proof of a former characterization of such polynomials [10, Theorem 3.3, page 4]. We also prove that this previous results holds for polynomials with at most two indeterminates (Lemma 29), which may explain why it was not detected earlier. Furthermore, these characterizations provide effective descriptions of polynomials that can be expressed in $\mathbb{Z}$Series as those obtained using integer combinations of products of *binomial coefficients* (called integer binomial polynomials, defined page 11) and similarly for $\mathbb{N}$Series by introducing the notion of strongly natural binomial polynomials (defined page 12), which we believe has its own interest. Finally, these characterizations demonstrate that polynomials expressible by $\mathbb{Z}$Series (resp. $\mathbb{N}$Series) are exactly those expressible by $\mathbb{Z}$SF (resp. $\mathbb{N}$SF), that is, polynomials are inherently *star free* functions.

Outline of the paper. In Section 2, we provide a combinatorial definition of $\mathbb{N}$-polyregular functions (resp. $\mathbb{Z}$-polyregular functions), show that one can decide if a function $f \in \mathbb{Z}$Poly is commutative (Lemma 8). In Section 3, we provide a counterexample to the flawed result of [10, Theorem 3.3, page 4] (Lemma 15), and correct it by providing effective characterizations of polynomials computed by $\mathbb{Z}$Series (Theorem 31) and $\mathbb{N}$Series (Theorem 34). Finally, in Section 4, we answer positively to [8, Open question 5.55] (Theorem 37) and to [8, Conjecture 7.61] (Theorem 40), both under the extra assumption of *commutativity*.

2 Preliminaries

The capital letters Σ, Γ denote fixed alphabets, i.e. finite set of letters, and Σ^*, Γ^* (resp. Σ^+, Γ^+) are the set of words (resp. non-empty words) over Σ, Γ. The empty word is written $\varepsilon \in \Sigma^*$. When $w \in \Sigma^*$ and $a \in \Sigma$, we let $|w| \in \mathbb{N}$ be the length of w, and $|w|_a$ be the number of occurrences of a in w.

We assume that the reader is familiar with the basics of automata theory, in particular the notions of monoid morphisms, idempotents in monoids, monadic second-order (MSO) logic and first-order (FO) logic over finite words (see e.g. [17]). As aperiodicity will be a central notion of this paper, let us recall that a monoid M is *aperiodic* whenever for all $x \in M$, there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. If the monoid M is finite, this n can be uniformly chosen for all elements in M.

We use the notation $\{\!\{\cdot\}\!\} \colon \Sigma^* \to \mathbb{N}^\Sigma$ for the map that counts occurrences of every letter in the input word (that is, computes the Parikh vector) namely: $\{\!\{w\}\!\} := (a \mapsto |w|_a)_{a \in \Sigma}$. Given a set X, a function $f \colon \Sigma^* \to X$ is *commutative* whenever for all $u \in \Sigma^*$, for all permutations σ of $\{1, \ldots, |w|\}$, $f(\sigma(u)) = f(u)$. Equivalently, it is *commutative* whenever there exists a map $g \colon \mathbb{N}^\Sigma \to X$ such that $g \circ \{\!\{\cdot\}\!\} = f$.

Let $k \in \mathbb{N}$, and let Σ be a finite alphabet. Given a function $\eta \colon \{1, \ldots, k\} \to \Sigma$, we define the $\eta^\dagger \colon \mathbb{N}^k \to \Sigma^*$ as $\eta^\dagger(\vec{x}) := \eta(1)^{x_1} \ldots \eta(k)^{x_k}$. A function $f \colon \mathbb{N}^k \to X$ is *represented* by a commutative function $g \colon \Sigma^* \to X$ if there exists a map $\eta \colon \{1, \ldots, k\} \to \Sigma$ such that $g \circ \eta^\dagger = f$. This notion will be useful to formally state that a polynomial "is" a commutative polyregular function. For instance, the polynomial function $P(X, Y) = X \times Y$ is represented by the commutative function $g \colon \{a, b\}^* \to \mathbb{Z}$ defined by $g(w) := |w|_a \times |w|_b$.

2.1 Polynomials

A polynomial $P \in \mathbb{Z}[X_1, \ldots, X_k]$ is *non-negative* when for all non-negative integer inputs $n_1, \ldots, n_k \geq 0$, the output $P(n_1, \ldots, n_k)$ of the polynomial is non-negative. In the case of at most three indeterminates, we use variables X, Y, Z instead of X_1, X_2, X_3 to lighten the notation. Beware that we do not consider negative values as input, as the numbers n_i will ultimately count the number of occurrences of a letter in a word. As an example, the polynomial $(X - Y)^2$ is non-negative, and so is the polynomial X^3, but the polynomial $X^2 - 2X$ is not.

A *monomial* is a product of indeterminates and integers. For instance, XY is a monomial, $3X$ is a monomial, $-Y$ is a monomial, but $X + Y$ and $2X^2 + XY$ are not. Every polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ decomposes uniquely into a sum of monomials. A monomial S *divides* a monomial T, when S divides T seen as polynomials in $\mathbb{Q}$. For instance, $2X$ divides XY, $-YZ$ divides X^2YZ^3, and Y does not divide X. In the decomposition of $P \in \mathbb{Z}[X_1, \dots, X_k]$, a monomial is a *maximal monomial* if it is a maximal element for the divisibility preordering of monomials. In the polynomial $P(X,Y) := X^2 - 2XY + Y^2 + X + Y$, the set of *maximal monomials* is $\{X^2, -2XY, Y^2\}$. For instance, the non-negative monomials of $P(X,Y) := (X-Y)^2$ are X^2 and Y^2.

2.2 Polyregular Functions

Because the functions of interest in this paper have output in $\mathbb{N}$ or $\mathbb{Z}$, we will only provide the definition of *polyregular functions* for these two output semigroups, and we refer the reader to [3] for the general definition of polyregular functions and their aperiodic counterpart, the *star-free polyregular functions*. We chose in this paper to provide a combinatorial description of polyregular functions with commutative outputs because it will play nicely with our analysis on polynomials. This description is very similar in shape to the *finite counting automata* introduced by [15].

▶ **Definition 3** ($\mathbb{Z}$-polyregular functions [5]). *Let $d \in \mathbb{N}$. The set $\mathbb{Z}\mathsf{Poly}_d$ of polyregular* $\mathbb{Z}$-polyregular functions *of degree at most d, is the set of functions $f \colon \Sigma^* \to \mathbb{Z}$ such that there exists a finite monoid M, a morphism $\mu \colon \Sigma^* \to M$, and a function $\pi \colon M^{d+1} \to \mathbb{Z}$ satisfying for all $w \in \Sigma^*$:*

$$f(w) = \pi^\dagger(w) := \sum_{w = u_1 \cdots u_{d+1}} \pi(\mu(u_1), \dots, \mu(u_{d+1})) \quad .$$

We call π the production function *of f. If the function π has codomain $\mathbb{N}$, then f is* $\mathbb{N}$-polyregular *of degree at most d, i.e., $f \in \mathbb{N}\mathsf{Poly}_d$. If the monoid M is aperiodic then the function f is* star-free $\mathbb{Z}$-polyregular *($\mathbb{Z}\mathsf{SF}_d$), resp.* star-free $\mathbb{N}$-polyregular *($\mathbb{N}\mathsf{SF}_d$).*

We complete Definition 3 by letting $\mathbb{N}\mathsf{Poly} := \bigcup_{d \in \mathbb{N}} \mathbb{N}\mathsf{Poly}_d$, and similarly for $\mathbb{Z}\mathsf{Poly}$, $\mathbb{N}\mathsf{SF}$, and $\mathbb{Z}\mathsf{SF}$. In order to illustrate these definitions, let us provide an example of an $\mathbb{N}$-polyregular function computed using a finite monoid in Example 4. Let us also introduce in Example 5 a function that serves as an example of *division* computed by a $\mathbb{N}$-polyregular function.

▶ **Example 4.** The map $f \colon w \mapsto |w| + 1$ belongs to $\mathbb{N}\mathsf{SF}_1$.

Proof. Let us define $M := (\{1\}, \times)$ which is a finite aperiodic monoid, $\mu \colon \Sigma^* \to M$ defined by $\mu(w) := 1$, and $\pi \colon M^2 \to \mathbb{N}$ that is the constant function equal to 1. We check that for all $w \in \Sigma^*$: $\pi^\dagger(w) = \sum_{uv=w} 1 = |w| + 1 = f(w)$. ◀

▶ **Example 5.** Let $f \colon \Sigma^* \to \mathbb{N}$ be the function that maps a word w to the number of distinct pairs of positions in w, i.e., $f(w) = \binom{|w|}{2} = |w|(|w|-1)/2$. Then, $f \in \mathbb{N}\mathsf{SF}_2$.

Proof. Let us remark that the set P_w of distinct pairs of positions $i < j$ in a word w is in bijection with the set D_w of decompositions of the form $w = xyz$, where x and y are non-empty, via the map $(i,j) \mapsto (w_{1,i}, w_{i+1,j}, w_{j+1,|w|})$. Let us write $M := (\{0,1\}, \max)$ which is a finite aperiodic monoid, and $\mu \colon \Sigma^* \to M$ that maps the empty word ε to 0 and the other words to 1. Then, let us define $\pi \colon M^3 \to \mathbb{N}$ via $\pi(x,y,z) = x \times y$. We conclude because:

$$\pi^\dagger(w) := \sum_{xyz=w} \pi(\mu(x), \mu(y), \mu(z)) = \sum_{xyz = w \wedge x \neq \varepsilon \wedge y \neq \varepsilon} 1 = |D_w| = |P_w| = f(w) \quad .$$ ◀

One of the appeals of $\mathbb{N}$Poly and $\mathbb{Z}$Poly are the numerous characterizations of these classes in terms of logic, weighted automata, and the larger class of polyregular functions [5, 8]. In this paper, the main focus will be the connection to weighted automata, which is based on the notion of *growth rate*. The *growth rate* of a function $f\colon \Sigma^* \to \mathbb{Z}$ is defined as the minimal d such that $|f(w)| = \mathcal{O}\left(|w|^d\right)$. If such a d exists, we say that the function f has *polynomial growth*. It turns out that for all $k \in \mathbb{N}$, $\mathbb{Z}\mathsf{Poly}_d$ (resp. $\mathbb{N}\mathsf{Poly}_d$) are precisely functions in $\mathbb{Z}$Series (resp. in $\mathbb{N}$Series) that have growth rate at most d.

▶ **Lemma 6** ([8, Theorem 5.22])**.** *Let $f \in \mathbb{Z}$Series. The following are equivalent:*
1. *$f \in \mathbb{Z}\mathsf{Poly}_d$.*
2. *f has polynomial growth of degree at most d.*

And similarly for $\mathbb{N}$Poly and $\mathbb{N}$Series.

Let us introduce some compositional properties of $\mathbb{Z}$-polyregular functions that will be used in this paper to construct $\mathbb{Z}$-polyregular functions.

▶ **Lemma 7** ([5, Theorem II.20])**.** *Let $d \geq 1$, $f, g \in \mathbb{N}\mathsf{Poly}_d$ (resp. $\mathbb{Z}\mathsf{Poly}_d$, $\mathbb{N}\mathsf{SF}_d$, $\mathbb{Z}\mathsf{SF}_d$), L be a star-free language over Σ^*, and $h\colon \Sigma^* \to \Gamma^*$ be a polyregular function (resp. a star-free polyregular function). Then, the following are also in $\mathbb{N}\mathsf{Poly}_d$ (resp. $\mathbb{Z}\mathsf{Poly}_d$, $\mathbb{N}\mathsf{SF}_d$, $\mathbb{Z}\mathsf{SF}_d$): $f \circ h$, $f + g := w \mapsto f(w) + g(w)$, $f \times g := w \mapsto f(w) \times g(w)$, $\mathbf{1}_L \times f$. Furthermore, the above constructions preserve commutativity.*

Let us briefly state that commutativity is a decidable property of $\mathbb{Z}$-rational series, hence of $\mathbb{Z}$-polyregular functions. As a consequence, we are working inside a relatively robust and decidable subclass of $\mathbb{Z}$-rational series.

▶ **Lemma 8.** *Let $f \in \mathbb{Z}$Series. One can decide if f is commutative.*

Proof. Remark that the group of permutations of $\{1, \ldots, n\}$ is generated by the cycle $c := (n, 1, \ldots, n-1)$ and the transposition $t := (1, 2)$. As a consequence, a function f is commutative if and only if $f \circ c = f = f \circ t$. When f is a rational series, $f \circ c$ and $f \circ t$ are both rational series that can be effectively computed from f,[1] and since equivalence of rational series is decidable [1, Corollary 3.6], we have obtained a decision procedure. ◀

3 $\mathbb{N}$-rational Polynomials

In this section, we will completely characterize which polynomials in $\mathbb{Q}[\vec{X}]$ are represented by $\mathbb{N}$-rational series (resp. $\mathbb{Z}$-rational series). To that end, we start by characterizing these classes for polynomials in $\mathbb{Z}[\vec{X}]$. We say that a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ is an $\mathbb{N}$-*rational polynomial* if it is represented by a $\mathbb{N}$-rational series. It is an easy check that polynomials with coefficients in $\mathbb{N}$ are $\mathbb{N}$-rational polynomials (Lemma 9). However, Example 10 provides a polynomial with negative coefficients that is an $\mathbb{N}$-rational polynomial. The problem of characterizing $\mathbb{N}$-rational polynomials was claimed to be solved in [10], using the Definition 11 to characterize $\mathbb{N}$-rational polynomials, as restated in Flawed Theorem 12.

▶ **Lemma 9.** *Let $P \in \mathbb{N}[\vec{X}]$. Then, P is an $\mathbb{N}$-rational polynomial.*

[1] This can be done by guessing the second (resp. last) letter of the input word, remembering the first letter in a state, and then running the original automaton for f on the modified input, checking at the second position (resp. the end of the word) if the guess was correct.

▶ **Example 10.** The polynomials X, $X^2 + 3$, and $X^2 - 2X + 2$ are ℕ-rational polynomials, but $-X$ is not an ℕ-rational polynomial.

▶ **Definition 11** ([10, Section 3, page 3])**.** *The class* PolyNNeg$[\vec{X}]$ *is the class of polynomials* $P \in \mathbb{Z}[\vec{X}]$ *that are non-negative and such that every maximal monomial is non-negative. When the indeterminates are clear from the context, we write this class* PolyNNeg*.*

▶ **Flawed Theorem 12** ([10, Theorem 3.3, page 4])**.** *Let* $P \in \mathbb{Z}[\vec{X}]$ *be a polynomial. Then,* P *is an* ℕ*-rational polynomial if and only if* $P \in$ PolyNNeg*.*

Before giving a counterexample to the above statement, let us first exhibit in Example 14 some non-negative polynomial that is not an ℕ-rational polynomial. While the example will not be in PolyNNeg, it illustrates the key difference between non-negative polynomials and ℕ-rational polynomials. In order to derive this example, we will need the following fundamental result about the pre-image of regular languages by polyregular functions.[2] Before that, let us remark that if a polynomial P is represented by a ℕ-rational series, then it is in fact represented by a ℕ-polyregular function thanks to Lemma 6.

▶ **Theorem 13** ([2, Theorem 1.7])**.** *The pre-image of a regular language by a (string-to-string) polyregular function is a regular language.*

▶ **Example 14.** Let $P(X, Y) := (X - Y)^2$. Then P is non-negative, but is not an ℕ-rational polynomial. Indeed, assume by contradiction that $f \in \mathbb{N}\mathsf{Poly}$ represents P over the alphabet $\Sigma := \{a, b\}$. Then, $f^{-1}(\{0\})$ is a regular language (Theorem 13), but $f^{-1}(\{0\}) = \{w \in \Sigma \mid |w|_a = |w|_b\}$ is not.

Please note that the same argument cannot be leveraged for proving that P is not representd by a ℤ-rational series: Theorem 13 only holds for *string-to-string* functions, and is applied to the specific case where the output alphabet is $\{1\}$, i.e., where the output of the function belongs to $\{1\}^*$ which is isomorphic to ℕ.

Let us now design a counterexample to Flawed Theorem 12 by suitably tweaking Example 14 to ensure that the polynomial not only is non-negative, but also belongs to PolyNNeg. We define $\mathsf{P_{bad}}(X, Y, Z) := Z(X + Y)^2 + 2(X - Y)^2$.

▶ **Lemma 15.** *The polynomial* $\mathsf{P_{bad}}$ *belongs to* PolyNNeg*, but is not an* ℕ*-rational polynomial. As a corollary, [10, Theorem 3.3], restated in Flawed Theorem 12, is false when allowing at least* 3 *indeterminates.*

Proof. It is clear that $\mathsf{P_{bad}}$ is non-negative. We can expand the expression of $\mathsf{P_{bad}}$ to obtain $\mathsf{P_{bad}} = ZX^2 + ZY^2 + 2ZXY + 2X^2 - 4XY + 2Y^2$. The maximal monomials of P are ZX^2, ZY^2, and $2ZXY$, all of which are non-negative.

Assume by contradiction that $\mathsf{P_{bad}}$ is an ℕ-rational polynomial. Let $\Sigma := \{a, b, c\}$ be a finite alphabet. There exists a commutative ℕ-polyregular function $f : \Sigma^* \to \mathbb{N}$ such that for all $w \in \Sigma^*$, $\mathsf{P_{bad}}(|w|_a, |w|_b, |w|_c) = f(w)$. Remark that for all $x, y, z \geq 0$, $\mathsf{P_{bad}}(x, y, z) = 0$ if and only if $z(x + y)^2 = -2(x - y)^2$. Hence, $\mathsf{P_{bad}}(x, y, z) = 0$ if and only if $z = 0$ and $x = y$, or $z \neq 0$, and $x = y = 0$. Now, let us consider the language $L := \{w \mid f(w) = 0\}$. By the above computation, we conclude that $L = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\} \cup \{c\}^*$. Because $L \cap \{a, b\}^*$ is not a regular language, we conclude that L is not a regular language. However, $L = f^{-1}(\{0\})$ is a regular language (Theorem 13). ◀

[2] In this particular case, one could have considered more generally ℕ-rational series, and replaced regular languages over a unary alphabet by semi-linear sets.

We will discuss at the end of Section 3.2 why Lemma 15 is minimal in the number of indeterminates, which first requires us to provide a *correct* analogue of Flawed Theorem 12. Our counterexample relies on the fact that PolyNNeg is not stable under fixing indeterminates, while $\mathbb{N}$-rational polynomials are. Indeed, the polynomial $\mathsf{P}_{\mathsf{bad}}$ satisfies $\mathsf{P}_{\mathsf{bad}}(X, Y, 1) = 3X^2 + 3Y^2 - 2XY$, which has a negative coefficient for a maximal monomial. Let us now prove that closing PolyNNeg under variable assignments is enough to recover from Flawed Theorem 12. We use the following notation to fix the value of some indeterminate, if $P(X, Y)$ is a polynomial in $\mathbb{Z}[X, Y]$, then $[P(X, Y)]_{X=1}$ is the polynomial $P(1, Y) \in \mathbb{Z}[Y]$. More generally, if ν is a partial function from $\vec{X}$ to $\mathbb{N}$, written $\nu \colon \vec{X} \rightharpoonup \mathbb{N}$, the restriction $[P(\vec{X})]_\nu$ is the polynomial with indeterminates $\vec{Y} := \vec{X} - \mathrm{dom}(\nu)$ obtained by fixing the variables of the domain of ν.

▶ **Definition 16.** *The class* PolyStrNNeg$[\vec{X}]$ *is the collection of polynomials $P \in \mathbb{Z}[\vec{X}]$ such that, for every partial function $\nu \colon \vec{X} \rightharpoonup \mathbb{N}$, every maximal monomial of $[P]_\nu$ is non-negative.*

First, let us remark that PolyStrNNeg $\subseteq$ PolyNNeg, because polynomials in PolyStrNNeg are non-negative. We also remarked at the beginning of this section that our counterexample $\mathsf{P}_{\mathsf{bad}}$ provided in Lemma 15 is not in PolyStrNNeg. The rest of the section is mainly concerned with proving the following corrected version of Flawed Theorem 12.

▶ **Theorem 17.** *Let $P \in \mathbb{Z}[\vec{X}]$. The following are equivalent:*
1. $P \in$ PolyStrNNeg*,*
2. *P is represented by a $\mathbb{N}$-rational series,*
3. *P is represented by a $\mathbb{N}$-polyregular function,*
4. *P is represented by a star-free $\mathbb{N}$-polyregular function,*

Furthermore, the properties are decidable, and conversions effective.

Theorem 17 is surprising given the fact that it is not possible to decide whether a polynomial $P \in \mathbb{Z}[\vec{X}]$ is non-negative or if a polynomial P belongs to PolyNNeg (Remark 18), by reduction to the undecidability of Hilbert's Tenth Problem [9, 12]. That is, PolyStrNNeg is a decidable class that strictly contains $\mathbb{N}[\vec{X}]$, and is contained in the undecidable classes PolyNNeg and the class of non-negative polynomials.

▶ Remark 18. Checking whether a polynomial $P \in \mathbb{Z}[\vec{X}]$ is non-negative is undecidable. Similarly, checking whether a polynomial $P \in \mathbb{Z}[\vec{X}]$ belongs to PolyNNeg is undecidable.

The proof of Theorem 17 is divided into two parts. First, we provide in Section 3.1 a fine combinatorial understanding of what functions can be computed in $\mathbb{N}$Poly and $\mathbb{Z}$Poly. This allows us to prove that $\mathbb{N}$-polyregular functions are in PolyStrNNeg (Corollary 22). Then, in Section 3.2 we will show how to compute polynomials in PolyStrNNeg using $\mathbb{N}$Poly (Lemma 26). Finally, we will next generalize Theorem 17 to polynomials in $\mathbb{Q}[\vec{X}]$ in Section 3.3.

3.1 From $\mathbb{N}$-polyregular functions to polynomials

Let us prove that $\mathbb{N}$-rational polynomials are in PolyStrNNeg. This fact follows from the correct implication in the statement of Flawed Theorem 12, but we provide a self-contained proof using a refinement of the classical combinatorial *pumping arguments* for $\mathbb{Z}$Poly [5, Lemma 4.16] and $\mathbb{N}$Poly [8, Lemma 5.37]. We take extra care to reprove in our upcoming Lemma 20 a strong statement that has two main goals. Our first goal is to highlight the role of commutative polyregular functions in the broader study of polyregular functions, which is done by reformulating the traditional pumping argument as a composition property involving said functions, which will be reused in the upcoming Definitions 35 and 39 of Section 4. Our second goal is to give a precise shape of the functions that arise from such *pumping arguments*, which was lacking in former similar statements.

To address our first goal, let us define that a function q is a *pumping pattern* from $\mathbb{N}^p$ to Σ^* whenever there exists words $\alpha_0, \dots, \alpha_p \in \Sigma^*$, and words $u_1, \dots, u_p \in \Sigma^*$, such that $q(X_1, \dots, X_p) = \alpha_0 \prod_{i=1}^{p} u_i^{X_i} \alpha_i$. That is, q is syntactically defined by a non-commutative monomial over the monoid Σ^*. Pumping patterns are commutative polyregular functions.

Our second goal is achieved by understanding that $\mathbb{N}$-polyregular functions essentially compute binomial coefficients, as illustrated by the polynomial $X(X-1)/2 = \binom{X}{2}$ of Example 5. A *simple binomial function* is a function of the form $\binom{X-\ell}{k}$, where ℓ and k are natural numbers. We extend this to *natural binomial functions* that are obtained by considering $\mathbb{N}$-linear combinations of products of simple binomial functions, that is, we consider functions that have the following shape: $f(x_1, \dots, x_k) = \sum_{i=1}^{n} n_i \prod_{j=1}^{k} \binom{x_j - p_{i,j}}{k_{i,j}}$. Beware that $\binom{X-\ell}{k}$ is defined to be 0 when $X \leq \ell$, and is therefore not a polynomial. Let us immediately prove that simple binomial functions can be represented in $\mathbb{N}\mathsf{SF}$, generalizing Example 5. Conversely, we prove in Lemma 20 that, when suitably pumping a $\mathbb{N}$-polyregular function, one always obtains natural binomial functions.

▶ **Lemma 19.** *Let F be a simple binomial function from $\mathbb{N}^k$ to $\mathbb{N}$. Then it is represented by a star-free polyregular function.*

Proof. Because of the stability properties of $\mathbb{N}\mathsf{SF}$ (Lemma 7), we only need to check that given $r, s \in \mathbb{N}$, the function $x \mapsto \binom{x-r}{s}$ is represented by a function $f_{r,s} \in \mathbb{N}\mathsf{SF}$. Let us prove it when $r = 0$, since the other functions can be obtained by translating $f_{r,s}$. By definition, $\binom{x}{s} = |P_{x,s}|$, where $P_{x,s} := \{(x_1, \dots, x_s) \in \mathbb{N}^s \mid 1 \leq x_1 < \dots < x_s \leq x\}$. Let us proceed as in Example 5 and define $D_{w,s} := \{(u_1, \dots, u_s, u_{s+1}) \in (\Sigma^+)^s \times \Sigma^* \mid w = u_1 u_2 \cdots u_s u_{s+1}\}$. It is clear that $D_{a^x,s}$ is in bijection with $P_{x,s}$ for all $x \in \mathbb{N}$ using the map $(x_1, \dots, x_s) \mapsto (a^{x_1}, \dots, a^{x_s})$. Now, using the monoid $M := (\{0,1\}, \max)$ and the morphism $\mu(\varepsilon) := 0$ and $\mu(a) := 1$, one can compute $|D_{w,s}|$ as $\pi^\dagger(w)$ where $\pi \colon M^{s+1} \to \mathbb{N}$ is defined by $\pi(m_1, \dots, m_s, m_{s+1}) := m_1 \times \dots \times m_s$. We conclude that $f_{0,s} \in \mathbb{N}\mathsf{SF}_s$ is a star-free polyregular function. ◀

▶ **Lemma 20.** *Let f be an $\mathbb{N}$-polyregular function. There exists a computable $\omega \in \mathbb{N}_{\geq 1}$ such that for all pumping patterns $q \colon \mathbb{N}^p \to \Sigma^*$, there exists a computable natural binomial function F such that:*

$$f \circ q(\omega X_1, \dots, \omega X_p) = F \quad \text{over } (\mathbb{N}_{\geq 1})^p \quad .$$

The multiplicative factor ω is necessary in Lemma 20. Indeed, the function $f \colon \{a\}^* \to \mathbb{N}$ defined as 0 when the input is of odd length and 1 when the input is of even length is $\mathbb{N}$-polyregular, but $f(a^X)$ is not a polynomial. We can trade off this multiplicative factor for a constant term addition under the extra assumption that the function is star-free polyregular, as described in the following Lemma 21. This lemma is not immediately of use, but is crucial for the upcoming characterization of $\mathbb{N}$-rational polynomials in Theorem 34, which in turn is a key ingredient of our main Theorem 40.

▶ **Lemma 21.** *Let f be a star-free $\mathbb{N}$-polyregular function. There exists a computable $s \in \mathbb{N}_{\geq 1}$ such that for all pumping patterns $q \colon \mathbb{N}^p \to \Sigma^*$, there exists a computable natural binomial function F such that:*

$$f \circ q(X_1 + s, \dots, X_p + s) = F \quad \text{over } \mathbb{N}^p \quad .$$

Because natural binomial functions behave as polynomials with non-negative maximal monomials on large enough inputs, we can conclude from Lemma 20 that $\mathbb{N}$-rational polynomials are in $\mathsf{PolyStrNNeg}$.

▶ **Corollary 22.** *Let $P \in \mathbb{Z}[X_1, \ldots, X_p]$ be an $\mathbb{N}$-rational polynomial. Then, $P \in \mathsf{PolyStrNNeg}$.*

Proof. Let f be a commutative $\mathbb{N}$-rational series with domain defined as $\Sigma := \{a_1, \ldots, a_p\}$ that represents P. Because f has polynomial growth, $f \in \mathbb{N}\mathsf{Poly}$ (Lemma 6). Using Lemma 20, there exists a number $\omega \in \mathbb{N}_{\geq 1}$ and natural binomial function Q such that for all $n_1, \ldots, n_p \geq 1$:

$$f\left(\prod_{i=1}^{p} a_i^{\omega n_i}\right) = Q(n_1, \ldots, n_p) = P(\omega n_1, \ldots, \omega n_p) \quad .$$

For large enough values of X, the simple binomial function $\binom{X-p}{k}$ coincides with a polynomial whose leading coefficient is $1/k!$ which is non-negative. We conclude that the maximal monomials of $P(\omega X_1, \ldots, \omega X_p)$ are non-negative, and since $\omega \geq 1$, we conclude that the maximal monomials of P have non-negative coefficients.

For every partial valuation $\nu \colon \vec{X} \rightharpoonup \mathbb{N}$, the polynomial $[P]_\nu$ continues to be represented by a $\mathbb{N}$-polyregular function, namely $f_u \colon w \mapsto f(uw)$ where w belongs to a restricted alphabet. As a consequence, the maximal monomials of $[P]_\nu$ are also non-negative, and we have proven that $P \in \mathsf{PolyStrNNeg}$. ◀

3.2 From polynomials to $\mathbb{N}$-polyregular functions

This section is devoted to proving that polynomials in $\mathsf{PolyStrNNeg}$ can be represented by star-free $\mathbb{N}$-polyregular functions. The key lemma of this section is Lemma 26, which is proved by induction on the number of indeterminates of a given polynomial P. In order to prove that result, we use the combinatorial Lemma 25 that allows us to transform a polynomial $P \in \mathsf{PolyStrNNeg}$ into a polynomial in $\mathbb{N}[\vec{X}]$ through a well-chosen translation of the indeterminates. This argument is based on the notion of *discrete derivative* which is built by translating the domain of the polynomial. To that end, let us write τ_K for the *translation function* that maps a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_k]$ to the polynomial $P(X_1 + K, \ldots, X_k + K)$.

▶ **Definition 23.** *Let $K \in \mathbb{N}$, and $P \in \mathbb{Z}[\vec{X}]$ be a polynomial, then $\Delta_K(P) := \tau_K(P) - P$.*

▶ **Lemma 24.** *Let $P \in \mathbb{N}[\vec{X}]$ that is non-constant, and $K \in \mathbb{N}$, then $\Delta_K(P) \in \mathbb{N}[\vec{X}]$ and all of its coefficients are (positive) multiples of K. Furthermore, every monomial that strictly divides some monomial of P appears in $\Delta_K(P)$.*

Proof. We prove the result for monomials, as it extends to $\mathbb{N}$-linear combinations by linearity. Let $P = \prod_{i=1}^{k} X_i^{\alpha_i}$ be a monomial. Notice that $\tau_K(P) = \prod_{i=1}^{k} (X_i + K)^{\alpha_i}$, and using a binomial expansion we list all the possible divisors of P, all of which with coefficients that are positive integers and multiples of K except the coefficient of the maximal monomial (equal to P itself) which is 1. As a consequence, $\tau_K(P) - P$ is simply obtained by removing this maximal monomial, which concludes the proof. ◀

▶ **Lemma 25.** *Let $P \in \mathsf{PolyStrNNeg}$, P_1 be the sum of maximal monomials of P, and $P_2 := P - P_1$ be the sum of non-maximal monomials of P. There exists a computable number $K \in \mathbb{N}$, such that $Q := (\Delta_K(P_1) + \tau_K(P_2)) \in \mathbb{N}[\vec{X}]$.*

Proof. Let us first tackle the specific case where P is a constant polynomial. In this case, $P_1 = P$ and $P_2 = 0$. Furthermore, $\Delta_K(P_1) = 0$ for all $K \in \mathbb{N}$. We conclude that $\Delta_K(P_1) + \tau_K(P_2) = 0$ for all $K \in \mathbb{N}$, hence belongs to $\mathbb{N}[\vec{X}]$. Selecting $K = 0$ we conclude. Assume now that P is not a constant polynomial. We will use Lemma 24 on a well-selected value of K. Let us write α to be the maximal absolute value of a coefficient in P. Let D be the number of unitary monomials that divide some monomial appearing in P. We can now define $K := D \times \alpha$, and let $Q := (\Delta_K(P_1) + \tau_K(P_2))$. Remark that $\Delta_K(P_1)$ is already in $\mathbb{N}[\vec{X}]$, and the constant coefficient of $\tau_K(P_2)$ is also in $\mathbb{N}$. For any other monomial of P_2, by the maximality of P_1, it strictly divides some monomial of P_1, and equals some monomial of $\Delta_K(P_1)$ up to a multiplication by a factor in $\mathbb{Q}$. Because every monomial of $\Delta_K(P_1)$ has a coefficient that is a multiple of $K = \alpha \times D$, we can compensate every monomial of P_2 by a monomial of $\Delta_K(P_1)$. Therefore, $Q \in \mathbb{N}[\vec{X}]$. ◀

▶ **Lemma 26.** *Let $P \in \mathbb{Z}[\vec{X}]$. If $P \in$ PolyStrNNeg, then P is represented by a star-free $\mathbb{N}$-polyregular function, which is computable given P.*

Proof. We prove the result by induction on the number of indeterminates of P. In the proof, we write $\vec{X}$ for the indeterminates appearing in P, that is, we assume without loss of generality that all indeterminates are used.

Base case: If the (unique) maximal monomial of P is a constant term. Since $P \in$ PolyStrNNeg, $P = n \in \mathbb{N}$, and therefore P is represented by a constant star-free $\mathbb{N}$-polyregular function.

Induction: Assume that P is not a constant polynomial, and let us write $P = P_1 + P_2$ where P_1 is the sum of the maximal monomials of P. We compute a bound K such that $Q := (\Delta_K(P_1) + \tau_K(P_2)) \in \mathbb{N}[\vec{X}]$ (Lemma 25). In particular, Q is represented by a star-free $\mathbb{N}$-polyregular function using Lemma 9, the latter being effectively computable from Q. Let us now remark that $P_1 \in \mathbb{N}[\vec{X}]$, and is therefore (effectively) represented by a star-free $\mathbb{N}$-polyregular function (using again Lemma 9). As a consequence, $\tau_K(P) = P_1 + Q$ is (effectively) represented by a function f_Δ.

For all partial valuations $\nu \colon \vec{X} \rightharpoonup \{0, \dots, K\}$ fixing at least one indeterminate, one can use the induction hypothesis to compute a star-free $\mathbb{N}$-polyregular function f_ν that represents $[P]_\nu$. This is possible because we assumed that all indeterminates in $\vec{X}$ are used in P.

Let us assume that the alphabet over which the (commutative) functions f_Δ and f_ν are defined is $\{a_1, \dots, a_k\}$, with a_i representing the indeterminate X_i of the polynomials. Now, let us define by case analysis the following commutative star-free $\mathbb{N}$-polyregular function, defined on words w of the form $w := a_1^{x_1} \cdots a_k^{x_k}$, with $x_1, \dots, x_k \geq 0$.

$$f(w) := \begin{cases} f_{[X_i \mapsto x_i]}(w) & \text{if } \exists i \in \{1, \dots, k\}, x_i \leq K \\ f_\Delta(a_1^{x_1 - K} \cdots a_k^{x_k - K}) & \text{otherwise} \end{cases} .$$

Remark that f is a commutative star-free $\mathbb{N}$-polyregular function that represents P. ◀

While Lemma 26 provides an effective conversion procedure, it does not explicitly state that the membership is decidable to keep the proof clearer. A similar proof scheme can be followed to conclude that membership is decidable, and even show that elements in PolyStrNNeg are, up to suitable translations, polynomials in $\mathbb{N}[\vec{X}]$ (Lemma 27). Beware that partial applications are still needed in this characterization, as Example 28 illustrates.

▶ **Lemma 27.** *Let $P \in \mathbb{Z}[\vec{X}]$. There exists a computable number $K \in \mathbb{N}$ such that the following are equivalent:*
1. $P \in$ PolyStrNNeg*,*
2. *for all partial functions $\nu \colon \vec{X} \rightharpoonup \mathbb{N}$, $\tau_K([P]_\nu) \in \mathbb{N}[\vec{X}]$,*
3. *for all partial functions $\nu \colon \vec{X} \rightharpoonup \{0, \dots, K\}$, $\tau_K([P]_\nu) \in \mathbb{N}[\vec{X}]$.*

In particular, the above properties are decidable.

▶ **Example 28.** The polynomial $\mathsf{P}_{\mathsf{bad}}$ is not a $\mathbb{N}$-rational polynomial, but is non-negative and satisfies $\tau_{10}(\mathsf{P}_{\mathsf{bad}}) \in \mathbb{N}[\vec{X}]$.

We now have all the tools to prove the corrected version of Flawed Theorem 12.

Proof of Theorem 17 on page 7. The implications Item 4 $\implies$ Item 3 $\implies$ Item 2 are obvious. Lemma 26 proves Item 1 $\implies$ Item 4, while Corollary 22 proves Item 2 $\implies$ Item 1. Note that the lemmas provide effective conversion procedures, and that Lemma 27 also provides a decision procedure. ◀

For completeness, let us remark that the counterexample of Lemma 15 uses three indeterminates, and this is not a coincidence: in the particular cases of one or two indeterminates, the classes PolyStrNNeg and PolyNNeg coincide. In particular, the examples appearing in [10] are not invalidated, as they all use at most two indeterminates. Note that the equivalence is clear for the univariate case, where being non-negative and having non-negative maximal coefficient clearly imply being an $\mathbb{N}$-rational polynomial.

▶ **Lemma 29.** $\mathsf{PolyStrNNeg}[X, Y] = \mathsf{PolyNNeg}[X, Y]$.

Proof. It is clear that $\mathsf{PolyStrNNeg}[X, Y] \subseteq \mathsf{PolyNNeg}[X, Y]$, by considering the empty valuation $\nu\colon \{X, Y\} \rightharpoonup \mathbb{N}$. For the converse inclusion, let us consider $P(X, Y)$ that is non-negative, such that the maximal monomials are non-negative.

If we fix none of the variables, then the maximal monomials are non-negative by assumption. If we fix one of the variables, we can assume without loss of generality that we fix $X = k$ for some $k \in \mathbb{N}$. Then $P(k, Y)$ is a non-negative *univariate* polynomial, and therefore must either have a positive leading coefficient (which is the unique maximal monomial in this case) or be constant equal to 0. In both cases, the maximal monomials have positive coefficient. The same reasoning applies *a fortiori* in the case where we fix the two indeterminates, leading to a constant polynomial. ◀

3.3 From $\mathbb{Z}$ to $\mathbb{Q}$

Let us complete our analysis of polynomials represented by $\mathbb{N}$Series or $\mathbb{Z}$Series by considering polynomials with coefficients in $\mathbb{Q}$, and justify that all the combinatorial work has already happened in $\mathbb{Z}$ and $\mathbb{N}$. From Lemma 21, we know that the polynomials that can be computed by star-free $\mathbb{N}$-polyregular functions are going to coincide (on large enough inputs) with natural binomial functions. For that reason, we introduce the following "polynomial counterpart" of a binomial coefficient: given two numbers $\ell, k \in \mathbb{N}$, $\bullet\!\binom{X-\ell}{k}\!\bullet$ defined as $(X - \ell) \cdots (X - \ell - k)/k!$,[3] that we call a *binomial monomial*, and we introduce *natural binomial polynomials* as $\mathbb{N}$-linear combinations of products of binomial monomials, i.e., of the shape: $P(X_1, \dots, X_k) = \sum_{i=1}^{n} n_i \prod_{j=1}^{k} \bullet\!\binom{X_j - p_{i,j}}{k_{i,j}}\!\bullet$. Similarly, we introduce the class of *integer binomial polynomials*, which are obtained by $\mathbb{Z}$-linear combinations of products of binomial monomials.

These definitions are justified by the classical result of Pólya that characterizes polynomials P in $\mathbb{Q}[X]$ that are *integer-valued* (i.e., are such that $P(\mathbb{Z}) \subseteq \mathbb{Z}$) as integer binomial polynomials [14, 4]. Note that this result straightforwardly extends to multiple indeterminates as we prove in Lemma 30.

[3] In particular, $\bullet\!\binom{X-\ell}{k}\!\bullet$ is defined to be 1 when $k = 0$, and $X - \ell$ when $k = 1$.

▶ **Lemma 30.** *Let $P \in \mathbb{Q}[X_1, \dots, X_k]$ be a polynomial. Then, P is an integer binomial polynomial if and only if $P(\mathbb{Z}^k) \subseteq \mathbb{Z}$, if and only if $P(\mathbb{N}^k) \subseteq \mathbb{Z}$.*

As an immediate corollary, we completely characterize the class of polynomials in $\mathbb{Q}[\vec{X}]$ that are represented by $\mathbb{Z}\mathsf{Poly}$ as the integer binomial polynomials.

▶ **Theorem 31.** *Let $P \in \mathbb{Q}[\vec{X}]$. Then, the following properties are equivalent:*
1. *P is integer-valued,*
2. *P is represented by a $\mathbb{Z}$-rational series,*
3. *P is represented by a $\mathbb{Z}$-polyregular function,*
4. *P is represented by a star-free $\mathbb{Z}$-polyregular function,*
5. *P is an integer binomial polynomial.*

These properties are furthermore decidable.

Proof. The implications Item 4 $\implies$ Item 3 $\implies$ Item 2 $\implies$ Item 1 are obvious. Now, Item 1 $\implies$ Item 5 is a direct consequence of Lemma 30. Finally, Item 5 $\implies$ Item 4 follows from the fact that $\bullet\binom{X-p}{k}\bullet$ is represented by a star-free $\mathbb{Z}$-polyregular function defined by hardcoding the output values (in $\mathbb{Z}$) when $0 \leq X \leq p$, and using a star-free $\mathbb{N}$-polyregular function when $X > p$ (Lemma 19). Because $\mathbb{Z}\mathsf{SF}$ is closed under products and $\mathbb{Z}$-linear combinations, we conclude. ◀

Obtaining an analogue of Theorem 31 for $\mathbb{N}$-polyregular functions requires a bit more work, as polynomials in $\mathbb{Q}[\vec{X}]$ that are represented by $\mathbb{N}\mathsf{Poly}$ are not exactly natural binomial polynomials (see Example 32). To address the issues raised by the former example, we introduce the notion of *strongly natural binomial polynomials*, as the polynomials $P \in \mathbb{Q}[X]$ such that for all partial valuation $\nu \colon \rightharpoonup \mathbb{N}$, $[P]_\nu$ is a natural binomial polynomial, which characterizes the class of polynomials that are represented by $\mathbb{N}\mathsf{Poly}$ (Theorem 34).

▶ **Example 32.** The polynomial $Q(X, Y, Z) := \bullet\binom{X-4}{1}\bullet \bullet\binom{Y}{1}\bullet \bullet\binom{Z}{1}\bullet + 8 \bullet\binom{Y}{2}\bullet + 8 \bullet\binom{Z}{2}\bullet + 4$ is a non-negative natural binomial polynomial in $\mathbb{Z}[X, Y, Z]$, but cannot be computed by a star-free $\mathbb{N}$-polyregular function. Indeed, $Q(0, Y, Z)$ has a negative maximal monomial, hence $Q \notin \mathsf{PolyStrNNeg}$, and we conclude using Theorem 17.

▶ **Lemma 33.** *Let $P \in \mathbb{Q}[\vec{X}]$ be an integer-valued polynomial, and $n \in \mathbb{N}_{\geq 1}$ be such that $nP \in \mathsf{PolyStrNNeg}$. Then, P is a strongly natural binomial polynomial.*

▶ **Theorem 34.** *Let $P \in \mathbb{Q}[\vec{X}]$ be a polynomial with* rational *coefficients and let α be the smallest number in $\mathbb{N}_{\geq 1}$ such that $\alpha P \in \mathbb{Z}[\vec{X}]$. Then, the following are equivalent:*
1. *$\alpha P \in \mathsf{PolyStrNNeg}$ and P is integer-valued,*
2. *P is represented by a $\mathbb{N}$-rational series,*
3. *P is represented by a $\mathbb{N}$-polyregular function,*
4. *P is represented by a star-free $\mathbb{N}$-polyregular function,*
5. *P is a strongly natural binomial polynomial.*

In particular, the properties are decidable.

Proof. Let us first remark that $\mathbb{N}\mathsf{Poly} \subseteq \mathbb{N}\mathsf{Series}$, and that if P is represented by a function $f \in \mathbb{N}\mathsf{Series}$, then said function has polynomial growth, and in particular $f \in \mathbb{N}\mathsf{Poly}$ thanks to Lemma 6. As a consequence, Item 2 $\iff$ Item 3. For the implication Item 3 $\implies$ Item 1, we obtain $\alpha P \in \mathsf{PolyStrNNeg}$ via Theorem 17 by remarking that $\mathbb{N}$-polyregular functions have output in $\mathbb{N}$ and are closed under multiplication by a constant $\alpha \in \mathbb{N}$. The fact that P is integer-valued follows from Theorem 31 and the fact that $\mathbb{N}\mathsf{SF} \subseteq \mathbb{Z}\mathsf{Poly}$. The implication Item 1 $\implies$ Item 5 is obtained thanks to Lemma 33.

Let us now prove by induction on the number of indeterminates that Item 5 $\implies$ Item 4. Note that by construction, there exists a number $K \in \mathbb{N}$ such that when the input values of P are all greater than K, P coincides with a natural binomial function, which is itself represented by a star-free $\mathbb{N}$-polyregular function. If some input value X_i is set to a number $x_i \leq K$, then one can leverage the fact that $[P]_{X_i=x_i}$ remains a strongly natural binomial polynomial to conclude by induction that $[P]_{X_i=x_i}$ is represented by a star-free $\mathbb{N}$-polyregular function. Combining these, we obtain a star-free $\mathbb{N}$-polyregular function representing P.

Finally, the implication Item 4 $\implies$ Item 3 is immediate as $\mathbb{N}\mathsf{SF} \subseteq \mathbb{N}\mathsf{Poly}$. ◀

Let us remark that Theorem 34 shows that the class of polynomials represented by $\mathbb{N}\mathsf{Poly}$ is the same as the class of polynomials represented by $\mathbb{N}\mathsf{SF}$, which is a non-trivial statement that will be reused in the study of more general commutative functions in $\mathbb{Z}\mathsf{Poly}$.

4 Beyond Polynomials

In this section, we leverage the decidability results of Section 3 to decide membership in $\mathbb{N}\mathsf{Poly}$ inside $\mathbb{Z}\mathsf{Poly}$ and membership in $\mathbb{N}\mathsf{SF}$ inside $\mathbb{N}\mathsf{Poly}$, both under the extra assumption of commutativity. To characterize $\mathbb{N}\mathsf{Poly}$ inside $\mathbb{Z}\mathsf{Poly}$ we introduce the notion of $(k, \mathbb{N})$-combinatorial function (Definition 35), following the spirit of revious characterizations of subclasses of $\mathbb{Z}\mathsf{Poly}$ in terms of *polynomial pumping arguments* [6, 7, 5].

▶ **Definition 35.** *Let $k \in \mathbb{N}$, and $f \colon \Sigma^* \to \mathbb{Z}$ be a $\mathbb{Z}$-polyregular function. The function f is* $(k, \mathbb{N})$-combinatorial *if there exists $\omega \in \mathbb{N}$, such that for all pumping patterns $q \colon \mathbb{N}^k \to \Sigma^*$, there exists a strongly natural binomial polynomial P satisfying:*

$$f \circ q(\omega X_1, \dots, \omega X_k) = P \qquad \text{over } (\mathbb{N}_{\geq 1})^k \quad .$$

Let us now introduce a decomposition of commutative $\mathbb{Z}$-polyregular functions into integer binomial polynomials. Given a number $\omega \in \mathbb{N}$, let us write $\omega\mathsf{Types}^k$ for the collection of pairs $(S, \vec{r})$ where $S \subseteq \{1, \dots, k\}$ and $r \in \{0, \dots, \omega - 1\}^k$. To a tuple $\vec{x} \in \mathbb{N}^k$, one can associate its *ω-type*, written $\omega\mathsf{type}(\vec{x})$, which is the pair $(S, \vec{r})$ where $S = \{i \in \{1, \dots, k\} \mid x_i \geq \omega\}$ and $\vec{r} = (x_i \mod \omega)_{i \in \{1,\dots,k\}}$.

▶ **Lemma 36.** *Let $f \colon \Sigma^* \to \mathbb{Z}$ be a commutative $\mathbb{Z}$-polyregular function, where we fix the alphabet $\Sigma = \{a_1, \dots, a_k\}$. There exists a computable $\omega \in \mathbb{N}_{\geq 1}$, and computable integer binomial polynomials $P_{(S,\vec{r})} \in \mathbb{Q}[(X_i)_{i \in S}]$ for $(S, \vec{r}) \in \omega\mathsf{Types}^k$, such that for all $\vec{x} \in \mathbb{N}^k$,*

$$f\left(\prod_{i=1}^{k} a_i^{x_i}\right) = P_{(S,\vec{r})}\left((\lfloor x_i/\omega \rfloor)_{i \in S}\right) \text{ where } (S, \vec{r}) = \omega\mathsf{type}(\vec{x}) \quad .$$

▶ **Theorem 37.** *Let $k, d \in \mathbb{N}$, and $f \in \mathbb{Z}\mathsf{Poly}_d$ be commutative over an alphabet of size k. Then, the following are equivalent:*

1. *f is $(k, \mathbb{N})$-combinatorial,*
2. *$f \in \mathbb{N}\mathsf{Poly}_d$,*

Furthermore, the properties are decidable, and conversions effective.

Proof. Let $f \in \mathbb{Z}\mathsf{Poly}_d$ be commutative over an alphabet of size k. We apply Lemma 36 to compute an $\omega \in \mathbb{N}$ and integer binomial polynomials $(P_{(S,\vec{r})})_{(S,\vec{r}) \in \omega\mathsf{Types}^k}$ such that for all $\vec{x} \in \mathbb{N}^k$, $f\left(\prod_{i=1}^{k} a_i^{x_i}\right) = P_{(S,\vec{r})}((\lfloor x_i/\omega \rfloor)_{i \in S})$, where $(S, \vec{r}) = \omega\mathsf{type}(\vec{x})$. We are first going to prove that $f \in \mathbb{N}\mathsf{Poly}_d$ if and only if $P_{(S,\vec{r})}$ is a strongly natural binomial polynomial for all $(S, \vec{r}) \in \omega\mathsf{Types}^k$. This will also provide decidability of Item 2, since one can decide whether a polynomial is strongly natural binomial polynomial using Theorem 34.

Assume that $f \in \mathbb{N}\mathsf{Poly}$, then by definition, the polynomials $P_{(S,\vec{r})}$ are represented by an ℕ-polyregular function, hence are strongly natural binomial polynomials (Theorem 34). Conversely, if $P_{(S,\vec{r})}$ is a strongly natural binomial polynomial for all $(S, \vec{r}) \in \omega\mathsf{Types}^k$, then $f \in \mathbb{N}\mathsf{Poly}$ because one can compute the ω-type of the input using a polyregular function, and then compute the suitable strongly natural binomial polynomial $P_{(S,\vec{r})}$ which is possible in $\mathbb{N}\mathsf{Poly}$ thanks to Theorem 34. The resulting composition belongs to $\mathbb{N}\mathsf{Poly}$ thanks to Lemma 7, and we conclude that $f \in \mathbb{N}\mathsf{Poly}_d$ because it has growth rate at most d (Lemma 6).

Note that the same proof scheme can be used to conclude that Item 2 implies Item 1. For the converse implication, we are going to introduce ω_2 associated to the fact that f is $(k, \mathbb{N})$-combinatorial. Because polynomials represented by ℕ-polyregular functions and integer binomial polynomials are both closed under multiplication of their input by a constant factor, we can assume that $\omega = \omega_2$ in the decomposition of f. Now, consider $(S, \vec{r}) \in \omega\mathsf{Types}^k$. Notice that for all vectors $\vec{x} \in (\mathbb{N}_{\geq 1})^k$, the vector $(x_1 \omega \mathbf{1}_S(1) + r_1, \ldots, x_k \omega \mathbf{1}_S(k) + r_k)$ has ω-type $(S, \vec{r})$. In particular, the following equality holds:

$$f\left(\prod_{i=1}^{k} a_i^{x_i \omega \mathbf{1}_S(i) + r_i}\right) = P_{(S,\vec{r})}((x_i)_{i \in S}) \quad \forall \vec{x} \in (\mathbb{N}_{\geq 1})^k \quad .$$

Let us therefore consider the pumping pattern $q\colon \mathbb{N}^k \to \Sigma^*$ that is simply defined as $q(X_1, \ldots, X_k) := \prod_{i=1}^{k} a_i^{X_i \mathbf{1}_S(i) + r_i}$. Because f is $(k, \mathbb{N})$-combinatorial with parameter ω, there exists a strongly natural binomial polynomial $P \in \mathbb{Q}[X_1, \ldots, X_k]$ such that $f \circ q(\omega X_1, \ldots, \omega X_k) = P(X_1, \ldots, X_k)$ over $(\mathbb{N}_{\geq 1})^k$. This proves that $P_{(S,\vec{r})}((X_i)_{i \in S})$ equals $P(X_1, \ldots, X_k)$ as polynomials, hence, that $P_{(S,\vec{r})}$ is a strongly natural binomial polynomial for all $(S, \vec{r}) \in \omega\mathsf{Types}^k$. We have proven that $f \in \mathbb{N}\mathsf{Poly}_d$. ◀

It was already known that ℤ-polyregular functions with unary input that are non-negative are ℕ-polyregular [1, Proposition 2.1 p 137]. Let us derive this fact from our Theorem 37.

▶ **Corollary 38.** *Let $f\colon \{a\}^* \to \mathbb{Z}$ be a non-negative ℤ-polyregular function, then $f \in \mathbb{N}\mathsf{Poly}$.*

Proof. Since f has unary input, it is commutative. Furthermore, f is $(1, \mathbb{N})$-combinatorial because for all $q\colon \mathbb{N} \to \{a\}$ and all $\omega \geq 1$, $f(q(\omega X))$ is non-negative. When it is a polynomial function, it therefore belongs to PolyStrNNeg, hence is a strongly natural binomial polynomial. We conclude using Theorem 37. ◀

Let us now prove that the above characterizations of commutative ℕ-polyregular functions can be combined with the recent advances in the study of ℤ-polyregular functions [5] allowing to decide the membership of ℤSF inside ℤPoly. The key ingredient of this study is the use of a semantic characterization of star-free ℤ-polyregular functions among ℤ-rational series that generalizes the notion of aperiodicity for languages to functions.

▶ **Definition 39** (Ultimately polynomial)**.** *Let Σ be a finite alphabet. A function $f\colon \Sigma^* \to \mathbb{Z}$ is* ultimately polynomial *when there exists $N_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, for all pumping pattern $q\colon \mathbb{N}^k \to \Sigma^*$, there exists a polynomial $P \in \mathbb{Q}[X_1, \ldots, X_k]$ such that:*

$$f \circ q = P \quad \text{over } (\mathbb{N}_{\geq N_0})^k \quad .$$

It was observed in [5, Claim V.6], and in [8, Claim 7.45, Lemma 7.53] that a regular language L is *star-free* if and only if its indicator function $\mathbf{1}_L$ is ultimately polynomial. We can now answer [8, Conjecture 7.61] positively, by proving that $\mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF} = \mathbb{N}\mathsf{SF}$.

▶ **Theorem 40.** *Let Σ be a finite alphabet, and $f\colon \Sigma^* \to \mathbb{Z}$ be a commutative ℕ-polyregular function. Then, the following are equivalent:*

1. *f is ultimately polynomial,*

2. $f \in \mathbb{Z}\mathsf{SF}$,
3. $f \in \mathbb{N}\mathsf{SF}$.

Furthermore, membership is decidable and conversions are effective.

Proof. The implication Item 3 ⇒ Item 2 is immediate since $\mathbb{N}\mathsf{SF} \subseteq \mathbb{Z}\mathsf{SF}$. Furthermore, Item 2 implies Item 1 following previous results for star-free $\mathbb{Z}$-polyregular functions [5, Theorem V.13].

For the implication Item 1 ⇒ Item 3, let us assume that f is ultimately polynomial. We prove the result by induction on the size of the alphabet Σ. By definition, there exists $N_0 \in \mathbb{N}$, and $P \in \mathbb{Q}[(X_a)_{a\in\Sigma}]$ such that:

$$f\left(\prod_{a\in\Sigma} a^{x_a}\right) = P((x_a)_{a\in\Sigma}) \quad \forall \vec{x} \in (\mathbb{N}_{\geq N_0})^{\Sigma} \quad .$$

It is clear that $\tau_{N_0}(P)$ is represented by an $\mathbb{N}$-polyregular function, namely, $f_u \colon w \mapsto f(uw)$ where $u := \prod_{a\in\Sigma} a^{N_0}$, and is therefore represented by a star-free $\mathbb{N}$-polyregular function thanks to Theorem 34. For every letter $a \in \Sigma$ and number $0 \leq n \leq N_0$, there exists, by induction hypothesis, a star-free $\mathbb{N}$-polyregular function g_{a^n} that represents the function $f_{a^n} \colon (\Sigma \setminus \{a\})^* \to \mathbb{Z}$ that maps $w \in (\Sigma \setminus \{a\})^*$ to $f(a^n w)$.

Let us conclude by computing f using the following star-free $\mathbb{N}$-polyregular function $g \colon \Sigma^* \to \mathbb{Z}$:

$$g \colon w \mapsto \begin{cases} g_{a^n}(w) & \text{if } |w|_a = n \text{ for some } a \in \Sigma \text{ and } n \leq N_0 \\ \tau_{N_0}(P)(((|w|_a - N_0)_{a\in\Sigma}) & \text{otherwise} \end{cases}$$ ◀

5 Outlook

Let us end on a more general discussion regarding the status of commutative input functions in the study of unary output polyregular functions. A *quantitative pumping argument* for polyregular function $f \colon \Sigma^* \to \mathbb{Z}$ states that f has property X if and only if for all pumping pattern $q \colon \mathbb{N}^k \to \Sigma^*$, $f \circ q$ has property X. Let us formalize such a statement for growth rate and aperiodicity respectively in Lemmas 41 and 42. Note that we generalized pumping patterns to commutative star-free polyregular functions to simplify the statements.

▶ **Lemma 41.** *Let $f \in \mathbb{Z}\mathsf{Series}$, and $d \in \mathbb{N}$. Then, $f \in \mathbb{Z}\mathsf{Poly}_d$ if and only if for every commutative star-free polyregular function h of growth rate $l \in \mathbb{N}$, $(f \circ h) \in \mathbb{Z}\mathsf{Poly}_{d\times l}$.*

▶ **Lemma 42.** *Let $f \in \mathbb{Z}\mathsf{Poly}$. Then, $f \in \mathbb{Z}\mathsf{SF}$, if and only if for every commutative star-free polyregular function h, $(f \circ h) \in \mathbb{Z}\mathsf{SF}$.*

Remark that if Lemma 42 were to hold for $\mathbb{N}$-polyregular functions, then the decidability of $\mathbb{N}\mathsf{Poly}$ inside $\mathbb{Z}\mathsf{Poly}$, and the decidability of $\mathbb{N}\mathsf{SF}$ inside $\mathbb{N}\mathsf{Poly}$ would immediately follow. On the one hand, one can guess a candidate function and check for equivalence, on the other hand, one can guess a commutative star-free polyregular function and check membership (which is decidable thanks to this paper). This is restated in our concluding conjecture.

▶ **Conjecture 43.** *Let $f \in \mathbb{Z}\mathsf{Poly}$. Then, $f \in \mathbb{N}\mathsf{Poly}$ if and only if for every commutative star-free polyregular function h, $(f \circ h) \in \mathbb{N}\mathsf{Poly}$.*

References

1 Jean Berstel and Christophe Reutenauer. *Noncommutative rational series with applications*, volume 137 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2010. doi:10.1017/CBO9780511760860.

2 Mikołaj Bojańczyk. Polyregular functions, 2018. arXiv:1810.08760.

3 Mikołaj Bojańczyk, Sandra Kiefer, and Nathan Lhote. String-to-String Interpretations With Polynomial-Size Output. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 106:1–106:14, Dagstuhl, Germany, 2019. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2019.106.

4 Paul-Jean Cahen and Jean-Luc Chabert. *Integer-Valued Polynomials*. American Mathematical Society, December 1996. doi:10.1090/surv/048.

5 Thomas Colcombet, Gaëtan Douéneau-Tabot, and Aliaume Lopez. Z-polyregular functions. In *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13, Los Alamitos, CA, USA, June 2023. IEEE Computer Society. doi:10.1109/LICS56636.2023.10175685.

6 Gaëtan Douéneau-Tabot. Pebble Transducers with Unary Output. In Filippo Bonchi and Simon J. Puglisi, editors, *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, volume 202 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 40:1–40:17, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.MFCS.2021.40.

7 Gaëtan Douéneau-Tabot. Hiding Pebbles When the Output Alphabet Is Unary. In Mikołaj Bojańczyk, Emanuela Merelli, and David P. Woodruff, editors, *49th International Colloquium on Automata, Languages, and Programming (ICALP 2022)*, volume 229 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 120:1–120:17, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ICALP.2022.120.

8 Gaëtan Douéneau-Tabot. *Optimization of string transducers*. PhD thesis, Université Paris Cité, 2023. URL: https://gdoueneau.github.io/pages/DOUENEAU-TABOT_Optimization-transducers_v2.pdf.

9 David Hilbert. Mathematical problems. *Bulletin of the American Mathematical Society*, 8(10):437–479, 1902. doi:10.1090/s0002-9904-1902-00923-3.

10 Juhani Karhumäki. Remarks on commutative N-rational series. *Theoretical Computer Science*, 5(2):211–217, 1977. doi:10.1016/0304-3975(77)90008-1.

11 Aliaume Lopez. Commutative n-polyregular functions, 2024. doi:10.48550/arXiv.2404.02232.

12 Yuri Vladimirovich Matiyasevich. The diophantineness of enumerable sets. *Doklady Akademii Nauk SSSR*, 191:279–282, 1970. in Russian.

13 Robert McNaughton and Seymour A. Papert. *Counter-Free Automata*. The MIT Press, 1971. doi:10.5555/1097043.

14 G. Pólya. Über ganzwertige ganze Funktionen. *Rend. Circ. Mat. Palermo*, 40:1–16, 1915. doi:10.1007/BF03014836.

15 Marcel P. Schützenberger. Finite counting automata. *Information and control*, 5(2):91–107, 1962. doi:10.1016/S0019-9958(62)90244-9.

16 Marcel P. Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8(2):190–194, 1965. doi:10.1016/S0019-9958(65)90108-7.

17 Wolfgang Thomas. Languages, automata, and logic. In Grzegorz Rozenberg and Arto Salomaa, editors, *Handbook of formal languages*, pages 389–455. Springer, 1997. doi:10.1007/978-3-642-59136-5.