Geometric Spanners of Bounded Tree-Width

Kevin Buchin
TU Dortmund, Germany

Carolin Rehs
TU Dortmund, Germany

Torben Scheele
TU Dortmund, Germany

Abstract

Given a point set P in the Euclidean space, a geometric t-spanner G is a graph on P such that for every pair of points, the shortest path in G between those points is at most a factor t longer than the Euclidean distance between those points. The value $t \geq 1$ is called the dilation of G. Commonly, the aim is to construct a t-spanner with additional desirable properties. In graph theory, a powerful tool to admit efficient algorithms is bounded tree-width. We therefore investigate the problem of computing geometric spanners with bounded tree-width and small dilation t.

Let d be a fixed integer and $P \subset \mathbb{R}^d$ be a point set with n points. We give a first algorithm to compute an $\mathcal{O}(n/k^{d/(d-1)})$-spanner on P with tree-width at most k. The dilation obtained by the algorithm is asymptotically worst-case optimal for graphs with tree-width k: We show that there is a set of n points such that every spanner of tree-width k has dilation $\mathcal{O}(n/k^{d/(d-1)})$. We further prove a tight dependency between tree-width and the number of edges in sparse connected planar graphs, which admits, for point sets in $\mathbb{R}^2$, a plane spanner with tree-width at most k and small maximum vertex degree.

Finally, we show an almost tight bound on the minimum dilation of a spanning tree of n equally spaced points on a circle.

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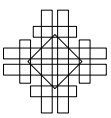
1 Introduction

Geometric spanners are an extensively studied area in computational geometry, see [7, 23] for surveys. A geometric spanner for a point set $P \in \mathbb{R}^d$ is a weighted graph on P, where the weight of an edge is the Euclidean distance between its endpoints, aiming for sparsity while avoiding long paths between two points in P. This is formalised by the dilation $t = \max\left\{\frac{d(p,p')}{|pp'|} \mid p, p' \in P\right\}$, where $d(p, p')$ is the distance between two points in G and $|pp'|$ is the Euclidean distance between p and p'. A geometric spanner with dilation at most t is also called a *t-spanner*.

When geometric spanners were introduced in 1986 [12], sparsity was obtained by requiring planarity. Since then, plane geometric spanners have been widely studied [7]. But also other measures are of interest: Especially spanners with a linearly bounded number of edges or a bounded vertex degree have been widely researched [20, 23, 30, 18, 6, 14, 19, 5].

In graph theory, an important tool to design efficient algorithms for in general NP-hard problems is to restrict these problems to graphs which are bounded in certain parameters. Especially graphs with bounded *tree-width* [25, 26] allow polynomial-time algorithms for many in general NP-hard problems using a dynamic programming algorithm on the tree decomposition. By Courcelle's Theorem, every problem that is describable in monadic second




order logic with vertex and edge quantification is solvable in polynomial time for graphs with bounded tree-width [13]. Thus, geometric spanners with bounded tree-width would allow polynomial time solutions for many in general NP-hard problems and thus be a strong tool for many geometric applications. For instance, Cabello and Knauer [9] recently gave some examples for efficient algorithms for geometric problems on graphs with bounded tree-width. They show that via orthogonal range searching, for graphs of n vertices and tree-width k with $k \geq 3$, the sum of the distances between all pairs of vertices can be computed in $\mathcal{O}(n \log^{k-1} n)$ time. Further, they show that the dilation of a geometric graph of bounded tree-width can be computed in $\mathcal{O}(n \log^{k+1} n)$ time. That makes it possible to compute the dilation of every tree-width bounded graph, but does not lead to tree-width bounded spanners with low dilation. In [8], Cabello shows how to compute the inverse geodesic length in graphs of bounded tree-width. Further, Cabello and Rote [10] show how to compute obnoxious centers in graphs with bounded tree-width in $\mathcal{O}(n \log n)$ time.

Therefore, we aim to construct spanners with bounded tree-width. There are spanners with sublinear tree-width. More specifically, since planar graphs have tree-width $\mathcal{O}(\sqrt{n})$ [22] (where n is the number of vertices), for point sets in the Euclidean plane, constant dilation spanners with tree-width $\mathcal{O}(\sqrt{n})$ can be obtained by plane spanner approaches such as the Delaunay triangulation or the greedy triangulation. For point sets in $\mathbb{R}^d$, we point out that the greedy spanner has tree-width $\mathcal{O}(n^{1-1/d})$, using separator results from [21].

While this gives some first results, constructing spanners with tree-width $k \in o(n^{1-1/d})$ will require other constructions. The only known construction for small tree-width and a guarantee on the dilation is the Euclidean minimum spanning tree: it has tree-width $k = 1$ and worst-case dilation $n - 1$.

We present a first algorithm that provides a trade-off between tree-width and the dilation:

▶ **Theorem 1.** *Given a set $P \subset \mathbb{R}^d$ of n points for some fixed d and a positive integer $k \leq n^{1-1/d}$. There is a geometric spanner of tree-width k on P with a dilation of $\mathcal{O}\left(n/k^{d/(d-1)}\right)$ and bounded degree that can be computed in time $\mathcal{O}(n^2 \log n)$.*

We complement our result with a matching lower bound.

▶ **Theorem 2.** *Let $d \geq 2$ be a fixed integer. For positive integers n and $k \leq n^{(d-1)/d} \cdot (5d)^{1/d-2}$ there is a set of n points in $\mathbb{R}^d$, so that every geometric spanner of tree-width k on this set has dilation $\Omega\left(n/k^{d/(d-1)}\right)$.*

This proves that our algorithm for Theorem 1 has a worst-case optimal trade off between tree-width and dilation. We therefore now turn our attention to bounded tree-width spanners with an additional property. The most commonly studied property of geometric spanners is planarity.

For point sets in the Euclidean plane, we can adapt the construction of the spanner for Theorem 1 to obtain a planar spanner with bounded tree-width (and bounded degree), that is not necessarily plane. As a general tool to obtain bounded tree-width plane spanners, we provide the following strong dependency between tree-width and the number of edges in sparse connected planar graphs:

▶ **Theorem 3.** *Let G be a planar connected graph with n vertices and $n + \lfloor (k-1)^2/72 \rfloor$ edges, where $k \geq 2$ is an integer. The tree-width of G is at most k.*

This result is quite surprising, since for non-planar graphs, arbitrarily placed edges increase the tree-width linearly in the number of added edges, as there are graphs with m edges and tree-width at least $m \cdot \varepsilon$ for a constant ε [17].

It further yields an alternative proof for Theorem 1 for $d = 2$, creating a bounded tree-width spanner with additional properties:

▶ **Corollary 4.** *Given a set $P \subset \mathbb{R}^2$ of n points and some positive integer $k \leq 12\sqrt{n-3}$, a plane spanner with tree-width k, maximum vertex degree 4 and dilation $\mathcal{O}(n/k^2)$ can be constructed in $\mathcal{O}(n \log n)$ time.*

This follows immediately using an adapted version of the algorithm provided in [3]: Instead of the Delaunay triangulation, use a plane constant dilation spanner with maximum vertex degree 4, as provided in [19] and the MST of this spanner instead of the EMST.

In the last section, we answer an open question stated in [3] by giving some smaller results for spanners of bounded tree-width on points in convex position. In particular, we show that, for point sets P which are equally spaced on a circle, there is a tree spanner with dilation $\leq \frac{2n}{\pi} + \frac{\pi}{2n}$. This complements a lower bound given in [3]. Further, since plane spanners on points in convex position are outerplanar, there is a 1.88-spanner of tree-width 2 for P. Further, we give a (straight-forward) lower bound: There are sets of points in convex position on which every spanner of tree-width 2 has dilation at least 1.43.

2 Preliminaries

We start giving some preliminary definitions to clarify the notations we use later on.

In this paper, all graphs G are simple and undirected. We refer to the vertex set of G by $V(G)$ and to the edge set of G by $E(G)$. Euclidean graphs are complete and weighted and the edge weight is given by the Euclidean distance of the incident vertices. We denote the *degree* of a vertex $v \in V(G)$ by $\Delta_G(v)$ and the *maximum vertex degree* of a graph by $\Delta(G)$. By K_n we denote the complete graph on n vertices.

A *Euclidean 2-dimensional $n \times n$-grid* is defined as a graph on the point set $\{0, \dots, n-1\} \times \{0, \dots, n-1\}$ in $\mathbb{R}^2$. We call two vertices (x, y) and (x', y') *neighbouring*, if $|x-x'|+|y-y'| = 1$. The edge set of the $n \times n$ grid is then $\{\{(x, y), (x', y')\} \mid (x, y)$ and (x', y') are neighbouring$\}$.

A *Euclidean d-dimensional n^d-grid* is defined as a graph on the point set $\{0, \dots, n-1\}^d$. Here, two vertices $(x_1, \dots, x_d)$ and $(x'_1, \dots, x'_d)$ are called *neighbouring* if $|x_1 - x'_1| + \dots + |x_d - x'_d| = 1$. The edge set of the n^d grid is then $\{\{(x_1, \dots x_d), (x'_1, \dots, x'_d)\} \mid (x_1, \dots x_d)$ and $(x'_1, \dots x'_d)$ are neighbouring$\}$.

We use a standard definition of tree-width [26] in the following notation: A *tree decomposition* (T, X) is a tree T over a set of nodes I and a mapping $X : I \to 2^V$, where $X(i)$ is called the *bag* of i, so that the following conditions hold:

1. $\bigcup_{i \in I} X(i) = V$
2. $\bigcup_{i \in I} E(i) = E$ where $E(i) = \{\{u, v\} \in E \mid u, v \in X(i)\}$
3. If $v \in X(i)$ and $v \in X(k)$ for some $i, k \in I$ then $v \in X(j)$ holds for every j on the path in T between i and k.

The *width* of a tree decomposition is defined as $\max_{i \in I} |X(i)| - 1$. The *tree-width* $\mathrm{tw}(G)$ of G is the minimum width over all tree decompositions of G.

A minor of a graph can be obtained by subgraph operations and edge contractions. As, however, edge contractions in general do not maintain any properties of the embedding, which is essential for geometric graphs, we give the definition of contracting one vertex to another: For a graph G, *contracting vertex u* to v or equivalently *contracting vertex u* along e, we create a minor $G' = (V', E')$ of G with $V' = V \setminus \{u\}$ and $E' = \{\{u', v'\} \mid u', v' \in V', \{u', v'\} \in E\} \cup \{\{u', v\} \mid u' \neq v, \{u', u\} \in E\}$. Note that this is just another way of defining an edge contraction. Given a graph $G = (V, E)$, some vertex $v \in V$ and a path

$\gamma = v, v_2, ..., v_\ell$ inside G, we define the *path contraction* of γ to v to be the series of vertex contractions starting with the contraction of v_2 to v up to the contraction of v_ℓ to v. A graph H is a *minor* of G if it can be obtained by vertex contractions and subgraph operations. A class of graphs $\mathcal{G}$ is considered *minor closed* if every minor H of a graph $G \in \mathcal{G}$ is also in $\mathcal{G}$.

A *t-balanced separator* of size s of a graph G is an edge set $S \subseteq V(G)$, $|S| = s$ such that $V - S$ can be partitioned into two sets A and B with no edges between the vertices of A and B and $|A| \leq t \cdot |V|$ as well as $|B| \leq t \cdot |V|$. If $t = \frac{2}{3}$, we call S a *separator* of G.

3 Bounded tree-width spanners in higher dimensions

In this section, we present an algorithm for constructing bounded tree-width spanners for points in $\mathbb{R}^d$ for constant dimension d. As we will see later on, the trade-off between tree-width and dilation of this spanner is worst-case optimal.

The algorithm makes use of two ingredients: a Euclidean minimum spanning tree ($\mathcal{EMST}$) and a class of spanners with sublinear tree-width. For $d = 2$ a class of planar spanners can be used (and will be used in Section 5). For general d we can use the greedy spanner [2] instead.

▶ **Lemma*** **5.** *Let $P \subset \mathbb{R}^d$ be a set of n points and G the greedy 3/2-spanner of P. G has tree-width at most $15\eta_d c_d 8^d \cdot n^{1-1/d}$, where η_d is the packing constant in d-dimensional Euclidean space and $c_d \leq 2^{\mathcal{O}(d)}$ is a constant.*

The following algorithm computes a bounded tree-width spanner. The constant C is defined as $C = 30\eta_d c_d 8^d$.

Algorithm 1 d-`DimensionalBoundedTreeWidthSpanner`(P, k).

Require: a set of n points $P \subset \mathbb{R}^d$ and a natural number $k \leq n^{1-1/d}$
Ensure: an $\mathcal{O}(n \cdot k^{d/(1-d)})$-spanner $G = (P, E)$ of tree-width k

1: $\mathcal{EMST} \leftarrow$ Euclidean minimum spanning tree of P
2: **if** $k = 1$ **then**
3: **return** $\mathcal{EMST}$
4: $m \leftarrow \left\lceil (k/C)^{d/(d-1)} + 1 \right\rceil$
5: Compute a set $\mathcal{T}$ of m disjoint subtrees of $\mathcal{EMST}$, each containing $\mathcal{O}(n/m)$ points, as explained in the proof of Lemma 9.
6: For $T \in \mathcal{T}$ let $R(T)$ be the vertices in T incident to edges removed by the previous step.
7: For each $T \in \mathcal{T}$ iterate through its edges e from long to short. If e lies on a path between two vertices in $R(T)$, remove the edge. Let $E'(T)$ be the set of remaining edges.
8: $E' \leftarrow \bigcup_{T \in \mathcal{T}} E'(T)$
9: $(R, E'') \leftarrow$ greedy 3/2-spanner for $R = \bigcup_{T \in \mathcal{T}} R(T)$
10: **return** $G = (P, E' \cup E'')$

We now start with a detailed description on how the subtrees (line 5) are constructed. The following is a folklore result and is for instance mentioned in [22].

▶ **Fact*** **6.** *For any tree T on n vertices there is a separator vertex that splits T into subtrees each at most of size $n/2$ and can be found in time $\mathcal{O}(n)$.*

* The proofs of the results marked with $*$ can only be found in the full version due to space restrictions.

Considering the separator vertex v and its edges, we observe that one of the by removal of v induced subtrees must have size at least $n \cdot 1/\Delta(v)$. Let e be the edge that connects v to this subtree. Then, e separates the tree into one subtree of size at least $n \cdot 1/\Delta(v)$, this being the aforementioned subtree, and one of size at most $n \cdot (\Delta(v) - 1)/\Delta(v)$.

▶ **Lemma 7.** *For any tree T of degree Δ with n vertices there is a separator edge, that separates T into two subtrees, one of size at least $n \cdot 1/(\Delta + 1)$ and one of size at most $n \cdot \Delta/(\Delta + 1)$. This edge can be found in time $\mathcal{O}(n)$.*

This result can be extended as follows:

▶ **Lemma 8.** *Given a tree T of degree Δ, that has n vertices and some integer $m \geq 1$, there is a set of $m - 1$ edges of T, whose removal lead to m disjoint subtrees of size $\mathcal{O}(n/m)$. This set of edges can be found in time $\mathcal{O}(n(\Delta + 1)(\log m + \log(\Delta + 1)))$.*

Proof sketch. By Lemma 7 for any tree of size n there is always an edge, whose removal splits the tree into one subtree of size at most $n \cdot \Delta/(\Delta+1)$ and one of size at least $n \cdot 1/(\Delta+1)$. The idea now is to split subtrees until their sizes are sufficiently small. So given a tree on n vertices and some $m \in \mathbb{N}$, we keep splitting the largest subtrees until there is no subtree of size more than $(\Delta + 1) \cdot n/m$ left. Note that this procedure can only lead to subtrees as small as n/m and therefore only to at most m subtrees. To obtain exactly m subtrees we can simply continue splitting the largest remaining subtrees until the number of subtrees is exactly m. The subtrees are now at least of size $n/((\Delta + 1) \cdot m)$ and at most of size $(\Delta + 1) \cdot n/m$ and therefore of size $\mathcal{O}(n/m)$. For the running time, we refer to the full version of the paper. ◀

Since the $\mathcal{EMST}$ is of constant degree, we obtain the following corollary:

▶ **Lemma 9.** *Given the $\mathcal{EMST}$ of n points in $\mathbb{R}^d$ and some integer $m \geq 1$, there is a set of $m - 1$ edges of T, whose removal lead to m disjoint subtrees of size $\mathcal{O}(n/m)$. This set of edges can be found in time $\mathcal{O}(n \log m)$.*

We call a vertex a *representative* of a subtree $T \in \mathcal{T}$ if it was incident to an edge of the $\mathcal{EMST}$ that was removed in the construction of Lemma 8. While bounding the dilation of the spanner requires more work, we can already derive the running time and tree-width.

▶ **Lemma 10.** *Algorithm 1 constructs in $\mathcal{O}(n^2 \log n)$ time a graph with tree-width $\leq k$ and bounded degree.*

Proof. The greedy spanners in line 9 of the algorithm can be computed in $\mathcal{O}(n^2 \log n)$ time [6]. The $\mathcal{EMST}$ can be computed in subquadratic time [1], and all remaining steps take $\mathcal{O}(n \log n)$ time. The greedy t-spanner and the $\mathcal{EMST}$ both have bounded degree for fixed dimension (and t) [27, 23].

The set of representatives of all subtrees has size at most $2m - 2$, since each representative is incident to at least one of the $m - 1$ edges removed. Thus, $\mathcal{GS}$ is the greedy spanner of at most $2m - 2$ points. By Lemma 5 and the choice of the constant C the tree-width of $\mathcal{GS}$ is at most k. After line 7 of the algorithm each of the subtrees of the subtrees $T \in \mathcal{T}$ contains only one representative. Thus, the graph G consists of $\mathcal{GS}$ with a tree attached to each of its vertices. These trees do not increase the tree-width, since every tree contains only one vertex from $\mathcal{GS}$. Thus the tree-width of G is k. ◀

Next we bound the dilation of the computed spanner.

▶ **Lemma*** **11.** *Given a set $P \subset \mathbb{R}^d$ of n points for some fixed d and a positive integer $k \leq n^{1-1/d}$, Algorithm 1 computes an $\mathcal{O}(n/k^{d/(d-1)})$-spanner for P.*

Proof idea. First ignore that we removed edges in line 7. For two points p, q in the same subtree the dilation is small, since the edges of the path between p and q in the $\mathcal{EMST}$ are short, and since the path contains only few edges. For points in different subtrees the path in the $\mathcal{EMST}$ contains a representative of each of the two subtrees. We can take the paths to these representatives and between the representatives the greedy spanner.

Removing edges in line 7 complicates the argument, but we can use the lengths of the edges removed to bound the distances to the representatives. ◀

From the previous lemmas we now directly obtain the main result of this section.

▶ **Theorem 1.** *Given a set $P \subset \mathbb{R}^d$ of n points for some fixed d and a positive integer $k \leq n^{1-1/d}$. There is a geometric spanner of tree-width k on P with a dilation of $\mathcal{O}\left(n/k^{d/(d-1)}\right)$ and bounded degree that can be computed in time $\mathcal{O}(n^2 \log n)$.*

4 Lower bound

In the previous section it was shown that Algorithm 1 gives an upper bound for the dilation of tree-width bounded spanners. In this section, we investigate corresponding lower bounds. We show that the bound given in Theorem 1 for the dilation of a tree-width bounded spanner on Euclidean point sets is asymptotically tight:

▶ **Theorem 2.** *Let $d \geq 2$ be a fixed integer. For positive integers n and $k \leq n^{(d-1)/d} \cdot (5d)^{1/d-2}$ there is a set of n points in $\mathbb{R}^d$, so that every geometric spanner of tree-width k on this set has dilation $\Omega\left(n/k^{d/(d-1)}\right)$.*

To obtain this result, we construct a set of points resembling a grid. While it is commonly known that a two-dimensional grid has high tree-width, the $\left(k^{1/(d-1)}\right)^d$-grid does not necessarily have tree-width k. We thus construct a set of points resembling the $(h+1)^d$-grid, where $h = \left\lceil (9d/2 \cdot (k+2))^{1/(d-1)} - 1 \right\rceil$. To lower bound the tree-width of this grid, we generalise a basic idea for the 3-dimensional grid given by Korhonen in an online forum[1]:

▶ **Lemma*** **12.** *The n^d-grid has tree-width $\geq \frac{2}{9d} \cdot n^{d-1} - 1$.*

Thus, the tree-width of our grid is at least

$$\begin{aligned}
\frac{2}{9d} \cdot (h+1)^{d-1} - 1 &= \frac{2}{9d} \cdot \left(\left\lceil (9d/2 \cdot (k+2))^{1/(d-1)} - 1 \right\rceil + 1 \right)^{d-1} - 1 \\
&\geq \frac{2}{9d} \cdot \left((9d/2 \cdot (k+2))^{1/(d-1)} \right)^{d-1} - 1 \\
&= k+1.
\end{aligned}$$

Using an inductive construction for the $(h+1)^d$-grid, by taking d copies of the $(h+1)^{d-1}$-grid and connecting them using $(d-1) \cdot (h+1)^{d-1}$ edges, we can prove that the $(h+1)^d$-grid has a total of $d \cdot (h+1)^d + d \cdot (h+1)^{d-1}$ edges.

[1] https://cstheory.stackexchange.com/questions/53029/what-is-the-treewidth-of-the-3d-grid-mesh-or-lattice-with-sidelength-n

The constructing of the grid-like set $P_{d,n,k}$ is as follows: Let $m = \lfloor n/(d \cdot (h+1)^d + d \cdot (h+1)^{d-1}) \rfloor$, be the number of points representing an edge of the grid in $P_{d,n,k}$. For every dimension $i \in \{1, \dots, d\}$, we define the set

$$P_i = \bigcup_{j_1=0}^{h} \cdots \bigcup_{j_d=0}^{h} \{(j_1 \cdot m, \dots, j_{i-1} \cdot m, x_i, j_{i+1} \cdot m, \dots, j_d \cdot m) \mid x_i \in \{0, \dots, h \cdot m\}\},$$

which consists of $(h+1)^{d-1}$ sets of collinear points representing the grid edges parallel to the axis of the i-th dimension. Each of these collinear sets of points consists of $h \cdot m + 1$ points and represents h edges of the grid. Hence $|P_i| = (h \cdot m + 1) \cdot (h+1)^{d-1}$ for every i. We can now define the set $P_{d,n,k} = \bigcup_{i=1}^{d} P_i$ which represents the whole $(h+1)^d$-grid. (In Figure 1 an example is shown for a 2-dimensional point set.) Notice that the $(h+1)^d$ points $P_{\boxplus} = \{(j_1 \cdot m, \dots, j_d \cdot m) \mid j_1, \dots, j_d \in \{0, \dots, h\}\}$ are contained in every P_i. We call these points the *grid points* of the set $P_{d,n,k}$ and two grid points are called *neighbouring* if the Euclidean distance between them is m, which corresponds to them being connected through an edge in the $(h+1)^d$-grid.

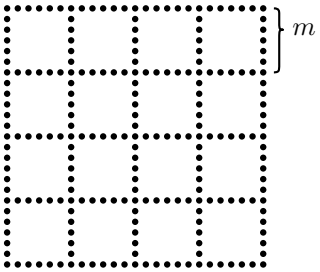


Figure 1 The set of points resembling the $(4+1)^2$-grid where $n = 240$ and $m = 6$.

▶ **Lemma*** **13.** *Let $d \geq 2$ be a fixed integer. The grid-like set $P_{d,n,k}$ is well-defined for every integer $n \geq d \cdot 64^{d \log d}$ and positive integer $k \leq n^{(d-1)/d} \cdot (5d)^{1/d-2}$ and is of size at most n.*

▶ **Lemma 14.** *If G is a geometric $o\left(n/k^{d/(d-1)}\right)$-spanner on $P_{d,n,k}$, then it must be of tree-width at least $k+1$.*

Proof. Let G be a $o(n/k^{d/(d-1)})$-spanner on $P_{d,n,k}$. Let $\mathcal{R}_x := x + [-m/4, m/4]^d$ denote the hypercube of side-length $m/2$ centered at x. For any pair $p, q \in P_{\boxplus}$ of neighbouring grid points, we consider the three hypercubes $\mathcal{R}_p, \mathcal{R}_q$, and $\mathcal{R}_s := s + [-m/4, m/4]^d$, where $s = \frac{p+q}{2}$ is the midpoint between p and q. We argue that there is a path in G from p to q such that: *(i)* the path only visits vertices in $\mathcal{R}_{p,q} := \mathcal{R}_p \cup \mathcal{R}_q \cup \mathcal{R}_s$ and *(ii)* for any $p' \in \mathcal{R}_p$ and $q' \in \mathcal{R}_q$, if the path visits p' and q' then it visits p' before it visits q'.

Let $p_0 = p, p_1, p_2, \dots p_m = q$ the ordered sequence of points representing the edge of the grid between p and q, and further let $s_i := \frac{p_i + p_{i+1}}{2}$. Since G is a $o(n/k^{d/(d-1)})$-spanner, we can assume that there is for every pair p_i, p_{i+1} a shortest path γ_i in G of length at most $m/4 = \Theta(n/k^{d/(d-1)})$. Such a path necessarily will remain with in $\mathcal{R}_{p_i} \cap \mathcal{R}_{p_{i+1}} \subset \mathcal{R}_{s_i}$. For $0 \leq i < m/2$ we have $\mathcal{R}_{s_i} \subset \mathcal{R}_p \cup \mathcal{R}_s$, and for $m/2 \leq i < m$ we have $\mathcal{R}_{s_i} \subset \mathcal{R}_q \cup \mathcal{R}_s$.

Consider the walk from p to q obtained by concatenating the paths $\gamma_0, \gamma_1, \dots \gamma_{m-1}$. It first only visits vertices in $\mathcal{R}_p \cup \mathcal{R}_s$ and then only vertices in $\mathcal{R}_q \cup \mathcal{R}_s$. By removing cycles we obtain a path $\gamma_{p,q}$ from p to q with this property. Thus, we have a path with the properties *(i)*+*(ii)* stated above.

We have such a path $\gamma_{p,q}$ for any pair $p, q \in P_{\boxplus}$. We claim that this implies that G must have a minor isomorphic to the $(h+1)^d$-grid and is therefore of tree-width $> k$. To show this, we perform a series of path contractions on G, obtaining a graph G'. More specifically, we first contract the edges of $\gamma_{p,q}$ starting from p, until we encounter as endpoint the last vertex in $\mathcal{R}_p$, identifying the merged vertices with p. After skipping an edge, we contract the remaining edges of $\gamma_{p,q}$, identifying the merged vertices with q. This is illustrated in Figure 2. After these contractions, the remaining edge of $\gamma_{p,q}$ is an edge from p to q.

Further we observe that *(i)* any vertex p' of $\gamma_{p,q}$ in $\mathcal{R}_p$ has been identified with p, *(ii)* any vertex q' of $\gamma_{p,q}$ in $\mathcal{R}_q$ has been identified with q, and *(iii)* any other vertex p'' of $\gamma_{p,q}$ does not also lie on another path $\gamma_{r,s}$ with $\{p,q\} \neq \{r,s\}$. The latter holds, since p'' cannot lie in $\mathcal{R}_{r,s}$. This means, that the paths $\gamma_{p,q}$ and $\gamma_{r,s}$ are contracted consistently. Thus, regardless of the order in which we contract the paths, we obtain a graph G' which has a minor isomorphic to the $(h+1)^d$-grid. In G', there is an edge between every neighbouring pair of grid points. Therefore the subgraph $G'[P_{\boxplus}]$ containing only the grid points is, if we remove edges between non-neighbouring grid points, isomorphic to the $(h+1)^d$-grid. We conclude that the spanner G has a minor isomorphic to the $(h+1)^d$-grid and by Lemma 12 therefore must be of tree-width at least $\frac{2}{9d}(h+1)^{d-1} - 1 \geq k+1$. ◀

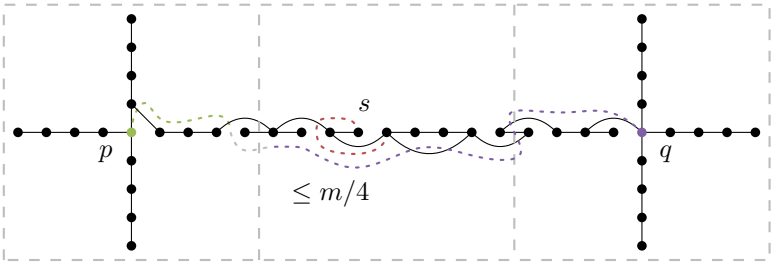


Figure 2 Grid points p and q as well as the three hypercubes centered at p, q and s. The subpath of $\gamma_{p,q}$ that is contracted to p is shown in green and the subpath that is contracted to q in purple. The red path is a detour of length $\leq m/4$ for points p_i and p_{i+1}.

5 Minor-3-cores and the tree-width of planar spanners

We now have given an asymptotically tight algorithm to obtain small bounded tree-width, with a dilation dependent on the chosen tree-width. We now focus on bounded tree-width spanners with the additional property of being plane.

In general, there are graphs with m edges and tree-width at least $m \cdot \varepsilon$ for a constant ε [17]. We show that this is not true for connected planar graphs, yielding a tight dependency between tree-width and the number of edges:

▶ **Theorem 3.** *Let G be a planar connected graph with n vertices and $n + \lfloor (k-1)^2/72 \rfloor$ edges, where $k \geq 2$ is an integer. The tree-width of G is at most k.*

5.1 Minor-3-cores

To obtain this Theorem, we first introduce the concept of *minor-3-cores*, in reference to the k-core which was established by Seidman [29] for the study of social networks. We then use this concept to obtain the tree-width of planar graphs with a certain fixed number of edges.

▶ **Definition 15** (Minor-3-Core). *Let G be a graph and H a minor of G with minimum degree 3. If there is no minor of minimum degree 3 that has more edges than H i.e., H is an edge maximal minor with this property, we call it a minor-3-core of G.*

This definition gives a very clear idea of what a minor-3-core should be, but unfortunately it does not lead to any structural insight concerning the concept. It especially leaves open whether minor-3-cores are unique or if there are multiple non-isomorphic minor-3-cores. We therefore need to further characterise minor-3-cores with regard to algorithmic properties and uniqueness. In order to answer these questions, we first define the disjoint-paths property.

▶ **Definition 16** (Disjoint-Paths Property). *Let G be a graph and let C be a minor-3-core of G as well as v be a vertex of the minor-3-core (and therefore a vertex of G that has not been contracted). We say that v fulfils the disjoint-paths property for C, if for every neighbour $u_1, ..., u_{\Delta_C(v)}$ of v in C, there is a path γ_i in G which leads from v to u_i and $\left(\bigcup_{u \text{ in } \gamma_i} u\right) \cap \left(\bigcup_{w \text{ in } \gamma_j} w\right) = \{v\}$ for every path γ_j for $j \neq i$.*

We now show that we can assume having the disjoint paths property in a minor-3-core:

▶ **Lemma*** **17.** *For every graph G and minor-3-core C of G, there is a minor-3-core isomorphic to C in which every vertex has the disjoint paths property.*

By Lemma 17 we now know that there is a minor-3-core, in which every vertex has the disjoint-paths property. We call a minor with this property a *canonical minor-3-core*:

▶ **Definition 18** (Canonical Minor-3-Core). *Let C be a minor-3-core of a graph G. We call C the canonical minor-3-core, if every vertex of C has the disjoint-paths property. The vertices of C are called the canonical vertices.*

We now show that the canonical minor-3-core is, as the name implies, unique:

▶ **Lemma*** **19.** *Let G be a graph, the canonical minor-3-core of G is unique.*

Combining Lemma 17 and Lemma 19, we can state that every minor-3-core C of G is isomorphic to the canonical minor-3-core of G. Thus we obtain uniqueness and can refer to *the minor-3-core*, instead of a minor-3-core, meaning the canonical minor-3-core.

We will now give an algorithmic characterisation for the minor-3-core. Similarly to k-cores [29] we can compute the minor-3-core by an iterative pruning process:

▶ **Theorem 20.** *Let G be a graph. The minor-3-core of G can be obtained by performing the following operations to exhaustion:*

- *Let e be an edge incident to a vertex v with degree ≤ 2. Contract v along e.*
- *If v is an isolated vertex, delete v.*

Proof. The graph H resulting from the iterative pruning process described above is clearly a minor of G, in which every vertex has degree at least 3. Let C be the canonical minor-3-core of G. Suppose that there is some edge in C that is not in H. For an edge to not be an edge of H one of the vertices of this edge needs to be contracted to one of its neighbours in the construction of H. Let u be the first vertex of C that is contracted in the construction of H. By definition of the pruning process u needs to be of degree ≤ 2 to be contracted. Since u by the definition of the disjoint-paths property starts off as a vertex of degree ≥ 3, there needs to be a contraction that reduces the degree of u to ≤ 2. For this to be possible, there needs to be a contraction of two neighbouring vertices of u, where both of these vertices are canonical vertices. This must be the case, since in G there are at least three disjoint

paths starting in u that lead to at least three different canonical vertices and there cannot be an edge between the vertices of the disjoint paths, since otherwise these vertices would be canonical vertices themselves. The vertices of this edge would need to be canonical as they are both connected to u, to one other canonical vertex each, as well as to each other by disjoint paths. But since we assumed u to be the first canonical vertex that gets contracted in the construction of H, we reach a contradiction. Hence every edge of C is also an edge of H and since C is edge maximal, H must be equal to the canonical minor-3-core C. ◀

In the following we use the minor-3-core in order to bound the tree-width of planar spanners. To begin the exploration of the relationship of minor-3-cores and tree-width we first make a couple of observations.

1. The minor-3-core of a tree is the empty graph.
2. The minor-3-core of a graph with at most one circle is the empty graph.
3. The K_4 is its own minor-3-core.

Even more, every graph without a K_4 minor has an empty minor-3-core.*

Considering a graph and then connecting a tree to some vertex of our graph, like a branch growing out of our graph, does not increase the tree-width of the graph. The same holds for edges that we replace with paths. So we can see that it is possible to increase the size of our graph without affecting the tree-width of the graph. This is a problem if we want to bound the tree-width of our graph through common separator arguments, for example using [24], since the bound on the tree-width there depends on the size of the graph. Hence it would be nice to have a tool that prunes our graph by removing "tree-like" structures from our graph. For that, we use the definition of minor-3-cores:

▶ **Theorem 21.** *Given a graph of tree-width $k \geq 3$, the tree-width of the minor-3-core of G is k as well.*

Proof. Let G be a graph of tree-width $k \geq 3$ and H its minor-3-core. Since H is a minor of G, the tree-width of H can be at most k. To prove that H must be of tree-width at least k, we prove that contracting a vertex of degree at most 2 does not decrease the tree-width. So let G be a graph of tree-width $k \geq 3$. Suppose that there is a vertex v of degree 2 and let u_1 and u_2 be the two vertices adjacent to v. Let G' be the graph resulting from the contraction of v to u_1. We now argue that G' must have tree-width k as well. Assume that G' has tree-width $k' < k$ and let (T', X') be a tree-decomposition of width k' for G'. We now construct a tree-decomposition (T, X) for G of width k' from (T', X'). In (T', X') there must be some bag $X(i)$ which contains both u_1 and u_2, since they are connected by an edge. We now create a new node, whose bag contains v, u_1 and u_2 and connect it to i by an edge. The tree-decomposition (T, X) is now a valid tree-decomposition for G with tree-width $k' < k$, which is a contradiction to G being of tree-width k. If v is of degree 1, the same argument holds, but we only consider u_1, which has to appear in at least one bag of (T', X'). ◀

As mentioned before we want to bound the tree-width of graphs (especially spanners) through the size of their minor-3-core. Therefore it would be useful to have a bound on the size of the minor-3-core of a graph. If a graph is very "tree-like", i.e. the graph can be decomposed into a set of trees by removing only a few edges, we expect it to have a small minor-3-core. We formalise this statement as follows:

▶ **Lemma*** **22.** *Let G be a connected graph with n vertices and $n-1+m$ edges, where $m \geq 1$ is some integer. The minor-3-core of G has at most $2 \cdot (m-1)$ vertices.*

This lemma is the last part we need to prove an upper bound for the tree-width of connected planar graphs. Intuitively the statement of the lemma could just as well be stated as a bound on the size of minor-3-cores of trees with an additional m edges. If we place these additional edges in a planar fashion, we can combine our bound on the size of minor-3-cores and the bound on the tree-width of planar graphs derived from Theorem 6.2 in [24] to obtain a bound on the tree-width of trees with additional edges that were placed without creating edge crossings. Using this, we can finally proof our second main result:

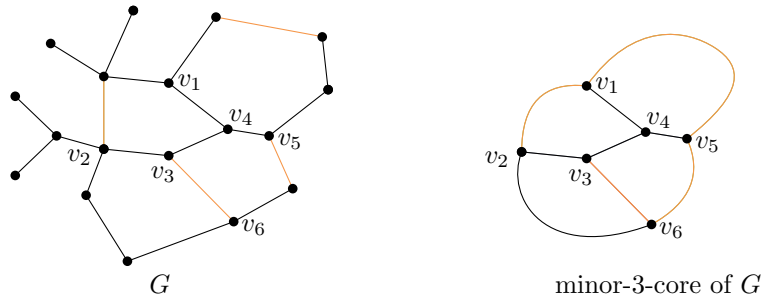


Figure 3 Minor-3-core of a graph G that consists of a tree with 4 additional edges (orange).

Proof of Theorem 3. Since G is a connected graph with $n + \lfloor (k-1)^2/72 \rfloor$ edges, its minor-3-core must by Lemma 22 have at most $\lfloor (k-1)^2/36 \rfloor$ vertices. Also, since the minor-3-core is a minor of G and G is planar, the minor-3-core must be planar as well. The tree-width of a planar graph with n vertices by [24] is at most $6\sqrt{n}+1$. The tree-width of the minor-3-core of G therefore is

$$6\sqrt{\lfloor (k-1)^2/36 \rfloor} + 1 \leq 6\sqrt{(k-1)^2/36} + 1 \leq k.$$

By Lemma 21 we can now conclude that the tree-width of G is either ≤ 2 or exactly k, so in both cases at most k. ◀

5.2 Tree-width of planar spanners

Theorem 3 allows us a stronger result than Theorem 1 in the Euclidean plane: Adjusting the algorithm given in [3], we obtain a plane bounded-degree spanner with bounded tree-width:

▶ **Corollary 4.** *Given a set $P \subset \mathbb{R}^2$ of n points and some positive integer $k \leq 12\sqrt{n-3}$, a plane spanner with tree-width k, maximum vertex degree 4 and dilation $\mathcal{O}(n/k^2)$ can be constructed in $\mathcal{O}(n \log n)$ time.*

Our main adjustment to the algorithm is to chose a plane constant-dilation spanner $\mathcal{S}$ of P with maximum degree 4, as provided in [19], instead of the Delaunay triangulation and to utilise a minimum spanning tree of $\mathcal{S}$ instead of the Euclidean MST. The full version of the algorithm can be found in the full version of the paper. Since the constructed spanner then is a subgraph of $\mathcal{S}$, it is plane and has maximum vertex degree 4. By Theorem 3, its tree-width can then be bounded by the given k. Further, the dilation is not affected, which follows directly by Lemma 4 in [3].

6 Bounded tree-width spanners on convex point sets

We have seen that for unrestricted sets of points in the plane the Delaunay triangulation is an 1.998-spanner of tree-width $6\sqrt{n}+1$. To obtain spanners with smaller tree-widths, we now look at sets of points in the plane that are convex or even lie on a circle.

6.1 Tree spanners

The Euclidean minimum spanning tree is an $(n-1)$-spanner for any set of n points in $\mathbb{R}^d$ of tree-width 1. Aronov et. al. [3] proved that there are sets of points, on which the dilation bound of the minimum spanning tree is almost optimal. To be more precise they showed that there is a set of points, on which any tree-spanner has dilation at least $\frac{2n}{\pi} - 1$. The set they used in their proof was a set of $n \geq 3$ points equally spaced on the unit circle. So, even for tree-spanners on convex sets and sets of points on a circle a dilation of $\Omega(n)$ may be unavoidable. The most obvious tree-spanner on a set of points equally spaced on the unit circle would probably be the spanner that consists of all the edges between points next to each other except for one edge. It is not hard to see that this spanner has dilation $n-1$. But we can show that for points equally spaced on the unit circle, there are tree-spanners with dilation almost $\frac{2n}{\pi} - 1$.

▶ **Lemma*** **23.** *Let P be a set of n points equally spaced on a circle. There is a tree-spanner on P with dilation $\leq \frac{2n}{\pi} + \frac{\pi}{2n}$.*

Proof Idea. Given some $n \geq 2$, let $p_i = \left(\sin\left(\frac{2\pi i}{n}\right), \cos\left(\frac{2\pi i}{n}\right)\right)$. We define $P = \{p_i \mid i \in \{0, \ldots, n-1\}\}$ which is the set of n points equally spaced on the unit circle, where one point is $(0, 1)$. Notice that for any i we have $p_i = p_{i+n}$.

We now construct a tree-spanner G for P with dilation $\leq \frac{2n}{\pi} + \frac{\pi}{2n}$. The spanner G is a path visiting each point on the unit circle in a sawtooth pattern. We define G as the path $p_0, p_{n-1}, p_1, \ldots, p_{\lfloor n/2 \rfloor}$ or more formally as the graph with edges

$$E = \{\{p_i, p_{n-1-i}\} \mid i \in \{0, \ldots, \lfloor n/2 \rfloor - 1\}\} \cup \{\{p_{i+1}, p_{n-i}\} \mid i \in \{0, \ldots, \lfloor (n-1)/2 \rfloor - 1\}\}.$$

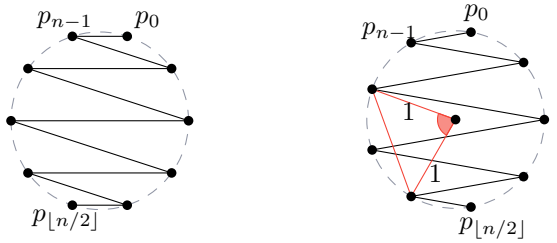


Figure 4 The spanner G on n points equally spaced on the unit circle for $n = 10$ (left) and for $n = 9$ (right).

The Euclidean distance between points p_i and p_j can be derived from the law of cosines:

$$|p_i p_j| = \sqrt{2 - 2\cos\left(\frac{2\pi}{n}(j-i)\right)} = \left|2\sin\left((j-i)\cdot\frac{\pi}{n}\right)\right|.$$

For the distance in G between two points we can add up the Euclidean distances between the points on the path connecting them. Since the spanner G is a path from p_0 to $p_{\lfloor n/2 \rfloor}$, we now let $q_1, ..., q_n$ be the vertices along the path G, where $q_1 = p_0$ and $q_n = p_{\lfloor n/2 \rfloor}$. The distance in G from some q_i to q_j for $i \leq j$ then is

$$d(q_i, q_j) = \sum_{k=i}^{j-1} 2\sin\left(\frac{\pi}{n}\cdot k\right).$$

To complete the proof, we still need to calculate $|q_iq_j|$ and then bound $\frac{d(q_i,q_j)}{|q_iq_j|}$. We can show that

$$|q_iq_j| = \left|2\sin\left(\left\lfloor\frac{j-i+1}{2}\right\rfloor\cdot\frac{\pi}{n}\right)\right|.$$

We then use trigonometric identities to bound $d(q_i,q_j)$ and $|q_iq_j|$ by simpler expressions, from which the lemma then follows. ◀

6.2 Spanners of tree-width 2

While for geometric spanners of tree-width 1, there are sets of points in convex position on which a dilation of $\Omega(n)$ is unavoidable, we can obtain some results for geometric spanners of tree-width 2 on sets of points in convex position.

As every plane graph on a set of points in convex position is outerplanar, and outerplanar graphs have tree-width ≤ 2 [15], every plane spanner on a set of points in convex position has tree-width ≤ 2. Let P be a point set with points in convex position. Since by [4], there is a plane 1.88-spanner for P, there is a 1.88-spanner of tree-width 2 for P. If the points are equally spaced on a circle, by [28] there is even a 1.4482-spanner with tree-width 2.

To show a lower bound, we build on the idea of [16]:

▶ **Lemma*** **24.** *Let P be the set of $n \geq 3$ points equally spaced on the unit circle. Every spanner on P with dilation < 2 must contain the convex hull of P.*

▶ **Lemma 25.** *Let P be the set of $n \geq 3$ points equally spaced on the unit circle. Every t-spanner for $t < 2$ of tree-width 2 on P is plane.*

Proof. Let G be a t-spanner for $t < 2$ of tree-width 2 on P. By Lemma 24, G contains the convex hull of P. For $n = 3$ the statement holds, since the K_3 is plane and has tree-width 2. So let $n \geq 4$ and suppose in the straight-line drawing of G there is an edge-crossing. Let these edges be $\{p,q\}$ and $\{p,q'\}$. On the convex hull of P there must be a path from p to p', not containing q and q', a path from p' to q, not containing p and q', a path from q to q' not containing p and p', and finally a path from q' to p, not containing p' and q. If we perform path-contractions along these paths, leaving only p, q, p' and p', the resulting graph is the K_4, which is of tree-width 3. We reach a contradiction and can conclude that G must be plane. ◀

The desired lower bound for spanners of tree-width 2 now follows from [16] combined with Lemma 25.

▶ **Proposition 26.** *Let P be a set of 23 points equally spaced on a circle. Every spanner of tree-width 2 on P has dilation at least 1.430814.*

7 Conclusion and open questions

In this paper, we give the first algorithm for computing geometric spanners with bounded tree-width for points in Euclidean space of constant dimension. Further, we show a strong dependency between the tree-width and the number of edges in connected planar graphs with few edges.

Our lower bound proves that the dilation of our spanner is worst-case optimal for tree-width k. This naturally leads to the question of optimisation: Given a set of points and a parameter k, can we compute the minimum-dilation spanner of tree-width k? Already for

$k = 1$, that is for minimum-dilation trees, the problem is NP-hard [20, 11]. This raises the question of approximation: Can we efficiently approximate the minimum-dilation spanner of tree-width k in terms of its dilation within a factor better than $\mathcal{O}(n/k^{d/d-1})$? As a bicriteria problem, the tree-width of the approximation may also be $f(k)$ for a suitable function f.

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