# Laplace Transform Interpretation of Differential Privacy

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**Abstract**

We introduce a set of useful expressions of Differential Privacy (DP) notions in terms of Laplace transformations. The underlying bare-form expressions for these transforms appear in several works on analyzing DP, either as an integral or an expectation. We show that recognizing these expressions as Laplace transformations unlocks a new way to reason about DP properties by exploiting the duality between time and frequency domains. Leveraging our interpretation, we connect the $(q, \rho(q))$-Rényi DP curve and the $(\varepsilon, \delta(\varepsilon))$-DP curve as being the Laplace and inverse-Laplace transforms of one another. Using our Laplace transform-based analysis, we also prove *an adaptive composition theorem for* $(\varepsilon, \delta)$*-DP guarantees that is exactly-tight* (i.e., matches even in constants) for all values of $\varepsilon$. Additionally, we resolve an issue regarding symmetry of $f$-DP on subsampling that prevented equivalence across all functional DP notions.

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## 1 Introduction

Differential privacy (DP) [12] is now the de facto standard for privacy quantification. Various functional interpretations of the concept of differential privacy have emerged over time – the $(\varepsilon, \delta(\varepsilon))$-DP curves aka. privacy profiles [4], the Rényi DP [20] function of the order $q$, the $f$-DP notion [10], and the privacy loss distribution (PLD)'s density function [23] – each having their own advantages and limitations.

However, they all have some shared difficulties – certain types of manipulations on them are harder to perform in the time-domain, but considerably simple to do in the frequency domain. For instance, composing two mechanisms involve convolving their PLDs, which is a lot easier to numerically approximate by multiplying the Discrete Fast-Fourier Transformation (DFFT) of the densities and then taking an Inverse-DFFT [18, 16]. Similarly, the Moments' accountant's $(\varepsilon, \delta)$-DP bound for the DP-SGD algorithm [1] composes (multiplies) the higher-order moments of subsampled Gaussian mechanisms induced by individual noisy gradient updates. Since the moment of a random variable $X$ is the two-sided Laplace transform of its density $f_X$, the Momements' accountant also uses the time-frequency domain dualities.

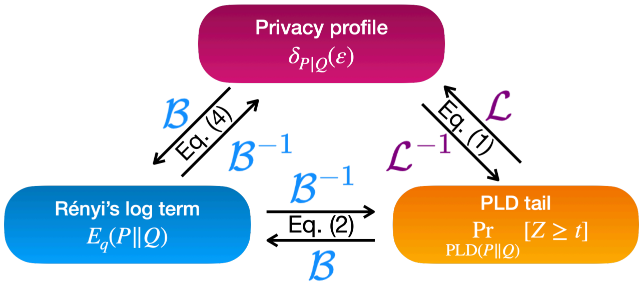


**Figure 1** Caricature of Laplace-transform links between three functional notions of differential privacy: log-term in Rényi divergence $\mathrm{E}_q(P\|Q)$ as a function of order $q$, privacy profile $\delta_{P|Q}(\varepsilon)$ as a function of $\varepsilon$, and the survival function (aka. the tail) of the privacy loss distribution $\mathrm{PLD}(P\|Q)$.

In this paper, we formalize these time-frequency domain dualisms enjoyed by several DP notions into a *new interpretation of differential privacy*. In addition to augmenting existing perspectives on DP, we show with examples that this interpretation extends our ability to reason about DP and its underpinnings. This interpretation is based on recognizing that the privacy-profile $\delta_{P|Q}(\varepsilon) := \sup_S P(S) - e^{\varepsilon} \cdot Q(S)$ and the Rényi-divergence $\mathrm{R}_q(P\|Q) := \frac{1}{q-1} \log \mathrm{E}_q(P\|Q)$ (where $\mathrm{E}_q(P\|Q) := \int_\Omega P^q Q^{1-q} \mathrm{d}\theta$) between any two distributions $P, Q$ on the same space $\Omega$ are both Laplace transforms[1] of the tail of privacy loss distribution $\mathrm{PLD}(P\|Q)$ – the distribution of privacy loss random variable $Z = L_{P|Q}(\Theta) := \log(P(\Theta)/Q(\Theta))$ where $\Theta \sim P$:

$$\forall \varepsilon \in \mathbb{R} \ : \ \delta_{P|Q}(\varepsilon) = \mathop{\mathbb{E}}_{Z \leftarrow \mathrm{PLD}(P\|Q)} \left[\max\{0, 1 - e^{\varepsilon - Z}\}\right] = \mathcal{L}\left\{\mathbb{P}\left[Z \geq t + \varepsilon\right]\right\}(1), \tag{1}$$

$$\forall q \in \mathbb{C} \ : \ \mathrm{E}_q(P\|Q) = \mathop{\mathbb{E}}_{Z \leftarrow \mathrm{PLD}(P\|Q)} \left[e^{(q-1)\cdot Z}\right] = (q-1) \cdot \mathcal{B}\left\{\mathbb{P}\left[Z \geq t\right]\right\}(1-q). \tag{2}$$

The first equality in (1) is a widely used way to represent the $\delta$ as a function of $\varepsilon$ in DP literature [23, 5, 4, 18, 16, 24, 8]. Similarly, the first equality in (2) represents the well-known moment-generating function of privacy loss [20, 1, 5]. The second equalities above are part of a set of Laplace transform expressions of differential privacy presented in this paper. Together, these expressions unlock a formal approach to perform a wide-variety of manipulations on DP formulae using the fundamental properties of the Laplace functional. Appendix A provides a non-exhaustive list of useful properties of the Laplace transforms. These properties capture the relationship between the time and frequency domain representations of a function. Relying on these properties often simplifies analysis as operations like differentiation, convolution, change of variables and integration by parts in the time-domain map to basic algebra in the frequency domain. Using this Laplace transform interpretation of DP extensively, our paper presents the following findings:

1. By eliminating $\mathrm{PLD}(P\|Q)$ in equations (1) and (2), we show that the privacy-profile and Rényi divergence between two distributions are also connected by a Laplace transform:

$$\forall q \in \mathbb{C} \ : \ e^{(q-1)\cdot \mathrm{R}_q(P\|Q)} = \mathrm{E}_q(P\|Q) = q(q-1) \cdot \mathcal{B}\left\{\delta_{P|Q}(t)\right\}(1-q). \tag{3}$$

2. We show that tightly converting a Rényi divergence curve into $(\varepsilon, \delta)$-DP requires taking an *inverse Laplace transform* of (3), which results in an expression of $\delta_{P|Q}(\varepsilon)$ that exclusively

[1] Laplace transform maps a time-domain function $g(t)$ with $t \in \mathbb{R}$ to a function $\mathcal{L}\{g\}(s) := \int_0^\infty e^{-st} g(t) \mathrm{d}t$ with $s \in \mathbb{C}$. Similarly, bilateral Laplace transform of $g(t)$ is $\mathcal{B}\{g\}(s) := \int_{-\infty}^\infty e^{-st} g(t) \mathrm{d}t$.

relies on complex orders ($q \in \mathbb{C}$):

$$\forall \varepsilon \in \mathbb{R} \ : \ \delta_{P|Q}(\varepsilon) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\gamma - i\omega}^{\gamma + i\omega} \frac{\exp[(q-1) \cdot (\mathrm{R}_q(P\|Q) - \varepsilon)]}{q(q-1)} \mathrm{d}q, \tag{4}$$

where $\gamma$ can be *any real in* $\mathbb{R} \setminus \{0,1\}$ when $P, Q$ are absolute continuous w.r.t. each other. In other words, tightly converting a $(q, \mathrm{R}_q(P\|Q))$ Rényi divergence curve into an $(\varepsilon, \delta_{P|Q}(\varepsilon))$-DP privacy-profile curve requires checking complex orders $\{q \in \mathbb{C} : \mathfrak{Re}(q) = \gamma\}$, not real orders $\{q \in \mathbb{R} : q > 1\}$ as prior works [5, 26, 2, 8] focus on.

3. We show that the stochastic dominance relationship between PLD, privacy-profile and Rényi divergence *holds only in one direction.* For any two absolutely continuous distribution pairs $(P_1, Q_1)$ and $(P_2, Q_2)$,

$$\forall t \in \mathbb{R} \ : \ \underset{\mathrm{PLD}(P_1\|Q_1)}{\mathbb{P}}[Z \geq t] \leq \underset{\mathrm{PLD}(P_2\|Q_2)}{\mathbb{P}}[Z \geq t] \tag{5}$$

$$\implies \forall \varepsilon \in \mathbb{R} \ : \ \delta_{P_1|Q_1}(\varepsilon) \leq \delta_{P_2|Q_2}(\varepsilon) \tag{6}$$

$$\implies \forall q > 1 \ : \ \mathrm{R}_q(P_1\|Q_1) \leq \mathrm{R}_q(P_2\|Q_2). \tag{7}$$

However, *none of the above implication applies in the other direction.*

4. For any two absolutely continuous product distributions $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$, the Rényi divergence composes linearly [25], i.e., $\mathrm{R}_q(P\|Q) = \mathrm{R}_q(P_1\|Q_1) + \mathrm{R}_q(P_2\|Q_2)$ for all $q > 0$. We provide an *exactly-tight and adaptive theorem for composing any two privacy profiles that mirrors the lossless composition of Rényi divergence*:

$$\forall \varepsilon \in \mathbb{R} \ : \ \delta_{P|Q}(\varepsilon) = \left(\dot{\delta}_{P_1|Q_1} \circledast \left(\dot{\delta}_{P_2|Q_2} - \delta_{P_2|Q_2}\right)\right)(\varepsilon), \tag{8}$$

where $\dot{\delta}_{P_1|Q_1}$ and $\dot{\delta}_{P_2|Q_2}$ are the respective derivatives and "$\circledast$" denotes the convolution operation. Like Rényi composition, our composition result also holds under adaptivity.

5. We apply our composition result (8) on $(\varepsilon, \delta)$-DP Randomized Response mechanism – whose privacy-profile dominates all other $(\varepsilon, \delta)$-DP mechanisms [17] – to derive an *exactly-tight composition theorem for* $(\varepsilon, \delta)$*-DP guarantees.* In particular, we show that composing $k$ many $(\varepsilon, \delta)$-DP mechanisms satisfies $(\varepsilon, \delta^{\otimes k}(\varepsilon))$-DP for all $\varepsilon \in \mathbb{R}$, where

$$\delta^{\otimes k}(t) := 1 - (1-\delta)^k \left(1 - \underset{Y \leftarrow \mathrm{Binomial}\left(k, \frac{e^{\varepsilon}}{e^{\varepsilon}+1}\right)}{\mathbb{E}} \left[1 - e^{t - \varepsilon \cdot (2Y - k)}\right]_+\right). \tag{9}$$

Our bound improves upon the previously best composition result in [17, Theorem 3.3].

6. We identify that a symmetry problem when working with DP curves – in many cases $\delta_{P|Q}(\varepsilon) \neq \delta_{Q|P}(\varepsilon)$ or $\mathrm{PLD}(P\|Q) \neq \mathrm{PLD}(Q\|P)$. Existing works handle asymmetry either by maintaining functional representation in both directions [16, 23], or by enforcing symmetry at the cost of overestimating the curve [10]. We highlight a *reversal property enjoyed by all function notions of DP* and use it to eliminate this asymmetry problem.

**Related work.** We build on interpretations of DP that appeared before [10, 14, 13, 20, 4, 5, 7, 23, 16, 18, 26]. In particular, [26] shares the most similarity and relevance with ours, as they were the first to observe that many functional notions of differential privacy appear to be linked via Laplace or Fourier transforms. However, their work centers on using the characteristic function of privacy loss as an intermediate representation connecting DP notions. In contrast, we examine the nature of these connections themselves to harness the perspective of Laplace transformations as an analytical tool for DP. Related studies on composition theorems include [15, 17, 21, 7, 20], along with numerical accounting methods [16, 18, 11].

# 2 Preliminaries

## 2.1 Background on Differential Privacy

Let $\mathcal{X}$ be a data universe, and consider datasets $D \in \mathcal{X}^n$ of size $n$, where $D = (x_1, \ldots, x_n)$. Algorithms $\mathcal{M} : \mathcal{X}^n \to \Omega$ output random variables in space $\Omega$. Two datasets $D$ and $D'$ are *neighboring*, denoted $D \simeq D'$, if they differ by a single record replacement. We denote the distributions of $\mathcal{M}(D)$ and $\mathcal{M}(D')$ as $P$ and $Q$, respectively. For a random variable $Z$, its *cumulative distribution function* (CDF) is defined as $F_Z(t) := \Pr[Z \leq t]$. We define the corresponding *generalized probability density function* (PDF) as $f_Z(t)dt := \lim_{a\to 0^+} \int_{t-a}^{t+a} F_Z(u)du$ to handle points of discontinuities in CDF with *Dirac delta masses.*

▶ **Definition 1** (Differential Privacy and Privacy Profiles [13, 4]). *An algorithm $\mathcal{M}$ is $(\varepsilon, \delta)$-differentially private ($(\varepsilon, \delta)$-DP) if for all $D \simeq D'$ and all $S \subset \Omega$,*

$$\mathbb{P}[\mathcal{M}(D) \in S] \leq e^{\varepsilon} \cdot \mathbb{P}[\mathcal{M}(D') \in S] + \delta. \tag{10}$$

*The* privacy profile *of $\mathcal{M}$ on neighbors $D \simeq D'$ is $\delta_{P|Q}(\varepsilon)$ for $\varepsilon \in \mathbb{R}$, defined as*

$$\delta_{P|Q}(\varepsilon) := \sup_{S\subset\Omega} [P(S) - e^{\varepsilon}Q(S)] = \int_{\Omega} \max\{0, P(\theta) - e^{\varepsilon}Q(\theta)\}d\theta. \tag{11}$$

▶ **Definition 2** (Rényi Differential Privacy [20]). *An algorithm $\mathcal{M}$ is $(q, \rho)$-*Rényi differentially private *($(q, \rho)$-RDP) if for all $D \simeq D'$, the $q$-Rényi divergence satisfies $\mathrm{R}_q(P\|Q) \leq \rho$, where*

$$\mathrm{R}_q(P\|Q) := \frac{1}{q-1} \log \mathrm{E}_q(P\|Q), \quad \textit{with} \quad \mathrm{E}_q(P\|Q) := \int_{\theta\in\Omega} P(\theta)^q Q(\theta)^{1-q} d\theta, \tag{12}$$

*defined for all orders $q \in \mathbb{C}$ where the integral converges.*

▶ **Definition 3** ($f$-Differential Privacy [10]). *An algorithm $\mathcal{M}$ is $f$-*differentially private *($f$-DP) if for all $\alpha \in [0, 1]$, the trade-off function $f_{P|Q}(\alpha) \geq f(\alpha)$, where $f_{P|Q}$ is defined as*

$$f_{P|Q}(\alpha) := \inf_{\phi:\Omega\to[0,1]} \{\beta_\phi : \alpha_\phi \leq \alpha\}, \quad \textit{with} \quad \alpha_\phi := \mathbb{E}_P[\phi] \ \textit{ and } \ \beta_\phi := 1 - \mathbb{E}_Q[\phi]. \tag{13}$$

▶ **Theorem 4** (Privacy of Gaussian Mechanism [13, 6, 20]). *Let $P = \mathcal{N}(\mu, \sigma^2 I_d)$ and $Q = \mathcal{N}(\mu', \sigma^2 I_d)$ be two multivariate Gaussians with $\kappa = \|\mu - \mu'\|_2^2/(2\sigma^2)$. Then, for all $\varepsilon \in \mathbb{R}$,*

$$\delta_{P|Q}(\varepsilon) = \overline{\Phi}\left(\frac{\varepsilon - \kappa}{\sqrt{2\kappa}}\right) - e^{\varepsilon}\overline{\Phi}\left(\frac{\varepsilon + \kappa}{\sqrt{2\kappa}}\right) = O(e^{-\varepsilon^2/4\kappa}), \tag{14}$$

*where $\overline{\Phi}(t) = \mathbb{P}[G > t]$ for $G \sim \mathcal{N}(0, 1)$. And for all $q > 1$, $\mathrm{R}_q(P\|Q) = \kappa \cdot q$.*

## 2.2 Background on Laplace Transforms

Laplace transforms map time-domain functions to frequency-domain functions. The one-sided and two-sided Laplace transforms of a function $g(t)$ with $(t \in \mathbb{R})$ are defined as:

$$\forall s \in \mathbb{C} \ : \ \mathcal{L}\{g(t)\}(s) := \int_0^{\infty} e^{-st}g(t)dt, \quad \text{and} \quad \mathcal{B}\{g(t)\}(s) := \int_{-\infty}^{\infty} e^{-st}g(t)dt. \tag{15}$$

The set of complex numbers $s \in \mathbb{C}$ for which the integrals in (15) converge is called the *region of convergence* (ROC) of the Laplace transforms. Through dominated convergence theorem, it can be shown that this region is always a vertical strip in the complex plane parallel to the imaginary axis [22]. We denote the ROC for the one-sided Laplace transform of $g$ as $\mathrm{ROC}_{\mathcal{L}}\{g\}$, and for the two-sided transform as $\mathrm{ROC}_{\mathcal{B}}\{g\}$.

**Uniqueness and Inversion.** Laplace transform are unique for continuous functions [9]. Therefore, a Laplace transform $\bar{g}(s) = \mathcal{L}\{g(t)\}(s)$ for a continuous $g(t)$ can be inverted via the inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\}(t) := \frac{1}{2\pi i} \lim_{\omega\to\infty} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{st}\bar{g}(s)ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\bar{g}(s)ds, \tag{16}$$

where $\gamma$ lies in the region of convergence $\bar{g}(s)$. Equation (16) also inverts two-sided transform [22], i.e. if $\bar{g}(s) = \mathcal{B}\{g(t)\}(s)$ then $\mathcal{B}^{-1}\{\bar{g}(s)\}(t) = g(t)$ where the operator $\mathcal{B}^{-1} = \mathcal{L}^{-1}$.

**Other Properties.** Laplace transforms simplify many time-domain operations by converting them into simpler frequency-domain operations. For detailed properties, see [9] for one-sided transforms and [22] for two-sided transforms. Appendix A summarizes the properties used in this paper, referenced as $\overset{(m)}{=}$, where $(m)$ is the equation number.

## 3 Laplace Transform Expressions of Differential Privacy

Differential privacy bounds the maximum divergence in the output distribution caused by including or omitting a data-point from the dataset. This principle is captured well by the *privacy loss distribution* formalism.

▶ **Definition 5** (Privacy Loss Distribution [23]). *The* privacy loss *of an observation $\theta \in \Omega$ from an algorithm $\mathcal{M}$, when comparing datasets $D \simeq D'$, is defined as $L_{P|Q}(\theta) := \log(P(\theta)/Q(\theta))$, where $P$ and $Q$ are the probability mass/density functions of $\mathcal{M}(D)$ and $\mathcal{M}(D')$, respectively. The* privacy loss distribution $\mathrm{PLD}(P\|Q)$ *is the distribution of $L_{P|Q}(\Theta)$ when $\Theta \sim P$.*

The $\mathrm{PLD}(P\|Q)$ describes how outputs arising from $D$ increase an observer's confidence that they did not in fact come from $D'$. Many prior works make use of the following set of DP expressions in terms of the privacy loss distribution [23, 5, 4, 18, 16, 24, 8]. Common ways to express the privacy profile $\delta_{P|Q}(\varepsilon)$ in terms of $Z \leftarrow \mathrm{PLD}(P\|Q)$ and $Z' \leftarrow \mathrm{PLD}(Q\|P)$ are

$$\delta_{P|Q}(\varepsilon) = \mathbb{E}\left[1 - e^{\varepsilon - Z}\right]_+ = \mathbb{E}\left[e^{-Z'} - e^{\varepsilon}\right]_+ = \Pr[Z \geq \varepsilon] - e^{\varepsilon} \cdot \Pr[Z' \leq -\varepsilon], \tag{17}$$

where $[\bullet]_+ := \max\{0, \bullet\}$. Similarly, the Rényi divergence $\mathrm{R}_q(P\|Q)$ is often expressed as

$$\mathrm{R}_q(P\|Q) = \frac{1}{q-1}\log\mathbb{E}\left[e^{(q-1)\cdot Z}\right] = \frac{1}{q-1}\log\mathbb{E}\left[e^{-q\cdot Z'}\right]. \tag{18}$$

In the following theorem, we present a more dynamic version of these relationships by expressing them as a set of *Laplace transforms* of the privacy loss distributions.

▶ **Theorem 6** ($\delta_{P|Q}(\varepsilon)$ and $\mathrm{E}_q(P\|Q)$ as Laplace Transforms of PLDs). *Let $P$ and $Q$ be probability distributions on the same space $\Omega$. If random variables $Z \sim \mathrm{PLD}(P\|Q)$ and $Z' \sim \mathrm{PLD}(Q\|P)$, then for all $\varepsilon \in \mathbb{R}$,*

$$\delta_{P|Q}(\varepsilon) = \mathcal{L}\left\{\mathbb{P}\left[Z \geq t + \varepsilon\right]\right\}(1) = e^{\varepsilon} \cdot \mathcal{L}\left\{\mathbb{P}\left[Z' \leq -(t+\varepsilon)\right]\right\}(-1). \tag{19}$$

*And, for all $q \in \mathrm{ROC}_{\mathcal{B}}\{f_{Z'}\}$ (or equivalently, $1 - q \in \mathrm{ROC}_{\mathcal{B}}\{f_Z\}$),*

$$\mathrm{E}_q(P\|Q) = (q-1) \cdot \mathcal{B}\left\{\mathbb{P}\left[Z \geq t\right]\right\}(1-q) = q \cdot \mathcal{B}\left\{\mathbb{P}\left[Z' \leq t\right]\right\}(q). \tag{20}$$

Laplace expressions in Theorem 6 often arise in their explicit integral forms within several proofs in related works on differential privacy, for instance in [8, Lemma 9], [24, Proposition 7], and [6, Theorem 5]. In their integral forms, they frequently undergo manipulations like integration-by-parts or change-of-variables which can quickly get complicated. Our Theorem 6 offers a way to simplify the complexity of such manipulations as one can express the concerned terms in their Laplace expressions and invoke its properties from Table 1, like (**??**) and (**??**) for shifting or scaling variable of integration and (**??**) and (**??**) for integrating-by-parts. Examples in this paper illustrates that reasoning about privacy this way through its Laplace transform interpretation both accurate and economical.

In the following theorem, we show that the Rényi divergence $\mathrm{R}_q(P\|Q)$ and the privacy profile $\delta_{P|Q}(\varepsilon)$ are connected through a Laplace transform as well.

▶ **Theorem 7** ($\mathrm{E}_q(P\|Q)$ as Laplace Transforms of $\delta_{P|Q}(\varepsilon)$)**.** *For any distributions $P$ and $Q$,*

$$\mathrm{E}_q(P\|Q) = q(q-1)\cdot\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(1-q), \tag{21}$$

*for all orders $q$ such that $1-q \in \mathrm{ROC}_{\mathcal{B}}\{\delta_{P|Q}\}$.*

**Proof.** Let $Z \sim \mathrm{PLD}(P\|Q)$ and $Z' \sim \mathrm{PLD}(Q\|P)$. From (17), we have

$$\delta_{P|Q}(\varepsilon) = \mathbb{P}[Z \geq \varepsilon] - e^{\varepsilon}\cdot\mathbb{P}[Z' \leq -\varepsilon]. \tag{22}$$

We apply the linearity, time-shifting, and reversal properties of two-sided Laplace transforms to simplify the Laplace transform of privacy profile as follows

$$\begin{aligned}
\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(1-q) &= \mathcal{B}\Big\{\mathbb{P}[Z\geq t] - e^{t}\cdot\mathbb{P}[Z'\leq -t]\Big\}(1-q) && (23)\\
&\overset{(\mathbf{??})}{=} \mathcal{B}\Big\{\mathbb{P}[Z\geq t]\Big\}(1-q) - \mathcal{B}\Big\{e^{t}\cdot\mathbb{P}[Z'\leq -t]\Big\}(1-q) && (24)\\
&\overset{(\mathbf{??})}{=} \mathcal{B}\Big\{\mathbb{P}[Z\geq t]\Big\}(1-q) - \mathcal{B}\Big\{\mathbb{P}[Z'\leq -t]\Big\}(-q) && (25)\\
&\overset{(\mathbf{??})}{=} \mathcal{B}\Big\{\mathbb{P}[Z\geq t]\Big\}(1-q) - \mathcal{B}\Big\{\mathbb{P}[Z'\leq t]\Big\}(q). && (26)
\end{aligned}$$

Recall from Theorem 6 (Eqn. (20)) that

$$\mathcal{B}\Big\{\mathbb{P}[Z\geq t]\Big\}(1-q) = \frac{\mathrm{E}_q(P\|Q)}{q-1} \quad \text{and} \quad \mathcal{B}\Big\{\mathbb{P}[Z'\leq t]\Big\}(q) = \frac{\mathrm{E}_q(P\|Q)}{q}. \tag{27}$$

Therefore, the Laplace transform of privacy profile simplifies to

$$\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(1-q) = \mathrm{E}_q(P\|Q)\left[\frac{1}{q-1} - \frac{1}{q}\right] = \frac{\mathrm{E}_q(P\|Q)}{q(q-1)}. \tag{28}$$

◀

The privacy profile $\delta_{P|Q}(\varepsilon)$ is a continuous function for all distributions pairs $P, Q$.[2] Therefore, *any two privacy profiles share the same Rényi divergence curve if and only if they are identical*, i.e.,

$$\forall\varepsilon\in\mathbb{R} \;:\; \delta_{P_1|Q_1}(\varepsilon) = \delta_{P_2|Q_2}(\varepsilon) \iff \forall q\in\mathbb{C} \;:\; \mathrm{R}_q(P_1\|Q_1) = \mathrm{R}_q(P_2\|Q_2). \tag{29}$$

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[2] We can write $\delta_{P|Q}(\varepsilon) = \sup_S \psi_S(\varepsilon)$ where $\psi_S(\varepsilon) := P(S) - e^{\varepsilon}\cdot Q(S)$ is a continuous decreasing function. Therefore, the supremum of $\phi_S$ over all $S$ is also a continuous decreasing function.

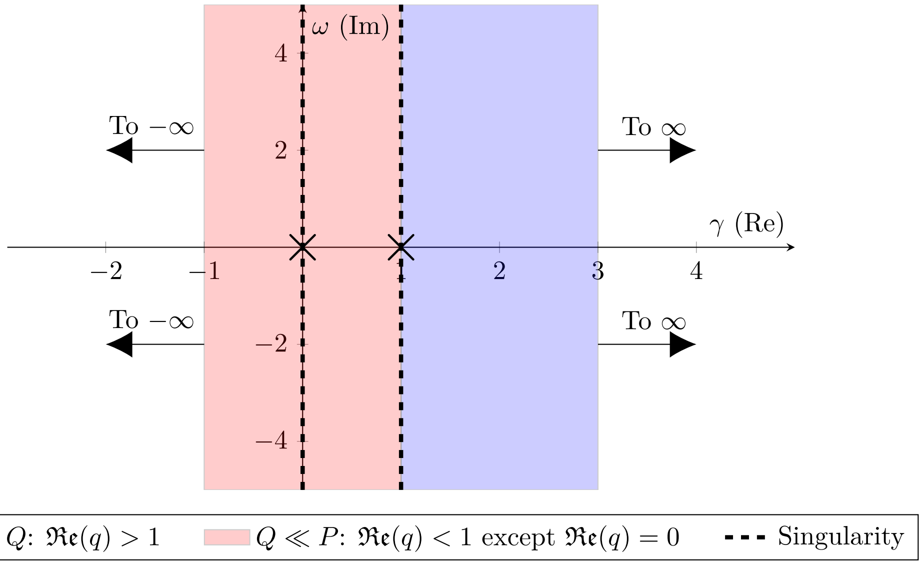


**Figure 2** Region of Convergence of $\mathcal{B}\left\{\delta_{P|Q}\right\}(1-q)$ at Rényi order $q = \gamma + i\omega$.

Theorem 7 connects Rényi divergence and privacy profiles for *all distributions* $P$ and $Q$, whether $P \ll Q$ (absolutely continuous with respect to $Q$) or not. The choice of $P$ and $Q$ affects the region of convergence (ROC) for the transform $\mathcal{B}\left\{\delta_{P|Q}\right\}(1-q)$ converges – if $P \ll Q$, it converges for $q > 1$; if $Q \ll P$, it converges for $q < 1$ (excluding $q = 0$). The ROC thus extends to the complex plane as a vertical strip as visualized above (cf. Section 2.2). At $q = 0$ or $q = 1$, singularities arise as the numerator $e^{(q-1)\cdot \mathrm{R}_q(P\|Q)} = \int P^q Q^{1-q} \mathrm{d}\theta = 1$, but the denominator $q(q-1)$ becomes zero.

As long as either $P \ll Q$ or $Q \ll P$, $\mathrm{ROC}_{\mathcal{B}}\{\delta_{P|Q}\}$ is non-empty, and $\mathrm{R}_q(P\|Q)$ can characterize $\delta_{P|Q}(\varepsilon)$. By substituting $1-q = s$ into (21) and taking the *inverse Laplace transform* (16), we derive the exact privacy profile $\delta_{P|Q}$ as follows:

$$\delta_{P|Q}(\varepsilon) = \mathcal{L}^{-1}\left\{\frac{\mathrm{E}_{1-s}(P\|Q)}{s(s-1)}\right\}(\varepsilon) = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{\gamma-i\omega}^{\gamma+i\omega}\frac{\exp[s(\varepsilon - \mathrm{R}_{1-s}(P\|Q))]}{s(s-1)}\mathrm{d}s, \tag{30}$$

where $\gamma \in \mathbb{R}$ can be *any* real point in $\mathrm{ROC}_{\mathcal{B}}\{\delta_{P|Q}\}$.

**Case Study.** We illustrate the Laplace transform identity from Theorem 7 by analyzing the privacy profile and Rényi divergence of the randomized response mechanism [17].

▶ **Theorem 8** (Privacy Profile of Randomized Response). *Let $\varepsilon > 0$ and $\delta \in [0,1]$. The randomized response mechanism $\mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta} : \{0,1\} \to \{0,1\} \times \{\bot, \top\}$ outputs:*

$$\mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}(0) = \begin{cases} (0,\bot) & \textit{with prob } \delta, \\ (0,\top) & \textit{with prob } \frac{(1-\delta)e^{\varepsilon}}{e^{\varepsilon}+1}, \\ (1,\top) & \textit{with prob } \frac{(1-\delta)}{e^{\varepsilon}+1}, \\ (1,\bot) & \textit{with prob } 0. \end{cases} \quad \mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}(1) = \begin{cases} (0,\bot) & \textit{with prob } 0, \\ (0,\top) & \textit{with prob } \frac{(1-\delta)}{e^{\varepsilon}+1}, \\ (1,\top) & \textit{with prob } \frac{(1-\delta)e^{\varepsilon}}{e^{\varepsilon}+1}, \\ (1,\bot) & \textit{with prob } \delta. \end{cases} \tag{31}$$

*For output distributions $P = \mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}(0)$ and $Q = \mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}(1)$, the privacy profile is:*

$$\delta_{\mathrm{RR}}^{\varepsilon,\delta}(t) = \delta_{P|Q}(t) = \delta_{Q|P}(t) = \begin{cases} \delta & \text{if } t > \varepsilon, \\ 1 - \dfrac{e^t+1}{e^\varepsilon+1}(1-\delta) & \text{if } -\varepsilon < t \le \varepsilon, \\ 1 - e^t(1-\delta) & \text{otherwise.} \end{cases} \tag{32}$$

**Proof.** Let $S_1 = \{(0,\bot)\}$, $S_2 = S_1 \cup \{(1,\bot)\}$, and $S_3 = S_2 \cup \{(1,\top)\}$. Let $Z \sim \mathrm{PLD}(P\|Q)$ and $Z' \sim \mathrm{PLD}(Q\|P)$. From (17),

$$\delta_{P|Q}(t) = \mathbb{P}\left[Z \ge t\right] - e^\varepsilon \cdot \mathbb{P}\left[Z' \le -t\right] \tag{33}$$

$$= \mathbb{P}_P\left[\log\frac{P(\Theta)}{Q(\Theta)} \ge t\right] - e^\varepsilon \cdot \mathbb{P}_Q\left[\log\frac{Q(\Theta)}{P(\Theta)} \le -t\right] \tag{34}$$

$$= \mathbb{P}_P\left[P(\Theta) \ge e^t \cdot Q(\Theta)\right] - e^\varepsilon \cdot \mathbb{P}_Q\left[P(\Theta) \le e^t \cdot Q(\Theta)\right] \tag{35}$$

$$= \begin{cases} P(S_1) - e^t \cdot Q(S_1) & \text{if } t > \varepsilon, \\ P(S_2) - e^t \cdot Q(S_2) & \text{if } -\varepsilon < t \le \varepsilon, \\ P(S_3) - e^t \cdot Q(S_3) & \text{otherwise.} \end{cases} \tag{36}$$

$$= \begin{cases} \delta & \text{if } t > \varepsilon, \\ \delta + \frac{1-\delta}{e^\varepsilon+1} \cdot (e^\varepsilon - e^t) & \text{if } -\varepsilon < t \le \varepsilon, \\ 1 - e^t \cdot (1-\delta) & \text{otherwise.} \end{cases} \tag{37}$$

The same holds for $\delta_{Q|P}$ due to symmetry of (31). ◀

Theorem 8 extends the privacy profile given in [4, Theorem 2] to all $(\varepsilon, \delta)$ regimes. Note that when $\delta > 0$, neither $P \ll Q$ nor $Q \ll P$, so the Laplace transform $\mathcal{B}\left\{\delta_{P|Q}\right\}(1-q)$ does not converge for any $q \in \mathbb{R}$. Consequently as per Theorem 7, the Rényi divergence cannot be defined in this case. However, when $\delta = 0$, both $P \ll Q$ and $Q \ll P$, and the Laplace transform exists for all $q \in \mathbb{R} \setminus \{0,1\}$. Computing the Laplace transform yields:

▶ **Theorem 9** (Rényi curve of Randomized Response)**.** *The Rényi divergence between output distributions $P = \mathcal{M}_{\mathrm{RR}}^{\varepsilon,0}(0)$ and $Q = \mathcal{M}_{\mathrm{RR}}^{\varepsilon,}(1)$ is as follows:*

$$\forall q \in \mathbb{C} \ s.t. \ \mathfrak{Re}(q) \notin \{0,1\} \ : \ \mathrm{R}_q\left(P\|Q\right) = \frac{1}{q-1}\log\left(\frac{e^\varepsilon}{e^\varepsilon+1}e^{-q\varepsilon} + \frac{1}{e^\varepsilon+1}e^{q\varepsilon}\right). \tag{38}$$

**Proof.** From Theorem 8, when $\delta = 0$, the privacy profile of the randomized response algorithm's output-distributions $P$ and $Q$ is

$$\delta_{P|Q}(t) = \begin{cases} 0 & \text{if } t > \varepsilon, \\ \frac{e^\varepsilon - e^t}{1+e^\varepsilon} & \text{if } -\varepsilon < t \le \varepsilon, \\ 1 - e^t & \text{otherwise.} \end{cases} \tag{39}$$

From the equivalence (21) of Theorem 7,

$$\frac{e^{(q-1)R_q(P\|Q)}}{q(q-1)} = \mathcal{B}\{\delta_{P|Q}(t)\}(1-q) \tag{40}$$

$$= \int_{-\infty}^{\infty} e^{(q-1)t}\delta_{P|Q}(t)\mathrm{d}t \tag{41}$$

$$= \int_{-\infty}^{-\varepsilon} e^{(q-1)t}\cdot(1-e^t)\mathrm{d}t + \int_{-\varepsilon}^{\varepsilon} e^{(q-1)t}\cdot\frac{e^\varepsilon - e^t}{1+e^\varepsilon}\mathrm{d}t \tag{42}$$

$$= \left[\frac{e^{(q-1)t}}{q-1} - \frac{e^{qt}}{q}\right]_{-\infty}^{-\varepsilon} + \frac{1}{1+e^\varepsilon}\left[\frac{e^\varepsilon\cdot e^{(q-1)t}}{q-1} - \frac{e^{qt}}{q}\right]_{-\varepsilon}^{\varepsilon} \tag{43}$$

$$= \frac{e^\varepsilon\cdot e^{-q\varepsilon}+e^{q\varepsilon}}{1+e^\varepsilon}\cdot\frac{1}{q(q-1)}. \qquad \text{(On simplification)}$$

Therefore, for any $q \in \mathbb{R}\setminus\{0,1\}$ we can cancel $q(q-1)$ from the denominator in both sides, which proves the theorem statement for real orders. Since the region of convergence has to be a vertical strip (cf. Subsection 2.2), the statement holds for all $q \in \mathbb{C}$ such that $\mathfrak{Re}(q) \notin \{0,1\}$. ◀

Theorem 9 generalizes the result in [20, Proposition 5], extending it to complex orders $q$.

## 3.1 Dominance: PLD vs. Privacy Profile vs. Rényi Divergence

Differential privacy is a study of distributional divergence between output distributions $P$ and $Q$ not just for a pair of neighboring inputs $D, D'$ but across *all* neighboring inputs. When considering functional notions of DP, comparing indistinguishability characteristics of two output-distribution pairs, say $(P_1, Q_1)$ and $(P_2, Q_2)$, to decide which one is tighter requires a notion of dominance. For the privacy loss random variables $Z_1 \sim \mathrm{PLD}(P_1\|Q_1)$ and $Z_2 \sim \mathrm{PLD}(P_2\|Q_2)$, the *first-order stochastic dominance* satisfies the needs.

▶ **Definition 10** (Dominance in PLD). *We say that $Z_2 \sim \mathrm{PLD}(P_2\|Q_2)$ dominates $Z_1 \sim \mathrm{PLD}(P_1\|Q_1)$ (denoted as $(P_1, Q_1) \preceq_{\mathrm{PLD}} (P_2, Q_2)$) if $\mathbb{P}[Z_1 \geq t] \leq \mathbb{P}[Z_2 \geq t]$ for all $t \in \mathbb{R}$.*

As for privacy profile and Rényi divergence, we define the following notions of dominance.

▶ **Definition 11** (Dominance in privacy-profile and Rényi divergence). *We say the distribution pair $(P_2, Q_2)$ dominates $(P_1, Q_1)$ in privacy profile (denote as $(P_1, Q_1) \preceq_\delta (P_2, Q_2)$) if*

$$\forall \varepsilon \in \mathbb{R} \ : \ \delta_{P_1|Q_1}(\varepsilon) \leq \delta_{P_2|Q_2}(\varepsilon). \tag{44}$$

*We say $(P_2, Q_2)$ dominates $(P_1, Q_1)$ in Rényi divergence (denote as $(P_1, Q_1) \preceq_R (P_2, Q_2)$) if*

$$\forall q > 1 \ : \mathrm{R}_q(P_1\|Q_1) \leq \mathrm{R}_q(P_2\|Q_2). \tag{45}$$

The following theorem shows that, while the PLD, Rényi divergence curve and privacy profile curve are equivalent for all absolutely continuous distribution pairs, this equivalence does not mean that their respective dominance definitions are also equivalent.

▶ **Theorem 12.** *For any two absolutely continuous distribution pairs $(P_1, Q_1)$ and $(P_2, Q_2)$,*

$$(P_1, Q_1) \preceq_{\mathrm{PLD}} (P_2, Q_2) \implies (P_1, Q_1) \preceq_\delta (P_2, Q_2) \implies (P_1, Q_1) \preceq_R (P_2, Q_2). \tag{46}$$

*However, the opposite direction does not hold:*

$$(P_1, Q_1) \preceq_R (P_2, Q_2) \not\Longrightarrow (P_1, Q_1) \preceq_\delta (P_2, Q_2) \not\Longrightarrow (P_1, Q_1) \preceq_{\mathrm{PLD}} (P_2, Q_2). \tag{47}$$

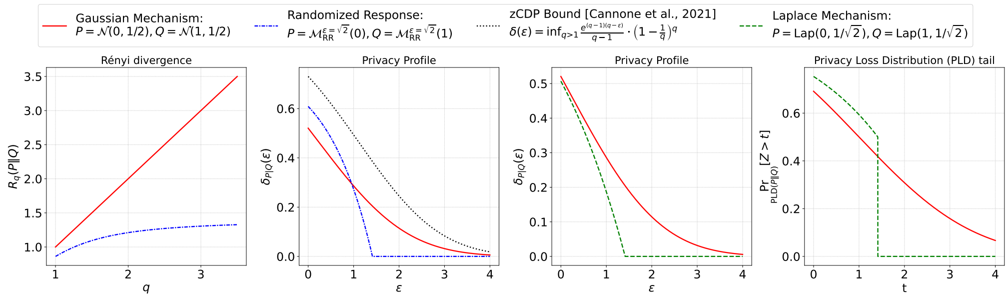


**Figure 3** Comparison of the Gaussian, Laplace and randomized-response mechanism illustrates key differences in their privacy characteristics. The first two figures shows that dominance in $\mathrm{R}_q\left(P_1\|Q_1\right) \leq \mathrm{R}_q\left(P_2\|Q_2\right)$ for $q>1$ does not imply $\delta_{P_1|Q_1}(\varepsilon) \leq \delta_{P_2|Q_2}(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$ and the last two figures shows that dominance in $\delta_{P_1|Q_1}(\varepsilon) \leq \delta_{P_2|Q_2}(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$ does not imply $\underset{\mathrm{PLD}(P_1\|Q_1)}{\mathbb{P}}[Z_1 \geq t] \leq \underset{\mathrm{PLD}(P_2\|Q_2)}{\mathbb{P}}[Z_2 \geq t]$ for all $t \in \mathbb{R}$.

**Proof.** First part directly follows from (19) and (21) by noting that

$$\delta_{P_1|Q_1}(\varepsilon)=\int_0^{\infty} e^{-t}\cdot\mathbb{P}\left[Z_1 \geq t+\varepsilon\right]\mathrm{d}t \leq \int_0^{\infty} e^{-t}\cdot\mathbb{P}\left[Z_2 \geq t+\varepsilon\right]\mathrm{d}t=\delta_{P_2|Q_2}(\varepsilon),$$

$$\frac{\mathrm{E}_q\left(P_1\|Q_1\right)}{q(q-1)}=\int_{-\infty}^{\infty} e^{-(1-q)t}\cdot\delta_{P_1|Q_1}(t)\mathrm{d}t \leq \int_{-\infty}^{\infty} e^{-(1-q)t}\cdot\delta_{P_2|Q_2}(t)\mathrm{d}t=\frac{\mathrm{E}_q\left(P_2\|Q_2\right)}{q(q-1)},$$

and that for $q>1$, $q(q-1)>0$. The proof of the second statement is based on the counterexamples[3] shown in Figure 3, comparing $\sqrt{2}$-GDP[4] Gaussian mechanism, $\sqrt{2}$-DP Randomized Response mechanism and $\sqrt{2}$-DP Laplace mechanisms ◀

Theorem 12 highlights a critical issue: while the $\mathrm{R}_q\left(P\|Q\right)$ curve, the $\delta_{P|Q}(\varepsilon)$ function, and the $\mathrm{PLD}(P\|Q)$ tail are equivalent characterizations of a distribution pair $(P,Q)$'s indistinguishability, their dominance relations differ. This distinction means that a tight upper bound on the privacy loss distribution's tail tightly bounds the privacy profile, which in turn tightly bounds the Rényi divergence. However, the reverse direction of deriving upper bounds inherently introduces a gap as we go from a weaker functional DP notion to a stronger one. This gap can be significant, as shown in Figure 3, where the (nearly) tight upper bound (dotted black line) derived from the Gaussian mechanism's Rényi divergence curve (red line) [8, Corollary 13] overestimates the Gaussian mechanism's actual privacy profile as it must cover the privacy-profiles of mechanisms like randomized response (in blue).

**Role of Complex Orders.** Achieving an exactly-tight upper-bound on $\delta_{P|Q}(\varepsilon)$ from an upper-bound on $\mathrm{R}_q\left(P\|Q\right)$ curve or on $\mathbb{P}_{\mathrm{PLD}(P\|Q)}[Z \geq t]$ from an upper-bound on $\delta_{P|Q}(\varepsilon)$ remains unresolved. The best available bounds for the former rely on taking an infimum over point-wise conversions from a bound on Rényi divergence at a single order $q>1$ to $(\varepsilon,\delta)$-DP [8, 7]. However, this approach is unlikely to give tight bound conversions because unlike functional DP notions, converting point guarantees of DP are lossy. We highlight that complex numbers likely plays an important role in tight conversions of upper-bounds on $\mathrm{R}_q\left(P\|Q\right)$ to $\delta_{P|Q}(\varepsilon)$. That is because the inverse Laplace transform for $\delta_{P|Q}$ depends on the behavior of $\mathrm{R}_q\left(P\|Q\right)$ along the complex line $\mathfrak{Re}(1-q)=\gamma$ (cf. Eqn. (30)), not on real values $q>1$. In fact, the choice of $\gamma \in \mathbb{R}\setminus\{0,1\}$ is irrelevant, provided it lies in $\mathrm{ROC}_{\mathcal{B}}\{\delta_{P|Q}\}$.

[3] The counterexample for showing $(P_1,Q_1) \preceq_R (P_2,Q_2) \not\Longrightarrow (P_1,Q_1) \preceq_\delta (P_2,Q_2)$ is from [26].
[4] $\mu$-GDP is a type of an $f$-DP guarantee with Gaussian trade-off function $f(\alpha):=\Phi(\Phi^{-1}(1-\alpha)-\mu)$.

## 4 Exactly-Tight Composition Theorems

Point DP guarantees like $(\varepsilon, \delta)$-DP or $(q, \rho)$-Rényi DP provide a lossy characterization of the indistinguishability between two distributions. While they suffice for reporting an algorithm's worst-case privacy – offering conservative but acceptable protection – the main issue arises when composing these point guarantees as the quantification loss compounds, significantly overestimating the actual privacy protection pretty quickly.

Consider the $k$-fold (non-adaptive) composition of a one-dimensional Gaussian mechanism with $L_2$ sensitivity 1 and noise variance $\sigma^2 = 1$. Individually, the mechanism satisfies a privacy profile $\delta_{P|Q}(\varepsilon) = O(e^{-\varepsilon^2/2})$ (cf. Theorem 4), corresponding to a point guarantee of $(O(\sqrt{\log(1/\delta)}), \delta)$-DP for any $\delta \in (0, 1]$. For the composed mechanism, the output distributions are $P^{\otimes k} = \mathcal{N}(\vec{0}, I_k)$ and $Q^{\otimes k} = \mathcal{N}(\vec{1}, I_k)$. Again from Theorem 4, the true privacy profile becomes $\delta_{P^{\otimes k}|Q^{\otimes k}}(\varepsilon) = O(e^{-\varepsilon^2/2k})$, yielding a point guarantee of $(O(\sqrt{k\log(1/\delta)}), \delta)$-DP. However, composing individual point guarantees using the previously-best composition theorem in [17, Theorem 3.3] results in a much worse bound: $(O(\sqrt{k}\log(1/\delta)), (k+1)\delta)$-DP. This overestimates the privacy loss by a factor of $O(\sqrt{\log(k/\delta)})$ in $\varepsilon$ for the same $\delta$.

Functional DP notions address this issue by accurately capturing indistinguishability between distributions [10, 18]. Notably among all functional notions of DP, *exactly tight* composition theorem (i.e. matching even in constants) that accommodates *adaptively chosen heterogeneous mechanisms* is only known for Rényi DP function of $q$.[5] In the following theorem, we provide an *exactly tight* composition result that applies to arbitrary privacy profiles $\delta_{P_1|Q_1}$ and $\delta_{P_2|Q_2}$. Later we also extend this theorem to handle adaptivity.

▶ **Theorem 13** (Exactly Tight Composition of Privacy Profiles). *If $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$ are two product distributions on $\Omega_1 \times \Omega_2$ such that $(P_1, Q_1)$ and $(P_2, Q_2)$ are absolutely continuous at least in one direction, then*

$$\delta_{P|Q}(\varepsilon) = \left(\dot{\delta}_{P_1|Q_1} \circledast \left(\dot{\delta}_{P_2|Q_2} - \delta_{P_2|Q_2}\right)\right)(\varepsilon) = \left(\delta_{P_1|Q_1} \circledast \left(\ddot{\delta}_{P_2|Q_2} - \dot{\delta}_{P_2|Q_2}\right)\right)(\varepsilon). \tag{48}$$

*where $\dot{\delta}_{P_1|Q_1}$ and $\dot{\delta}_{P_2|Q_2}$ are the gradient functions of $\delta_{P_1|Q_1}$ and $\delta_{P_2|Q_2}$ respectively and $\ddot{\delta}_{P_2|Q_2}$ is the second-order gradient function of $\delta_{P_2|Q_2}$.*

**Proof.** Let's consider the random variables $Z \sim \mathrm{PLD}(P\|Q)$, $Z_1 \sim \mathrm{PLD}(P_1\|Q_1)$, and $Z_2 \sim \mathrm{PLD}(P_2\|Q_2)$. For a pair $(\Theta_1, \Theta_2) \sim P$, the privacy loss random variable $Z$ is given by $L_{P|Q}(\Theta_1, \Theta_2)$, which simplifies to:

$$\log \frac{P_1(\Theta_1)P_2(\Theta_2)}{Q_1(\Theta_1)Q_2(\Theta_2)} = L_{P_1|Q_1}(\Theta_1) + L_{P_2|Q_2}(\Theta_2). \tag{49}$$

This decomposition implies that $Z$ can be expressed as the sum of $Z_1$ and $Z_2$, i.e., $Z = Z_1 + Z_2$. Consequently, the probability density of $Z$, $f_Z$, is the convolution of $f_{Z_1}$ and $f_{Z_2}$:

$$f_Z(t) = \int_{-\infty}^{\infty} f_{Z_1}(\tau) f_{Z_2}(t-\tau) d\tau = (f_{Z_1} \circledast f_{Z_2})(t). \tag{50}$$

[5] The work in [10] explores tight composition in $f$-DP, introducing it as the operation $f_1 \otimes f_2$ but detailing its explicit form only for the Gaussian trade-off function $G_\mu$, not for general $f$. And, works in [16, 23, 18] explore composition of PLDs, showing it to be the convolution of the underlying PDFs, but only in non-adaptive scenarios. Only Rényi divergence curves have an adaptive composition theorem: for all $q > 1$, $\mathrm{R}_q(P\|Q) \leq \mathrm{R}_q(P_1\|Q_1) + \sup_x \mathrm{R}_q(P_2^x\|Q_2^x)$, where $P_1, P_2$ are marginals $P_2^x, Q_2^x$ are conditionals.

From Theorem 6, we have:

$$\mathrm{E}_q(P\|Q) = (q-1)\cdot\mathcal{B}\Big\{\mathbb{P}[Z\geq t]\Big\}(1-q) \tag{51}$$

$$\overset{(??)}{=} \mathcal{B}\{f_Z(t)\}(1-q) \tag{52}$$

$$\overset{(??)}{=} \mathcal{B}\{f_{Z_1}(t)\}(1-q)\cdot\mathcal{B}\{f_{Z_2}(t)\}(1-q) = \mathrm{E}_q(P_1\|Q_1)\cdot\mathrm{E}_q(P_2\|Q_2). \tag{53}$$

From Definition 2, this directly implies that the Rényi divergence of order $q$ for the pair $(P, Q)$ is the sum of the Rényi divergences for the pairs $(P_1, Q_1)$ and $(P_2, Q_2)$ as $\mathrm{R}_q(P\|Q) = \mathrm{R}_q(P_1\|Q_1) + \mathrm{R}_q(P_2\|Q_2)$. Now suppose $s = 1-q < 0$. Thanks to absolute continuity, we can express the Rényi divergences in terms of the Laplace transform using Theorem 7 as:

$$s(s-1)\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(s) = s(s-1)\mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot s(s-1)\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s). \tag{54}$$

Since $q = 1-s$ cannot be 0 or 1, we can divide by $s(s-1)$ on both sides:

$$\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(s) = s\mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot(s-1)\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s) \tag{55}$$

$$= s\mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot\left(s\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s) - \mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s)\right) \tag{56}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\dot{\delta}_{P_1|Q_1}(t)\right\}(s)\cdot\left(\mathcal{B}\left\{\dot{\delta}_{P_2|Q_2}(t)\right\}(s) - \mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s)\right) \tag{57}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\dot{\delta}_{P_1|Q_1}(t)\right\}(s)\cdot\mathcal{B}\left\{\dot{\delta}_{P_2|Q_2}(t) - \delta_{P_2|Q_2}(t)\right\}(s) \tag{58}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\left(\dot{\delta}_{P_1|Q_1}\circledast\left(\dot{\delta}_{P_2|Q_2} - \delta_{P_2|Q_2}\right)\right)(t)\right\}(s). \tag{59}$$

Similarly, with a slight variation, we have

$$\mathcal{B}\left\{\delta_{P|Q}(t)\right\}(s) = \mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot s(s-1)\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s) \tag{60}$$

$$= \mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot\left(s^2\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s) - s\mathcal{B}\left\{\delta_{P_2|Q_2}(t)\right\}(s)\right) \tag{61}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot\left(\mathcal{B}\left\{\ddot{\delta}_{P_2|Q_2}(t)\right\}(s) - \mathcal{B}\left\{\dot{\delta}_{P_2|Q_2}(t)\right\}(s)\right) \tag{62}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\delta_{P_1|Q_1}(t)\right\}(s)\cdot\mathcal{B}\left\{\ddot{\delta}_{P_2|Q_2}(t) - \dot{\delta}_{P_2|Q_2}(t)\right\}(s) \tag{63}$$

$$\overset{(??)}{=} \mathcal{B}\left\{\left(\delta_{P_1|Q_1}\circledast\left(\ddot{\delta}_{P_2|Q_2} - \dot{\delta}_{P_2|Q_2}\right)\right)(t)\right\}(s). \tag{64}$$

From the uniqueness of Laplace transform, we therefore have that for all $\varepsilon \in R$,

$$\delta_{P|Q}(\varepsilon) = \left(\dot{\delta}_{P_1|Q_1}\circledast\left(\dot{\delta}_{P_2|Q_2} - \delta_{P_2|Q_2}\right)\right)(\varepsilon) = \left(\delta_{P_1|Q_1}\circledast\left(\ddot{\delta}_{P_2|Q_2} - \dot{\delta}_{P_2|Q_2}\right)\right)(\varepsilon). \tag{65}$$

◀

Theorem 13 provides an *exactly tight* composition theorem – not only because the terms in (48) are equal, but also because the privacy profiles $\delta_{P_1|Q_1}$ and $\delta_{P_2|Q_2}$ precisely capture the indistinguishability of their respective distributions. This theorem mirrors the composition of Rényi divergence but works in the time domain of $\varepsilon$ instead of the frequency domain of $q$.

Note that Theorem 13 assumes absolute continuity in at least one direction ($P_i \ll Q_i$ or $Q_i \ll P_i$) for $i \in \{1, 2\}$. However, the result in (48) appears to hold even when absolute continuity fails in either directions. In the following result we apply (48) to compose the privacy profiles of the randomized mechanisms $\delta_{\mathrm{RR}}^{\varepsilon_1,\delta_1} \otimes \delta_{\mathrm{RR}}^{\varepsilon_2,\delta_2}$ and get a tight expression for $\delta_1, \delta_2 > 0$ without running into problems. This happens because even when the Laplace transform is undefined everywhere, the frequency-domain manipulations performed on it still correspond to valid manipulation steps in the time domain. Since our interest lies in the time domain function – the composed privacy profile – taking advantage of this flexibility proves beneficial.

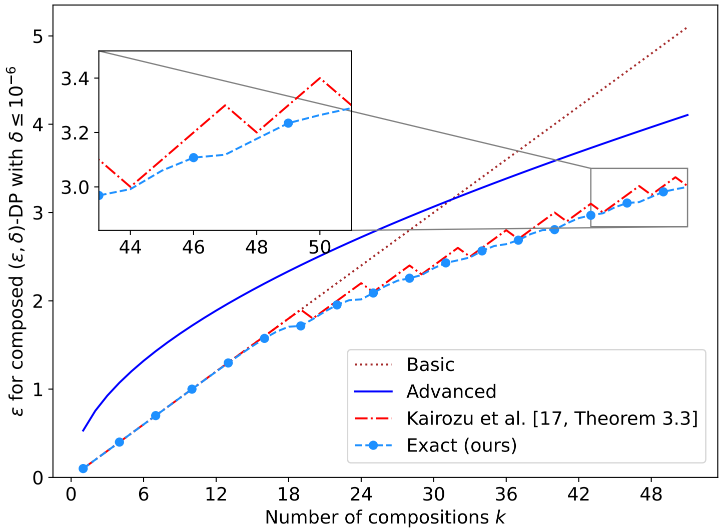


**Figure 4** Comparison of $(\varepsilon, \delta)$-DP bounds from various composition theorems for $k$-fold composition of a $(0.1, 10^{-8})$-DP guarantee, under the constraint $\delta \leq 10^{-6}$.

▶ **Theorem 14** (Tight Composition of $(\varepsilon, \delta)$-DP)**.** *For any $\varepsilon_i \geq 0$, $\delta_i \in [0,1]$ for $i \in \{1, \cdots, k\}$, the (non-adaptive) composition of a sequence $(\varepsilon_1, \delta_1), \cdots, (\varepsilon_k, \delta_k)$-DP mechanisms satisfies $(\varepsilon, \delta^{\otimes k}(\varepsilon))$-DP for all $\varepsilon \in \mathbb{R}$, where $\delta^{\otimes l}(\varepsilon)$ is defined recursively as*

$$\forall t \in \mathbb{R} \ : \ \delta^{\otimes l}(t) := \delta_l + \frac{(1-\delta_l)}{e^{\varepsilon_l}+1}\left[e^{\varepsilon_l} \cdot \delta^{\otimes l-1}(t-\varepsilon_l) + \delta^{\otimes l-1}(t+\varepsilon_l)\right], \ \textit{with } \delta^{\otimes 0}(t) := [1-e^t]_+.$$

Theorem 14 gives an *exactly-tight non-adaptive composition theorem* for *any sequence of heterogeneous $(\varepsilon_i, \delta_i)$-DP mechanisms.* We prove this theorem by applying our composition result in Theorem 13 to the privacy profile of a sequence $(\mathcal{M}_{\mathrm{RR}}^{\varepsilon_1,\delta_1}, \cdots, \mathcal{M}_{\mathrm{RR}}^{\varepsilon_k,\delta_k})$ of randomized response mechanisms, and using the fact that the randomized response mechanism $\mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}$ subtends the worst-case privacy profile among all $(\varepsilon, \delta)$-DP mechanisms [17]. The recursion in Theorem 14 simplifies to the following expression for homogeneous $(\varepsilon, \delta)$-DP composition.

▶ **Corollary 15.** *For any $\varepsilon \geq 0$, $\delta \in [0,1]$, the composition of $k$ many $(\varepsilon, \delta)$-DP mechanisms satisfies $(\varepsilon, \delta^{\otimes k}(\varepsilon))$-DP for all $\varepsilon \in \mathbb{R}$, where*

$$\forall t \in \mathbb{R} \ : \ \delta^{\otimes k}(t) = 1 - (1-\delta)^k \left(1 - \mathop{\mathbb{E}}_{Y \leftarrow \mathrm{Binomial}\left(k, \frac{e^{\varepsilon}}{1+e^{\varepsilon}}\right)} \left[1 - e^{t - \varepsilon \cdot (2Y - k)}\right]_+\right). \tag{66}$$

While our bound in Corollary 15 matches each of the $\lfloor k/2 \rfloor + 1$ discrete $(\varepsilon, \delta)$-DP values given by the composition theorem in [17, Theorem 3.3], our result offers a continuous curve over all $\varepsilon \in \mathbb{R}$. So, our bound yields a tighter $\varepsilon$ for a given $\delta$ than [17], as shown in Figure 4. We also compare our bound with popular numerical accountants [18, 16] in Figure 6.

**Adaptive Composition.** When $\mathcal{M}_2$ sees the output of $\mathcal{M}_1$, the output distributions of $\mathcal{M}_1$ and $\mathcal{M}_2$ become dependent. However, Theorem 13 still applies if a pair $P_2, Q_2$ exists that dominates the privacy profile of $\mathcal{M}_2(D, \theta)$ and $\mathcal{M}_2(D', \theta)$ across all $\theta$ for $D \simeq D'$ – a standard assumption for adaptive composition. The following result ensures this.

▶ **Lemma 16** ([26, Theorem 27])**.** *Let $P(x,y) = P_1(x) \cdot P_2^x(y)$ and $Q(x,y) = Q_1(x) \cdot Q_2^x(y)$ be two joint distributions on $\Omega_1 \times \Omega_2$. Then for any distributions $P_2$ and $Q_2$ on $\Omega_2$ such that $\delta_{P_2^x|Q_2^x}(\varepsilon) \leq \delta_{P_2|Q_2}(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$ and $x \in \Omega_1$, we have $\delta_{P|Q}(\varepsilon) \leq \delta_{P_1 \times P_2|Q_1 \times Q_2}(\varepsilon)$.*

## 5 Dealing with Asymmetry for DP Notion Equivalences

Functional notions of DP can be *asymmetric*, meaning that switching $P \leftrightarrow Q$ may yield a different curve $\delta_{Q|P}$ than $\delta_{P|Q}$. For a mechanism $\mathcal{M}$, this implies a sample $\Theta \sim P$ may reveal more (or less) about originating from $D$ than $\Theta' \sim Q$ reveals about $D'$. Asymmetry often arises from *subsampling*, a common technique to enhance privacy [4, 1]. For example, Poisson subsampling under the *add or remove adjacency* relationship, as used in DP-SGD [1], skews the privacy profile of any symmetric mechanism as detailed below.

**Effect of Poisson Subsampling on $\delta_{P|Q}$.** Without loss of generality, assume datasets $D \simeq D'$ are such that the record at index $i$ is present in $D$ but empty in $D'$, i.e., $D[i] \neq D'[i] = \perp$. If we randomly filter the records using an iid selection mask $U \sim \text{Bernoulli}(\lambda)^{\otimes n}$, the subsampled datasets $D_U$ and $D'_U$ are defined as follows for all $i \in [n]$:

$$D_U[i] := \begin{cases} D[i] & \text{if } U_i = 1 \\ \perp & \text{otherwise} \end{cases} \quad \text{and} \quad D'_U[i] := \begin{cases} D'[i] & \text{if } U_i = 1 \\ \perp & \text{otherwise} \end{cases}. \tag{67}$$

The distributions $P_\lambda$ and $Q_\lambda$ of the outputs $\mathcal{M}(D_U)$ and $\mathcal{M}(D'_U)$ for any algorithm $\mathcal{M}$ will be identical with probability $1-\lambda$, which amplifies privacy considerably. More precisely, let $P_{\text{IN}}$ and $Q_{\text{IN}}$ be the distributions of $\mathcal{M}(D_U)$ and $\mathcal{M}(D'_U)$ conditioned on $i \in U$, and $P_{\text{OUT}}$ and $Q_{\text{OUT}}$ be the distributions conditioned on $i \notin U$. For any event $S \subseteq \Omega$, we have:

$$\begin{aligned} P_\lambda(S) &= \Pr[i \notin U] \cdot P_{\text{OUT}}(S) + \Pr[i \in U] \cdot P_{\text{IN}}(S) \\ &= (1-\lambda) \cdot Q_{\text{OUT}}(S) + \lambda \cdot P_{\text{IN}}(S) \qquad (68) \\ &= (1-\lambda) \cdot Q_{\text{IN}}(S) + \lambda \cdot P_{\text{IN}}(S). \qquad (69) \end{aligned}$$

Equation (68) holds because if $i \notin U$, then $D_U = D'_U$, and equation (69) holds because the $i$th record in $D'$ is empty, so conditioning on $i \in U$ or $i \notin U$ does not affect the output distribution, i.e., $Q_{\text{IN}} = Q_{\text{OUT}}$. Using this fact, the following theorem shows the *exact effect* Poisson subsampling has on the privacy profile.

▶ **Theorem 17** (Poisson Subsampling with Add/Remove Adjacency). *Let $0 < \lambda \leq 1$. For any distributions $P_\lambda$, $Q_\lambda$, $P$ and $Q$ such that $P_\lambda = \lambda P + (1-\lambda)Q$ and $Q_\lambda = Q$,*

$$\forall \varepsilon \in \mathbb{R} \ : \ \delta_{P_\lambda|Q_\lambda}(\varepsilon) = \begin{cases} \lambda \delta_{P|Q}(\log(1 + (e^\varepsilon - 1)/\lambda)) & \text{if } \varepsilon > \log(1-\lambda), \\ 1 - e^\varepsilon & \text{otherwise.} \end{cases} \tag{70}$$

Theorem 17, originally from [19] and appearing in several works [4, 3, 1, 24], is presented here as a single curve that exactly captures the impact of subsampling in both directions, since negative values of $\varepsilon$ gives the flipped profile $\delta_{Q_\lambda|P_\lambda}$ (see reversal property discussed next). Note the asymmetry between $\delta_{P_\lambda|Q_\lambda}$ and $\delta_{Q_\lambda|P_\lambda}$ in Figure 5 (orange vs. blue curves).

**Imprecise Handling of Asymmetry.** Many works handle asymmetric DP notions rather imprecisely, introducing slackness in analysis or approximating with numerical bounds. For instance, the work in [10, Theorem 4.2] uses the biconjugation $\min\{f_{P_\lambda|Q_\lambda}, f_{Q_\lambda|P_\lambda}\}^{**}$ to quantify subsampling amplification for the trade-off curve $f_{P|Q}$, overestimating the actual trade-off (see the bottom-right plot in Figure 5, red vs. dashed orange line). Similarly, the work in [26, Proposition 30] takes $\max\{\delta_{P_\lambda|Q_\lambda}, \delta_{Q_\lambda|P_\lambda}\}$ to dominate the subsampled profiles in both direction, overestimating the actual profile (see the top-left plot, red vs. orange/blue).

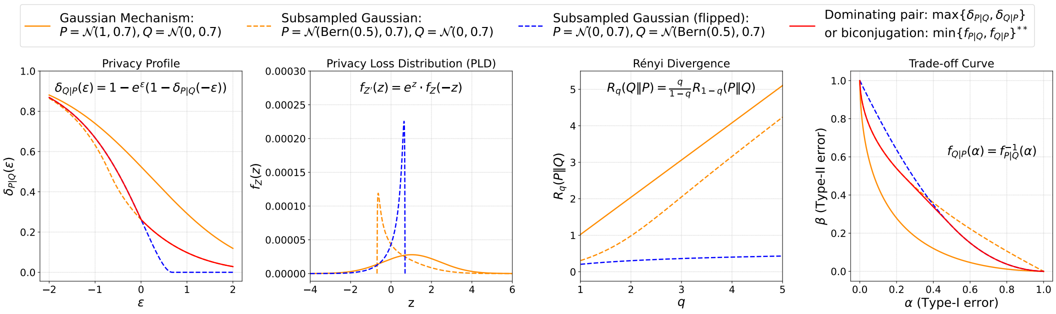


**Figure 5** Visualization of the effect of Poisson subsampling as well as the reversal theorems for all four functional notions of DP, namely privacy profile $\delta_{P|Q}(\varepsilon)$ as a function of $\varepsilon$, the generalized density function of privacy loss distribution $\text{PLD}(P\|Q)$, the Rényi divergence $\text{R}_q\,(P\|Q)$ as a function of order $q$, and the trade-off function $f_{P|Q}(\alpha)$ for hypothesis testing between $P$ and $Q$.

While these operations ensure worst-case guarantees in both directions, they introduce gaps that disrupt equivalence between privacy notions. That is, after the symmetrization step or the biconjugation operation, converting to another functional notions would no longer match the true privacy characteristics. Furthermore, these gaps tend to compound significantly on composition, which causes an overestimation of the actual composed profile.

**Reversal Properties and Proposed Solution.** All functional DP notions exhibit a reversal property that is often overlooked – privacy profiles $\delta_{Q|P}$, Rényi divergence $\text{R}_q\,(Q\|P)$ and trade-off curve $f_{Q|P}$ are expressible as follows [26, 25, 10]:

$$\delta_{Q|P}(\varepsilon) = 1 - e^{\varepsilon}(1 - \delta_{P|Q}(-\varepsilon)), \quad \text{R}_q\,(Q\|P) = \frac{q}{1-q}\text{R}_{1-q}\,(P\|Q)\,, \quad f_{Q|P}(\alpha) = f^{-1}_{P|Q}(\alpha).$$

The PLD formalism also follows the reversal $f_{Z'}(z) = e^{z} \cdot f_Z(-z)$, linking the densities of $Z \leftarrow \text{PLD}(P\|Q)$ and $Z' \leftarrow \text{PLD}(Q\|P)$ [16]. These reversal properties eliminate the need for symmetrization or biconjugation. Instead of such steps, we propose retaining the functional notion of choice *only in one direction without controlling asymmetry*, which avoids the overhead of maintaining both directions (e.g., as done in Google's PLDAccountant [11]). For tight composition, we propose avoiding changing adjacency notions or directions. Doing so enables lossless composition via Theorem 13 or Rényi DP composition [20]. When a pointwise guarantees is needed, the reversal property allows querying both directions $\delta_{P|Q}, \delta_{Q|P}$ and taking the maximum, which yields exactly-tight DP guarantees. Figure 5 visualizes the equivalences between these four notions of DP and their reversal properties.

## 6 Conclusion

In summary, this paper presents a novel interpretation of differential privacy via Laplace transforms that formalizes the time-frequency duality enjoyed by various functional notions of differential privacy. The Laplace transform identities derived in this paper form a set of versatile analytical expressions that simplify formal reasoning and proofs of properties related to differential privacy. Our formalism provides a precise and notationally economical language to reason about differential privacy, which we apply to prove lossless conversion results between different formalisms, establish orderings on functional notions based on dominance, improve upon existing adaptive composition bounds for $(\varepsilon, \delta)$-DP, and addresses limitations in tightness due to asymmetry in privacy curves.

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# A Properties of Laplace Transform

**Table 1** Properties of the Laplace Transform. Let $g(t)$ and $h(t)$ be two functions defined for $t \in \mathbb{R}$ and let $a, b \in \mathbb{R}$ be arbitrary constants.

| Property | Expression | |
|---|---|---|
| Linearity : | $\mathcal{L}\{ag(t)+bh(t)\}(s) = a\mathcal{L}\{g(t)\}(s) + b\mathcal{L}\{h(t)\}(s)$ <br> $\mathcal{B}\{ag(t)+bh(t)\}(s) = a\mathcal{B}\{g(t)\}(s) + b\mathcal{B}\{h(t)\}(s)$ | (71) |
| Time-Shifting : | $\mathcal{L}\{g(t-a)\mathbb{I}\{t>a\}\}(s) = e^{-as}\mathcal{L}\{g(t)\}(s),\ \text{for } a>0$ <br> $\mathcal{B}\{g(t-a)\}(s) = e^{-as}\mathcal{B}\{g(t)\}(s),\ \text{for } a\in\mathbb{R}$ | (72) |
| Frequency-Shifting : | $\mathcal{L}\{e^{at}g(t)\}(s) = \mathcal{L}\{g(t)\}(s-a)$ <br> $\mathcal{B}\{e^{at}g(t)\}(s) = \mathcal{B}\{g(t)\}(s-a)$ | (73) |
| Time-Scaling : | $\mathcal{L}\{g(at)\}(s) = \frac{1}{a}\mathcal{L}\{g(t)\}\left(\frac{s}{a}\right)\ \text{ for } a>0$ <br> $\mathcal{B}\{g(at)\}(s) = \frac{1}{\lvert a\rvert}\mathcal{B}\{g(t)\}\left(\frac{s}{a}\right)\ \text{ for } a\in\mathbb{R}$ | (74) |
| Reversal : | $\mathcal{B}\{g(-t)\}(s) = \mathcal{B}\{g(t)\}(-s)$ | (75) |
| Derivative : | $\mathcal{L}\{\dot{g}(t)\}(s) = s\mathcal{L}\{g(t)\}(s) - g(0)$ <br> $\mathcal{B}\{\dot{g}(t)\}(s) = s\mathcal{B}\{g(t)\}(s)$ | (76) |
| Integration : | $\mathcal{L}\left\{\int_0^t g(t)dt\right\}(s) = \frac{1}{s}\mathcal{L}\{g(t)\}(s),\ \text{for } \mathfrak{Re}(s)>0$ <br> $\mathcal{B}\left\{\int_{-\infty}^t g(t)dt\right\}(s) = \frac{1}{s}\mathcal{B}\{g(t)\}(s),\ \text{for } \mathfrak{Re}(s)>0$ | (77) |
| Convolution : | $\mathcal{L}\left\{\int_0^t g(\tau)h(t-\tau)d\tau\right\}(s) = \mathcal{L}\{g(t)\}(s)\cdot\mathcal{L}\{h(t)\}(s)$ <br> $\mathcal{B}\left\{\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau\right\}(s) = \mathcal{B}\{g(t)\}(s)\cdot\mathcal{B}\{h(t)\}(s)$ | (78) |

# B Deferred Proofs for Section 3

▶ **Theorem 6** ($\delta_{P|Q}(\varepsilon)$ and $\mathrm{E}_q(P\|Q)$ as Laplace Transforms of PLDs)**.** *Let $P$ and $Q$ be probability distributions on the same space $\Omega$. If random variables $Z \sim \mathrm{PLD}(P\|Q)$ and $Z' \sim \mathrm{PLD}(Q\|P)$, then for all $\varepsilon \in \mathbb{R}$,*

$$\delta_{P|Q}(\varepsilon) = \mathcal{L}\Big\{\mathbb{P}[Z \geq t+\varepsilon]\Big\}(1) = e^{\varepsilon} \cdot \mathcal{L}\Big\{\mathbb{P}[Z' \leq -(t+\varepsilon)]\Big\}(-1). \tag{19}$$

*And, for all $q \in \mathrm{ROC}_{\mathcal{B}}\{f_{Z'}\}$ (or equivalently, $1-q \in \mathrm{ROC}_{\mathcal{B}}\{f_Z\}$),*

$$\mathrm{E}_q(P\|Q) = (q-1)\cdot \mathcal{B}\Big\{\mathbb{P}[Z \geq t]\Big\}(1-q) = q \cdot \mathcal{B}\Big\{\mathbb{P}[Z' \leq t]\Big\}(q). \tag{20}$$

**Proof.** Denote the set where privacy loss $L_{P|Q}(\theta)$ exceeds $\varepsilon$ as $S^*_{\geq\varepsilon} := \{\theta \in \Omega : P(\theta) \geq e^{\varepsilon}Q(\theta)\}$. Note from this definition that for all $\varepsilon \in \mathbb{R}$, $\delta_{P|Q}(\varepsilon) = \sup_{S\subset\Omega} P(S) - e^{\varepsilon}Q(S) = P(S^*_{\geq\varepsilon}) - e^{\varepsilon}Q(S^*_{\geq\varepsilon})$, because $S^*_{\geq\varepsilon}$ includes any and all points $\theta$ where $P(\theta) \geq e^{\varepsilon}Q(\theta)$.

$$\begin{aligned}
P(S^*_{\geq\varepsilon}) - e^{\varepsilon}Q(S^*_{\geq\varepsilon}) &= \int_{S^*_{\geq\varepsilon}} (1 - e^{\varepsilon}\frac{Q(\theta)}{P(\theta)})P(\theta)\mathrm{d}\theta && (79)\\
&= \int_{S^*_{\geq\varepsilon}} (1 - e^{\varepsilon - L_{P|Q}(\theta)})P(\theta)\mathrm{d}\theta && (80)\\
&= \int_{\varepsilon}^{\infty} (1 - e^{\varepsilon - t})f_Z(t)\mathrm{d}t && (81)\\
&= \mathbb{P}[Z \geq \varepsilon] - \int_0^{\infty} e^{-t'} f_Z(t'+\varepsilon)\mathrm{d}t' && (\text{Change } t = t'+\varepsilon)\\
&= \mathbb{P}[Z \geq \varepsilon] - \mathcal{L}\{f_Z(t+\varepsilon)\}(1) && (82)\\
&\overset{(??)}{=} \mathcal{L}\Big\{\mathbb{P}[Z \geq t+\varepsilon]\Big\}(1). && (\because \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}[Z \geq t+\varepsilon] = -f_Z(t+\varepsilon))
\end{aligned}$$

Similarly, we express the privacy profile $\delta_{P|Q}(\varepsilon)$ in terms of $Z'$ as well:

$$\begin{aligned}
P(S^*_{\geq\varepsilon}) - e^{\varepsilon}Q(S^*_{\geq\varepsilon}) &= \int_{S^*_{\geq\varepsilon}} (\frac{P(\theta)}{Q(\theta)} - e^{\varepsilon})Q(\theta)\mathrm{d}\theta && (83)\\
&= \int_{S^*_{\geq\varepsilon}} (e^{L_{P|Q}(\theta)} - e^{\varepsilon})Q(\theta)\mathrm{d}\theta && (84)\\
&= \int_{\varepsilon}^{\infty} (e^{t} - e^{\varepsilon})f_{Z'}(-t)\mathrm{d}t && (85)\\
&= \int_{\varepsilon}^{\infty} e^{t}\cdot f_{Z'}(-t)\mathrm{d}t - e^{\varepsilon}\cdot \mathbb{P}\big[Z' \leq -\varepsilon\big] && (86)\\
&= e^{\varepsilon}\left(\int_0^{\infty} e^{t'} f_{Z'}(-t'-\varepsilon)\mathrm{d}t' - \mathbb{P}\big[Z' \leq -\varepsilon\big]\right) && (\text{Change } t = t'+\varepsilon)\\
&= e^{\varepsilon}\Big(\mathcal{L}\{f_{Z'}(-t-\varepsilon)\}(-1) - \mathbb{P}\big[Z' \leq -\varepsilon\big]\Big) && (87)\\
&\overset{(??)}{=} e^{\varepsilon}\cdot\mathcal{L}\Big\{\mathbb{P}\big[Z' \leq -(t+\varepsilon)\big]\Big\}(-1).\\
& && (\because \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}[Z' \leq -(t+\varepsilon)] = -f_{Z'}(-t-\varepsilon))
\end{aligned}$$

For the second part, we show the following from the definition of $\mathrm{E}_q(P\|Q)$:

$$\mathrm{E}_q\left(P\|Q\right) = \int_\Omega \left(\frac{P(\theta)}{Q(\theta)}\right)^{q-1} P(\theta)d\theta \tag{88}$$
$$= \int_\Omega e^{(q-1)L_{P|Q}(\theta)} P(\theta)d\theta \tag{89}$$
$$= \int_{-\infty}^{\infty} e^{(q-1)t} f_Z(t)dt \tag{90}$$
$$= \mathcal{B}\left\{f_Z(t)\right\}(1-q) \tag{91}$$
$$\overset{(??)}{=} (q-1)\cdot\mathcal{B}\left\{\mathbb{P}\left[Z \geq t\right]\right\}(1-q).\mathrm{E}_q\left(P\|Q\right) \qquad = \int_\Omega \left(\frac{P(\theta)}{Q(\theta)}\right)^{q} Q(\theta)d\theta$$
$$= \int_\Omega e^{-qL_{Q|P}(\theta)} Q(\theta)d\theta \tag{92}$$
$$= \int_{-\infty}^{\infty} e^{-qt} f_{Z'}(t)dt \tag{93}$$
$$= \mathcal{B}\left\{f_{Z'}(t)\right\}(q) \tag{94}$$
$$\overset{(??)}{=} q\cdot\mathcal{B}\left\{\mathbb{P}\left[Z' \leq t\right]\right\}(q).$$ ◀

## C Deferred Proofs for Section 4

We use the following lemma to prove Theorem 14.

▶ **Lemma 18** (Dominating Privacy Profile under $(\varepsilon, \delta)$-DP [17]). *Fix $\varepsilon \geq 0$ and $\delta \in [0, 1]$. Suppose distributions $P$ and $Q$ over $\Omega$ satisfy $(\varepsilon, \delta)$-differential privacy. Then,*

$$\forall \varepsilon \in \mathbb{R} \;:\; \delta_{P|Q}(\varepsilon) \leq \delta_{\mathrm{RR}}^{\varepsilon,\delta}(\varepsilon) \quad \textit{and} \quad \delta_{Q|P}(\varepsilon) \leq \delta_{\mathrm{RR}}^{\varepsilon,\delta}(\varepsilon), \tag{95}$$

*where $\delta_{\mathrm{RR}}^{\varepsilon,\delta}(t)$ is the privacy profile of the randomized response mechanism $\mathcal{M}_{\mathrm{RR}}^{\varepsilon,\delta}$, as described in Theorem 8.*

▶ **Theorem 14** (Tight Composition of $(\varepsilon, \delta)$-DP). *For any $\varepsilon_i \geq 0$, $\delta_i \in [0, 1]$ for $i \in \{1, \cdots, k\}$, the (non-adaptive) composition of a sequence $(\varepsilon_1, \delta_1), \cdots, (\varepsilon_k, \delta_k)$-DP mechanisms satisfies $(\varepsilon, \delta^{\otimes k}(\varepsilon))$-DP for all $\varepsilon \in \mathbb{R}$, where $\delta^{\otimes l}(\varepsilon)$ is defined recursively as*

$$\forall t \in \mathbb{R} \;:\; \delta^{\otimes l}(t) := \delta_l + \frac{(1-\delta_l)}{e^{\varepsilon_l}+1}\left[e^{\varepsilon_l}\cdot\delta^{\otimes l-1}(t-\varepsilon_l) + \delta^{\otimes l-1}(t+\varepsilon_l)\right], \textit{ with } \delta^{\otimes 0}(t) := [1-e^t]_+.$$

**Proof.** Let $P_{1:k}, Q_{1:k}$ be the output product distributions of the $k$ composed mechanism on neighboring inputs. To prove the statement, we need to show that $\delta_{P_{1:k}|Q_{1:k}}(t) \leq \delta^{\otimes k}(t)$ for all $t \in \mathbb{R}$. We show this inductively.

**Base step.** Let's denote the Heaveside step function as $H(t) := \mathbb{I}\{t > 0\}$. Then, we can write $\delta^{\otimes 0}(t) = (1 - H(t)) \cdot (1 - e^t)$. Using this, we can express

$$
\begin{aligned}
\delta^{\otimes 1}(t) &= \delta_1 + \frac{1-\delta_1}{e^{\varepsilon_1}+1}\left[e^{\varepsilon_1}\cdot\delta^{\otimes 0}(t-\varepsilon_1) + \delta^{\otimes 0}(t+\varepsilon_1)\right] \\
&= \delta_1 + \frac{1-\delta_1}{e^{\varepsilon_1}+1}\left[e_1^{\varepsilon}\cdot(1-H(t-\varepsilon_1))\cdot(1-e^{t-\varepsilon_1}) + (1-H(t+\varepsilon_1))\cdot(1-e^{t+\varepsilon_1})\right] \\
&= 1 + e^{x}(1-\delta_1) + H(t-\varepsilon_1)\cdot\frac{(1-\delta_1)(e^{t}-e^{\varepsilon_1})}{e^{\varepsilon_1}+1} + H(t+\varepsilon_1)\cdot\frac{(1-\delta_1)(e^{t+\varepsilon_1}-1)}{e^{\varepsilon_1}+1} \\
&= \begin{cases} \delta_1 & \text{if } \varepsilon_1 < t, \\ 1-\frac{(e^{t}+1)(1-\delta_1)}{e_1^{\varepsilon}+1} & \text{if } -\varepsilon_1 < t \le \varepsilon_1, \\ 1-e^{t}(1-\delta_1) & \text{if } t \le -\varepsilon_1. \end{cases} \\
&= \delta_{\mathrm{RR}}^{\varepsilon_1,\delta_1}(x) \ge \delta_{P_1|Q_1}(x) \qquad \text{(From Lemma 18)}
\end{aligned}
$$

**Induction step.** Suppose for any $l \in \{2,\cdots,k\}$ the composition of first $l-1$ mechanisms have a privacy profile dominated by $\delta^{\otimes l-1}$. More precisely, assume that $\delta_{P_{1:l-1}|Q_{1:l-1}}(t) \le \delta^{\otimes l-1}(t)$ for all $t \in \mathbb{R}$. We need to show that $\delta_{P_{1:l}|Q_{1:l}}(t) \le \delta^{\otimes l}(t)$ for all $t \in \mathbb{R}$. From Theorem 13, and the induction assumption, we have that

$$
\begin{aligned}
\delta_{P_{1:l}|Q_{1:l}}(t) &= \left(\delta_{P_l|Q_l} \circledast \left(\ddot{\delta}_{P_{1:l-1}|Q_{1:l-1}} - \dot{\delta}_{P_{1:l-1}|Q_{1:l-1}}\right)\right) \\
&\le \left(\delta_{\mathrm{RR}}^{\varepsilon_l,\delta_l} \circledast \left(\ddot{\delta}_{P_{1:l-1}|Q_{1:l-1}} - \dot{\delta}_{P_{1:l-1}|Q_{1:l-1}}\right)\right) && \text{(From Lemma 18)} \\
&= \left(\delta_{P_{1:l-1}|Q_{1:l-1}} \circledast \left(\ddot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l} - \dot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}\right)\right)(t) && \text{(Integration by parts swap)} \\
&\le \left(\delta^{\otimes l-1} \circledast \left(\ddot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l} - \dot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}\right)\right)(t) && \text{(Induction assumption)} \\
&= \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) \times \left(\ddot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}(\tau) - \dot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}(\tau)\right) \mathrm{d}\tau.
\end{aligned}
$$

To differentiate in a manner that handles discontinuity, we first express the function $\delta_{\mathrm{RR}}^{\varepsilon,\delta}(t)$ (cf. (32) in Theorem 8) in terms of Heaviside functions as follows:

$$
\delta_{\mathrm{RR}}^{\varepsilon,\delta}(t) = \underbrace{1-e^{t}(1-\delta)}_{I_1(t)} + H(t-\varepsilon)\cdot\underbrace{\frac{(e^{t}-e^{\varepsilon})(1-\delta)}{e^{\varepsilon}+1}}_{I_2(t)} + H(t+\varepsilon)\cdot\underbrace{\frac{(e^{t+\varepsilon}-1)(1-\delta)}{e^{\varepsilon}+1}}_{I_3(t)}.
$$

We denote the Dirac delta function as $\triangle(x)$ to avoid confusion. By chain rule, the first and second derivatives of $\delta_{\mathrm{RR}}^{\varepsilon,\delta}(t)$ are as follows:

$$
\begin{aligned}
\dot{\delta}_{\mathrm{RR}}^{\varepsilon,\delta}(t) &= \dot{I}_1(t) + \left(H(t-\varepsilon)\cdot\dot{I}_2(t) + \underbrace{\triangle(t-\varepsilon)\cdot I_2(t)}_{J_2(t)}\right) + \left(H(t+\varepsilon)\cdot\dot{I}_3(t) + \underbrace{\triangle(t+\varepsilon)\cdot I_3(t)}_{J_3(t)}\right) \\
\ddot{\delta}_{\mathrm{RR}}^{\varepsilon,\delta}(t) &= \ddot{I}_1(t) + \left(H(t-\varepsilon)\cdot\ddot{I}_2(t) + \triangle(t-\varepsilon)\cdot\dot{I}_2(t) + \dot{J}_2(t)\right) \\
&\quad + \left(H(t+\varepsilon)\cdot\ddot{I}_3(t) + \triangle(t+\varepsilon)\cdot\dot{I}_3(t) + \dot{J}_3(t)\right)
\end{aligned}
$$

Note that $\dot{I}_1(t) = \ddot{I}_1(t) = -e^{t}(1-\delta)$, $\dot{I}_2(t) = \ddot{I}_2(t) = e^{t}\cdot\frac{(1-\delta)}{e^{\varepsilon}+1}$ and $\dot{I}_3(t) = \ddot{I}_3(t) = e^{t}\cdot\frac{e^{\varepsilon}(1-\delta)}{e^{\varepsilon}+1}$. Therefore, on subtracting the two, a lot of terms cancel out, and we get:

$$
\begin{aligned}
\ddot{\delta}_{\mathrm{RR}}^{\varepsilon,\delta}(t) - \dot{\delta}_{\mathrm{RR}}^{\varepsilon,\delta}(t) &= \triangle(t-\varepsilon)\cdot(\dot{I}_2(t) - I_2(t)) + \dot{J}_2(t) + \triangle(t+\varepsilon)\cdot(\dot{I}_3(t) - I_3(t)) + \dot{J}_3(t) \\
&= \triangle(t-\varepsilon)\cdot\frac{(1-\delta)e^{\varepsilon}}{e^{\varepsilon}+1} + \dot{J}_2(t) + \triangle(t+\varepsilon)\cdot\frac{(1-\delta)}{e^{\varepsilon}+1} + \dot{J}_3(t).
\end{aligned}
$$

Note that $J_2(t) = 0$ everywhere except at $t = \varepsilon$. Similarly, $J_3(t) = 0$ everywhere except $t = -\varepsilon$. On substituting and convolving, we get

$$
\begin{aligned}
\delta_{P_{1:l}|Q_{1:l}}(t) &\leq \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) \times \left(\ddot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}(\tau) - \dot{\delta}_{\mathrm{RR}}^{\varepsilon_l,\delta_l}(\tau)\right) \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) \times \left( \triangle(\tau-\varepsilon_l)\cdot \frac{(1-\delta_l)e^{\varepsilon_l}}{e^{\varepsilon_l}+1} + \triangle(\tau+\varepsilon_l)\cdot \frac{(1-\delta_l)}{e^{\varepsilon_l}+1} + \dot{J}_2(\tau) + \dot{J}_3(\tau) \right) \mathrm{d}\tau \\
&= \delta^{\otimes l-1}(t-\varepsilon_l)\cdot \frac{(1-\delta_l)e^{\varepsilon_l}}{e^{\varepsilon_l}+1} + \delta^{\otimes l-1}(t+\varepsilon_l)\cdot \frac{1-\delta_l}{e^{\varepsilon_l}+1} \\
&\quad + \underbrace{\int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) \times \dot{J}_2(\tau)\mathrm{d}\tau}_{K_2(t)} + \underbrace{\int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) \times \dot{J}_3(\tau)\mathrm{d}\tau}_{K_3(t)}
\end{aligned}
$$

For the last integral, apply integration by parts to get

$$
\begin{aligned}
K_2(t) &= \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau)\dot{I}_2(\tau)\cdot\triangle(\tau-\varepsilon_l)\mathrm{d}\tau + \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) I_2(\tau)\cdot\dot{\triangle}(\tau-\varepsilon_l)\mathrm{d}\tau \\
&= \delta^{\otimes l-1}(t-\varepsilon_l)\dot{I}_2(\varepsilon_l) - \left(\delta^{\otimes l-1}(t-\varepsilon_l)\dot{I}_2(\varepsilon_l) + \dot{\delta}^{\otimes l-1}(t-\varepsilon_l)I_2(\varepsilon_l)\right) = 0 - 0
\end{aligned}
$$

since $I_2(\varepsilon_l) = 0$ and for any function $f$ on $\mathbb{R}$ it holds that, $\int_{\mathbb{R}} f\cdot\dot{\triangle}\mathrm{d}\tau = -\int_{\mathbb{R}} \dot{f}\cdot\triangle\mathrm{d}\tau$. Similarly,

$$
\begin{aligned}
K_3(t) &= \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau)\dot{I}_3(\tau)\cdot\triangle(\tau+\varepsilon_l)\mathrm{d}\tau + \int_{-\infty}^{\infty} \delta^{\otimes l-1}(t-\tau) I_3(\tau)\cdot\dot{\triangle}(\tau+\varepsilon_l)\mathrm{d}\tau \\
&= \delta^{\otimes l-1}(t+\varepsilon_l)\dot{I}_3(-\varepsilon_l) - \delta^{\otimes l-1}(t+\varepsilon_l)\dot{I}_3(-\varepsilon_l) - \dot{\delta}^{\otimes l-1}(t-\varepsilon_l)I_3(-\varepsilon_l) = 0 - 0
\end{aligned}
$$

since $I_3(-\varepsilon_l) = 0$. Therefore, we have

$$
\delta_{P_{1:l}|Q_{1:l}}(t) \leq \frac{(1-\delta_l)}{e^{\varepsilon_l}+1}\left[e^{\varepsilon_l}\cdot\delta^{\otimes l-1}(t-\varepsilon_l) + \delta^{\otimes l-1}(t+\varepsilon_l)\right] \leq \delta^{\otimes l}(t). \tag{96}
$$

Hence, the induction statement holds. ◀

▶ **Corollary 15.** *For any $\varepsilon \geq 0$, $\delta \in [0,1]$, the composition of $k$ many $(\varepsilon,\delta)$-DP mechanisms satisfies $(\varepsilon, \delta^{\otimes k}(\varepsilon))$-DP for all $\varepsilon \in \mathbb{R}$, where*

$$
\forall t \in \mathbb{R} \;:\; \delta^{\otimes k}(t) = 1 - (1-\delta)^k \left(1 - \mathop{\mathbb{E}}_{Y \leftarrow \mathrm{Binomial}\left(k, \frac{e^{\varepsilon}}{1+e^{\varepsilon}}\right)} \left[1 - e^{t-\varepsilon\cdot(2Y-k)}\right]_+ \right). \tag{66}
$$

**Proof.** We show that the recurrence relationship from Theorem 14:

$$
\forall t \in \mathbb{R} \;:\; \delta^{\otimes k}(t) = \delta_k + \frac{(1-\delta_k)}{e^{\varepsilon_k}+1}\left[e^{\varepsilon_k}\cdot\delta^{\otimes k-1}(t-\varepsilon_k) + \delta^{\otimes k-1}(t+\varepsilon_k)\right], \text{ with } \delta^{\otimes 0}(t) = [1-e^t]_+,
$$

simplifies to the theorem statement when $\varepsilon_i = \varepsilon_j = \varepsilon$ and $\delta_i = \delta_j = \delta$ for all $i, j \in \{1, \cdots, k\}$. Let's define $p = \frac{e^{\varepsilon}}{e^{\varepsilon}+1}$. This recurrence can then be stated in terms of Bernoulli random variables $Y_1, \cdots, Y_k \sim \mathrm{Bernoulli}(p)$ as follows:

$$\begin{aligned}
\delta^{\otimes k}(t) &= \delta + (1-\delta)\mathbb{E}\left[\delta^{\otimes k-1}(t - \varepsilon(2Y_k - 1))\right] \\
&= \delta + (1-\delta)\mathbb{E}\left[\delta + (1-\delta) \underset{Y_{k-1}\leftarrow \text{Bernoulli}(p)}{\mathbb{E}} \left[\delta^{\otimes k-2}(t - \varepsilon(2Y_k - 1) - \varepsilon(2Y_{k-1} - 1))\right]\right] \\
&= \delta \sum_{i=1}^{2} (1-\delta)^{i-1} + (1-\delta)^2 \mathbb{E}\left[\delta^{\otimes k-2}(t - \sum_{i=k-1}^{k} \varepsilon(2\cdot Y_i - 1))\right] \\
&= \delta \sum_{i=1}^{k} (1-\delta)^{i-1} + (1-\delta)^k \mathbb{E}\left[\delta^{\otimes 0}(t - \sum_{i=1}^{k} \varepsilon(2\cdot Y_i - 1))\right] && \text{(On unrolling)} \\
&= 1 - (1-\delta)^k + (1-\delta)^k \underset{Y\leftarrow \text{Binomial}(k,p)}{\mathbb{E}} \left[\delta^{\otimes 0}(t - \varepsilon(2\cdot Y - k))\right] \\
&= 1 - (1-\delta)^k \left(1 - \underset{Y\leftarrow \text{Binomial}(k,p)}{\mathbb{E}} \left[1 - e^{t-\varepsilon\cdot(2Y-k)}\right]_+\right).
\end{aligned}$$

◀

## D Deferred Proofs and Figures for Section 5

▶ **Theorem 17** (Poisson Subsampling with Add/Remove Adjacency). *Let $0 < \lambda \le 1$. For any distributions $P_\lambda$, $Q_\lambda$, $P$ and $Q$ such that $P_\lambda = \lambda P + (1-\lambda)Q$ and $Q_\lambda = Q$,*

$$\forall \varepsilon \in \mathbb{R} \;:\; \delta_{P_\lambda|Q_\lambda}(\varepsilon) = \begin{cases} \lambda \delta_{P|Q}(\log(1 + (e^\varepsilon - 1)/\lambda)) & \text{if } \varepsilon > \log(1-\lambda), \\ 1 - e^\varepsilon & \text{otherwise.} \end{cases} \tag{70}$$

**Proof.** Recall from Definition 5 that $\mathrm{PLD}(Q\|P)$ and $\mathrm{PLD}(Q_\lambda\|P_\lambda)$ are the distributions of $L_{Q|P}(\Theta)$ and $L_{Q|P\lambda+(1-\lambda)Q}(\Theta)$ respectively with $\Theta \sim Q$. Since for any $\theta \in \Omega$,

$$L_{Q|\lambda P+(1-\lambda)Q}(\theta) = -\log \frac{\lambda P(\theta) + (1-\lambda)Q(\theta)}{Q(\theta)} = -\log(1 - \lambda + \lambda \cdot e^{-L_{Q|P}(\theta)}), \tag{97}$$

the random variables $Z'_\lambda \sim \mathrm{PLD}(Q_\lambda\|P_\lambda)$ and $Z' \sim \mathrm{PLD}(Q\|P)$ are related as $Z'_\lambda = -\log(1 + \lambda(e^{-Z'} - 1))$. Therefore, from Equation 17, we get:

$$\begin{aligned}
\delta_{P_\lambda|Q_\lambda}(\varepsilon) &= \mathbb{E}\left[e^{-Z'_\lambda} - e^\varepsilon\right]_+ && (98) \\
&= \mathbb{E}\left[1 - \lambda + \lambda e^{-Z'} - e^\varepsilon\right]_+ && (99) \\
&= \lambda \mathbb{E}\left[e^{-Z'} - \frac{e^\varepsilon + \lambda - 1}{\lambda}\right]_+ && (100) \\
&= \begin{cases} \lambda \mathbb{E}\left[e^{-Z'} - e^{\log(1+(e^\varepsilon-1)/\lambda)}\right]_+ & \text{if } \varepsilon > \log(1-\lambda) \\ \lambda \mathbb{E}\left[e^{-Z'}\right] + 1 - \lambda - e^\varepsilon & \text{otherwise} \end{cases} && (101) \\
&= \begin{cases} \lambda \delta_{P|Q}(\log(1 + (e^\varepsilon - 1)/\lambda)) & \text{if } \varepsilon > \log(1-\lambda) \\ 1 - e^\varepsilon & \text{otherwise} \end{cases}. && (\because \mathbb{E}\left[e^{-Z'}\right] = 1)
\end{aligned}$$

◀

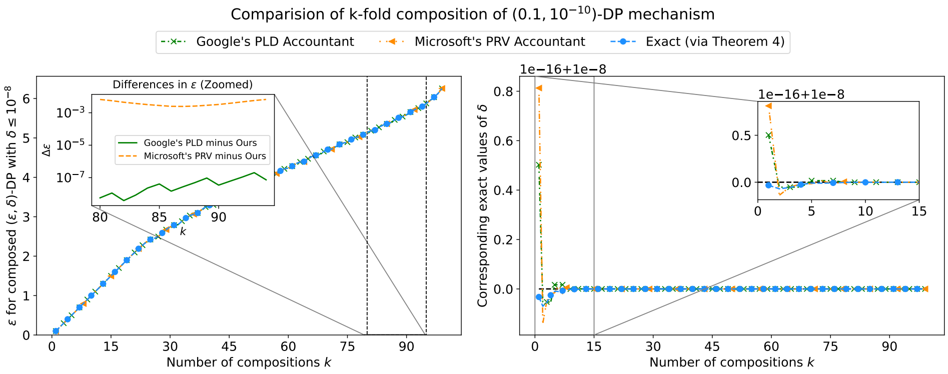


**Figure 6** Comparison of $(\varepsilon, \delta)$-DP bounds between numerical accountants and our Corollary 15 for 100-fold composition of a $(0.1, 10^{-10})$-DP point guarantee, with the budget constraint $\delta < 10^{-8}$. Note that at $k = 100$, the constraint on $\delta$ cannot be satisfied and so the corresponding $\varepsilon = \infty$ at that value. We note that at smaller values of $k$, numerical accountants can sometimes over exceed the budget constraints on $\delta$. Additionally, the gap for $\varepsilon$ between our exact bound and those approximated by numerical accountant tend to be of the order $\approx 10^{-7}$ for Google's PLDAccountant and $\approx 10^{-3}$ for Microsoft's PRVAccountant.