On the Twin-Width of Smooth Manifolds

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Abstract

Building on Whitney's classical method of triangulating smooth manifolds, we show that every compact d-dimensional smooth manifold admits a triangulation with dual graph of twin-width at most $d^{O(d)}$. In particular, it follows that every compact 3-manifold has a triangulation with dual graph of bounded twin-width. This is in sharp contrast to the case of treewidth, where for any natural number n there exists a closed 3-manifold such that every triangulation thereof has dual graph with treewidth at least n. To establish this result, we bound the twin-width of the dual graph of the d-skeleton of the second barycentric subdivision of the $2d$-dimensional hypercubic honeycomb. We also show that every compact, piecewise-linear (hence smooth) d-dimensional manifold has triangulations where the dual graph has an arbitrarily large twin-width.

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1 Introduction

Structural graph parameters have become increasingly important in computational topology in the past two decades. This is mainly due to the emergence of fixed-parameter tractable (FPT) algorithms for problems on knots, links [13, 36, 37, 39] and 3-manifolds [15, 17, 18, 19, 20],[1] most of which are known to be NP-hard in general. Although these FPT algorithms may have exponential worst-case running time, on inputs with bounded *treewidth* they are guaranteed to terminate in polynomial (or even in linear) time.[2] In addition, some of these algorithms have been implemented in software packages such as `Regina`, providing practical tools for researchers in topology [12, 14].[3]

[1] Also see [2] for an FPT algorithm checking tightness of (weak) pseudomanifolds in arbitrary dimensions.
[2] Here the term *input* refers either to a link diagram $\mathcal{D}$, or to a 3-manifold triangulation $\mathcal{T}$. In the first case the treewidth means the treewidth of $\mathcal{D}$ considered as a 4-regular graph, in the second case it means the treewidth of the dual graph $\Gamma(\mathcal{T})$ of $\mathcal{T}$. The running times are measured in terms of the *size* of the input. The size of a link diagram is defined as the number of its crossings, and the size of a 3-manifold triangulation is the number of its tetrahedra. More definitions are given in Section 2.
[3] See [3] for an implemented algorithm to effectively compute certain Khovanov homology groups of knots. This algorithm is conjectured to be FPT in the *cutwidth* of the input knot diagram, cf. [3, Section 6].

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The success of the above algorithms naturally leads to the following question. Given a 3-manifold $\mathcal{M}$ (resp. knot $\mathcal{K}$), what is the smallest treewidth that the dual graph of a triangulation of $\mathcal{M}$ (resp. a diagram of $\mathcal{K}$) may have?[4] Motivated by this challenge, in recent years several results have been obtained that reveal quantitative connections between *topological invariants* of knots and 3-manifolds, and *width parameters* associated with their diagrams [22, 35] and triangulations [26, 27, 28, 29, 30, 40], respectively. It turns out that topological properties of 3-manifolds may prohibit the existence of "thin" triangulations.[5,6] At the same time, geometric or topological descriptions of 3-manifolds can also give strong hints on how to triangulate them so that their dual graphs have constant pathwidth or treewidth, or at least bounded in terms of a topological invariant of these 3-manifolds.[7]

In this work we establish similar results for another graph parameter called *twin-width*. Introduced in [11], this notion has been subject of growing interest and found many algorithmic applications in recent years.[8] Namely, on classes of effectively[9] bounded twin-width, first-order properties can be decided efficiently[10] [11] (also see [8] for improved running times on specific problems definable in first-order logic), first-order queries can be enumerated fast [24], and enhanced approximation algorithms can be designed for several graph optimization problems [4]. Besides, classes of bounded twin-width are fairly general and broad. They for instance include classes of bounded tree-width, and even its dense analogue, clique-width, classes excluding a minor, proper permutation classes, d-dimensional grid graphs [11], some classes of cubic expanders [9], segment intersection graphs without biclique subgraphs of a fixed size [7], and modifications definable in first-order logic (called first-order *transductions*) of all these classes [11]. We observe that the definition of twin-width can readily be lifted to binary structures (i.e., edge-colored multigraphs).

Our first result shows that a compact d-dimensional smooth manifold always has a triangulation with dual graph of twin-width bounded in terms of d.

▶ **Theorem 1.** *Any compact d-dimensional smooth manifold $\mathcal{M}$ admits a triangulation (more specifically, a* Whitney triangulation[11]*) with dual graph of twin-width at most $d^{O(d)}$.*

Let us emphasize that, in an appropriate computational model, Whitney triangulations of smooth manifolds can be constructed algorithmically [5]. As every 3-manifold is smooth [41] (see also [38]), the next corollary follows immediately from Theorem 1. We recall that $\mathrm{tww}(G)$ denotes the twin-width of a graph G, and $\Gamma(\mathcal{T})$ denotes the dual graph of a triangulation $\mathcal{T}$.

▶ **Corollary 2.** *There exists a universal constant $C > 0$ such that every compact 3-dimensional manifold $\mathcal{M}$ admits a triangulation $\mathcal{T}$ with $\mathrm{tww}(\Gamma(\mathcal{T})) \leqslant C$.*

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[4] For links and knots, this question was respectively asked in [37, Section 4] and [16, p. 2694].

[5] For non-Haken 3-manifolds of large Heegaard genus [30] or Haken 3-manifolds with a "complicated" JSJ decomposition [29], the dual graph of *any* triangulation must also have large treewidth.

[6] For results where the treewidth of a knot diagram is bounded below by topological properties of the underlying knot, see [22, 35].

[7] This is case with Seifert fibered spaces [28] or hyperbolic 3-manifolds [27, 40].

[8] For an introduction to twin-width and an overview of its applications, see [6] and the references therein.

[9] A class has *effectively bounded twin-width* if it has bounded twin-width, and contraction sequences of width $O(1)$ (objects witnessing the twin-width upper bound) can be found in polynomial time; see Section 2.1 for the definitions of *contraction sequences* and *twin-width*.

[10] More precisely, there is a *fixed-parameter tractable* algorithm that, given a first-order sentence φ and an n-vertex graph G with a contraction sequence of width d, decides if G satisfies φ in time $f(\varphi, d) \cdot n$, for some computable function f.

[11] We call a triangulation $\mathcal{T}$ of a compact smooth manifold $\mathcal{M}$ a *Whitney triangulation* of $\mathcal{M}$, if $\mathcal{T}$ is obtained via Whitney's method discussed in Section 3.

This is in sharp contrast to the case of treewidth, for which it is known that for every $n \in \mathbb{N}$ there are infinitely many 3-manifolds where the smallest treewidth of the dual graph of *every* triangulation is at least n [29, 30]. Complementing Theorem 1, we also show that for any fixed $d \geqslant 3$, the d-dimensional triangulations of large twin-width are abundant (Theorem 19). Moreover we show that any piecewise-linear (hence smooth) manifold of dimension at least three admits triangulations with dual graph of arbitrarily large twin-width.

▶ **Theorem 3.** *Let $d \geqslant 3$ be an integer. For every compact d-dimensional piecewise-linear manifold $\mathcal{M}$ and natural number $n \in \mathbb{N}$, there is a triangulation $\mathcal{T}$ of $\mathcal{M}$ with* $\operatorname{tww}(\Gamma(\mathcal{T})) \geqslant n$.

▶ Remark 4. The assumption of $d \geqslant 3$ in Theorem 3 is essential. Indeed, the dual graph of any triangulation of the genus-g surface S_g is, in particular, a graph that embeds into S_g and such graphs are known to have twin-width bounded above by $c(\sqrt{g}+1)$ for some universal constant $c > 0$ [32].[12]

Outline of the paper. In Section 2 we review the relevant notions from graph theory and topology. In Section 3 we recall Whitney's seminal work on triangulating smooth manifolds. This is followed by a detailed proof of Theorem 1 in Section 4. Finally, in Section 5 we prove the complementary results about triangulations with dual graph of large twin-width.

2 Preliminaries

Basic notation. For a finite set S we let $|S|$ denote its cardinality, while for a real number x we let $|x|$ denote its absolute value. For a positive integer $k \leqslant |S|$, we let $\binom{S}{k}$ denote the set of k-element subsets of S. For a positive integer n, we let $[n]$ denote the set of all positive integers up to n, and for two real numbers $a \leqslant b$, we use $[a, b]$ to denote the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$. For a vector $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ we use $\|y\|$ to denote its Euclidean norm, i.e, $\|y\|^2 = \sum_{i=1}^{d} y_i^2$. However, if $\mathcal{X}$ is a (cubical or simplicial) complex, then $\|\mathcal{X}\|$ refers to its geometric realization.

2.1 Trigraphs, contraction sequences and twin-width

Following [11, Sections 3 and 4], in this section we review the graph-theoretic notions central to our work and collect some basic, yet important facts about twin-width.

Trigraphs. A *trigraph* G is a triple $G = (V, E, R)$, where V is a finite set of *vertices*, and $E, R \subseteq \binom{V}{2}$ are two disjoint subsets of pairs of vertices called *black edges* and *red edges*, respectively. We also refer to the sets of vertices, black edges and red edges of a given trigraph G as $V(G)$, $E(G)$ and $R(G)$, respectively. Any simple graph $G = (V, E)$ may be regarded as a trigraph (V, E, R) with $R = \emptyset$. For a vertex $v \in V(G)$ the *degree* $\deg(v)$ *of* v is the number of edges incident to it, i.e., $\deg(v) = |\{e \in E \cup R : v \in e\}|$. Additionally, the *red degree* $\deg_R(v)$ *of* v is the number of red edges incident to it, i.e., $\deg_R(v) = |\{e \in R : v \in e\}|$. A trigraph G for which $\deg_R(G) = \max_{v \in V(G)} \deg_R(v) \leqslant b$ is called a *b-trigraph*. Given two trigraphs $G = (V, E, R)$ and $G' = (V', E', R')$, we say that G' is a *subtrigraph* of G, if $V' \subseteq V$, $E' \subseteq E \cap \binom{V'}{2}$ and $R' \subseteq R \cap \binom{V'}{2}$.[13] In addition, if $E' = E \cap \binom{V'}{2}$ and $R' = R \cap \binom{V'}{2}$, then we say that G' is an *induced subtrigraph* of G. For a trigraph G and a subset $S \subseteq V(G)$ of its vertices, $G - S$ denotes the induced subtrigraph of G with vertex set $V(G) \setminus S$.

[12] This bound is sharp up to a constant multiplicative factor [32]. For $g = 0$, we know that planar graphs have twin-width at most eight [25], and there are planar graphs with twin-width equal to seven [31].
[13] As usual, subtrigraphs of graphs (those without any red edges) will also be called subgraphs.

Contraction sequences and twin-width. Let $G = (V, E, R)$ be a trigraph and $u, v \in V$ be two arbitrary distinct vertices of G. We say that the trigraph $G/u, v = (V', E', R')$ is obtained from G by *contracting* u and v into a new vertex w if **1.** $V' = (V \setminus \{u, v\}) \cup \{w\}$, **2.** $G - \{u, v\} = (G/u, v) - \{w\}$ and **3.** for any $x \in V' \setminus \{w\} = V \setminus \{u, v\}$ we have

- $\{w, x\} \in E'$ if and only if $\{u, x\} \in E$ and $\{v, x\} \in E$,
- $\{w, x\} \notin E' \cup R'$ if and only if $\{u, x\} \notin E \cup R$ and $\{v, x\} \notin E \cup R$, and
- $\{w, x\} \in R'$ otherwise.

We call the trigraph $G/u, v$ a *contraction* of G. A sequence $\mathfrak{S} = (G_1, \dots, G_m)$ of trigraphs is a *contraction sequence* if G_{i+1} is a contraction of G_i for every $1 \leqslant i \leqslant m - 1$. Note that $|V(G_{i+1})| = |V(G_i)| - 1$. We use the notation "$\mathfrak{S}\colon G_1 \rightsquigarrow G_m$" to indicate that the trigraphs G_1 and G_m are initial and terminal entries of the contraction sequence $\mathfrak{S}$. The *width* $\mathrm{w}(\mathfrak{S})$ of a contraction sequence $\mathfrak{S} = (G_1, \dots, G_m)$ is defined as $\mathrm{w}(\mathfrak{S}) = \max_{1 \leqslant i \leqslant m} \deg_R(G_i)$, i.e., the largest red degree of any vertex of any trigraph in $\mathfrak{S}$. Now, the *twin-width* $\mathrm{tww}(G)$ of a trigraph G is defined as the smallest width of any contraction sequence $(G_1, \dots, G_{|V(G)|})$ with $G_1 = G$ and $G_{|V(G)|} = \bullet$, where $\bullet$ denotes the trigraph consisting of a single vertex.

Some properties of twin-width; grid graphs

We conclude this section by collecting some properties of twin-width that we will rely on later. The first one states that twin-width is monotonic under taking induced subtrigraphs and is a simple consequence of the definitions (cf. [11, Section 4.1]).

▶ **Proposition 5.** *If H is an induced subtrigraph of a trigraph G, then* $\mathrm{tww}(H) \leqslant \mathrm{tww}(G)$.

Smallness. An infinite class $\mathcal{G}$ of graphs is *small* if there exists a constant $c > 1$ such that for every $n \in \mathbb{N}$ the class $\mathcal{G}$ contains at most $n!c^n$ labeled graphs on n vertices. The next theorem says that every graph class of bounded twin-width – i.e., for which there exists a constant $C > 0$, such that $\mathrm{tww}(G) \leqslant C$ for every graph G in the class – is small.

▶ **Theorem 6** ([9, Theorem 2.5])**.** *Every graph class with bounded twin-width is small.*

Now let s be a non-negative integer. The *s-subdivision* of G is the graph $\mathrm{subd}_s(G)$ obtained from G by subdividing each edge in $E(G)$ exactly s times. A simple counting argument together with Theorem 6 yields the following:

▶ **Proposition 7.** *For any fixed integers $k \geqslant 4$ and $s \geqslant 0$, the class* $\mathrm{subd}_s(\mathcal{G}_k)$ *of s-subdivisions of k-regular*[14] *simple graphs is not small, hence has unbounded twin-width.*

This proposition follows from the adaptation of an argument given in the first paragraph of [23, Section 3]. For completeness, we spell out this proof below.

Proof of Proposition 7. Let $N_k(m)$ be the number of labeled k-regular simple graphs on m vertices. Note that if $N_k(m) > 0$, then km is even; which we now assume. It is known (cf. [42, Section 6.4.1]) that, asymptotically

$$N_k(m) \sim \exp\left(\frac{1-k^2}{4}\right) \frac{(km)!}{(km/2)! \cdot 2^{km/2} \cdot (k!)^m} = \Omega\left(\frac{(km/2)!}{(k!)^m}\right), \tag{1}$$

which readily implies that k-regular graphs (for $k \geqslant 3$) do not form a small class.

[14] A simple graph $G = (V, E)$ is *k-regular* if every vertex $v \in V$ has degree $\deg(v) = k$.

Now, let $N_k^{(s)}(n)$ be the number of n-vertex graphs in the class $\mathrm{subd}_s(\mathcal{G}_k)$ of s-subdivisions of k-regular simple graphs. If a graph $G \in \mathrm{subd}_s(\mathcal{G}_k)$ has n vertices, then $n = m + \frac{km}{2}s$, where m is the number of vertices of G of degree k. Note that such a graph G can be obtained by first choosing a k-regular labeled graph H on m vertices, then ordering the remaining $n - m = kms/2$ vertices arbitrarily and evenly distributing them to the edges of H according to some fixed ordering of $E(H)$. It follows that

$$N_k^{(s)}(n) \geqslant \binom{n}{m} \cdot N_k(m) \cdot (n-m)! = n! \frac{N_k(m)}{m!} \overset{(1)}{=} n! \cdot \Omega\left(\frac{(km/2)!}{m! \cdot (k!)^m}\right). \tag{2}$$

Since $k \geqslant 4$, we have $km/2 \geqslant 2m$. This, together with $(2m)! \geqslant 2^m (m!)^2$ and (2) implies

$$N_k^{(s)}(n) \geqslant n! \cdot \Omega\left(\frac{m!}{(2 \cdot k!)^m}\right). \tag{3}$$

Recall that $m = 2n/(2+ks)$. In particular, m is proportional to n. Hence $m!/(2 \cdot k!)^m$ grows faster than c^n for any fixed constant $c > 1$. Thus the graph class $\mathrm{subd}_s(\mathcal{G}_k)$ is not small. ◀

Grid graphs. The *d-dimensional n-grid* P_n^d is the graph with vertex set $V(P_n^d) = [n]^d$, and $\{u, v\} \in E(P_n^d)$ for two vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ if and only if $\sum_{i=1}^d |u_i - v_i| = 1$. The next result states that $\mathrm{tww}(P_n^d) \leqslant 3d$ irrespective of the value of n.

▶ **Theorem 8** (Theorem 4 in [11]). *For every positive d and n, the d-dimensional n-grid P_n^d has twin-width at most $3d$.*

The *d-dimensional n-grid with diagonals* $D_{n,d}$ is the graph with $V(D_{n,d}) = [n]^d$, and $\{u, v\} \in E(D_{n,d})$ for two vertices $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ if and only if $\max_{i=1}^d |u_i - v_i| \leqslant 1$. Now, for a given trigraph $G = (V, E, R)$ we set $\mathrm{red}(G) = (V, \emptyset, E \cup R)$. In words, $\mathrm{red}(G)$ is the trigraph obtained from G by replacing every black edge of G by a red edge between the same vertices. With this notation $\mathrm{red}(D_{n,d})$ is the *d-dimensional **red** n-grid with diagonals*, i.e., $\mathrm{red}(D_{n,d}) = ([n]^d, \emptyset, E(D_{n,d}))$. Clearly, $\mathrm{tww}(D_{n,d}) \leqslant \mathrm{tww}(\mathrm{red}(D_{n,d}))$.

▶ **Theorem 9** (Lemma 4.4 in [11]). *For every positive d and n, every subtrigraph of the d-dimensional red n-grid with diagonals $\mathrm{red}(D_{n,d})$ has twin-width at most $2(3^d - 1)$.*

2.2 Background in topology

For general background in (combinatorial and differential) topology we refer to [43].

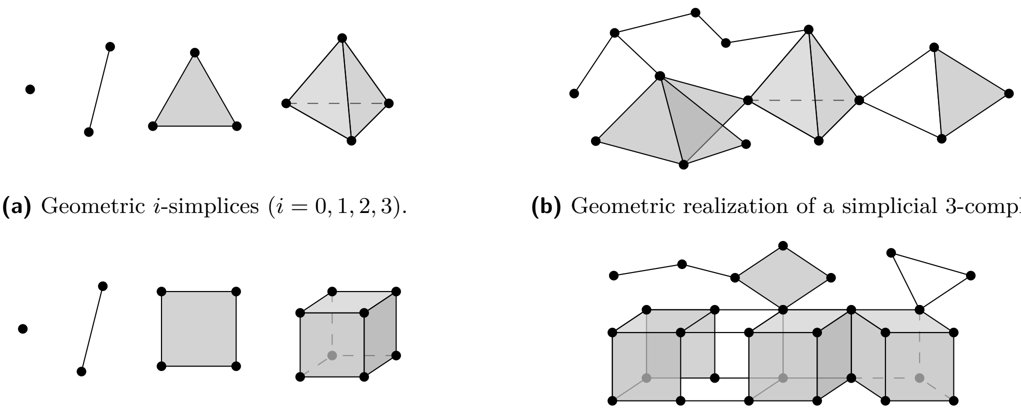
2.2.1 Simplicial and cubical complexes

Abstract simplicial complexes. Given a finite ground set $\mathcal{S}$, an *abstract simplicial complex* (or *simplicial complex*, for short) $\mathcal{X}$ over $\mathcal{S}$ is a downward closed subset of the power set $2^{\mathcal{S}}$, i.e., $\mathcal{F} \in \mathcal{X}$ and $\mathcal{F}' \subset \mathcal{F}$ imply $\mathcal{F}' \in \mathcal{X}$. Any element of $\mathcal{X}$ is called a *face* or *simplex* of $\mathcal{X}$, and for $\sigma \in \mathcal{X}$ the *dimension of* σ is defined as $\dim \sigma = |\sigma| - 1$. The *dimension of* $\mathcal{X}$, denoted by $\dim \mathcal{X}$, is then the maximum dimension of a face of $\mathcal{X}$. If $\dim \mathcal{X} = d$, we also say that $\mathcal{X}$ is a *simplicial d-complex*. For $0 \leqslant i \leqslant \dim \mathcal{X}$, we let $\mathcal{X}(i) = \{\sigma \in \mathcal{X} : \dim \sigma = i\}$ denote the set of i-dimensional faces (or i-faces, or i-simplices) of $\mathcal{X}$. The *i-skeleton* $\mathcal{X}_i = \bigcup_{j=0}^i \mathcal{X}(j)$ of $\mathcal{X}$ is the union of all faces of $\mathcal{X}$ up to dimension i. Note that any simple graph $G = (V, E)$ can be regarded as a 1-dimensional simplicial complex $\mathcal{X}_G$ with $\mathcal{X}_G(0) = \{\{v\} : v \in V\}$ and $\mathcal{X}_G(1) = E$. For $0 \leqslant i \leqslant 3$ the i-simplices of a simplicial complex $\mathcal{X}$ are respectively called the *vertices*, *edges*, *triangles*, and *tetrahedra* of $\mathcal{X}$.

Geometric realization of simplicial complexes. Every abstract simplicial complex $\mathcal{X}$ may be realized geometrically as follows. To each abstract i-simplex $\sigma = \{v_0, v_1, \dots, v_i\} \in \mathcal{X}$ we associate a *geometric i-simplex* $\|\sigma\| = [v_0, v_1, \dots, v_i] \subset \mathbb{R}^i$ defined as the convex hull of $i+1$ affinely independent points in $\mathbb{R}^i$. We equip $\|\sigma\|$ with the subspace topology inherited from $\mathbb{R}^i$. Next, we consider the disjoint union $\bigsqcup_{\sigma \in \mathcal{X}} \|\sigma\|$ of these geometric simplices, and perform identifications along their faces that reflect their relationship in $\mathcal{X}$. The resulting space $\|\mathcal{X}\|$, equipped with the quotient topology, is called the *geometric realization of* $\mathcal{X}$, see Figure 1. The geometric realization of a simplicial complex is unique up to homeomorphism.

Cubical complexes. Analogous to simplicial complexes, a *cubical complex* $\mathcal{X}$ over a ground set $\mathcal{S}$ is a set system $\mathcal{X} \subset 2^{\mathcal{S}}$ that consists of "cubes" instead of simplices. The terminology is the same as in the simplicial case. The only difference is that a *geometric i-cube* is a topological space homeomorphic to $[0,1]^i$, where $[0,1]$ denotes the closed unit interval.

Cubical or simplicial complexes in this paper will typically be defined geometrically, and as such they will naturally come with a geometric realization.



(a) Geometric i-simplices ($i = 0, 1, 2, 3$).

(b) Geometric realization of a simplicial 3-complex.

(c) Geometric i-cubes ($i = 0, 1, 2, 3$).

(d) Geometric realization of a cubical 3-complex.

Figure 1 The geometric perspective on simplicial and cubical complexes.

The hypercubic honeycomb. Let n and d be positive integers and consider the d-dimensional cube $[1,n]^d \subset \mathbb{R}^d$. The *$d$-dimensional hypercubic honeycomb* $\mathbf{H}^{d,n}$ is a cubical d-complex that decomposes $[1,n]^d$ into $(n-1)^d$ geometric cubes. The properties of this familiar object play an important role in this work, so we describe it for completeness. We define $\mathbf{H}^{d,n}$ geometrically, in a bottom-up fashion. First, the vertex set $\mathbf{H}^{d,n}(0)$ consists of precisely those points in $[1,n]^d$, which have only integral coordinates. Next, a 1-cube (i.e., an edge) is attached along its endpoints to vertices $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in $\mathbf{H}^{d,n}(0)$ if and only if $\sum_{i=1}^{d} |u_i - v_i| = 1$. Thus the 1-skeleton $\mathbf{H}^{d,n}_1$ is just the d-dimensional grid graph P_n^d encountered in the end of Section 2.1. Finally, the higher dimensional skeleta of $\mathbf{H}^{d,n}$ are *induced* by its 1-skeleton: for each subcomplex $\mathcal{Y} \subset \mathbf{H}^{d,n}_i$ isomorphic to the boundary of a $(i+1)$-cube, we attach an $(i+1)$-cube to $\mathbf{H}^{d,n}_i$ along $\mathcal{Y}$. In words, starting from the 1-skeleton $\mathbf{H}^{d,n}_1$, whenever we have the possibility to attach a cube of dimension at least two (because its boundary is already present), we attach it. See, e.g., Figure 2.

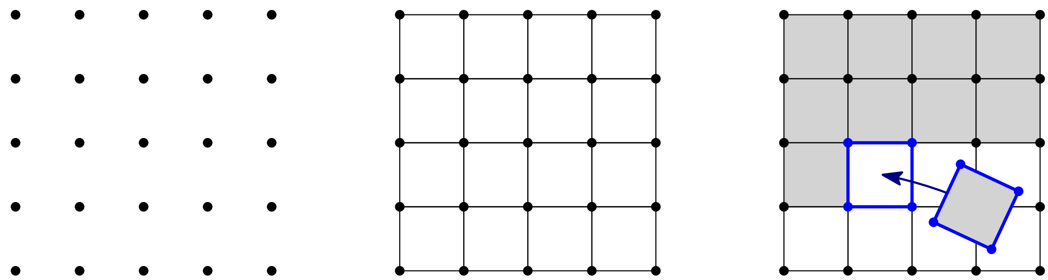
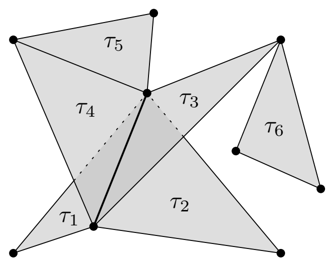
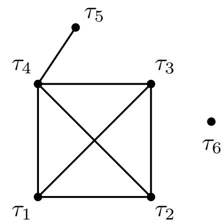


Figure 2 Constructing the complex $\mathbf{H}^{2,5}$. Its 1-skeleton $\mathbf{H}^{2,5}_1$ is isomorphic to the grid P_5^2.

Pure complexes and their dual graphs. A (cubical or simplicial) complex $\mathcal{X}$ is *pure* if every face of $\mathcal{X}$ is contained in a face of dimension $\dim \mathcal{X}$. It follows that, for every i with $0 \leqslant i \leqslant \dim \mathcal{X}$, the i-skeleton $\mathcal{X}_i$ of a pure complex $\mathcal{X}$ is also pure. Examples of pure complexes include nonempty graphs without isolated vertices, or triangulations of manifolds (see Section 2.2.3). Given a pure complex $\mathcal{X}$, the *dual graph* $\Gamma(\mathcal{X}_i) = (V, E)$ of its i-skeleton is defined as the graph, where the vertex set V corresponds to the set $\mathcal{X}(i)$ of i-faces, and $\{\sigma, \tau\} \in E$ if and only if σ and τ share an $(i-1)$-dimensional face in $\mathcal{X}$, see Figure 3.



(a) A pure simplicial 2-complex $\mathcal{X}$ formed by six triangles τ_1, τ_2, τ_3, τ_4, τ_5, and τ_6, four of which meet along a single edge.



(b) The dual graph $\Gamma(\mathcal{X}_2)$ of $\mathcal{X}_2$. As τ_6 shares no edge with any other triangle in $\mathcal{X}$, its corresponding vertex is isolated.

Figure 3 Example of a pure simplicial 2-complex and its dual graph.

2.2.2 Barycentric subdivisions

Given a (cubical or simplicial) complex $\mathcal{X}$, its *barycentric subdivision* is a **simplicial** complex $\mathcal{X}'$ defined abstractly as follows. For the ground set $\mathcal{S}'$ of $\mathcal{X}'$ we have $\mathcal{S}' = \mathcal{X}$. A $(k+1)$-tuple $\{\sigma_0, \dots, \sigma_k\} \subset \mathcal{S}'$ forms a k-simplex of $\mathcal{X}'$ if and only if $\sigma_i \subset \sigma_j$ for every $0 \leqslant i < j \leqslant k$. We denote the 2^{nd} (resp. ℓ^{th}) iterated barycentric subdivision of a complex $\mathcal{X}$ by $\mathcal{X}''$ (resp. $\mathcal{X}^{(\ell)}$). See [10, Appendix A] for examples. The following are simple consequences of the definitions.

▶ **Observation 10.** *For any $0 \leqslant l \leqslant k$, a k-simplex has $\binom{k}{l}$ l-faces.*

▶ **Observation 11.** *The barycentric subdivision of a k-simplex contains $(k+1)!$ k-simplices.*

▶ **Observation 12.** *The barycentric subdivision of a k-cube contains $2^k k!$ k-simplices.*

See [10, Appendix A] for a proof of Observation 12.

2.2.3 Manifolds and their triangulations

Manifolds can be regarded as higher dimensional analogs of surfaces. A *d-dimensional topological manifold with boundary* (or *d-manifold*, for short) is a topological space[15] $\mathcal{M}$, where every point has an open neighborhood homeomorphic to $\mathbb{R}^d$, or to the closed upper half-space $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \geqslant 0\}$. The points of $\mathcal{M}$ that do **not** have a neighborhood homeomorphic to $\mathbb{R}^d$ constitute the *boundary* $\partial\mathcal{M}$ of $\mathcal{M}$. If a manifold $\mathcal{M}$ satisfies $\partial\mathcal{M} = \emptyset$, then $\mathcal{M}$ is called a *closed* manifold. In this paper $\mathcal{M}$ always denotes a compact manifold.

Smooth manifolds. The main result of this paper (Theorem 1) applies for manifolds that have an additional property, namely *smoothness*. As we will not need to work with the actual definition of smoothness, but merely rely on it, we only give a brief definition here. For more background on smooth manifolds, we refer to [43, Chapter 5] and [33].

Given a connected and open subset $U \subset \mathcal{M}$ and a homeomorphism $\varphi\colon U \to \varphi(U)$ onto an open subset of $\mathbb{R}^d$, the pair (U, φ) is called a *chart*. Given two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the map $\tau_{\alpha,\beta}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ defined via $\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is called a *transition map*. A *smooth structure* on a topological manifold $\mathcal{M}$ with $\partial\mathcal{M} = \emptyset$ is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts that satisfies the following three properties.

1. The sets U_α cover $\mathcal{M}$, that is $\bigcup_{\alpha \in A} U_\alpha = \mathcal{M}$.
2. For any $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\tau_{\alpha,\beta}$ is smooth.[16]
3. The collection $\mathcal{A}$ is maximal in the sense that if (U, φ) is a chart and for every $\alpha \in A$ with $U \cap U_\alpha \neq \emptyset$ the transition maps $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are smooth, then $(U, \varphi) \in \mathcal{A}$.

A topological manifold together with a smooth structure is called a *smooth manifold*. By an appropriate modification of property **2** above, the definition extends to manifolds with non-empty boundary as well. We refer to [43, Section 5.1.1] for details.

Triangulations. Let $\mathcal{M}$ be a compact topological d-manifold. A simplicial complex $\mathcal{T}$ whose geometric realization $\|\mathcal{T}\|$ is homeomorphic to $\mathcal{M}$ is called a *triangulation* of $\mathcal{M}$. It follows that $\mathcal{T}$ is a pure simplicial complex of dimension d (see Section 2.2.1). For $d \leqslant 3$, every topological d-manifold admits a triangulation [41, 44], however, for $d > 3$ this is generally not true (see [38] for an overview). Smooth manifolds can nevertheless always be triangulated, irrespective of their dimension, e.g., by work of Whitney [48, Chapter IV.B] (cf. Section 3).

Given a d-dimensional triangulation $\mathcal{T}$, recall that its *dual graph* $\Gamma(\mathcal{T})$ is the graph with vertices corresponding to the d-simplices of $\mathcal{T}$, and edges to the face gluings, i.e, those $(d-1)$-simplices of $\mathcal{T}$ that are contained in precisely two d-simplices. Note that $\deg(v) \leqslant d+1$ for any vertex v of $\Gamma(\mathcal{T})$. The following proposition is a consequence of the definitions.

▶ **Proposition 13.** *Let $\mathcal{T}$ be a triangulation of a d-manifold $\mathcal{M}$ and $\mathcal{U}$ be a collection of d-simplices of $\mathcal{T}$ that define a submanifold of $\mathcal{M}$. Then $\Gamma(\mathcal{U})$ is an induced subgraph of $\Gamma(\mathcal{T})$.*

3 Whitney's method

A seminal result of Whitney states that a smooth d-dimensional manifold $\mathcal{M}$ always admits a smooth embedding into a $2d$-dimensional Euclidean space.

[15] As a topological space a manifold is required to be *second countable* [43, p. 2] and *Hausdorff* [43, p. 87].
[16] See [43, p. 185] for a discussion of (smooth) maps of manifolds.

▶ **Theorem 14** (strong Whitney embedding theorem [47, Theorem 5], cf. [33, Theorem 6.19]). *For $d > 0$, every smooth d-manifold admits a smooth embedding into $\mathbb{R}^{2d}$.*

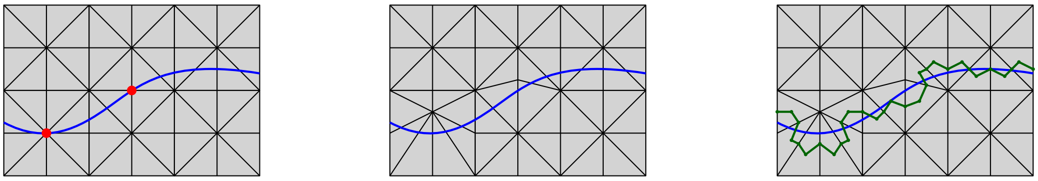
An important consequence of Theorem 14 is that smooth manifolds can be triangulated.

▶ **Theorem 15** (triangulation theorem [48, Chapter IV.B], cf. [5, Theorem 1.1]). *Every compact, smooth d-manifold $\mathcal{M}$ embedded in some Euclidean space $\mathbb{R}^m$ admits a triangulation.*

Next, we give a very brief and high-level overview of Whitney's method of triangulating smooth manifolds based on [48, Chapter IV.B] sufficient for our purposes. To the interested reader we also recommend [5], where Whitney's method is recast in a computational setting.

Triangulating smooth manifolds. Let $\mathcal{M}$ be a compact smooth d-manifold. By Theorem 14 there exists a smooth embedding $\iota\colon \mathcal{M} \to \mathbb{R}^{2d}$. Given such an embedding, we choose a sufficiently fine (with respect to ι) hypercubic honeycomb decomposition of $\mathbb{R}^{2d}$, denoted by L_0. Next, we pass to the first barycentric subdivision L of L_0. (Geometrically, L is obtained from L_0 by subdividing each k-dimensional hypercube of L_0 into $(2k)!!$ simplices.) By slightly perturbing the vertices of L we obtain a triangulation L^* of $\mathbb{R}^{2d}$, which is combinatorially isomorphic to L, but is in *general position* with respect to $\iota(\mathcal{M}) \subset \mathbb{R}^{2d}$. Now, by the work of Whitney, L^* induces a triangulation $\mathcal{T}$ of $\mathcal{M}$, where, importantly, $\mathcal{T}$ is a subcomplex of the d-skeleton of the barycentric subdivision $(L^*)'$ of L^*.

See Figure 4 for an illustration of this procedure for $d = 1$.



(a) The triangulation L and the image $\iota(\mathcal{M})$ (blue). The red points indicate the vertices of L contained by $\iota(\mathcal{M})$.

(b) The perturbed triangulation L^*, which in general position with respect to $\iota(\mathcal{M})$.

(c) The resulting triangulation $\mathcal{T}$ of $\mathcal{M}$ (dark green). $\mathcal{T}$ is a subcomplex of $(L^*)'$.

Figure 4 Illustration of Whitney's method of triangulating ambient submanifolds ($d = 1$).

Similar to Proposition 13, the next proposition is a direct consequence of the definitions.

▶ **Proposition 16.** *For any Whitney triangulation $\mathcal{T}$ of a compact smooth d-manifold $\mathcal{M}$, we have that the dual graph $\Gamma(\mathcal{T})$ is an induced subgraph of $\Gamma(((\mathbf{H}^{2d,n})'')_d)$.*

4 The proof of Theorem 1

In this section we prove our main result, i.e., every compact smooth d-manifold $\mathcal{M}$ has twin-width $\operatorname{tww}(\mathcal{M}) \leqslant d^{O(d)}$. To streamline the notation, we let $G_{d,n} = \Gamma(((\mathbf{H}^{2d,n})'')_d)$, i.e., $G_{d,n}$ denotes the dual graph of the d-skeleton[17] of the second barycentric subdivision of the hybercubic honeycomb $\mathbf{H}^{2d,n}$. The result is based on the following property of $G_{d,n}$.

[17] Recall that the dual graph $\Gamma(\mathcal{X})$ of a pure k-dimensional complex $\mathcal{X}$ has vertices corresponding to the k-faces of $\mathcal{X}$ and two vertices are connected if and only if their corresponding k-faces share a $(k-1)$-face.

▶ **Theorem 17.** *For the twin-width of the dual graph $G_{d,n} = \Gamma(((\mathbf{H}^{2d,n})'')_d)$ of the d-skeleton of the second barycentric subdivision of the hypercubic honeycomb $\mathbf{H}^{2d,n}$ we have*

$$\mathrm{tww}(G_{d,n}) \leqslant d^{O(d)}.$$

Before proving Theorem 17, we show how it implies Theorem 1.

Proof of Theorem 1. Let $\mathcal{M}$ be a compact, smooth d-dimensional manifold. Consider a Whitney triangulation $\mathcal{T}$ of $\mathcal{M}$. By Proposition 16, $\Gamma(\mathcal{T})$ is an induced subgraph of $G_{d,n}$ and by Theorem 17, $\mathrm{tww}(G_{d,n}) \leqslant d^{O(d)}$. Hence, since twin-width is monotone under taking induced subtrigraphs (Proposition 5), we obtain $\mathrm{tww}(\Gamma(\mathcal{T}))) \leqslant \mathrm{tww}(G_{d,n}) \leqslant d^{O(d)}$. ◀

To complete the proof of Theorem 1, it remains to show Theorem 17.

Proof of Theorem 17. We establish the theorem by exhibiting a $d^{O(d)}$-contraction sequence $\mathfrak{S}\colon G_{d,n} \rightsquigarrow \bullet$. We will obtain $\mathfrak{S}$ by concatenating two contraction sequences $\mathfrak{S}_1\colon G_{d,n} \rightsquigarrow G^*_{d,n}$ and $\mathfrak{S}_2\colon G^*_{d,n} \rightsquigarrow \bullet$, referred to as the *first* and the *second epoch*, where $G^*_{d,n}$ is an appropriate subtrigraph of $\mathrm{red}(D_{n,2d})$, the $2d$-dimensional red n-grid with diagonals. In the following we regard $\mathbf{H}^{2d,n}$ and its subdivisions mainly as abstract complexes, but we will also take advantage of their geometric nature.

Preparations. Consider the $(2d+1)$-coloring $\mathcal{C}\colon \mathbf{H}^{2d,n} \to \{0,\dots,2d\}$, where we color the cubes of $\mathbf{H}^{2d,n}$ by their dimension, that is, for $c \in \mathbf{H}^{2d,n}$ we set $\mathcal{C}(c) = \dim(c)$ (Figure 5a).

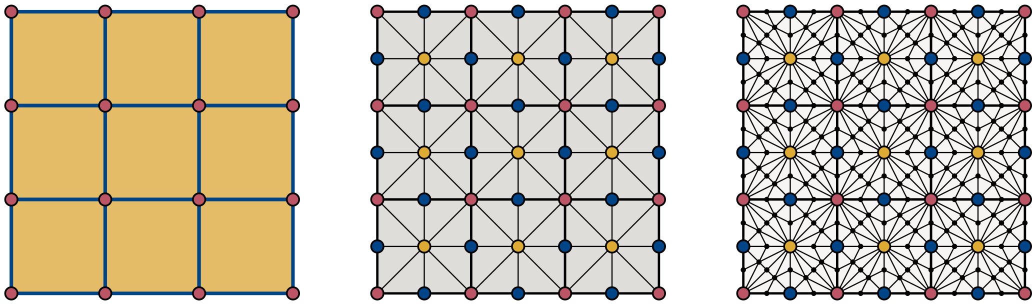
The coloring $\mathcal{C}$ induces a $(2d+1)$-coloring $\mathcal{C}''\colon (\mathbf{H}^{2d,n})'' \to \{0,\dots,2d\}$ of the simplices of the second barycentric subdivision $(\mathbf{H}^{2d,n})''$ as follows. The vertices of the first barycentric subdivision $(\mathbf{H}^{2d,n})'$ are in one-to-one correspondence with the cubes in $\mathbf{H}^{2d,n}$, hence $\mathcal{C}$ induces a coloring $\mathcal{C}'_0\colon (\mathbf{H}^{2d,n})'(0) \to \{0,\dots,2d\}$ via $\mathcal{C}'_0(v_c) = \mathcal{C}(c)$, where v_c denotes the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to the cube $c \in \mathbf{H}^{2d,n}$ (Figure 5b). Geometrically, v_c is in the barycenter of the cube c. When we pass to the second barycentric subdivision, the vertices of $(\mathbf{H}^{2d,n})'$ become vertices of $(\mathbf{H}^{2d,n})''$, thus there is a natural inclusion $\iota\colon (\mathbf{H}^{2d,n})'(0) \to (\mathbf{H}^{2d,n})''(0)$. Let $\mathcal{V} = \mathrm{im}(\iota) \subset (\mathbf{H}^{2d,n})''(0)$ be the image of $(\mathbf{H}^{2d,n})'(0)$ under this inclusion ι. We color the elements of $\mathcal{V}$ identically to $\mathcal{C}'_0$, that is, we consider the coloring $\mathcal{C}''_{\mathcal{V}}\colon \mathcal{V} \to \{0,\dots,2d\}$ defined as $\mathcal{C}''_{\mathcal{V}}(v) = \mathcal{C}'_0(\iota^{-1}(v))$, see Figure 5c. Now, pick a simplex $\sigma \in (\mathbf{H}^{2d,n})''$ and note that $\bigcup_{v\in\mathcal{V}} \overline{\mathrm{st}}(v) = (\mathbf{H}^{2d,n})''$, i.e., the closed stars of the vertices $v \in \mathcal{V}$ cover $(\mathbf{H}^{2d,n})''$. Let $\mathcal{V}_\sigma = \{v \in \mathcal{V} : \sigma \in \overline{\mathrm{st}}(v)\}$. We now define $\mathcal{C}''(\sigma)$ as

$$\mathcal{C}''(\sigma) = \min\{\mathcal{C}''_{\mathcal{V}}(v) : v \in \mathcal{V}_\sigma\}. \tag{4}$$

In words, $\mathcal{C}''(\sigma)$ is defined as the smallest dimension of any cube $c \in \mathbf{H}^{2d,n}$ such that σ belongs to the closed star of the vertex $\iota(v_c)$ in $(\mathbf{H}^{2d,n})''$, where v_c is the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to c. We refer to Figure 5d for an example.

The first epoch. We start the description of the *first epoch* $\mathfrak{S}_1\colon G_{d,n} \rightsquigarrow G^*_{d,n}$ by considering the restriction $\mathcal{C}''_d\colon (\mathbf{H}^{2d,n})''(d) \to \{0,\dots,2d\}$ of the coloring $\mathcal{C}''$ to the d-simplices of $(\mathbf{H}^{2d,n})''$, see Figure 5e. Note that for each $i \in \{0,\dots,2d\}$, the i-colored d-simplices of $(\mathbf{H}^{2d,n})''$ form a family $\mathcal{F}_i = \{F_{i,c} : c \in \mathbf{H}^{2d,n}(i)\}$ of pairwise-disjoint connected subcomplexes of the d-skeleton of $(\mathbf{H}^{2d,n})''$, where $F_{i,c}$ denotes the subcomplex induced by the i-colored d-simplices of $(\mathbf{H}^{2d,n})''$ belonging to the closed star of the vertex $\iota(v_c)$, where v_c is the vertex of $(\mathbf{H}^{2d,n})'$ corresponding to the i-cube c in $\mathbf{H}^{2d,n}$ (Figure 5e). We also let $\mathcal{F} = \bigcup_{i=0}^{2d} \mathcal{F}_i$.

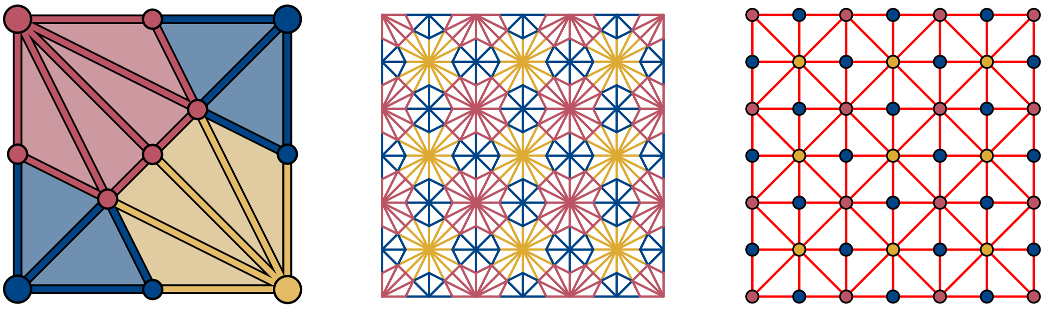
Since $G_{d,n}$ is defined as the dual graph of the d-skeleton of $(\mathbf{H}^{2d,n})''$, the nodes of $G_{d,n}$ are in one-to-one correspondence with the d-simplices of $(\mathbf{H}^{2d,n})''$. Let $\gamma\colon (\mathbf{H}^{2d,n})''(d) \to V(G_{d,n})$ denote the bijection realizing this correspondence. Henceforth, by a slight abuse of notation,



(a) The coloring $\mathcal{C}$ of the cubes of $\mathbf{H}^{2d,n}$ by their dimension.

(b) The complex $(\mathbf{H}^{2d,n})'$ with its vertices colored by $\mathcal{C}'_0$.

(c) The complex $(\mathbf{H}^{2d,n})''$ with its vertices in $\mathcal{V}$ colored by $\mathcal{C}''_{\mathcal{V}}$.



(d) The coloring $\mathcal{C}''$ near the upper left corner of $(\mathbf{H}^{2d,n})''$. The four larger disks represent vertices that belong to $\mathcal{V}$.

(e) The coloring $\mathcal{C}''$ restricted to the d-simplices of $(\mathbf{H}^{2d,n})''$. The color classes correspond to the families $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

(f) The trigraph $G^*_{d,n}$ obtained from $G_{d,n}$ by contracting each subtrigraph $C_{i,c}$ to a single node. All edges in $G^*_{d,n}$ are red.

Figure 5 (a)–(c) Illustrations of the cubical complex $\mathbf{H}^{2d,n}$ for $d = 1$ and $n = 4$, and of its first and second barycentric subdivisions (which are simplicial complexes) displaying the colorings described in the proof of Theorem 17. **(d)–(f)** Three essential steps in the proof of Theorem 17.

we also consider $\mathcal{C}''_d$ to be a $(2d+1)$-coloring of $V(G_{d,n})$ via $\mathcal{C}''_d(v) = \mathcal{C}''_d(\gamma^{-1}(v))$. Let $C_{i,c}$ denote the subtrigraph of $G_{d,n}$ induced by the nodes $\{\gamma(\sigma) : \sigma \in F_{i,c}\}$. Note that $\mathcal{C}''_d$ assigns the color i to all nodes of $C_{i,c}$. The first epoch $\mathfrak{S}_1 : G_{d,n} \rightsquigarrow G^*_{d,n}$ is merely the contraction sequence, where we contract each $C_{i,c}$ (where $0 \leqslant i \leqslant 2d$ and c is running over $\mathbf{H}^{2d,n}$) to a single node, in any order, obtaining a trigraph $G^*_{d,n}$ (Figure 5f).

The second epoch. $\mathfrak{S}_2 : G^*_{d,n} \rightsquigarrow \bullet$ is defined as an optimal contraction sequence of the trigraph $G^*_{d,n}$ to a single vertex. Since $G^*_{d,n}$ is a subtrigraph of $\mathrm{red}(D_{n,2d})$, the *$2d$-dimensional red n-grid with diagonals*, by Theorem 9 the width of $\mathfrak{S}_2$ is at most $2(3^{2d} - 1)$.

Bounding the width of the first epoch. We now give a rough estimate to show that the first epoch $\mathfrak{S}_1 : G_{d,n} \rightsquigarrow G^*_{d,n}$ is a $d^{O(d)}$-contraction sequence. The estimate is based on Claim 18 below, which follows from elementary properties of the hypercubic honeycomb and barycentric subdivisions.

▷ Claim 18. For any color $i \in \{0, \dots, 2d\}$ and cube $c \in \mathbf{H}^{2d,n}$, the subcomplex $F_{i,c}$ of $\mathbf{H}^{2d,n}$ (resp. the subtrigraph $C_{i,c}$ of $G_{d,n}$) defined above

1. has less than $h_1(d) = 4^d((2d)!)^2\binom{2d}{d}$ d-simplices (resp. nodes), and
2. less than $h_2(d) = 9^d$ incident subcomplexes $F_{i',c'} \in \mathcal{F}$ (resp. adjacent subtrigraphs $C_{i',c'}$).

Indeed, these two facts imply that by sequentially contracting each $C_{i,c}$ into a single node, the maximum red degree remains bounded by $O(h_1(d)^3 h_2(d))$ throughout the first epoch.

Proof of Claim 18. To bound the number of d-simplices in $F_{i,c}$, note that

- $F_{i,c}$ is covered by an appropriate translate of a $2d$-dimensional cube of $\mathbf{H}^{2d}$,
- the barycentric subdivision of a $2d$-cube contains $2^{2d}(2d)!$ $2d$-simplices (Observation 12),
- the barycentric subdivision of a $2d$-simplex contains $(2d)!$ $2d$-simplices (Observation 11),
- a $2d$-simplex has $\binom{2d}{d}$ faces of dimension d (Observation 10).

Multiplying these numbers, we obtain the first part of the claim.

To bound the number of subcomplexes $F_{i',c'} \in \mathcal{F}$ incident to $F_{i,c}$, note that the incidences between these subcomplexes reflect those between the handles in the canonical handle decomposition of $\mathbf{H}^{2d,n}$ obtained by "thickening up" its cubical cells. Thus, the number of subcomplexes in $\mathcal{F}$ incident to $F_{i,c}$ equals $3^{2d} - 1$ if $i \in \{0, 2d\}$ and $3^i + 3^{2d-i} - 2$ otherwise. Both of these numbers are less than 9^d, so the second part of the claim also holds. ◁

The width of $\mathfrak{S}$ is the maximum of the widths of $\mathfrak{S}_1$ and $\mathfrak{S}_2$, hence $\operatorname{tww}(G_{d,n}) \leqslant d^{O(d)}$. ◀

5 Triangulations with dual graph of arbitrary large twin-width

In Section 4 we showed that every compact, smooth d-manifold admits a triangulation with dual graph of twin-width at most $d^{O(d)}$. In this section we prove complementary results, showing that triangulations with dual graphs of large twin-width are abundant. We shed light on this fact in two ways. First, we show that for any fixed dimension $d \geqslant 3$, the class of $(d+1)$-regular graphs that can be dual graphs of triangulated d-manifolds is *not* small. Second, we show that the d-dimensional ball admits triangulations with a dual graph of arbitrarily large twin-width, which extends to every piecewise-linear (hence smooth) d-manifold. Both of these results rely on counting arguments, and thus are not constructive.

5.1 The class of dual graphs of triangulations is not small

Let us fix an integer $d \geqslant 3$. Following [21], we let $M_d(n)$ denote the number of colored[18] triangulations of closed orientable d-dimensional manifolds consisting of n d-simplices labeled from 1 to n. We assume n to be even. By [21, Theorem 1.1] we have[19]

$$n! \cdot n^{n/(2d)} \preceq M_d(n). \tag{5}$$

Ignoring the colors gives an uncolored triangulation, but any such d-dimensional triangulation can be obtained from $(d+1)!$ colored triangulations, since the $d+1$ colors can always be permuted. Now, any $(d+1)$-regular graph G with n vertices can be the dual graph of at most $d!^{n(d+1)/2}$ different d-dimensional triangulations. Indeed, if G is the dual graph of some d-dimensional triangulation, then each of the $n(d+1)/2$ edges of G corresponds to a face gluing, i.e., an identification of two $(d-1)$-dimensional simplices via a simplicial isomorphism, of which there are $d!$ many.[20] Hence by (5), for the number $\Gamma_d(n)$ of $(d+1)$-regular graphs on n labeled vertices that are dual graphs of some d-dimensional triangulations, we have

$$\frac{n! \cdot n^{n/(2d)}}{(d+1)! \cdot d!^{n(d+1)/2}} \preceq \Gamma_d(n). \tag{6}$$

As the left-hand side of (6) grows super-exponentially in n, the next theorem directly follows.

[18] We refer to [21, Section 2.1] for the precise definitions.

[19] Here "$\preceq$" denotes a comparison where exponential factors are ignored: more precisely, $f(n) \preceq g(n)$ means that there exists some constant $K > 0$, such that for n large enough, we have $f(n) \leqslant K^n g(n)$.

[20] This is because a simplicial isomorphism between two $(d-1)$-simplices σ and τ is determined by a perfect matching between the d vertices of σ and the d vertices of τ.

▶ **Theorem 19.** *For every $d \geqslant 3$, the class of $(d+1)$-regular graphs that are dual graphs of triangulations of d-manifolds is not small. In particular, there are graphs with arbitrarily large twin-width in this class.*

5.2 Complicated triangulations of the d-dimensional ball

In this section we show Theorem 3, according to which every piecewise-linear (PL) d-manifold ($d \geqslant 3$) admits triangulations with a dual graph of arbitrary large twin-width. To show the existence of such triangulations for every PL-manifold, we rely on the monotonicity of twin-width with respect to taking induced subtrigraphs (Proposition 5), the fact that the class of d-subdivisions of $(d+1)$-regular graphs is *not* small (Proposition 7) together with the following classical result from the theory of PL-manifolds. See [45] for an introduction.

▶ **Theorem 20** ([1, Corollary 1])**.** *Any triangulation of the boundary of a compact piecewise-linear (PL) manifold can be extended to a triangulation of the whole manifold.*

We first show that already the d-dimensional ball $\mathcal{B}^d = \{x \in \mathbb{R}^d : \|x\| \leqslant 1\}$ admits triangulations with dual graph of arbitrary large twin-width. More precisely, we prove:

▶ **Theorem 21.** *For every $m \in \mathbb{N}$ there is a triangulation $\mathcal{T}_m$ of the d-dimensional ball $\mathcal{B}^d$, such that $\operatorname{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$ and $\partial\mathcal{T}_m = \partial\Delta^d$, the boundary of the standard d-simplex Δ^d.*

Proof. Let G_m be a d-subdivision of a $(d+1)$-regular graph G such that $\operatorname{tww}(G_m) \geqslant m$. The existence of such a graph is guaranteed by Proposition 7. Let $\mathcal{N}$ be a d-manifold homeomorphic to a closed regular neighborhood of a straight-line embedding of G in $\mathbb{R}^d$. Informally, $\mathcal{N}$ can be seen as a d-dimensional thickening of the graph G.

Construct an abstract triangulation of $\mathcal{N}$ as follows. Take a d-simplex σ_v for each node v of G, and fix a one-to-one correspondence between the $d+1$ facets of σ_v and the $d+1$ arcs incident to v in G. For every arc $\{u,v\} \in E(G)$, take a simplicial d-prism $P_{\{u,v\}} = \sigma_{\{u,v\}} \times [0,1]$, where $\sigma_{\{u,v\}}$ is a $(d-1)$-simplex, and attach $P_{\{u,v\}}$ to the simplices σ_u and σ_v by identifying $\sigma_{\{u,v\}} \times \{0\}$ (resp. $\sigma_{\{u,v\}} \times \{1\}$) with the facet of σ_u (resp. σ_v) that corresponds to the arc $\{u,v\}$. Now triangulate each prism $P_{\{u,v\}}$ with a minimal triangulation consisting of d d-simplices stacked onto each other, see [10, Appendix B].

Let $\mathcal{T}_\mathcal{N}$ denote the resulting triangulation of $\mathcal{N}$.

▷ Claim 22. For the dual graph of the triangulation $\mathcal{T}_\mathcal{N}$ we have $\Gamma(\mathcal{T}_\mathcal{N}) = G_m$.

Proof. The claim follows immediately from the facts that the dual graph of the constructed triangulation of the simplicial d-prism is merely a path of length d, and G_m is the d-subdivision of the $(d+1)$-regular graph G on which the triangulation $\mathcal{T}_\mathcal{N}$ is modeled. ◁

Pick a sufficiently large $\ell \in \mathbb{N}$ such that the ℓ^{th} iterated barycentric subdivision $\mathcal{T}_\mathcal{N}^{(\ell)}$ of $\mathcal{T}_\mathcal{N}$ embeds linearly in $\mathbb{R}^d$ and fix a simplex-wise linear embedding $\mathcal{E}\colon \|\mathcal{T}_\mathcal{N}^{(\ell)}\| \to \mathbb{R}^d$. Consider a large geometric d-simplex $\Sigma \subset \mathbb{R}^d$ that contains the image $\operatorname{im}(\mathcal{E})$ of $\mathcal{E}$ in its interior. Now $\Sigma_\circ = \Sigma \setminus \operatorname{int}(\operatorname{im}(\mathcal{E}))$ is a PL manifold[21] with triangulated boundary, so by Theorem 20 this boundary triangulation can be extended to a triangulation $\mathcal{T}_{\Sigma_\circ}$ of the entire manifold $\Sigma_\circ$. The boundary of $\mathcal{T}_{\Sigma_\circ}$ has two connected components: $\partial_1\mathcal{T}_{\Sigma_\circ} \cong \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$ and $\partial_2\mathcal{T}_{\Sigma_\circ} \cong \partial\Delta^d$.

Let $\mathcal{Q}$ be a d-manifold homeomorphic to $\partial\mathcal{T}_\mathcal{N} \times [0,1]$. Take a triangulation $\mathcal{T}_\mathcal{Q}$ of $\mathcal{Q}$, such that for the two boundary components we have $\partial_1\mathcal{T}_\mathcal{Q} \cong \partial\mathcal{T}_\mathcal{N}$ and $\partial_2\mathcal{T}_\mathcal{Q} \cong \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$. One way to construct such a triangulation is as follows. First, consider the decomposition $\mathcal{P}_\mathcal{Q}$ of $\mathcal{Q}$

[21] It is folklore that every codimension zero submanifold of a Euclidean space is a PL manifold, see, e.g., [34, p. 118] or [46, Remark 1.1.10].

into simplicial d-prisms induced by the product structure $\partial\mathcal{T}_\mathcal{N} \times [0,1]$. That is, $\mathcal{P}_\mathcal{Q}$ consists of d-prisms $P_\sigma \cong \sigma \times [0,1]$, one for each $(d-1)$-simplex σ of $\partial\mathcal{T}_\mathcal{N}$, glued together along their vertical boundary prisms the same way as the simplices of $\partial\mathcal{T}_\mathcal{N}$. The boundary $\partial\mathcal{P}_\mathcal{Q}$ of $\mathcal{P}_\mathcal{Q}$ has two connected components: $\partial_1\mathcal{P}_\mathcal{Q}$ and $\partial_2\mathcal{P}_\mathcal{Q}$, each combinatorially isomorphic to $\partial\mathcal{T}_\mathcal{N}$. Next, pass to the ℓ^{th} iterated barycentric subdivision of $\partial_2\mathcal{P}_\mathcal{Q}$. This operation turns each prism P_σ of $\mathcal{P}_\mathcal{Q}$ into a polyhedral cell R_σ. These cells form a polyhedral decomposition $\mathcal{R}_\mathcal{Q}$ of $\mathcal{Q}$, where $\partial_1\mathcal{R}_\mathcal{Q} \cong \partial\mathcal{T}_\mathcal{N}$ and $\partial_2\mathcal{R}_\mathcal{Q} \cong (\partial\mathcal{T}_\mathcal{N})^{(\ell)} = \partial(\mathcal{T}_\mathcal{N}^{(\ell)})$. Triangulate R_σ as follows. Consider an order $c_1 \prec c_2 \prec \cdots$ of the vertical cells[22] of R_σ, where $c_i \prec c_j$ implies $\dim(c_i) \leqslant \dim(c_j)$. Place a new vertex v_i in the barycenter of c_i and, iterating over the vertical cells in the above order, triangulate c_i by coning from v_i over its (already triangulated) boundary ∂c_i. It is clear that the resulting triangulation of R_σ is symmetric with respect to the symmetries of its base simplex σ. Applying this procedure for each polyhedral cell R_σ of $\mathcal{R}_\mathcal{Q}$ yields a triangulation $\mathcal{T}_\mathcal{Q}$ of $\mathcal{Q}$ with the desired properties.

Now the triangulation $\mathcal{T}_m$ of $\mathcal{B}^d$ is obtained by gluing together $\mathcal{T}_\mathcal{N}$, $\mathcal{T}_\mathcal{Q}$ and $\mathcal{T}_{\Sigma_\circ}$ via the identity maps along the isomorphic boundary-pairs $\partial\mathcal{T}_\mathcal{N} \cong \partial_1\mathcal{T}_\mathcal{Q}$ and $\partial_2\mathcal{T}_\mathcal{Q} \cong \partial_1\mathcal{T}_{\Sigma_\circ}$.

To conclude, note that the d-simplicies of $\mathcal{T}_\mathcal{N}$ triangulate a submanifold of $\|\mathcal{T}_m\|$, hence by Proposition 13 the graph $\Gamma(\mathcal{T}_\mathcal{N})$ is an induced subgraph of $\Gamma(\mathcal{T}_m)$. By Claim 22, $\Gamma(\mathcal{T}_\mathcal{N}) = G_m$ and by the initial assumption $\operatorname{tww}(G_m) \geqslant m$, hence by Proposition 5, $\Gamma(\mathcal{T}_m) \geqslant m$ as well. ◀

Proof of Theorem 3. Let $\mathcal{M}$ be an arbitrary PL-manifold possibly with non-empty boundary and $\mathcal{T}_\circ$ be a simplicial triangulation of $\mathcal{M}$. Consider a triangulation $\mathcal{T}_m$ of the d-ball with $\operatorname{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$ as in Theorem 21. Let Δ be a d-simplex of $\mathcal{T}_\circ$ that is disjoint from $\partial\mathcal{M}$ (if $\mathcal{T}_\circ$ does not contain such a simplex, just replace $\mathcal{T}_\circ$ with its second barycentric subdivision). Since $\mathcal{T}_\circ$ is simplicial, Δ is embedded in $\mathcal{T}_\circ$ and is topologically a d-ball. Now replace Δ with $\mathcal{T}_m$ by first removing Δ from $\mathcal{T}_\circ$, thereby creating a boundary component isomorphic to $\partial\Delta$, then gluing $\partial\mathcal{T}_m$ to this new boundary component via a simplicial isomorphism. (Note that this is possible since $\partial\mathcal{T}_m \cong \partial\Delta$.) Let $\mathcal{T}$ denote the resulting triangulation of $\mathcal{M}$. By Propositions 5 and 13, it follows that $\operatorname{tww}(\Gamma(\mathcal{T})) \geqslant \operatorname{tww}(\Gamma(\mathcal{T}_m)) \geqslant m$. ◀

References

1. M. A. Armstrong. Extending triangulations. *Proc. Amer. Math. Soc.*, 18:701–704, 1967. doi:10.2307/2035442.
2. B. Bagchi, B. A. Burton, B. Datta, N. Singh, and J. Spreer. Efficient algorithms to decide tightness. In *32nd Int. Symp. Comput. Geom. (SoCG 2016)*, volume 51 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 12:1–12:15. Schloss Dagstuhl–Leibniz-Zent. Inf., 2016. doi:10.4230/LIPIcs.SoCG.2016.12.
3. D. Bar-Natan. Fast Khovanov homology computations. *J. Knot Theory Ramifications*, 16(3):243–255, 2007. doi:10.1142/S0218216507005294.
4. P. Bergé, É. Bonnet, H. Déprés, and R. Watrigant. Approximating highly inapproximable problems on graphs of bounded twin-width. In *40th Int. Symp. Theor. Aspects Comput. Sci. (STACS 2023)*, volume 254 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 10:1–10:15. Schloss Dagstuhl. Leibniz-Zent. Inform., 2023. doi:10.4230/lipics.stacs.2023.10.
5. J.-D. Boissonnat, S. Kachanovich, and M. Wintraecken. Triangulating submanifolds: an elementary and quantified version of Whitney's method. *Discrete Comput. Geom.*, 66(1):386–434, 2021. doi:10.1007/s00454-020-00250-8.
6. É. Bonnet. *Twin-width and contraction sequences.* Habilitation thesis, ENS de Lyon, April 2024. URL: https://perso.ens-lyon.fr/edouard.bonnet/text/hdr.pdf.
7. É. Bonnet, D. Chakraborty, E. J. Kim, N. Köhler, R. Lopes, and S. Thomassé. Twin-width VIII: delineation and win-wins. In *17th Int. Symp. Parametr. Exact Comput. (IPEC 2022)*, volume 249 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 9:1–9:18. Schloss Dagstuhl. Leibniz-Zent. Inform., 2022. doi:10.4230/LIPIcs.IPEC.2022.9.

[22] These are precisely those cells that are not contained in the boundary of $\mathcal{R}_\mathcal{Q}$.

8 É. Bonnet, C. Geniet, E. J. Kim, S. Thomassé, and R. Watrigant. Twin-width III: max independent set, min dominating set, and coloring. In *48th Int. Colloq. Autom. Lang. Prog. (ICALP 2021)*, volume 198 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 35:1–35:20. Schloss Dagstuhl. Leibniz-Zent. Inform., 2021. doi:10.4230/LIPIcs.ICALP.2021.35.

9 É. Bonnet, C. Geniet, E. J. Kim, S. Thomassé, and R. Watrigant. Twin-width II: small classes. *Comb. Theory*, 2(2):Paper No. 10, 42, 2022. doi:10.5070/C62257876.

10 É. Bonnet and K. Huszár. On the twin-width of smooth manifolds, 2024. arXiv:2407.10174.

11 É. Bonnet, E. J. Kim, S. Thomassé, and R. Watrigant. Twin-width I: Tractable FO model checking. *J. ACM*, 69(1):Art. 3, 46, 2022. doi:10.1145/3486655.

12 B. A. Burton. Computational topology with Regina: algorithms, heuristics and implementations. In *Geometry and Topology Down Under*, volume 597 of *Contemp. Math.*, pages 195–224. Am. Math. Soc., Providence, RI, 2013. doi:10.1090/conm/597/11877.

13 B. A. Burton. The HOMFLY-PT polynomial is fixed-parameter tractable. In *34th Int. Symp. Comput. Geom. (SoCG 2018)*, volume 99 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 18:1–18:14. Schloss Dagstuhl–Leibniz-Zent. Inf., 2018. doi:10.4230/LIPIcs.SoCG.2018.18.

14 B. A. Burton, R. Budney, W. Pettersson, et al. Regina: Software for low-dimensional topology, 1999–2023. Version 7.3. URL: https://regina-normal.github.io.

15 B. A. Burton and R. G. Downey. Courcelle's theorem for triangulations. *J. Comb. Theory, Ser. A*, 146:264–294, 2017. doi:10.1016/j.jcta.2016.10.001.

16 B. A. Burton, H. Edelsbrunner, J. Erickson, and S. Tillmann, editors. *Computational Geometric and Algebraic Topology*, volume 12 of *Oberwolfach Rep.* EMS Publ. House, 2015. doi:10.4171/OWR/2015/45.

17 B. A. Burton, T. Lewiner, J. Paixão, and J. Spreer. Parameterized complexity of discrete Morse theory. *ACM Trans. Math. Softw.*, 42(1):6:1–6:24, 2016. doi:10.1145/2738034.

18 B. A. Burton, C. Maria, and J. Spreer. Algorithms and complexity for Turaev–Viro invariants. *J. Appl. Comput. Topol.*, 2(1–2):33–53, 2018. doi:10.1007/s41468-018-0016-2.

19 B. A. Burton and W. Pettersson. Fixed parameter tractable algorithms in combinatorial topology. In *Proc. 20th Int. Conf. Comput. Comb. (COCOON 2014)*, volume 8591 of *Lect. Notes Comput. Sci.*, pages 300–311. Springer, 2014. doi:10.1007/978-3-319-08783-2_26.

20 B. A. Burton and J. Spreer. The complexity of detecting taut angle structures on triangulations. In *Proc. 24th Annu. ACM-SIAM Symp. Discrete Algorithms (SODA 2013)*, pages 168–183, 2013. doi:10.1137/1.9781611973105.13.

21 G. Chapuy and G. Perarnau. On the number of coloured triangulations of d-manifolds. *Discrete Comput. Geom.*, 65(3):601–617, 2021. doi:10.1007/s00454-020-00189-w.

22 A. de Mesmay, J. Purcell, S. Schleimer, and E. Sedgwick. On the tree-width of knot diagrams. *J. Comput. Geom.*, 10(1):164–180, 2019. doi:10.20382/jocg.v10i1a6.

23 Z. Dvořák and S. Norine. Small graph classes and bounded expansion. *J. Combin. Theory Ser. B*, 100(2):171–175, 2010. doi:10.1016/j.jctb.2009.06.001.

24 J. Gajarský, M. Pilipczuk, W. Przybyszewski, and Sz. Toruńczyk. Twin-width and types. In *49th Int. Conf. Autom. Lang. Prog. (ICALP 2022)*, volume 229 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 123:1–123:21. Schloss Dagstuhl. Leibniz-Zent. Inform., 2022. doi:10.4230/lipics.icalp.2022.123.

25 P. Hliněný and J. Jedelský. Twin-width of planar graphs is at most 8, and at most 6 when bipartite planar. In *50th Int. Colloq. Autom. Lang. Prog. (ICALP 2023)*, volume 261 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 75, 18. Schloss Dagstuhl. Leibniz-Zent. Inform., 2023. doi:10.4230/lipics.icalp.2023.75.

26 K. Huszár. *Combinatorial width parameters for* 3*-dimensonal manifolds.* PhD thesis, IST Austria, June 2020. doi:10.15479/AT:ISTA:8032.

27 K. Huszár. On the pathwidth of hyperbolic 3-manifolds. *Comput. Geom. Topol.*, 1(1):1–19, 2022. doi:10.57717/cgt.v1i1.4.

28 K. Huszár and J. Spreer. 3-Manifold triangulations with small treewidth. In *35th Int. Symp. Comput. Geom. (SoCG 2019)*, volume 129 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 44:1–44:20. Schloss Dagstuhl–Leibniz-Zent. Inf., 2019. doi:10.4230/LIPIcs.SoCG.2019.44.

29 K. Huszár and J. Spreer. On the width of complicated JSJ decompositions. In *39th Int. Symp. Comput. Geom. (SoCG 2023)*, volume 258 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 42:1–42:18. Schloss Dagstuhl–Leibniz-Zent. Inf., 2023. doi:10.4230/LIPIcs.SoCG.2023.42.

30 K. Huszár, J. Spreer, and U. Wagner. On the treewidth of triangulated 3-manifolds. *J. Comput. Geom.*, 10(2):70–98, 2019. doi:10.20382/jogc.v10i2a5.

31 D. Kráľ and A. Lamaison. Planar graph with twin-width seven. *Eur. J. Comb.*, page 103749, 2023. doi:10.1016/j.ejc.2023.103749.

32 D. Kráľ, K. Pekárková, and K. Štorgel. Twin-Width of Graphs on Surfaces. In *49th Int. Symp. Math. Found. Comput. Sci. (MFCS 2024)*, volume 306 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 66:1–66:15. Schloss Dagstuhl–Leibniz-Zent. Inf., 2024. doi:10.4230/LIPIcs.MFCS.2024.66.

33 J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Grad. Texts Math.* Springer, New York, second edition, 2013. doi:10.1007/978-1-4419-9982-5.

34 N. Levitt and A. Ranicki. Intrinsic transversality structures. *Pacific J. Math.*, 129(1):85–144, 1987. doi:10.2140/pjm.1987.129.85.

35 C. Lunel and A. de Mesmay. A Structural Approach to Tree Decompositions of Knots and Spatial Graphs. In *39th Int. Symp. Comput. Geom. (SoCG 2023)*, volume 258 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 50:1–50:16. Schloss Dagstuhl–Leibniz-Zent. Inf., 2023. doi:10.4230/LIPIcs.SoCG.2023.50.

36 J. A. Makowsky. Coloured Tutte polynomials and Kauffman brackets for graphs of bounded tree width. *Discrete Appl. Math.*, 145(2):276–290, 2005. doi:10.1016/j.dam.2004.01.016.

37 J. A. Makowsky and J. P. Mariño. The parametrized complexity of knot polynomials. *J. Comput. Syst. Sci.*, 67(4):742–756, 2003. Special issue on parameterized computation and complexity. doi:10.1016/S0022-0000(03)00080-1.

38 C. Manolescu. Lectures on the triangulation conjecture. In *Proc. 22nd Gökova Geom.-Topol. Conf. (GGT 2015)*, pages 1–38. Int. Press Boston, 2016. URL: https://gokovagt.org/proceedings/2015/manolescu.html.

39 C. Maria. Parameterized complexity of quantum knot invariants. In *37th Int. Symp. Comput. Geom. (SoCG 2021)*, volume 189 of *LIPIcs. Leibniz Int. Proc. Inf.*, pages 53:1–53:15. Schloss Dagstuhl–Leibniz-Zent. Inf., 2021. doi:10.4230/LIPIcs.SoCG.2021.53.

40 C. Maria and J. Purcell. Treewidth, crushing and hyperbolic volume. *Algebr. Geom. Topol.*, 19(5):2625–2652, 2019. doi:10.2140/agt.2019.19.2625.

41 E. E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. Math. (2)*, 56:96–114, 1952. doi:10.2307/1969769.

42 M. Noy. Graphs. In *Handbook of Enumerative Combinatorics*, Discrete Math. Appl., pages 397–436. Chapman & Hall / CRC, 2015. doi:10.1201/b18255-12.

43 V. V. Prasolov. *Elements of combinatorial and differential topology*, volume 74 of *Grad. Stud. Math.* Am. Math. Soc., Providence, RI, 2006. Translated from the 2004 Russian original by Olga Sipacheva. doi:10.1090/gsm/074.

44 T. Radó. Über den Begriff der Riemannschen Fläche. *Acta Sci. Math.*, 2(2):101–121, 1925.

45 C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology.* Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. Reprint. doi:10.1007/978-3-642-81735-9.

46 F. Waldhausen, B. Jahren, and J. Rognes. *Spaces of PL manifolds and categories of simple maps*, volume 186 of *Annals Math. Stud.* Princeton Univ. Press, Princeton, NJ, 2013. doi:10.1515/9781400846528.

47 H. Whitney. The self-intersections of a smooth n-manifold in $2n$-space. *Ann. of Math. (2)*, 45:220–246, 1944. doi:10.2307/1969265.

48 H. Whitney. *Geometric Integration Theory.* Princeton Univ. Press, Princeton, NJ, 1957. doi:10.1515/9781400877577.