The Maximum Number of Digons Formed by Pairwise Intersecting Pseudocircles

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Abstract

In 1972, Branko Grünbaum conjectured that any nontrivial arrangement of $n > 2$ pairwise intersecting pseudocircles in the plane can have at most $2n-2$ digons (regions enclosed by exactly two pseudoarcs), with the bound being tight. While this conjecture has been confirmed for cylindrical arrangements of pseudocircles and more recently for geometric circles, we extend these results to any simple arrangement of pairwise intersecting pseudocircles.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics

Keywords and phrases pairwise intersecting arrangement, arrangement of pseudocircles, counting digons, tangencies, Grünbaum's conjecture

Digital Object Identifier 10.4230/LIPIcs.SoCG.2025.2

Funding *Gábor Damásdi*: Research supported by ERC grant No. 882971, "GeoScape" and by the Eköp-24 university excellence scholarship program of the Ministry for Culture and Innovation from the source of the National Research, Development and Innovation Fund.
Balázs Keszegh: Research supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, by the National Research, Development and Innovation Office – NKFIH under the grant K 132696 and FK 132060, by the ÚNKP-23-5 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund and by the ERC Advanced Grant "ERMiD". This research has been implemented with the support provided by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, financed under the ELTE TKP 2021-NKTA-62 funding scheme.

1 Introduction

In this paper we focus on families of pairwise intersecting pseudocircles. A *family of pseudocircles* is a set $\mathcal{F}$ of simple closed Jordan curves in the plane such that every two of them are either disjoint, meet at exactly one point in which they touch or intersect at exactly two points in which they properly cross each other. The *arrangement* $\mathcal{A}(\mathcal{F})$ is the cell complex into which the plane is decomposed by the pseudocircles in $\mathcal{F}$. It consists of vertices, edges and faces. The arrangement is called *trivial* if there are two points that lie on every pseudocircle in $\mathcal{F}$. If there is no point that lies on three pseudocircles, then both the arrangement and $\mathcal{F}$ are called *simple*. Note that every simple arrangement of more than two pseudocircles is nontrivial. The arrangement is called *pairwise intersecting* if any two pseudocircles intersect in exactly two points.

A *digon* is a face in the arrangement that is bounded by exactly two edges. If the two edges of a digon are contained in pseudocircles c_1 and c_2, respectively, then we say that c_1 and c_2 *form* that digon and that each of c_1 and c_2 *supports* it.

A trivial arrangement of n pseudocircles contains $2n$ digons for $n > 1$. In his 1972 monograph "Arrangements and Spreads" [11] Branko Grünbaum conjectured that for nontrivial arrangements of pairwise intersecting pseudocircles the maximum number of digons is $2n - 2$.

▶ **Conjecture 1** (Grünbaum's digon conjecture [11, Conjecture 3.6]). *Every nontrivial arrangement of $n > 2$ pairwise intersecting pseudocircles has at most $2n - 2$ digons.*

This conjecture was accompanied by a construction with exactly $2n - 2$ digons, see Figure 1 for an example with six pseudocircles. This example can be easily extended by adding new pseudocircles to the four in the middle, which illustrates that the bound in Conjecture 1 is tight. With some care, this construction can even be realized with geometric circles. A different construction appears in [8].

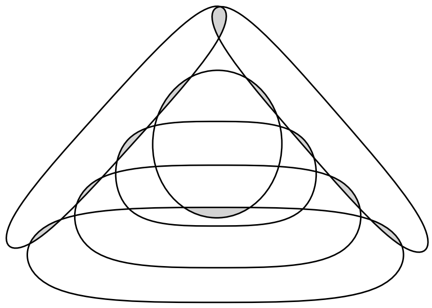


Figure 1 A family of 6 pseudocircles forming 10 digons.

Conjecture 1 was shown to be true in several special cases (assuming simple arrangements). Agarwal et al. [3] showed that Conjecture 1 is true for *cylindrical* arrangements, these are arrangements for which there is a point that is surrounded by each of the pseudocircles. More recently, Conjecture 1 was studied and advertised by Felsner, Roch and Scheucher [8] who showed that it holds for arrangements in which there are three pseudocircles such that every two of them form a digon. This new interest in Grünbaum's conjecture has also motivated us to study the problem. First, we were able to show in [1] that Grünbaum's conjecture is true for simple arrangements of pairwise intersecting geometric circles. Here, using ideas from [1] and [3] we prove:

▶ **Theorem 2.** *Every simple arrangement of $n > 2$ pairwise intersecting pseudocircles has at most $2n - 2$ digons. This bound is tight.*

Our proof is inspired by the proof technique introduced in [3] for bounding the number of digons in cylindrical arrangements of n pairwise intersecting pseudocircles. In that paper, a bipartite graph is associated to a cylindrical arrangement in a way that each pseudocircle is represented by a single point which is its intersection with a common fixed transversal line. The edges of this graph correspond to digons and are drawn according to a simple yet clever rule such that every two independent edges cross an even number of times. The Strong Hanani-Tutte Theorem (see below) then implies that the drawn (bipartite) graph is planar and hence by Euler's formula has at most $2n - 4$ edges.

For the general case of Grünbaum's conjecture, we combine a modification of this graph drawing technique and a graph *doubling* technique that was used for the case of geometric circles in [1]. Namely, a bipartite graph is drawn such that every pseudocircle is represented by *two* points which are its intersection with a common transversal pseudocircle. Furthermore, every digon is represented by *two* edges that are drawn following a rule in the spirit of [3] such that every two independent edges cross an even number of times. Therefore, the drawn (bipartite) graph is planar and hence has at most $2(2n) - 4 = 4n - 4$ edges, representing at most $2n - 2$ digons.

Note that our result can be applied for bounding the number of *touching points* in simple arrangements where any two pseudocircles intersect or touch. In any such arrangement of $n > 2$ pseudocircles, one can turn each touching point into exactly one digon between the two pseudocircles that support the touching. Therefore, for simple arrangements, an upper bound on the number of digons translates directly to a bound on the number of touching pairs or equivalently (in this case) touching points. One can also turn each digon into exactly one touching point in the case of simple arrangements if $n > 3$, but for $n = 3$, two pseudocircles may touch the third one at separate points forming a single digon (the region bounded by the entire third pseudocircle), see Figure 2. In this scenario, turning the digon into a touching point would reduce the third pseudocircle to a single point, which is degenerate, and the final arrangement would violate the assumptions on the number of crossings.

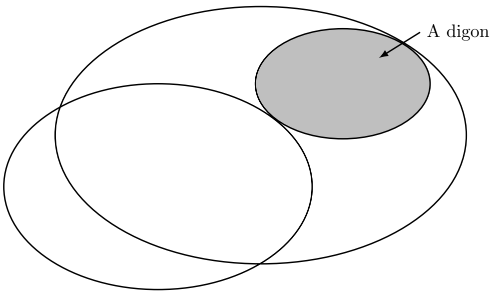


Figure 2 If we allow touchings and $n = 3$, then a digon might be surrounded by only one pseudocircle.

The transformation from digons to touching points or vice versa keeps their sum constant at every step (except for $n = 2$ or in the above-mentioned case), therefore they are equivalent for $n > 3$. From Theorem 2, we have:

▶ **Corollary 3.** *Every simple arrangement of pseudocircles where any two pseudocircles intersect or touch has at most $2n - 2$ touching pairs of pseudocircles. For $n > 3$, the sum of digons and touching pairs is at most $2n - 2$.*

1.1 Outline

In Section 2 we collect useful observations and results that we need for the main proof. In Section 3 we prove Theorem 2, and in Section 4 we discuss some open questions.

2 Some terminology and tools

We begin with some simple definitions and observations, some of which also appear in [1] in the context of circles.

Let $\mathcal{F}$ be a simple family of $n > 2$ pairwise intersecting pseudocircles in the plane. We may assume (and we will assume from now on) that every pseudocircle in $\mathcal{F}$ supports at least one digon. Otherwise, we can remove any pseudocircle from $\mathcal{F}$ that does not satisfy this condition and argue for the remaining family of pseudocircles.

As touchings are forbidden in a pairwise intersecting family, each digon is supported by exactly two pseudocircles. A digon is called a *lens* if it is surrounded by both supporting pseudocircles, and it is called a *lune* otherwise. A lune must be surrounded by precisely one of its supporting pseudocircles, while lying outside the region surrounded by the other (see Figure 3). Here we ignore the case where the unbounded face is a digon. This indeed can be avoided by means of a simple inversion of the plane.

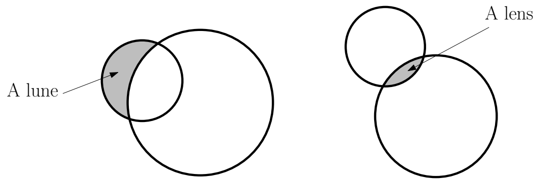


Figure 3 A lens and a lune.

A pseudocircle α is *internal* if it supports a digon that is surrounded by α and it is called *external* if it supports a digon (necessarily a lune) that is not surrounded by α. The following simple observation is important for the proof.

▶ **Proposition 4.** *A pseudocircle in $\mathcal{F}$ cannot be both internal and external.*

Proof. Assume to the contrary that there is a pseudocircle $c \in \mathcal{F}$ which is both external and internal. This means that there exist $c_1, c_2 \in \mathcal{F}$ such that c and c_1 form a digon not surrounded by c while c and c_2 form a digon that is surrounded by c (see Figure 4).

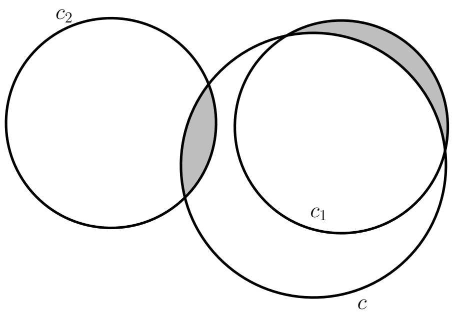


Figure 4 A pseudocircle cannot be both internal and external.

In such a case c_1 and c_2 cannot intersect, contradicting our assumption that we have an arrangement of pairwise intersecting pseudocircles. Indeed, c_1 and c_2 cannot cross in the region surrounded by c because that subarc of c_2 is an edge of a digon. Similarly, c_1 and c_2 cannot intersect in the complementary region of the plane because that subarc of c_1 is an edge of an other digon. ◀

Any simple closed Jordan curve in the plane divides the plane into two parts, one bounded (the *interior*) and the other unbounded (the *exterior*). We will refer to these two parts as the *regions* of the curve. For a pseudocircle $\alpha \in \mathcal{F}$, we will call the region that contains the digons supported by α the *digon-region* of α. By Proposition 4 and because we assume that every pseudocircle supports at least one digon, every pseudocircle in $\mathcal{F}$ has exactly one digon-region. One advantage of this definition is that we can treat lunes and lenses in a uniform way: every digon is just the intersection of the digon-regions of the two corresponding pseudocircles supporting it.[1] This will simplify the case-analysis in some of the proofs.

It is well-known that any family of pairwise intersecting pseudocircles can be extended to a larger family of pairwise intersecting pseudocircles, and the following lemma shows that we can also do it without destroying the existing digons.

▶ **Lemma 5.** *Let $\mathcal{F}$ be a simple family of $n > 2$ pairwise intersecting pseudocircles. Then there is a closed curve c such that $\mathcal{F} \cup \{c\}$ is a simple family of pairwise intersecting pseudocircles, and furthermore c does not intersect any of the digons in $\mathcal{A}(\mathcal{F})$.*

Proof. Let α be a pseudocircle in $\mathcal{F}$. From Proposition 4 we know that one of the two regions of α (the one that is not the digon-region of α) does not contain any digon supported by α. Let c be a curve that is running very close to α outside of the digon-region of α. Then c intersects each pseudocircle in $\mathcal{F} \setminus \{\alpha\}$ exactly twice and it intersects no digon of $\mathcal{A}(\mathcal{F})$. However, c does not intersect α. In order to fix this, we notice that because $\mathcal{F}$ is a simple arrangement of pseudocircles, there is a subarc s of α that is disjoint from any digon supported by α and consequently it is disjoint from any digon in $\mathcal{A}(\mathcal{F})$. We can modify c along s such that the resulting simple closed curve crosses α at two points yielding the desired extension of $\mathcal{F}$ (see Figure 5). ◀

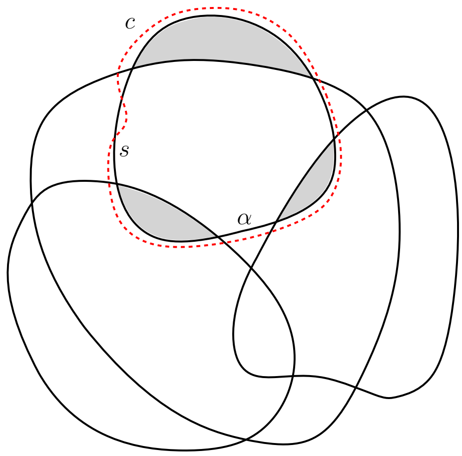


Figure 5 Extending $\mathcal{F}$ with a curve c which avoids the digons in $\mathcal{A}(\mathcal{F})$ and intersects every pseudocircle in $\mathcal{F}$ exactly twice.

For a pseudocircle α with a given orientation as a closed curve and points v_1, v_2 on α we denote by $[v_1, v_2]_\alpha$ (resp., $(v_1, v_2)_\alpha$) the closed (resp., open) segment of α from v_1 to v_2 following α in the given orientation.

▶ **Proposition 6.** *Let $\{\alpha_1, \alpha_2, c\}$ be a simple family of pairwise intersecting pseudocircles such that α_1 and α_2 form a digon. Suppose that c is oriented and let α_j^{in} (resp., α_j^{out}) be the intersection point of c and α_j in which c enters (resp., leaves) the digon-region of α_j, for $j = 1, 2$. Then $\alpha_1^{\text{in}}, \alpha_1^{\text{out}}, \alpha_2^{\text{in}}, \alpha_2^{\text{out}}$ is the cyclic order along c of the four intersection points of α_1 and α_2 with c.*

[1] Another way to think about this is to imagine that we work on a sphere, and we have caps corresponding to the digon-regions.

Proof. If $\alpha_2^{\text{in}} \in (\alpha_1^{\text{in}}, \alpha_1^{\text{out}})_c$ or $\alpha_2^{\text{out}} \in (\alpha_1^{\text{in}}, \alpha_1^{\text{out}})_c$, then we have a point from $\alpha_2 \cap c$ which lies in the digon-region of α_1. However, the intersection of α_2 with the digon-region of α_1 consists only of the subarc of α_2 which bounds the digon formed by α_1 and α_2. This implies that c intersects this digon which is impossible. Therefore, α_1^{out} must follow α_1^{in} in the cyclic order and, similarly, α_2^{out} must follow α_2^{in}. ◀

A *topological graph* is a graph drawn in the plane such that its vertices are drawn as distinct points and its edges are drawn as Jordan arcs connecting the corresponding points. Apart from its endpoints, an edge of a topological graph cannot contain any drawn vertex. Furthermore, every two edges in a topological graph intersect at a finite number of points, each of which is either a common endpoint or a crossing point. A graph is *planar* if it can be drawn as a topological graph where no pair of edges crosses. By the Strong Hanani-Tutte Theorem, it is enough to require an even number of crossings between independent edges.[2]

▶ **Theorem 7** (Strong Hanani-Tutte Theorem [20]). *A graph is planar if and only if it can be drawn as a topological graph in which every two independent edges cross an even number of times.*

3 Proof of Theorem 2

As we noted in the beginning of Section 2, we can suppose that $\mathcal{F}$ is a simple family of $n > 2$ pairwise intersecting pseudocircles in the plane such that every pseudocircle in $\mathcal{F}$ supports at least one digon. By Lemma 5 there is a simple closed curve c that intersects each pseudocircle in $\mathcal{F}$ twice but does not intersect the digons of the arrangement $\mathcal{A}(\mathcal{F})$. We fix the counterclockwise orientation on c. Whenever we traverse c or consider a cyclic order of points on it, we do it according to this orientation. Recall that c is an auxiliary pseudocircle and it is not a member of $\mathcal{F}$; in particular we do not care about digons supported by c.

Every pseudocircle $\alpha \in \mathcal{F}$ intersects c at two points such that at one of them c enters the digon-region of α and at the other c leaves this digon-region. We denote these points by α^{in} and α^{out}, respectively. See for example Figure 6. In this figure, the digon-regions of α and γ are the interior regions of these curves, whereas the digon-region of β is its exterior region.

3.1 The digon graph and its double cover

In order to bound from above the number of digons in $\mathcal{A}(\mathcal{F})$ we consider the graph G_d whose vertex set corresponds to the pseudocircles in $\mathcal{F}$ and whose edge set corresponds to pairs of pseudocircles which form a digon in $\mathcal{A}(\mathcal{F})$ (since $\mathcal{A}(\mathcal{F})$ is nontrivial and intersecting a pair of pseudocircles may not form more than one digon). We call G_d the *digon graph* of the arrangement. Bounding the number of edges in G_d is equivalent to bounding the number of digons in $\mathcal{A}(\mathcal{F})$.

For our proof, we consider the *bipartite double cover* of G_d - this graph, denote it by G, has two vertices, α^{in} and α^{out} for every vertex α of G_d, and two edges, $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\beta^{\text{out}}, \alpha^{\text{in}})$ for every edge (α, β) of G_d. Thus, the number of edges of G is twice the number of digons in $\mathcal{A}(\mathcal{F})$. By construction G is bipartite. It remains to show that G is also planar and therefore has at most $2|V(G)| - 4 = 4n - 4$ edges. This will prove Theorem 2.

[2] Two edges are *independent* if they do not share an endpoint.

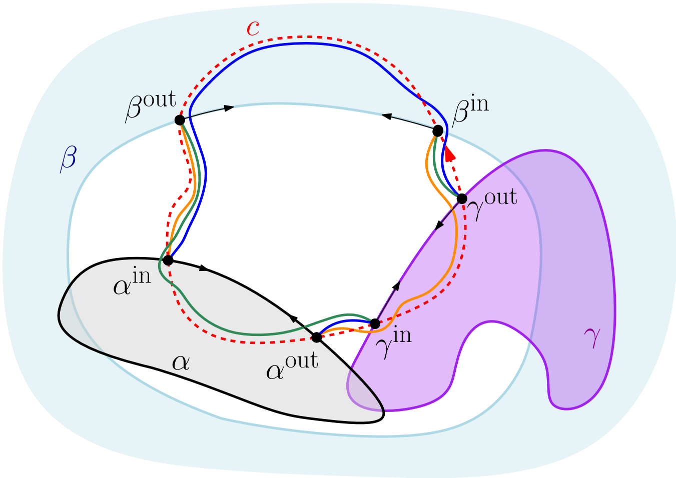


Figure 6 Three pseudocircles α, β and γ and a drawing of the corresponding bipartite double cover graph G. For example, α and γ form a digon, hence we draw the edges $(\alpha^{\text{out}}, \gamma^{\text{in}})$ and $(\gamma^{\text{out}}, \alpha^{\text{in}})$ along c. Considering the (blue) edge $(\gamma^{\text{out}}, \alpha^{\text{in}})$, starting at β^{in} (resp., β^{out}) and following β towards the interior of c, we first encounter α (resp., γ). Therefore, the drawn edge $(\gamma^{\text{out}}, \alpha^{\text{in}})$ "passes by" β^{in} (resp., β^{out}) within the exterior (resp., interior) of c. Each pseudocircle's label is placed within the digon-region associated with that pseudocircle.

3.2 Drawing G in the plane

We draw G as a topological graph in the plane. By abuse of notation, we will not distinguish between G and its drawing. The vertices of G are represented by the points $\{\alpha^{\text{in}} \mid \alpha \in \mathcal{F}\} \cup \{\alpha^{\text{out}} \mid \alpha \in \mathcal{F}\}$ as defined above. Thus, for every two pseudocircles $\alpha, \beta \in \mathcal{F}$ that form a digon in $\mathcal{A}(\mathcal{F})$ there should be two drawn edges $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\beta^{\text{out}}, \alpha^{\text{in}})$. These edges will satisfy the following properties:

(i) An edge $(\alpha^{\text{out}}, \beta^{\text{in}})$ connects α^{out} and β^{in}, while avoiding all other vertices of G;
(ii) every two edges intersect finitely many times; and
(iii) an edge $(\alpha^{\text{out}}, \beta^{\text{in}})$ follows very closely the subarc of c starting at α^{out} and ending at β^{in}, counterclockwise along c. It may cross the curve c several times, but it does not intersect the digon-regions of neither α nor β.

Properties (i), (ii), and (iii) can be easily satisfied. For the second part of (iii) note that Proposition 6 implies that when following c from α^{out} to β^{in} we remain outside the digon-regions of α and β.

We need one more property that the drawing of $(\alpha^{\text{out}}, \beta^{\text{in}})$ should satisfy. This is the most crucial property for the proof and it has to do with how $(\alpha^{\text{out}}, \beta^{\text{in}})$ is drawn with respect to the intermediate vertices of G along the arc on c from α^{out} to β^{in}. We use a modification of the drawing rule that appears in [3].

Let $v \in V$ be an intermediate vertex of G on c along the arc from α^{out} to β^{in}. We will need to indicate whether the edge $(\alpha^{\text{out}}, \beta^{\text{in}})$ passes near v in the interior region of c or in its exterior region. In the first case, we say that the edge *passes v from the inside* whereas in the second case it *passes v from the outside*. Other than these inside/outside decisions and properties (i)-(iii), $(\alpha^{\text{out}}, \beta^{\text{in}})$ can be drawn arbitrarily.

We determine whether $(\alpha^{\text{out}}, \beta^{\text{in}})$ passes v from the inside or from the outside according to the value $d(v, (\alpha^{\text{out}}, \beta^{\text{in}}))$ that we define in the following way. Let γ be the pseudocircle in $\mathcal{F}$ that intersects c at v. It follows from Proposition 6 that γ must be different from α and β.

We follow the curve γ starting from v in the direction *entering the interior region of c*. If we intersect α sooner than we intersect β, then $d(v,(\alpha^{\text{out}},\beta^{\text{in}}))=1$, otherwise $d(v,(\alpha^{\text{out}},\beta^{\text{in}}))=2$.[3] Since $\mathcal{F}$ is a simple family of pairwise intersecting curves, $d(v,(\alpha^{\text{out}},\beta^{\text{in}}))$ is well-defined. If $d(v,(\alpha^{\text{out}},\beta^{\text{in}}))=1$, then we pass v from the inside. Otherwise $d(v,(\alpha^{\text{out}},\beta^{\text{in}}))=2$ and we pass v from the outside. See Figure 6 for examples.

Having described the drawing of the graph G in the plane, we will conclude the proof by showing that every two independent edges of G cross an even number of times which implies that G is planar by the Strong Hanani-Tutte Theorem (Theorem 7).

3.3 Independent edges cross evenly

Let $(\alpha^{\text{out}},\beta^{\text{in}})$ and $(\gamma^{\text{out}},\delta^{\text{in}})$ be two independent edges of G. The parity of the number of crossings between two edges depends solely on how one edge passes by a vertex of the other edge. For example, if the cyclic order of their endpoints is $\alpha^{\text{out}},\gamma^{\text{out}},\delta^{\text{in}},\beta^{\text{in}}$, then the two edges cross evenly if and only if $(\alpha^{\text{out}},\beta^{\text{in}})$ passes by both of γ^{out} and δ^{in} from the inside or both of them from the outside. Therefore, it is enough to consider all the possible topologically different arrangements of the at most four involved pseudocircles (actually five because of c) and verify that in each of them $(\alpha^{\text{out}},\beta^{\text{in}})$ and $(\gamma^{\text{out}},\delta^{\text{in}})$ cross an even number of times. Using a computer we have verified this, still, for the remainder of the section (and of the proof) we provide a non-computer-aided proof.

We begin by observing that it is enough to consider the case that the involved pseudocircles α, β, γ, and δ are distinct. Indeed, $(\alpha^{\text{out}},\beta^{\text{in}})$ and $(\gamma^{\text{out}},\delta^{\text{in}})$ are independent edges, therefore, $\alpha\neq\gamma$ and $\beta\neq\delta$. Still, it is possible that $\alpha=\delta$ or $\beta=\gamma$. If $\alpha=\delta$ and $\beta=\gamma$ then by Proposition 6 the edges do not cross. Otherwise, we add a new pseudocircle as a substitute in the following way. Suppose without loss of generality that $\beta=\gamma$ (if $\alpha=\delta$ we switch the labels α and γ and the labels β and δ). Then we add a new pseudocircle γ' that goes very close to $\beta=\gamma$ outside of the digon-region of $\beta=\gamma$ except for near the vertices of the digon formed by $\beta=\gamma$ and δ. There, the pseudocircle γ' goes into the digon-region of $\beta=\gamma$ thus destroying the digon formed by $\beta=\gamma$ and δ and creating a new digon with δ that is fully contained in the digon that was destroyed. See Figure 7 for an example.

The edge $(\gamma'^{\text{out}},\delta^{\text{in}})$ starts at γ'^{out} and follows the edge $(\gamma^{\text{out}},\delta^{\text{in}})$, while $(\alpha^{\text{out}},\beta^{\text{in}})$ passes by γ'^{in} according the above rules. The number of crossings of the two independent edges $(\alpha^{\text{out}},\beta^{\text{in}})$ and $(\gamma'^{\text{out}},\delta^{\text{in}})$ is equal to that of the original edges because $(\gamma'^{\text{out}},\delta^{\text{in}})$ and $(\gamma^{\text{out}},\delta^{\text{in}})$ are drawn in precisely the same way as far as passing from the inside or from the outside of intermediate vertices of G. Note that the new vertex γ'^{in} becomes an intermediate vertex for $(\alpha^{\text{out}},\beta^{\text{in}})$, however, it is not an endpoint of $(\gamma'^{\text{out}},\delta^{\text{in}})$, thus it does not matter whether $(\alpha^{\text{out}},\beta^{\text{in}})$ passes it from the inside or from the outside.

Henceforth we assume that the pseudocircles α, β, γ and δ are distinct. We proceed by considering the six possible cyclic orders of the vertices α^{out}, β^{in}, γ^{out} and δ^{in}. If the cyclic order is $(\alpha^{\text{out}},\beta^{\text{in}},\gamma^{\text{out}},\delta^{\text{in}})$, then $(\alpha^{\text{out}},\beta^{\text{in}})$ and $(\gamma^{\text{out}},\delta^{\text{in}})$ do not intersect. The cyclic orders $(\alpha^{\text{out}},\gamma^{\text{out}},\beta^{\text{in}},\delta^{\text{in}})$ and $(\alpha^{\text{out}},\delta^{\text{in}},\beta^{\text{in}},\gamma^{\text{out}})$ are symmetric (by switching the labels α and γ and the labels β and δ), and the same goes for the cyclic orders $(\alpha^{\text{out}},\beta^{\text{in}},\delta^{\text{in}},\gamma^{\text{out}})$ and $(\alpha^{\text{out}},\gamma^{\text{out}},\delta^{\text{in}},\beta^{\text{in}})$. Therefore, it is enough to consider the cyclic orders $(\alpha^{\text{out}},\gamma^{\text{out}},\beta^{\text{in}},\delta^{\text{in}})$, $(\alpha^{\text{out}},\gamma^{\text{out}},\delta^{\text{in}},\beta^{\text{in}})$ and $(\alpha^{\text{out}},\delta^{\text{in}},\gamma^{\text{out}},\beta^{\text{in}})$.

[3] Thus, the value 1 or 2 represents whether we encounter the first or the second pseudocircle among the two pseudocircles of the ordered pair $(\alpha^{\text{out}},\beta^{\text{in}})$.

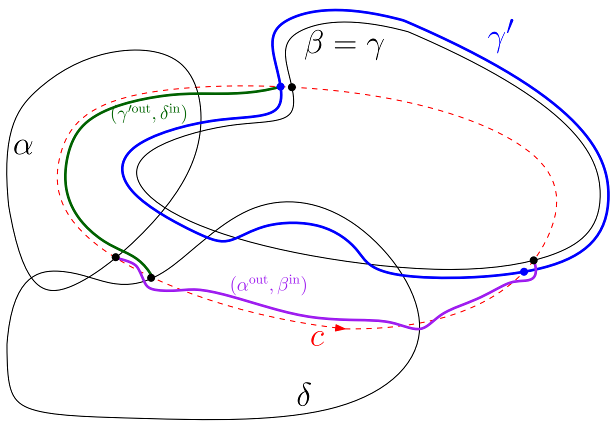


Figure 7 If $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\gamma^{\text{out}}, \delta^{\text{in}})$ are edges and $\beta = \gamma$, then we can replace γ with γ' such that $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\gamma'^{\text{out}}, \delta^{\text{in}})$ maintain the same number of crossings as the original edges. Each pseudocircle's label is placed within the digon-region associated with that pseudocircle.

Recall that we only care about how each edge passes by the vertices of the other edge. The following proposition summarizes the desired properties.

▶ **Proposition 8.** *Assume that the following conditions are satisfied:*

(A) *If $\alpha^{\text{out}}, \gamma^{\text{out}}, \beta^{\text{in}}$ and δ^{in}, appear in this cyclic order along c, then $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$;*

(B) *if $\alpha^{\text{out}}, \gamma^{\text{out}}, \delta^{\text{in}}$ and β^{in} appear in this cyclic order along c, then $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}}))$; and*

(C) *if $\alpha^{\text{out}}, \delta^{\text{in}}, \gamma^{\text{out}}$ and β^{in} appear in this cyclic order along c, then $d(\alpha^{\text{out}}, (\gamma^{\text{out}}, \delta^{\text{in}})) \neq d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}}))$ and $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$.*

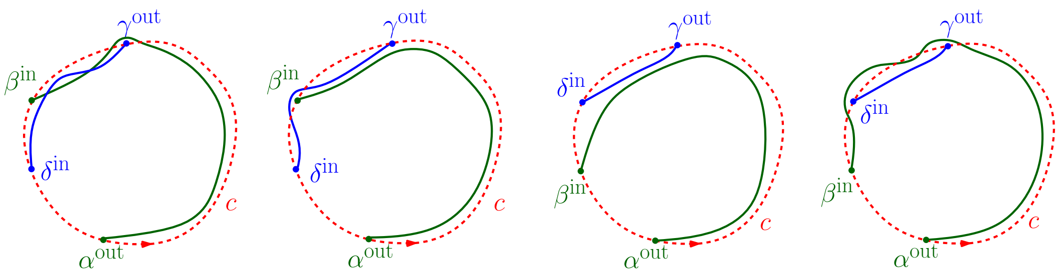
Then $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\gamma^{\text{out}}, \delta^{\text{in}})$ cross an even number of times.

Proof. From properties (i), (ii) and (iii) it follows that the parity of the number of crossings between $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\gamma^{\text{out}}, \delta^{\text{in}})$ depends only on the inside/outside decisions at the vertices $\alpha^{\text{out}}, \beta^{\text{in}}, \gamma^{\text{out}}$ and δ^{in}. This means that it is enough to check a single drawing for each set of inside/outside decisions that are allowed by the claim. The proof follows by a case analysis that is easy to verify by inspection, see Figure 8 for illustrations of the possible cases. ◀

It remains to prove that the drawing of G satisfies the conditions stated in Proposition 8. Conditions (A) and (C) are shown to hold in Proposition 9 while (B) is handled in Proposition 10. Note that in Condition (C) there is a symmetry between the two edges, namely, the cyclic order can also be written as $(\gamma^{\text{out}}, \beta^{\text{in}}, \alpha^{\text{out}}, \delta^{\text{in}})$. Therefore it is enough to prove, say, that $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$ holds for the given cyclic order of the endpoints, this is shown in the following Proposition 9.

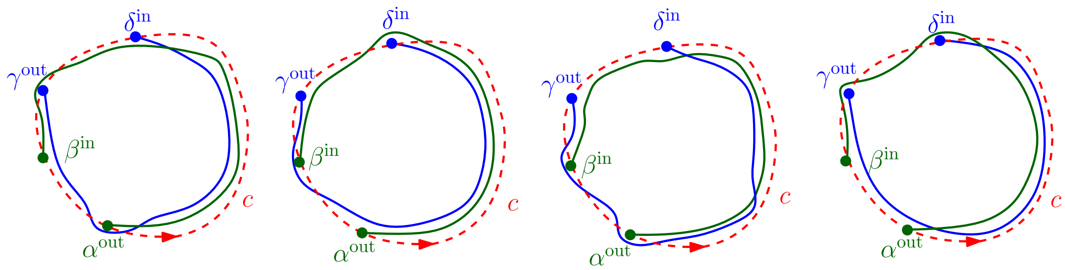
▶ **Proposition 9.** *If the cyclic of order of $\alpha^{\text{out}}, \beta^{\text{in}}, \gamma^{\text{out}}$ and δ^{in} along c is either $(\alpha^{\text{out}}, \delta^{\text{in}}, \gamma^{\text{out}}, \beta^{\text{in}})$ or $(\alpha^{\text{out}}, \gamma^{\text{out}}, \beta^{\text{in}}, \delta^{\text{in}})$, then $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$.*

Proof. Orient β and γ such that they enter the interior of c at β^{in} and γ^{out}, respectively (recall that $\beta \neq \gamma$). We will show that if $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 2$, then $d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}})) = 1$. A symmetric argument (reflect the arrangement and switch the role of β and γ and also of α and δ) shows that if $d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}})) = 1$, then $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 2$. Together, these two assertions imply the claim.



(a) The cyclic order $(\alpha^{\text{out}}, \gamma^{\text{out}}, \beta^{\text{in}}, \delta^{\text{in}})$ implies $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$.

(b) The cyclic order $(\alpha^{\text{out}}, \gamma^{\text{out}}, \delta^{\text{in}}, \beta^{\text{in}})$ implies $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}}))$.



(c) The cyclic order $(\alpha^{\text{out}}, \delta^{\text{in}}, \gamma^{\text{out}}, \beta^{\text{in}})$ implies $d(\alpha^{\text{out}}, (\gamma^{\text{out}}, \delta^{\text{in}})) \neq d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}}))$ and $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) \neq d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}}))$.

Figure 8 $(\alpha^{\text{out}}, \beta^{\text{in}})$ and $(\gamma^{\text{out}}, \delta^{\text{in}})$ cross evenly if the conditions of Proposition 8 hold.

Suppose that $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 2$. This means that as we follow γ starting from γ^{out} it intersects β before it intersects α. Let x denote this first intersection point of γ and β (see Figure 9). We will show that $[\beta^{\text{in}}, x]_\beta$ does not intersect δ, and hence $d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}})) = 1$.

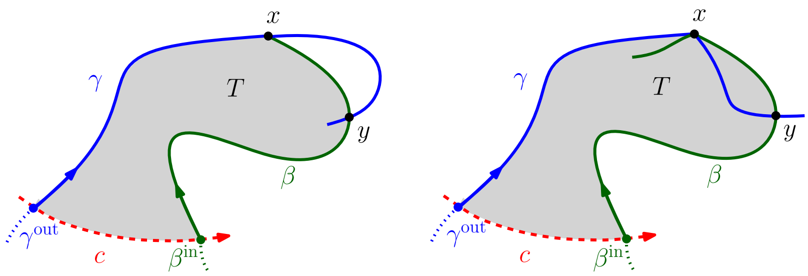
Let t be the closed curve which consists of $[\gamma^{\text{out}}, x]_\gamma$, $[\beta^{\text{in}}, x]_\beta$ and $[\gamma^{\text{out}}, \beta^{\text{in}}]_c$. We claim that t is a simple curve that is not crossed by any of α, β, and γ.

Consider first β and observe that it does not cross itself nor can it cross $[\gamma^{\text{out}}, x]_\gamma$ by the definition of x. From Proposition 6, it follows that $\beta^{\text{out}} \in (\beta^{\text{in}}, \alpha^{\text{out}})_c$ and therefore β does not cross $[\gamma^{\text{out}}, \beta^{\text{in}}]_c$, and hence it does not cross t.

Next, consider γ and observe that it does not cross itself nor can it cross $[\gamma^{\text{out}}, \beta^{\text{in}}]_c$ since it follows from Proposition 6 that $\gamma^{\text{in}} \in (\delta^{\text{in}}, \gamma^{\text{out}})_c$. Therefore, t is a simple curve (recall that β and $(\gamma^{\text{out}}, x)_\gamma$ cannot intersect by the definition of x). Let T be the "triangular" region which is bounded by t and is to the right of γ (and to the left of β and c).

Returning to γ, it remains to show that it cannot intersect $(\beta^{\text{in}}, x)_\beta$. Suppose for contradiction that it does intersect $(\beta^{\text{in}}, x)_\beta$ at a point y. Then either γ enters T at y or it leaves T at y. In the first case, following γ from y, it must leave T since the part of γ just before γ^{out} lies outside T. However, this means that γ must cross t and hence $(\beta^{\text{in}}, x)_\beta$ at another point which is impossible since β and γ already intersect at x and y, see Figure 9a. Considering the second case, if γ leaves T at y, then it implies that γ enters T at x, which in turn implies that β is also entering T at x (otherwise β and γ would touch at x which is impossible), see Figure 9b. However, following β from x, we would have to cross t since the part of β just before β^{in} lies outside T. Since we have already observed that β does not cross t we conclude that γ does not cross t either.

We now show that α cannot intersect t. By the definition of x it cannot intersect $[\gamma^{\text{out}}, x]_\gamma$. By Proposition 6 we have $\alpha^{\text{in}} \in (\beta^{\text{in}}, \alpha^{\text{out}})_c$ and it follows that α cannot intersect $[\gamma^{\text{out}}, \beta^{\text{in}}]_c$. Finally, if α crosses $[\beta^{\text{in}}, x]_\beta$, then it must cross it twice since α does not cross the other



(a) If γ enters T at y, then it must intersect $(\beta^{\text{in}}, x)_\beta$ once more.

(b) If γ leaves T at y, then β enters T at x and must cross t at some other point.

Figure 9 Illustrations for the proof of Proposition 9: when following γ into the interior of c we meet β sooner than α at a point x. Suppose for contradiction that γ intersects $[\beta^{\text{in}}, x]_\beta$ at another point y. Each pseudocircle's label is placed within the digon-region associated with that pseudocircle.

parts of t. The digon formed by α and β must lie outside of T since T lies outside the digon-region of β. Consequently, apart from its subarc that bounds this digon α is contained in T. However, this is impossible since then γ and α do not intersect.

We are now ready to show that δ cannot intersect $[\beta^{\text{in}}, x]_\beta$ which would imply that $d(\beta^{\text{in}}, (\gamma^{\text{out}}, \delta^{\text{in}})) = 1$. Assume to the contrary that δ intersects $[\beta^{\text{in}}, x]_\beta$. Notice that $\delta^{\text{in}}, \delta^{\text{out}}, \gamma^{\text{out}}$, and β^{in} appear in this cyclic order on c. This follows from Proposition 6 and the assumptions of the claim. This implies that δ does not intersect $[\gamma^{\text{out}}, \beta^{\text{in}}]_c$. It follows that the two simple closed curves δ and t must either cross twice or four times, since crossing six times or more would imply that δ intersects γ or β more than twice. If δ and t cross four times, then δ crosses twice each of $[\beta^{\text{in}}, x]_\beta$ and $[\gamma^{\text{out}}, x]_\gamma$. Therefore, the only part of δ that is outside the digon-region of γ and the digon-region of β must be in T. The curves α and δ cannot cross in the digon-region of γ, because δ and γ form a digon. Similarly, α and δ cannot cross in the digon-region of β because α and β form a digon. Since α is disjoint from T we conclude that α and δ do not intersect which is a contradiction. We reach a similar contradiction in the case where δ crosses t precisely at two points that happen to be on $[\beta^{\text{in}}, x]_\beta$, once again α and δ cannot intersect.

It remains to consider the case where δ crosses $[\beta^{\text{in}}, x]_\beta$ once and crosses $[\gamma^{\text{out}}, x]_\gamma$ once. The latter implies that either x or γ^{out} belong to the digon formed by δ and γ, which is impossible since neither β nor c intersects this digon. ◀

▶ **Proposition 10.** *If $\alpha^{\text{out}}, \gamma^{\text{out}}, \delta^{\text{in}}$ and β^{in} appear in this cyclic order along c, then we have $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}}))$.*

Proof. Orient γ and δ such that they enter the interior region of c at γ^{out} and δ^{in}, respectively. Recall that c is oriented counterclockwise and its interior region lies to its left. Thus, the digon-region of γ lies to its left whereas the digon-region of δ lies to its right. Let $\{A'_1, A'_2\} = \delta \cap \alpha$ and $\{B'_1, B'_2\} = \delta \cap \beta$, such that the cyclic order of these four points along δ is A'_1, A'_2, B'_1, B'_2 (since α and β form a digon it follows from Proposition 6 that A'_1 and A'_2 must be consecutive and so are B'_1 and B'_2). Similarly, let $\{A''_1, A''_2\} = \gamma \cap \alpha$ and $\{B''_1, B''_2\} = \gamma \cap \beta$, such that the cyclic order of these four points along γ is $A''_1, A''_2, B''_1, B''_2$.

Considering the cyclic order of A'_1, A'_2, A''_1 and A''_2 along α (under some orientation of α), it follows from Proposition 6 that the first two are consecutive and so are the last two (since γ and δ form a digon). Therefore, if we orient α such that A'_2 follows A'_1, then the cyclic order of those four points is either A'_1, A'_2, A''_1, A''_2 or A'_1, A'_2, A''_2, A''_1.

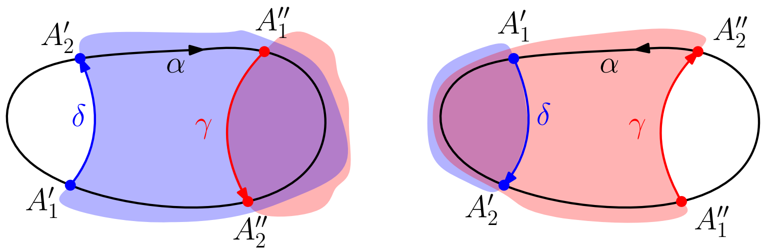


Figure 10 Suppose that A_1', A_2', A_1'', A_2'' appear in this cyclic order on α and B_1', B_2', B_1'' and B_2'' are in the exterior of α. Then the intersection of the digon-region of δ (which is to its right) and the digon-region of γ (which is to its left) contains either $[A_1'', A_2'']_\alpha$ or $[A_1', A_2']_\alpha$ which is impossible.

We claim that the former is impossible. Indeed, suppose that A_1', A_2', A_1'' and A_2'' appear in this cyclic order on α. Note that it follows from the cyclic order on δ (resp., γ) that both of B_1' and B_2' (resp., B_1'' and B_2'') must be in the same region (interior or exterior) of α. If B_1', B_2', B_1'' and B_2'' are in the exterior of α, then either $[A_1'', A_2'']_\alpha$ or $[A_1', A_2']_\alpha$ is contained in the intersection of the digon-regions of γ and δ (see Figure 10), which is impossible. By a symmetric argument, it is impossible that B_1', B_2', B_1'', and B_2'' are in the interior of α. Therefore, either B_1' and B_2' are in the digon-region of α or B_1'' and B_2'' are there. However, this implies that the subarc of β in the digon-region of α intersects either γ or δ which is impossible since α and β form a digon.

Thus A_1', A_2', A_2'', A_1'' appear in this cyclic order on α, and similarly, B_1', B_2', B_2'', B_1'' appear in this cyclic order on β if it is oriented such that B_2' immediately follows B_1'. Next, we redraw α close to the digon it forms with β such that these two pseudocircles become disjoint (see Figure 11). In a similar way, we redraw δ such that δ and γ become disjoint. Note that no other intersection points but $\alpha \cap \beta$ and $\gamma \cap \delta$ are destroyed and no new intersection points are introduced. Therefore, the vertex set of $\mathcal{A}(\{\alpha, \beta, \gamma, \delta\})$ is precisely $\{A_1', A_2', A_1'', A_2'', B_1', B_2', B_1'', B_2''\}$ and its edge set consists of the edges $[A_1', A_2']_\alpha$, $[A_2', A_2'']_\alpha$, $[A_2'', A_1'']_\alpha$, $[A_1'', A_1']_\alpha$, $[B_1', B_2']_\beta$, $[B_2', B_2'']_\beta$, $[B_2'', B_1'']_\beta$, $[B_1'', B_1']_\beta$, $[A_1'', A_2'']_\gamma$, $[A_2'', B_1'']_\gamma$, $[B_1'', B_2'']_\gamma$, $[B_2'', A_1'']_\gamma$, $[A_1', A_2']_\delta$, $[A_2', B_1']_\delta$, $[B_1', B_2']_\delta$ and $[B_2', A_1']_\delta$. It is not hard to see that this implies that the face set of $\mathcal{A}(\{\alpha, \beta, \gamma, \delta\})$ consists of four digons whose boundaries are $\{[A_1', A_2']_\alpha, [A_1', A_2']_\delta\}$, $\{[A_1'', A_2'']_\alpha, [A_1'', A_2'']_\gamma\}$, $\{[B_1', B_2']_\beta, [B_1', B_2']_\delta\}$ and $\{[B_1'', B_2'']_\beta, [B_1'', B_2'']_\gamma\}$, and six faces of size four whose boundaries are (refer to Figure 11 for an illustration):
$\{[A_1', A_2']_\delta, [A_2', A_2'']_\alpha, [A_2'', A_1'']_\gamma, [A_1'', A_1']_\alpha\}$, $\{[A_1', A_2']_\alpha, [A_2', B_1']_\delta, [B_1', B_2']_\beta, [B_2', A_1']_\delta\}$,
$\{[B_1', B_2']_\delta, [B_2', B_2'']_\beta, [B_2'', B_1'']_\gamma, [B_1'', B_1']_\beta\}$, $\{[A_1'', A_2'']_\alpha, [A_2'', B_1'']_\gamma, [B_1'', B_2'']_\beta, [B_2'', A_1'']_\delta\}$,
$\{[A_1', A_1'']_\alpha, [A_1'', B_2'']_\gamma, [B_2'', B_2']_\beta, [B_2', A_1']_\delta\}$ and $\{[A_2', A_2'']_\alpha, [A_2'', B_1'']_\gamma, [B_1'', B_1']_\beta, [B_1', A_2']_\delta\}$.

Note that $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 1$ and $d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 2$ iff $\gamma^{\text{out}} \in (B_2'', A_2'')_\gamma$ and $\delta^{\text{in}} \in (A_2', B_2')_\delta$. Similarly, $d(\gamma^{\text{out}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 2$ and $d(\delta^{\text{in}}, (\alpha^{\text{out}}, \beta^{\text{in}})) = 1$ iff $\gamma^{\text{out}} \in (A_2'', B_2'')_\gamma$ and $\delta^{\text{in}} \in (B_2', A_2')_\delta$. Suppose for contradiction that one of these cases holds. Then, on the one hand, there is a curve, namely $[\gamma^{\text{out}}, \delta^{\text{in}}]_c$, which connects γ^{out} and δ^{in} while not crossing any of α, β, γ and δ (this follows from the given cyclic order and Proposition 6). However, on the other hand, there is no face of $\mathcal{A}(\{\alpha, \beta, \gamma, \delta\})$ whose boundary contains segments of both $(B_2'', A_2'')_\gamma$ and $(A_2', B_2')_\delta$ or segments of both $(A_2'', B_2'')_\gamma$ and $(B_2', A_2')_\delta$, contradicting the existence of a curve as above. ◀

This completes the proof of Theorem 2.

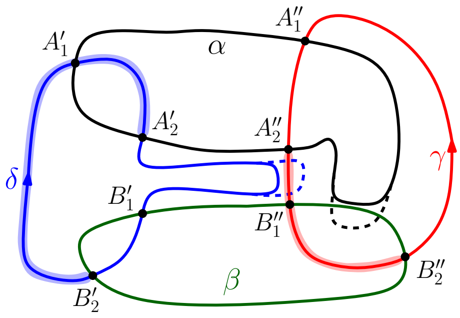


Figure 11 $\mathcal{A}(\{\alpha,\beta,\gamma,\delta\})$ after redrawing α and δ such that they do not form digons with β and γ, respectively. If $d(\gamma^{\text{out}},(\alpha^{\text{out}},\beta^{\text{in}}))=2$ and $d(\delta^{\text{in}},(\alpha^{\text{out}},\beta^{\text{in}}))=1$, then $\gamma^{\text{out}}\in(A_2'',B_2'')_\gamma$ and $\delta^{\text{in}}\in(B_2',A_2')_\delta$.

4 Concluding remarks

What can we say about drawings on other surfaces? For example, we can draw K_7 on the torus, which implies the existence of 7 curves such that they pairwise form a digon, giving us 21 digons in total. This suggests that we need a significantly different approach since an argument based on the embeddability of the double cover would give roughly $2n$ as an upper bound.

▶ **Problem 11.** *What is the maximum number of digons in a pairwise intersecting family of pseudocircles on a closed orientable connected surface M_g of genus g?*

We note that graphs with planar bipartite double covers were extensively studied in topological graph theory, see for example the works of Negami [14, 15]. Negami also considered general covers of graphs and formulated the following nice conjecture:

▶ **Conjecture 12** (Negami 1988)**.** *A graph G has a finite planar cover if and only if G embeds in the projective plane.*

Indeed, with some extra work, one can show that digon graphs are embeddable in the projective plane, but we omit the proof.

Another natural question is to consider the number of triangles in intersecting pseudocircle arrangements. Felsner and Scheucher [9] showed that the maximal number of triangles is $2n^2/3+O(n)$, and this is asymptotically tight. In their proof, they use the bound $O(n)$ from [3] for the maximum number of digons in pairwise intersecting arrangements of pseudocircles. By applying Theorem 2 instead, we achieve the following improvement for simple arrangements.

▶ **Corollary 13.** *Every simple arrangement of $n \geqslant 4$ pairwise intersecting pseudocircles has at most $\frac{2}{3}n^2$ triangles.*

On the other hand, triangles are unavoidable, they showed that there are at least $2n/3$ triangles and made the subsequent conjecture.

▶ **Conjecture 14.** *Every pairwise intersecting arrangement of $n \geqslant 3$ pseudocircles has at least $n-1$ triangles.*

On a final remark, we were able to extend our proof of Theorem 2 to nontrivial pairwise intersecting arrangements of pseudocircles. However, it involves dealing with several delicate subcases. Therefore, we have chosen to omit the full proof here.

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