On Zarankiewicz's Problem for Intersection Hypergraphs of Geometric Objects

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Abstract

In this paper we study the hypergraph Zarankiewicz's problem in a geometric setting – for r-partite intersection hypergraphs of families of geometric objects. Our main results are essentially sharp bounds for families of axis-parallel boxes in $\mathbb{R}^d$ and families of pseudo-discs. For axis-parallel boxes, we obtain the sharp bound $O_{d,t}(n^{r-1}(\frac{\log n}{\log\log n})^{d-1})$. The best previous bound was larger by a factor of about $(\log n)^{d(2^{r-1}-2)}$. For pseudo-discs, we obtain the bound $O_t(n^{r-1}(\log n)^{r-2})$, which is sharp up to logarithmic factors. As this hypergraph has no algebraic structure, no improvement of Erdős' 60-year-old $O(n^{r-(1/t^{r-1})})$ bound was known for this setting. Futhermore, even in the special case of discs for which the semialgebraic structure can be used, our result improves the best known result by a factor of $\tilde{\Omega}(n^{\frac{2r-2}{3r-2}})$.

To obtain our results, we use the recently improved results for the graph Zarankiewicz's problem in the corresponding settings, along with a variety of combinatorial and geometric techniques, including shallow cuttings, biclique covers, transversals, and planarity.

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1 Introduction

1.1 Background

Zarankiewicz's problem for graphs. A central research area in extremal combinatorics is *Turán-type questions*, which ask for the maximum number of edges in a graph on n vertices that does not contain a copy of a fixed graph H. This research direction was initiated in 1941 by Turán, who showed that the maximum number of edges in a K_r-free graph on n vertices is $(1-\frac{1}{r-1}+o(1))\frac{n^2}{2}$. Soon after, Erdős, Stone and Simonovits solved the problem for all non-bipartite graphs H. They showed that the maximum number is $(1-\frac{1}{\chi(H)-1}+o(1))\frac{n^2}{2}$, where $\chi(H)$ is the chromatic number of H.

The bipartite case turned out to be significantly harder. In 1951, Zarankiewicz raised the following Turán-type question for complete bipartite graphs: Given $n, t \in \mathbb{N}$, what is the maximum number of edges in a *bipartite*[1] graph G on n vertices that does not contain a copy of the complete bipartite graph $K_{t,t}$? This question, known as "Zarankiewicz's problem", has become one of the central open problems in extremal graph theory (see [18]). In one of the cornerstone results of extremal graph theory, Kővári, Sós and Turán [15] proved an upper bound of $O(n^{2-\frac{1}{t}})$. This bound is sharp for $t = 2, 3$ and known matching lower bound constructions use geometric bipartite intersection graphs [3]. The question whether this bound is tight for $t \geq 4$ is widely open.

In recent years, numerous works obtained improved bounds on the number of edges in various algebraic and geometric settings (e.g., [2, 5, 9, 11, 12, 13, 14, 17, 20, 21]). Several of these works studied Zarankiewicz's problem for *intersection graphs* of two families of geometric objects. In such a graph $G(A_1, A_2)$, the vertices are two families A_1, A_2 of geometric objects, and for $a_1 \in A_1$ and $a_2 \in A_2$, (a_1, a_2) is an edge of G if $a_1 \cap a_2 \neq \emptyset$. In particular, Chan and Har-Peled [5] obtained an $O(tn(\frac{\log n}{\log \log n})^{d-1})$ bound for intersection graphs of points vs. axis-parallel boxes in $\mathbb{R}^d$, improving upon results of [2, 21], and observed that a matching lower bound construction appears in a classical paper of Chazelle [6]. They also obtained an $O(tn \log \log n)$ bound for intersection graphs of points vs. pseudo-discs in the plane, and bounds for points vs. halfspaces, balls, shapes with "low union complexity", and more. Keller and Smorodinsky [14] proved a bound of $O_t(n\frac{\log n}{\log \log n})$ for intersection graphs of two families of axis-parallel rectangles in the plane and a bound of $O_t(n)$ for intersection graphs of two families of pseudo-discs. Both bounds are sharp, up to the dependence on t.

Zarankiewicz's problem for hypergraphs. The "hypergraph analogue" of Zarankiewicz's problem asks for the maximum number of hyperedges in an r-uniform hypergraph on n vertices that does not contain $K^r_{t,t,\ldots,t}$ (i.e., the complete r-partite r-uniform hypergraph with all parts having size t) as a subhypergraph.[2] This question was raised in 1964 by Erdős [10], who obtained an upper bound of $O(n^{r-\frac{1}{t^{r-1}}})$ and an almost matching lower bound of $\Omega(n^{r-\frac{c}{t^{r-1}}})$ for general hypergraphs. Note that $\Omega(n^{r-1})$ is a trivial lower bound for this problem, as the complete r-partite graph $K^r_{n,n,\ldots,n,t-1}$ has $(t-1)n^{r-1}$ edges and is clearly $K^r_{t,t,\ldots,t}$-free.

In recent years, a number of works obtained improvements of the bound of Erdős under various algebraic assumptions on the hypergraph. Do [8] obtained a bound of the form $O_{D,d,t,r}(n^{r-\alpha})$ for semialgebraic hypergraphs in $\mathbb{R}^d$, where $\alpha = \alpha(r, d) < 1$ and D is the description complexity. (Here and in the sequel, the asymptotic notation $g = O_x(f)$ means that the dependence on the parameter x, which is assumed to be constant, is not specified). Improved bounds in the same setting were later obtained by Do [9] and by Tidor and Yu [19]. In particular, the results of [19] yield a bound of the form $O_{t,r}(n^{r-\frac{r}{3r-2}})$ for r-partite intersection hypergraphs of r families of discs in the plane. Tong [22] generalized the results of [8] to classes of hypergraphs that satisfy the distal regularity lemma. Basit, Chernikov, Starchenko, Tao, and Tran [2] obtained an improved bound for semilinear hypergraphs. In particular, their technique yields the bound $O_{d,t,r}(n^{r-1}(\log n)^{d(2^{r-1}-1)})$, for r-partite intersection hypergraphs of r families of axis-parallel boxes in $\mathbb{R}^d$ (see definition below).

[1] Note that asking the question for bipartite graphs G like Zarankiewicz did, leads to the same order of magnitude of the size of the extremal graph like in Turán's original question which considers general graphs G, as any graph G with e edges contains a bipartite subgraph with at least $e/2$ edges.

[2] Like in the case of graphs, asking the question for r-partite r-uniform hypergraphs leads to the same order of magnitude of the size of the extremal hypergraph, as any r-uniform hypergraph H with e hyperedges contains an r-partite sub-hypergraph with at least $e \cdot \frac{r!}{r^r}$ hyperedges.

All these improved bounds use in a crucial way the algebraic structure of the hypergraph. No improvements of the bound of Erdős in non-algebraic settings are known.

1.2 Our results

In this paper we study Zarankiewicz's problem in intersection hypergraphs of families of geometric objects. In such an r-uniform r-partite hypergraph $H(A_1, A_2, \ldots, A_r)$, the vertices are r families $A_1, A_2, \ldots, A_r$ of geometric objects, and for $a_1 \in A_1, \ldots, a_r \in A_r$, $(a_1, a_2, \ldots, a_r)$ is a hyperedge of H if $a_1 \cap \ldots \cap a_r \neq \emptyset$. The number of hyperedges in H is denoted by $\mathcal{E}(H)$.

Intersection hypergraphs of axis-parallel boxes. Our first main result is the following upper bound for Zarankiewicz's problem for r-uniform intersection hypergraphs of axis-parallel boxes in $\mathbb{R}^d$. For convenience, we assume that all boxes involved are in a general position, meaning that no two ℓ-dimensional faces in the same directions lie on the same ℓ-flat.

▶ **Theorem 1.** *Let $r, t, d \geq 2$, and let H be the intersection hypergraph of families $A_1, \ldots, A_r$ of n axis-parallel boxes in $\mathbb{R}^d$, such that $A_1 \cup \ldots \cup A_r$ is in a general position. If H is $K^r_{t,\ldots,t}$-free, then*

$$|\mathcal{E}(H)| = O_{r,d}\left(t^2 n^{r-1} \left(\frac{\log n}{\log \log n}\right)^{d-1}\right).$$

This result, which improves the aforementioned result of Basit et al. [2] by a factor of about $(\log n)^{d(2^{r-1}-2)}$, is sharp, up to a factor of $O_{r,d}(t)$. Indeed, as was mentioned above, Chan and Har-Peled [5, Appendix B] showed that for $r = 2$, a lower bound of $\Omega_d(tn(\frac{\log n}{\log \log n})^{d-1})$ for the intersection graph of n points and n axis-parallel boxes in $\mathbb{R}^d$ follows from an old result of Chazelle [6]. This bound clearly holds also for the intersection graph of two families of n axis-parallel boxes in $\mathbb{R}^d$, and thus, it trivially yields an $\Omega_d(tn^{r-1}(\frac{\log n}{\log \log n})^{d-1})$ lower bound for r families, by adding $r-2$ families of n large boxes that contain all the boxes of the two first families.

Our proof exploits geometric properties of axis-parallel boxes and builds upon the upper bounds for Zarankiewicz's problem for intersection graphs of two families of axis-parallel boxes, obtained in [5] for intersections of points and boxes in $\mathbb{R}^d$, and in [14] for intersections of two families of rectangles in $\mathbb{R}^2$. In fact, a direct application of the result of [14] yields a weaker bound with t^6 instead of t^2. We improve the bound to t^2, and show that this is the best bound that can be obtained with the general strategy of [14]. However, we believe that the right dependence on t is linear.

It is somewhat surprising that the power of $\log n$ in the bound does not depend on r. Showing this is the most complex part of our proof, which uses biclique covers [7] and a non-standard inductive argument.

Intersection hypergraphs of pseudo-discs. Our second main result concerns intersection hypergraphs of pseudo-discs. A family of simple Jordan regions in the plane is called *a family of pseudo-discs* if the boundaries of every two regions intersect at most twice. For technical reasons, it is convenient to assume that the pseudo-discs are y-monotone, namely, that the intersection of any vertical line with a region from the family is either empty or an interval. In addition, we assume that the pseudo-discs are in general position, namely, no three boundaries intersect in a point.

▶ **Theorem 2.** *Let $r, t \geq 2$, and let H be the intersection hypergraph of families $A_1, \ldots, A_r$ of n y-monotone pseudo-discs in general position in the plane. If H is $K^r_{t,\ldots,t}$-free, then $|\mathcal{E}(H)| = O_r(t^6 n^{r-1} (\log n)^{r-2})$.*

This result is obviously sharp up to a factor of $O(t^5(\log n)^{r-2})$. Our proof builds upon the upper bound for Zarankiewicz's problem for intersection graphs of two families of pseudo-discs obtained in [14], and uses geometric properties of pseudo-discs and shallow cuttings [16]. As general intersection hypergraphs of pseudo-discs have no algebraic structure, no improvement of the 60-year-old $O(n^{r-\frac{1}{t^{r-1}}})$ bound of Erdős [10] was known in this setting. Furthermore, in the special case of discs whose semialgebraic structure allows applying the previous algebraic works, our result improves over the best known previous result of Tidor and Yu [19] by a factor of $\tilde{\Omega}(n^{\frac{2r-2}{3r-2}})$.

Organization of the paper. In Section 2 we present the proof of Theorem 1. In Section 3 we present the proof of Theorem 2. In the appendices we prove several lemmas that are used in the proofs of the theorems.

2 Intersection Hypergraphs of Axis-Parallel Boxes

In this section we prove our new bound for Zarankiewicz's problem for intersection hypergraphs of axis-parallel boxes in $\mathbb{R}^d$ – namely, Theorem 1. First, in Section 2.1 we present a bound for the *lopsided* Zarankiewicz problem for intersection *graphs* of two families of boxes. Then, in Section 2.2 we use the lopsided version to handle r-partite intersection hypergraphs of boxes.

2.1 Lopsided version of Zarankiewicz's problem for two families of boxes

In this subsection we bound the number of edges in a $K_{t,gt}$-free intersection graph of two families of axis-parallel boxes in $\mathbb{R}^d$. We prove the following:

▶ **Theorem 3.** *Let A, B be two multisets of boxes in $\mathbb{R}^d$ where $|A| = n, |B| = m$ and $m = \text{poly}(n)$. If their intersection graph $G_{A,B}$ is $K_{t,gt}$-free (t on the side of A), then*

$$|E(G_{A,B})| \leq O_d(gtnf_{d,t}(n) + mf_{d,t}(m)),$$

where $f_{d,t}(n) = t(\frac{\log n}{\log\log n})^{d-1}$.

Theorem 3 is a generalization and an improvement of the bound on intersections between two families of axis-parallel rectangles in $\mathbb{R}^2$ proved in [14, Theorem 1.7]. Compared to the result of [14], Theorem 3 applies in $\mathbb{R}^d$ for all $d \geq 2$, is not restricted to the symmetric case of $K_{t,t}$, and has a better dependence on t.

▶ Remark 4. We state the theorem in terms of the auxiliary function $f_{d,t}(n)$, as in the sequel it will be used in a blackbox manner. We conjecture that $f_{d,t}(n)$ can be replaced with $(\frac{\log n}{\log\log n})^{d-1}$, which would lead to a bound of $O_{d,r}(tn^{r-1}(\frac{\log n}{\log\log n})^{d-1})$ for Zarankiewicz's problem for r-partite intersection hypergraphs of axis-parallel boxes in $\mathbb{R}^d$, which is sharp in terms of both n and t.

The proof of Theorem 3 is divided into two main cases, based on the following observation. Any intersection between two axis-parallel boxes belongs to one of two types:

1. *Vertex containment intersections*, in which a vertex of one box is contained in the other box;

2. *Facet intersections*, in which a facet of one box intersects a facet of the other box.

We bound each type of intersections with a different divide-and-conquer argument, as presented below.

2.1.1 Bounding vertex containment intersections

Due to the lack of symmetry between the two sides, we have to consider separately two types of intersections:

- Intersections in which a vertex of a box in A is contained in a box in B;
- Intersections in which a vertex of a box in B is contained in a box in A.

For each type, the problem reduces to bounding the number of edges in $K_{t,gt}$-free intersection graphs of points and axis-parallel boxes. Indeed, for the first case we can bound the number of intersections by $|E(G_{A',B})|$, where A' is the set of vertices of the boxes in A. The second case can be handled similarly. We prove the following proposition, which is a lopsided version of the bound of [5, Theorem 4.5] on the number of edges in $K_{t,t}$-free bipartite intersection graphs of points and axis-parallel boxes in $\mathbb{R}^d$ (though, with a weaker dependence on t).

► **Proposition 5.** *Let A be a multiset of points in $\mathbb{R}^d$ and let B be a multiset of axis-parallel boxes, where $|A| = n$ and $|B| = m$. Then for any $\epsilon > 0$,*

1. *If the bipartite intersection graph $G_{A,B}$ is $K_{t,gt}$-free (t on the side of A) then it has $O_\epsilon\left(gt^2n(\frac{\log n}{\log\log n})^{d-1} + tm(\frac{\log n}{\log\log n})^{d-2+\epsilon}\right)$ edges.*
2. *If the bipartite intersection graph of $G_{A,B}$ is $K_{gt,t}$-free (t on the side of B) then it has $O_\epsilon\left(tn(\frac{\log n}{\log\log n})^{d-1} + gt^2m(\frac{\log n}{\log\log n})^{d-2+\epsilon}\right)$ edges.*

As we show below, this provides a sufficiently strong bound on the vertex containment intersections, for the needs of Theorem 3.

Due to space constraints, the proof of Proposition 5 is presented in the full version of the paper [4].

2.1.2 Bounding facet intersections

We claim that for a large n, the number of facet intersections is significantly smaller than the number of vertex containment intersections.

► **Proposition 6.** *Let A, B be multisets of axis-parallel boxes in $\mathbb{R}^d$, where $|A| = n$ and $|B| = m$. If the bipartite intersection graph $G_{A,B}$ is $K_{t,gt}$-free (t on the side of A) then the number of facet intersections between a box in A and a box in B is*

$$O_d\left(gt^2n(\log n)^{d-2} + tm(\log n)^{d-2}\right).$$

We use the following lemma.

► **Proposition 7.** *Let A be a multiset of horizontal segments in $\mathbb{R}^2$ and let B be a multiset of vertical segments in $\mathbb{R}^2$, where $|A| = n$ and $|B| = m$. Let $g > 0$. If the bipartite intersection graph $G_{A,B}$ is $K_{t,gt}$-free (where the t is on the side of A) then it has $O(gt^2n + tm)$ edges.*

Due to space constraints, the proof of Proposition 7 is presented in the full version of the paper [4].

Proof of Proposition 6. The proof is by induction on d. The base case $d = 2$ amounts to bounding the number of edges in a $K_{2t,2gt}$-free bipartite intersection graph of a multiset A of horizontal segments in $\mathbb{R}^2$ and a multiset B of vertical segments in $\mathbb{R}^2$. Indeed, each "facet intersection" in the plane is either an intersection between a horizontal edge of a rectangle in A with a vertical edge of a rectangle in B, or vice versa. We can bound the number of facet intersections of the first type by $|E(G_{A',B'})|$, where A' is the multiset of horizontal edges of rectangles in A and B' is the multiset of vertical edges of rectangles in B. Note that this intersection graph is $K_{2t,2gt}$-free, as otherwise, $G_{A,B}$ would contain $K_{t,gt}$. The second type can be handled similarly. Hence, Proposition 7 yields a bound of $O(gt^2n + tm)$ in this setting, which proves the induction basis.

In the induction step, we assume that for $(d-1)$-dimensional boxes, the number of facet intersections is bounded by

$$O_d(gt^2n(\log n)^{d-3} + tm(\log n)^{d-3}).$$

We bound the number of facet intersections between $x \in A$ and $y \in B$, such that the intersecting facets are othogonal to the $(d-1)$'th and the d'th axis, respectively. A bound on the total number of facet intersections clearly follows by multiplying with d^2. In the rest of the proof, we call such special intersections "good facet intersections", or just "intersections" for brevity. We make sure that in the dimension reductions in the induction process presented below, every time the first coordinate is the one that is removed, and thus, the good facet intersections are not affected.

We use the following auxiliary notion. Let $I_d(n,m)$ be the maximum number of good facet intersections between multisets A, B of axis-parallel boxes inside a "vertical strip" $\mathcal{U} = \{x \in \mathbb{R}^d : u_L < x_1 < u_R\}$ of $\mathbb{R}^d$, where:

1. Each box in A, B has either half of its vertices or all of its vertices in $\mathcal{U}$;
2. The total number of vertices of boxes in A (resp., boxes in B) inside $\mathcal{U}$ is $n \cdot 2^{d-1}$ (resp., $m \cdot 2^{d-1}$);
3. The bipartite intersection graph of A, B is $K_{t,gt}$-free.

We shall prove that

$$I_d(n,m) \leq O_d(gt^2n(\log n)^{d-2} + tm(\log n)^{d-2}).$$

This clearly implies the assertion of the proposition, as by considering a vertical strip $\mathcal{U}$ that fully contains all boxes in A and B (where $|A| = n$ and $|B| = m$), we get that the number of good facet intersections between A and B is at most $I_d(2n, 2m)$. (Note that we neglect factors of $O_d(1)$).

Let A, B be multisets of boxes that satisfy assumptions (1)–(3) with respect to a strip $\mathcal{U} \subset \mathbb{R}^d$. We divide $\mathcal{U}$ into two vertical sub-strips $\sigma_1 = \{x \in \mathbb{R}^d : u_L < x_1 < u'\}$ and $\sigma_2 = \{x \in \mathbb{R}^d : u' < x_1 < u_R\}$, such that each sub-strip contains $n \cdot 2^{d-2}$ vertices of boxes in A. For $i = 1, 2$, we denote by A_i (resp., B_i) the boxes in A (resp., B) that have at least one vertex in σ_i, and by A'_i (resp., B'_i) the boxes in A (resp., B) that intersect σ_i but do not have vertices in it. Note that $A'_1 \subset A_2, A'_2 \subset A_1$, and similarly for B. We also denote the number of vertices of boxes in B contained in σ_i by $2^{d-1} \cdot m_i$.

The number of good facet intersections in $\mathcal{U}$ between boxes in A and boxes in B is at most

$$I(A_1, B_1) + I(A_2, B_2) + I(A_1, B'_1) + I(A'_2, B_2) + I(A_2, B'_2) + I(A'_1, B_1), \tag{1}$$

where $I(X, Y)$ denotes the number of good facet intersections between an element of X and an element of Y.

Indeed, $I(A_1, B_1)$ (resp., $I(A_2, B_2)$) counts intersections between pairs of boxes that have at least one vertex in σ_1 (resp., σ_2). $I(A_1, B_1') + I(A_2', B_2)$ upper bounds the number of intersections between a box in A that has a vertex in σ_1 and a box in B that has a vertex in σ_2. $I(A_2, B_2') + I(A_1', B_1)$ upper bounds the number of intersections between a box in A that has a vertex in σ_2 and a box in B that has a vertex in σ_1.

By the definitions, we have $I(A_1, B_1) \le I_d(\frac{n}{2}, m_1)$ (where σ_1 is taken as the vertical strip instead of $\mathcal{U}$). Similarly, $I(A_2, B_2) \le I_d(\frac{n}{2}, m_2)$.

To handle the other types of intersections, we observe that a box in A_1 intersects a box in B_1' if and only if their projections on the hyperplane $\mathcal{H} = \{x \in \mathbb{R}^d : x_1 = u'\}$ (which are $(d-1)$-dimenstional boxes) intersect. Let $\bar{A}_1$ and $\bar{B}_1'$ denote the corresponding families of projections. We have $|\bar{A}_1| = |A_1| = \frac{n}{2}$ and $|\bar{B}_1'| = |B_1'| \le m_2$. The multisets $\bar{A}_1$ and $\bar{B}_1'$ are multisets of axis-parallel boxes in $\mathbb{R}^{d-1}$ whose bipartite intersection graph is $K_{t,gt}$-free. Hence, by the induction hypothesis we have

$$I(A_1, B_1') = I(\bar{A}_1, \bar{B}_1') \le O_d\left(gt^2 \tfrac{n}{2}\left(\log \tfrac{n}{2}\right)^{d-3} + tm_2\left(\log \tfrac{n}{2}\right)^{d-3}\right)$$
$$\le O_d\left(gt^2 \tfrac{n}{2}(\log n)^{d-3} + tm_2(\log n)^{d-3}\right),$$

Applying the same argument to $I(A_2', B_2), I(A_1', B_1)$ and $I(A_2, B_2')$, we get

$$\begin{aligned} &I(A_1, B_1') + I(A_2', B_2) + I(A_2, B_2') + I(A_1', B_1) \\ &\quad \le O_d(4gt^2 \tfrac{n}{2}(\log n)^{d-3} + 2tm_2(\log n)^{d-3} + 2tm_1(\log n)^{d-3}) \\ &\quad = O_d(2gt^2 n(\log n)^{d-3} + 2tm(\log n)^{d-3}). \end{aligned}$$

Combining this with the bounds on $I(A_1, B_1)$ and $I(A_2, B_2)$ and substituting to (1), we obtain the recursive formula

$$I_d(n, m) \le \max_{m_1+m_2=m}\left(I_d(\tfrac{n}{2}, m_1) + I_d(\tfrac{n}{2}, m_2) + O_d(2gt^2 n(\log n)^{d-3} + 2tm(\log n)^{d-3})\right),$$

which solves to

$$I_d(n, m) \le \log n \cdot O_d(gt^2 n(\log n)^{d-3} + tm(\log n)^{d-3}) \le O_d(gt^2 n(\log n)^{d-2} + tm(\log n)^{d-2}).$$

This completes the proof. ◀

2.1.3 Completing the proof of Theorem 3

Now we are ready to wrap up the proof of Theorem 3.

Proof of Theorem 3. Let A, B be multisets of axis-parallel boxes in $\mathbb{R}^d$ that satisfy the assumptions of the theorem. As written above, the intersections between A and B can be divided into *vertex containment intersections* and *facet intersections*.

To bound the number of vertex containment intersections, we apply Proposition 5, with $\epsilon = \frac{1}{2}$. Specifically, we use Proposition 5(1) to bound the number of intersections in which a vertex of $a \in A$ is contained in a box $b \in B$. Furthermore, we use Proposition 5(2) *with the roles of n and m reversed* to bound the number of intersections in which a vertex of $b \in B$ is contained in a box $a \in A$. We get that the number of vertex containment intersections is at most

$$O_d\left(gt^2 n(\tfrac{\log n}{\log\log n})^{d-1} + tm(\tfrac{\log n}{\log\log n})^{d-1.5} + tm(\tfrac{\log m}{\log\log m})^{d-1} + gt^2 n(\tfrac{\log m}{\log\log m})^{d-1.5}\right).$$

By Proposition 6, the number of facet intersections is at most

$$O_d(gt^2n(\log n)^{d-2} + tm(\log n)^{d-2}).$$

Note that for a large n, we have $(\log n)^{d-2} = o((\frac{\log n}{\log\log n})^{d-1.5})$, and hence, the number of facet intersections is dominated by the number of vertex containment intersections.

Finally, since $m = \text{poly(n)}$, the terms $tm(\frac{\log n}{\log\log n})^{d-1.5}$ and $gt^2n(\frac{\log m}{\log\log m})^{d-1.5}$ are dominated by the two other terms. Therefore, we have

$$|E(G_{A,B})| \le O_d\left(gt^2n(\frac{\log n}{\log\log n})^{d-1} + tm(\frac{\log m}{\log\log m})^{d-1}\right) = O_d(gtnf_{d,t}(n) + mf_{d,t}(m)),$$

as asserted. This completes the proof. ◀

2.2 Zarankiewicz problem for r-partite intersection hypergraph of boxes

Using Theorem 3, we can relatively easily obtain Proposition 8 below, which is a weaker version of Theorem 1. Since the proof of the weaker result may serve as a good introduction to the proof of Theorem 1, we present it in Section 2.2.1, and then we pass to the proof of Theorem 1 in Section 2.2.2.

2.2.1 Proof of a weaker variant of Theorem 1

We prove the following.

▶ **Proposition 8.** *Let $A_1, A_2, \ldots, A_r$ be families of n axis-parallel boxes in $\mathbb{R}^d$, and let H be their r-partite intersection hypergraph. If H is $K^r_{t,t,\ldots,t}$-free, then $|\mathcal{E}(H)| = O_r\left(t\,(nf_{d,t}(n))^{r-1}\right)$.*

The proof of Proposition 8 is by induction on r. For the sake of simplicity, we present here the proof for $r = 3$, and complete the argument for a general r in the full version of the paper [4].

Proof of Proposition 8 for $r = 3$. Let $A_1 = A, A_2 = B, A_3 = C$. Let AB be the following multiset of axis-parallel boxes: $AB = \{a \cap b : a \in A, b \in B, a \cap b \neq \emptyset\}$. (Note that the intersection of two axis-parallel boxes is indeed an axis-parallel box). Let G be the bipartite intersection graph of the families C and AB. It is clear that $|E(G)| = |\mathcal{E}(H)|$, since there is a clear one-to-one correspondence between edges of G and hyperedges of H.

We claim that for a sufficiently large constant M, the graph G is $K_{t,Mtnf_{d,t}(n)}$-free. Indeed, assume to the contrary that G contains a copy of $K_{t,Mtnf_{d,t}(n)}$, for a "large" M. This means that there exist t boxes $c_1, c_2, \ldots, c_t \in C$ which all have a non-empty intersection with certain $Mtnf_{d,t}(n)$ axis-parallel boxes of the form $a_i \cap b_i$, with $a_i \in A$, $b_i \in B$. Denote by A' the set of all $a_i \in A$ that participate in such intersections, and by B' the set of all $b_i \in B$ that participate in such intersections. Let G' be the bipartite intersection graph of A', B'. We have $|E(G')| \ge Mtnf_{d,t}(n)$, and hence, by Theorem 3 (applied with $m = n$ and $g = 1$), it contains a $K_{t,t}$ (using the assumption that M is sufficiently large). This means that there exist $a_1, a_2, \ldots, a_t \in A$ and $b_1, b_2, \ldots, b_t \in B$ such that for all $1 \le i, j \le t$, we have $a_i \cap b_j \neq \emptyset$, and both a_i and b_j have a non-empty intersection with each of $c_1, \ldots, c_t$ (since they participate in pairs whose intersection with each of $c_1, \ldots, c_t$ is non-empty). As any three pairwise intersecting axis-parallel boxes have a non-empty intersection, this implies that $a_i \cap b_j \cap c_k \neq \emptyset$ for all $1 \le i, j, k \le t$, and consequently, H contains a $K_{t,t,t}$, a contradiction.

We have thus concluded that G is $K_{t,Mtnf_{d,t}(n)}$ free, for a sufficiently large constant M. By Theorem 3, applied with n, $m = n^2$, and $g = Mnf_{d,t}(n)$, this implies that

$$|\mathcal{E}(H)| = |E(G)| \leq O\left(t\,(nf_{d,t}(n))^2\right).$$

This completes the proof. ◀

Note that in this proof, we used the fact that axis-parallel boxes in $\mathbb{R}^d$ have *Helly number 2*, namely, that any pairwise intersecting family of axis-parallel boxes has a non-empty intersection. Without this property, the existence of $K_{t,t}$ in G' does not necessarily imply the existence of $K_{t,t,t}$ in H.

The proof for a general r is an inductive process, in which at the ℓ'th step, one considers an auxiliary bipartite intersection graph of boxes, whose vertex sets are the multiset $A_1A_2\ldots A_{\ell+1} = \{(a_1 \cap \ldots \cap a_{\ell+1}) : \forall i, a_i \in A_i, \cap_{i=1}^{\ell+1} a_i \neq \emptyset\}$, where the multiplicity of each element is the number of $(a_1, \ldots, a_{\ell+1})$ tuples that lead to it, and $A_{\ell+2}$. At each step, Theorem 3 is applied once again, and the fact that axis-parallel boxes have Helly number 2 is deployed once again. As this part of the proof is a bit more cumbersome and is not needed for the proof of Theorem 1 presented below, we provide it in the full version of the paper [4].

2.2.2 Proof of Theorem 1

In Proposition 8, the polylogarithmic factor in the upper bound increases with r, since at each step of the inductive process we apply Theorem 3 and "pay" another $f_{d,t}(n)$-factor. In this subsection, we show how to avoid this dependency on r. Let us restate Theorem 1.

▶ **Theorem 9** (Theorem 1 restated). *Let $r, d, t \geq 2$ and let H be the intersection hypergraph of r families $A_1, \ldots, A_r$ of n axis-parallel boxes in general position in $\mathbb{R}^d$. If H is $K^r_{t,\ldots,t}$-free, then $|\mathcal{E}(H)| = O_{r,d}(tn^{r-1} f_{d,t}(n))$, where $f_{d,t}(n) = t(\frac{\log n}{\log\log n})^{d-1}$.*

In the proof, we use the following modification of the *biclique cover theorem* (see [7]) for axis-parallel boxes, that may be of independent interest. We note that the relation between Zarankiewicz's problem and biclique covers was already observed in [5, 9].

▶ **Definition 10.** *A biclique cover of a graph $G = (V, E)$ is a collection of pairs of vertex subsets $\{(A_1, B_1), \ldots, (A_l, B_l)\}$ such that $E = \bigcup_{i=1}^{l}(A_i \times B_i)$.*

▶ **Lemma 11.** *Let $n, m, b \in \mathbb{N}$ be such that $b \leq \min(n, m)$, and let A, B be families of axis-parallel boxes in $\mathbb{R}^d$, with $|A| = n$ and $|B| = m$. Then the bipartite intersection graph of A, B can be partitioned into a union of $O(b \log^{d-1} b)$ bicliques (with no restriction on their size) and $O(b \log^{d-1} b)$ "partial bicliques" (namely, subgraphs of bicliques), each of size at most $\frac{n}{b} \cdot \frac{m}{b}$.*

Due to space constraints, the proof of the lemma is presented in the full version of the paper [4].

The motivation behind our proof of Theorem 1 is as follows: Assume that we add an additional assumption that the $(r-1)$-partite intersection graph of the families $A_1, \ldots, A_{r-1}$ is complete. Namely, that for all $1 \leq i < j \leq r-1$, any box in A_i intersects any box in A_j. Furthermore, assume that by applying (only once!) Theorem 3, we show that the bipartite intersection graph of the multiset $A_1A_2\ldots A_{r-1} = \{a_1 \cap \ldots \cap a_{r-1} : \forall i, a_i \in A_i, \cap_{i=1}^{r-1} a_i \neq \emptyset\}$ and the family A_r contains $K_{gt,t}$, for $g = n^{r-2}$. This means that some t boxes from A_r intersect $gt = tn^{r-2}$ boxes of the type $\{a_1 \cap \ldots \cap a_{r-1} : a_i \in A_i\}$. These gt boxes involve at least t boxes from each A_i ($1 \le i \le r-1$). Thus, by our additional assumption and by the fact that axis-parallel boxes admit Helly number 2, the existence of $K_{gt,t}$ in the bipartite intersection graph, implies the existence of $K^r_{t,\dots,t}$ in the original r-partite intersection hypergraph. Therefore, the suggested additional assumption enables avoiding repeated applications of Theorem 3, and hence obtaining a bound with a logarithmic factor that does not increase with r.

In order to show that we can indeed make the additional assumption described above without affecting the final bound, we define a constraints graph G, not to be confused with the auxiliary graph G from Section 2.2.1.

▶ **Definition 12.** *For an r-tuple of set families $A_1, \dots, A_r$, the* constraints graph $G = G_{A_1,\dots,A_r}$ *is defined as follows. The vertex set of G is $V(G) = \{1, 2, \dots, r\}$, and $(i, j) \in G$ if $\exists a_i \in A_i, a_j \in A_j$ such that $a_i \cap a_j = \emptyset$.*

This graph G represents the "distance" of the r-tuple of families $A_1, \dots, A_r$ from the desired setting in which there exists i such that any box from A_j intersects any box from A_k, for all $j, k \neq i$. Our goal in the proof below is to remove all the edges from G, except for those emanating from a single vertex i. This will show that one can indeed make the additional assumption described above without affecting the final bound.

Proof of Theorem 9. Let $T_G(n_1, \dots, n_r)$ be the maximum number of hyperedges in a $K^r_{t,\dots,t}$-free r-partite intersection hypergraph H of r families $A_1, \dots, A_r$ of axis-parallel boxes in $\mathbb{R}^d$, where $|A_i| = n_i$, and the constraints graph of r-tuple $A_1, \dots, A_r$ is G. Denote $T_G(n) = T_G(n, \dots, n)$. In order to prove the theorem, it is clearly sufficient to prove that for any constraint graph G, we have $T_G(n) \le O(tn^{r-1} f_{d,t}(n))$. We prove this claim by induction on $|E(G)|$.

If $E(G) = \emptyset$, then for all $i \neq j$, any $a_i \in A_i$ intersects any $a_j \in A_j$. Since H is $K^r_{t,\dots,t}$-free, this implies $n < t$ and we are done.

In the induction step, we assume that we have already proved the claim for any constraints graph with a smaller number of edges, and we now consider a constraints graph G. We consider three cases:

- Case A: G contains two non-adjacent edges;
- Case B: G is a triangle;
- Case C: G is a star.

Clearly, any graph G belongs to one of the three cases.

Case A: *G* contains two non-adjacent edges. Say these two non-adjacent edges are $(1, 2)$ and $(3, 4)$. Our goal now is to "remove" the edge $(1, 2)$ (and later, also the edge $(3, 4)$) from G, by partitioning the intersection graph of A_1 and A_2 into bicliques. To this end, we use Lemma 11 with a large constant b to partition the entire hypergraph into smaller hypergraphs, induced by a partition of the bipartite intersection of A_1 and A_2 to "full" bicliques and "partial" bicliques. We obtain the recursion:

$$T_G(n) \le O(b \log^{d-1} b) T_G(\tfrac{n}{b}, \tfrac{n}{b}, n, \dots, n) + O(b \log^{d-1} b) O(tn^{r-1} f_{d,t}(n)), \tag{2}$$

where the right term comes from applying the induction hypothesis on the hypergraphs induced by full bicliques, whose constraints graph is $G \setminus \{(1, 2)\}$, and the left term comes from the partial bicliques.

Now, to bound $T_G(\frac{n}{b}, \frac{n}{b}, n, \dots, n)$, we do the same with the sets A_3, A_4 and obtain

$$T_G(\tfrac{n}{b}, \tfrac{n}{b}, n, \dots, n) \le O(b \log^{d-1} b) T_G(\tfrac{n}{b}, \tfrac{n}{b}, \tfrac{n}{b}, \tfrac{n}{b}, n, \dots, n) + O(b \log^{d-1} b) O(tn^{r-1} f_{d,t}(n)). \tag{3}$$

Combining (2) and (3) together, we get

$$T_G(n) \le O(b^2 \log^{2d-2} b) \left(T_G(\tfrac{n}{b}, \tfrac{n}{b}, \tfrac{n}{b}, \tfrac{n}{b}, n, \ldots, n) + O(tn^{r-1} f_{d,t}(n))\right). \quad (4)$$

To bound the left term in the right hand side, we partition the hypergraph into b^{r-4} hypergraphs which r sides of size $\frac{n}{b}$ by partitioning each of the sets $A_5, \ldots, A_r$ arbitrarily into b parts, each of size $\frac{n}{b}$. We obtain

$$T_G(n) \le O(b^2 \log^{2d-2} b) \left(b^{r-4} T_G(\tfrac{n}{b}) + O(tn^{r-1} f_{d,t}(n))\right). \quad (5)$$

The recursion (5) solves to $T_G(n) = O(tn^{r-1} f_{d,t}(n))$, and we are done.

Case B: G is a triangle. Say $E(G) = \{(1,2),(2,3),(1,3)\}$. In this case, we first split A_1 and A_2 by Lemma 11, then we split A_2 and A_3 and then we split A_1 and A_3.

Using the induction hypothesis, by splitting A_1, A_2 we obtain

$$T_G(n) \le O(b \log^{d-1} b) \left(T_G(\tfrac{n}{b}, \tfrac{n}{b}, n, \ldots, n) + O(tn^{r-1} f_{d,t}(n))\right). \quad (6)$$

By splitting A_2, A_3, we get

$$T_G(\tfrac{n}{b}, \tfrac{n}{b}, n, \ldots, n) \le O(b \log^{d-1} b) \left(T_G(\tfrac{n}{b}, \tfrac{n}{b^2}, \tfrac{n}{b}, n, \ldots, n) + O(tn^{r-1} f_{d,t}(n))\right), \quad (7)$$

and by splitting A_1, A_3 we have

$$T_G(\tfrac{n}{b}, \tfrac{n}{b^2}, \tfrac{n}{b}, n, \ldots, n) \le O(b \log^{d-1} b) \left(T_G(\frac{n}{b^2}, \frac{n}{b^2}, \frac{n}{b^2}, n, \ldots, n) + O(tn^{r-1} f_{d,t}(n))\right). \quad (8)$$

Combining (6)-(8), we obtain

$$T_G(n) \le O(b^3 \log^{3d-3} b) \left(T_G(\tfrac{n}{b^2}, \tfrac{n}{b^2}, \tfrac{n}{b^2}, n, \ldots, n) + O(tn^{r-1} f_{d,t}(n))\right), \quad (9)$$

and by partitioning each of $A_4, \ldots, A_r$ arbitrarily into b^2 equal parts, we get

$$T_G(n) \le O(b^{2r-3} \log^{3d-3} b) T_G(\tfrac{n}{b^2}) + O(b^3 \log^{3d-3} b) O(tn^{r-1} f_{d,t}(n)). \quad (10)$$

As above, this recursion solves to $T_G(n) = O(tn^{r-1} f_{d,t}(n))$.

Case C: G is a star. Say G is a star centered at A_r. In this case we don't apply the induction hypothesis. Instead, we apply Theorem 3 (only once, in contrast to the argument in Section 2.2.1).

Assume on the contrary that $T_G(n) > Ctn^{r-1} f_{d,t}(n)$ for some large constant C. Consider the intersection graph between the multiset $A_1 A_2 \ldots A_{r-1} = \{a_1 \cap \ldots \cap a_{r-1} : \forall i, a_i \in A_i, \cap_{i=1}^{r-1} a_i \ne \emptyset\}$ and the family A_r. Since the number of edges in this graph is at least $\Omega(tn^{r-1} f_{d,t}(n))$, by Theorem 3 (with the parameters $n, m = n^{r-1}$ and $g = n^{r-2}$), this intersection graph contains $K_{gt,t}$.

This means that some t specific boxes from A_r intersect $gt = n^{r-2}t$ boxes of the type $a_1 \cap \ldots \cap a_{r-1}$ ($a_i \in A_i$). These gt boxes involve at least t boxes from each A_i ($1 \le i \le r-1$). Hence, there exist $A'_1 \subset A_1, \ldots, A'_r \subset A_r$, each of size t, such that any $a_1, \ldots, a_r$, where $a_1 \in A'_1, \ldots, a_r \in A'_r$, are pairwise intersecting. (Here we use the setting of case C in which for all $1 \le i < j \le r-1$, any two boxes $a_i \in A_i$ and $a_j \in A_j$ intersect.) Since axis-parallel boxes have Helly number 2, this implies that for all $a_1 \in A'_1, \ldots, a_r \in A'_r$ we have $a_1 \cap \ldots \cap a_r \ne \emptyset$, and hence, the restriction of H to $A'_1 \cup \ldots \cup A'_r$ is a copy of $K^r_{t,t,\ldots,t}$, a contradiction. This completes the proof of Theorem 9, and thus of Theorem 1. ◀

3 Intersection Hypergraphs of Pseudo-discs

In this section we prove Theorem 2. Let us recall the statement of the theorem.

▶ **Theorem 2.** Let $r, t \geq 2$ and let H be the r-partite intersection hypergraph of families $A_1, \ldots, A_r$ of n y-monotone pseudo-discs in general position in the plane. If H is $K^r_{t,\ldots,t}$-free, then $|\mathcal{E}(H)| = O(t^6 n^{r-1} (\log n)^{r-2})$.

A tool crucially used in the proof is a lopsided version of the following result from [5].

▶ **Theorem 13** ([5, Corollary 5.1]). *Let P be a set of n points in the plane, and let $\mathcal{F}$ be a family of m y-monotone pseudo-discs in general position in the plane. If the bipartite intersection graph $G(P, \mathcal{F})$ is $K_{t,t}$-free ($t \geq 2$), then*

$$|E(G(P, \mathcal{F}))| = O(tn + tm \log \log m + \log t).$$

The lopsided variant of Theorem 13 reads as follows.

▶ **Theorem 14** (Lopsided variant of Theorem 13). *Let P be a multiset of n points in the plane, and let $\mathcal{F}$ be a multiset of m y-monotone pseudo-discs in general position in the plane. If the bipartite intersection graph $G(P, \mathcal{F})$ is $K_{gt,t}$-free ($t, g \geq 2$, gt on the side of the points), then*

$$|E(G(P, \mathcal{F}))| = O(tn + gtm \log m).$$

Due to space constraints, the proof of Theorem 14, which uses the geometric technique of *shallow cuttings* [16], is presented in the full version of the paper [4]. Note that for $g = 1$, the bound we obtain is weaker than the bound of Theorem 13. The reason for this is explained in the full version of the paper [4].

The proof of Theorem 2 uses induction on r, where the induction basis ($r = 2$) is the following theorem from [14]:

▶ **Theorem 15** ([14, Theorem 1.6]). *Let $t \geq 2$ and let G be the bipartite intersection graph of two families of pseudo-discs, each of size n. If G is $K_{t,t}$-free then $|E(G)| = O(t^6 n)$.*

Proof of Theorem 2. The proof is by induction on r, the base case being Theorem 15 above. In the induction step, we use the following observation: If r simple Jordan regions in $\mathbb{R}^2$ intersect, then either

- There is a pair of regions a_1, a_2 such that an intersection point of their boundaries is contained in all other $r - 2$ regions, or
- There is a region a_1 which is fully contained in all other $r - 1$ regions.

(To see why this holds, take a point in the intersection and look at all points on the boundaries of the regions which can be reached from it without crossing a boundary of one of the regions. If all these points belong to the same region, the second case holds. If not, by the assumptions one of these points is an intersection point of two regions and the first case holds).

We call an intersection of the first type a *type A* - intersection, and an intersection of the second type a *type B* - intersection.

For each intersecting pair $\{a_1, a_2\}$ of pseudo-discs we define a special point $p(a_1, a_2)$ as follows: If the intersection of a_1 and a_2 is of type A, then $p(a_1, a_2)$ is the left intersection point of the boundaries of a_1 and a_2. If this intersection is of type B, where $a_1 \subset a_2$, then $p(a_1, a_2)$ is the leftmost point of a_1.

Without loss of generality, both for Type A and for Type B, we count intersections of the type $a_1 \cap \ldots \cap a_r \neq \emptyset$ ($a_i \in A_i$), where $p(a_1, a_2)$ is contained in $a_3 \cap \ldots \cap a_r$. This affects the final bound by a multiplicative factor of $O_r(1)$.

Let P be the multiset $P = \{p(a_1, \ldots, a_{r-1}) : a_i \in A_i\}$, where $p(a_1, \ldots, a_{r-1})$ is $p(a_1, a_2)$ with multiplicity which is determined by the number of other tuples of a_i's whose intersection contains it. Clearly, $|P| \leq n^{r-1}$. Let G be the bipartite intersection graph of the multiset of points P and the family of pseudo-discs A_r. If $|\mathcal{E}(H)| > C_r t^6 n^{r-1} (\log n)^{r-2}$ (for a sufficiently large constant C_r), then by Theorem 14, G contains $K_{gt,t}$ for $g = C_{r-1} t^5 n^{r-2} (\log n)^{r-3}$ (where C_{r-1} will be specified below, and C_r is taken to be sufficiently large for the statement to hold by Theorem 14). This $K_{gt,t}$ comes from a set S of $gt = C_{r-1} t^6 n^{r-2} (\log n)^{r-3}$ distinct tuples of the form $(a_1, \ldots, a_{r-1})$ where $a_i \in A_i$, and a set T of t pseudo-discs from A_r. Let $Z = \bigcap \{a_r : a_r \in T\}$ be the intersection of all the pseudo-discs in T. Since all the elements of T contain a common point (e.g., each point in P), Z is non-empty, and it follows from properties of pseudo-discs families that Z is connected. (For a proof of this geometric fact, see [1, Theorem 4.4]).

Now, we clip each pseudo-disc $s \in A_1 \cup \ldots \cup A_{r-1}$ to $s' = s \cap Z$, and slightly perturb the boundaries such that the family $\{s' : s \in A_1 \cup \ldots \cup A_{r-1}\}$ is still a family of pseudo-discs. This perturbation can be performed as follows: Define a partial ordering on $A_1 \cup \ldots \cup A_{r-1}$, by considering the intersection of each pseudo-disc with the boundary of Z and ordering by inclusion. Extend the ordering into a linear ordering arbitrarily. Clip the pseudo-discs close to the boundary of Z from the inside, in such a way that a "smaller" pseudo-disc (according to the linear ordering) is clipped closer to the boundary. In this way, an intersection is added only to pairs of pseudo-discs whose boundaries have only one intersection point inside Z, and therefore, any two boundaries of $\{s' : s \in A_1 \cup \ldots \cup A_{r-1}\}$ intersect at most twice.

Consider the $(r-1)$-partite intersection hypergraph H' of the families $A'_1, \ldots, A'_{r-1}$, where $A'_i = \{s' : s \in A_i\}$. H' has at least $|S|$ hyperedges, since for each $(a_1, \ldots, a_{r-1}) \in S$, the point $p(a_1, \ldots, a_{r-1})$ is contained in Z, and hence, $a'_1 \cap a'_2 \cap \ldots \cap a'_{r-1} \neq \emptyset$. Since $|S| > C_{r-1} t^6 n^{r-2} (\log n)^{r-3}$, by the induction hypothesis H' contains a $K^{r-1}_{t,\ldots,t}$ all of whose elements are fully contained in $Z = \bigcap \{a : a \in T\}$. (Here, C_3 is chosen to be sufficiently large for applying the inductive hypothesis.) Together with the set $T \subset A_r$, we obtain $K^r_{t,\ldots,t}$ in H. This completes the proof. ◀

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