# Apex Representatives

**Tamal K. Dey**
Purdue University, West Lafayette, IN, USA

**Tao Hou**
University of Oregon, Eugene, OR, USA

**Dmitriy Morozov**
Lawrence Berkeley National Laboratory, Berkeley, CA, USA

## Abstract

Given a zigzag filtration, we want to find its barcode representatives, i.e., a compatible choice of bases for the homology groups that diagonalize the linear maps in the zigzag. To achieve this, we convert the input zigzag to a levelset zigzag of a real-valued function. This function generates a Mayer–Vietoris pyramid of spaces, which generates an infinite strip of homology groups. We call the origins of indecomposable (diamond) summands of this strip their apexes and give an algorithm to find representative cycles in these apexes from ordinary persistence computation. The resulting representatives map back to the levelset zigzag and thus yield barcode representatives for the input zigzag. Our algorithm for lifting a $p$-dimensional cycle from ordinary persistence to an apex representative takes $O(p \cdot m \log m)$ time. From this we can recover zigzag representatives in time $O(\log m + C)$, where $C$ is the size of the output.

**2012 ACM Subject Classification** Theory of computation → Computational geometry; Mathematics of computing → Algebraic topology

**Keywords and phrases** zigzag persistent homology, Mayer–Vietoris pyramid, cycle representatives

**Digital Object Identifier** 10.4230/LIPIcs.SoCG.2025.40

**Related Version** *Full Version*: `https://arxiv.org/abs/2502.17704` [8]

**Funding** *Tamal K. Dey*: partially supported by NSF grants CCF-2437030 and DMS-2301360.
*Tao Hou*: supported by NSF fund CCF 2439255.
*Dmitriy Morozov*: supported by the Director, Office of Science, Office of Advanced Scientific Computing Research, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

## 1 Introduction

In topological data analysis, one often wants to not only describe the distribution of topological features in data, but also to identify individual features directly in the input. For persistent homology, a natural starting point are the cycle representatives of homology classes. Specifically, one wants to find *barcode representatives*, a consistent choice of bases for the input sequence of homology groups. For ordinary persistence, this comes directly from the computation: a representative at the start of the bar includes into every consecutive space and therefore remains a valid choice of the basis element.

For zigzag persistence, the situation is more complicated. Because simplices can enter and leave the complex, a representative cycle at one step in the zigzag filtration may not exist at a later step. So even though zigzag persistence algorithms keep track of representative cycles, the specific choices that they make do not always produce compatible barcode representatives. To cope with this problem, Dey et al. [9] introduced an algorithm that keeps track of additional information that allows it to report consistent barcode representatives. For a zigzag filtration with $m$ insertions and deletions, the algorithm requires $O(m^3)$ preprocessing and $O(m^2)$ time to report one or all representatives for any single bar, and $O(m^3)$ time to report all representatives.

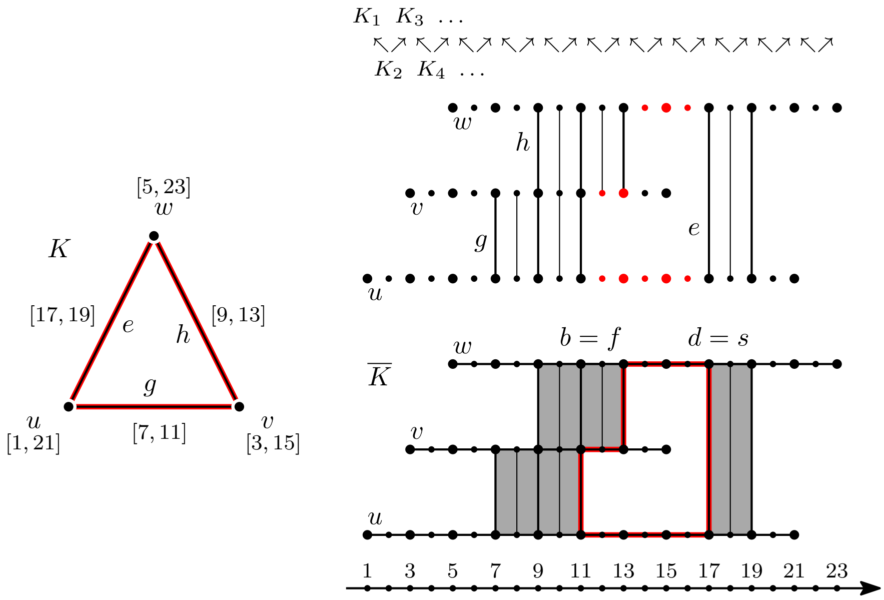



**Figure 1** A zigzag on the top. A total complex $K$ on the left. The support of each simplex in a zigzag, $T(\sigma)$, shown as an interval. The corresponding prism $\overline{K}$ on the bottom. A 1-cycle in $K[11, 17]$ and its lift in $\overline{K}[11, 17]$ are highlighted in red. The lifted cycle is an apex representative; its slices (pairs of vertices in red) are representatives of a bar in the original zigzag.

On the surface, this is as well as one can hope to do: there are $O(m)$ bars, over $O(m)$ spaces, with each representative consisting of $O(m)$ simplices. But we can do better. Every zigzag filtration is isomorphic to a levelset zigzag filtration of some function. The function generates the Mayer–Vietoris pyramid [5], which decomposes into diamond summands [2]. Each diamond has a single space as its origin, its apex: this space maps into every other space in the diamond. By finding the diamond's representative in the apex – an apex representative – we can quickly recover its representative in every other space, including those that fall in the levelset zigzag, and therefore in the original zigzag; see Figure 1. All the information can be recovered from an ordinary persistence computation. It takes matrix multiplication time, $O(m^\omega)$, to compute persistence [16], and an extra $O(p \cdot m \log m)$ time to recover a $p$-dimensional apex representative of size $O(p \cdot m)$. After the preprocessing, zigzag representatives can be recovered in $O(\log m + C)$ time, where $C$ is the size of the output.

## 2 Background

We assume familiarity with simplicial and cell complexes, homology, persistent homology, including zigzag persistence. We only recap select topics to establish notation. Any gaps can be filled with the standard literature [14, 12, 13].

Throughout the paper, we take care to work with arbitrary field coefficients, but abuse notation as follows. Cell $\sigma \in z$ means the coefficient of $\sigma$ in $z$ is non-zero, $\langle \sigma, z \rangle \neq 0$. If $\alpha$ is a chain in some cell complex $K$, then $\alpha \subseteq L$ means that $\alpha$ is supported on the subcomplex $L \subseteq K$, i.e., $\forall \sigma \in (K - L), \langle \sigma, \alpha \rangle = 0$. Similarly, $z \cap L$ refers to the restriction of chain $z$ to $L$, i.e., $z \cap L = \sum_{\sigma \in L} \langle \sigma, z \rangle \cdot \sigma$.

**Persistence.** Given a filtered cell complex $K$, let $D$ be a matrix that represents its boundary operator, with rows and columns ordered according to the filtration. To find ordinary persistence pairing, one can compute a decomposition, $R = DV$, where matrix $R$ is reduced, meaning its pivots – lowest non-zero elements in its columns – appear in unique rows, and matrix $V$ is full-rank upper-triangular. We index the columns and rows of the matrices by the (totally ordered) cells of the input complex. The persistence pairing is given by the pivots of matrix $R$: a $p$-dimensional homology class created by the addition of simplex $\sigma$ is destroyed by the addition of simplex $\tau$ iff $\operatorname{low} R[\tau] = \sigma$, where $\operatorname{low} R[\tau]$ returns the pivot in the column $R[\tau]$. By definition, the corresponding column $V[\tau]$ stores a $(p+1)$-dimensional chain whose boundary is the cycle $R[\tau]$.

The decomposition $R = DV$ is not unique: multiple matrices $R$ and $V$ satisfy it. However, the original persistence algorithm [10] performs operations in a lazy fashion, subtracting columns from left to right only if their pivots collide. We call this a *lazy reduction*. It has the following special properties that simplify our algorithms.

▶ **Lemma 1** (Lemma 1 in [17]). *Assume decomposition $R = DV$ is obtained via the lazy reduction, and $\sigma_i = \operatorname{low} R[\tau_i]$ and $\sigma_j = \operatorname{low} R[\tau_j]$ are such that $\tau_i < \tau_j$. If $\sigma_i < \sigma_j$, then entry $V[\tau_i, \tau_j] = 0$.*

**Proof.** Induction on $V$; see [17]. ◀

▶ Remark 2. The authors of [17] simplify the statement of the preceding lemma by abusing the notation. They assume that if $R[\tau] = 0$, then $\operatorname{low} R[\tau] = \bar{\sigma}$, a special imaginary cell that precedes every cell in the complex. In particular, it follows that in a lazy reduction, entry $V[\tau_i, \tau_j] = 0$ if $R[\tau_i] = 0$.

The contrapositive of the lemma is the following corollary.

▶ **Corollary 3.** *Assume decomposition $R = DV$ is obtained via the lazy reduction, and $\sigma_i = \operatorname{low} R[\tau_i]$ and $\sigma_j = \operatorname{low} R[\tau_j]$ are such that $\tau_i < \tau_j$. If $V[\tau_i, \tau_j] \neq 0$, then $\sigma_j < \sigma_i < \tau_i < \tau_j$.*

**Zigzag persistence.** We start with a zigzag of simplicial complexes,

$$K_0 \to K_1 \leftarrow K_2 \to K_3 \leftarrow K_4 \to \ldots \leftarrow K_n, \tag{1}$$

where every consecutive pair of complexes vary by at most one simplex, i.e., $K_i$ can be equal to $K_{i+1}$ in the sequence. It is convenient to assume that every simplex $\sigma$ appears and disappears exactly once in this sequence, and therefore is present during some interval $T(\sigma) = [i, j]$, i.e., $\sigma \in K_k$ iff $k \in T(\sigma)$. This assumption can be made without loss of generality by assuming that the union $K$ of all simplices across all times, $K = \bigcup_i K_i$ forms a $\Delta$-complex. The full definition of this object is too verbose – we refer the reader to [7], which adapts the classical concept of $\Delta$-complexes in [14, p. 103] to our case of zigzag filtration – but informally it relaxes the requirement that in a simplicial complex two simplices intersect in a common face. All the familiar notions of simplicial homology work the same, but now instead of having a simplex supported over multiple intervals in the zigzag, we can have multiple simplices, with the same boundary, each supported on a single interval. We use $m = |K|$ to denote the size of the input complex.

▶ Remark. To simplify the notation we let $\min(\sigma) = \min T(\sigma)$ and $\max(\sigma) = \max T(\sigma)$.

▶ Remark 4. We assume that $\max(\sigma) > \min(\sigma)$, i.e., $T(\sigma)$ is not a single point. We can assume this without loss of generality because we can always pad the zigzag with extra copies of a space.

Applying homology, we get a zigzag of homology groups, where we suppress dimension for simplicity,

$$\mathsf{H}(K_0) \to \mathsf{H}(K_1) \leftarrow \mathsf{H}(K_2) \to \mathsf{H}(K_3) \leftarrow \mathsf{H}(K_4) \to \ldots \leftarrow \mathsf{H}(K_n). \tag{2}$$

Just like with ordinary persistence, this zigzag decomposes into $k$ bars or persistence intervals [3]. Let $\{\alpha_i^1, \ldots, \alpha_i^k\}$ be a choice of elements in $\mathsf{H}(K_i)$ such that the non-zero elements form a basis for $\mathsf{H}(K_i)$. We say that such bases are *compatible* across the zigzag if the maps in Equation (2) diagonalize, i.e., $\alpha_{2i-1}^j \leftarrow\!\shortmid \alpha_{2i}^j \mapsto \alpha_{2i+1}^j$. For every $j$, $\alpha_*^j$ are non-zero over a single persistence interval $[b^j, d^j]$. Cycles $z_i^j$ that give a set of compatible bases $\alpha_i^j = [z_i^j]$ are called *zigzag representatives*, with $z_*^j$ being the representatives of the $j$-th bar in the zigzag barcode.

**Real-valued function.** A convenient setting for persistence is that of a Morse-like [4] real-valued function $f : \mathbb{X} \to \mathbb{R}$. We denote with $\mathbb{X}_a^b = f^{-1}[a, b]$ the preimage of an interval and allow the endpoints to be infinite, $a, b = \pm\infty$, in which case the interval is understood to be open at the infinite ends. We denote the following pairs of spaces:

$$\mathbb{X}[b,d] = (\mathbb{X}_b^d, \emptyset), \qquad \mathbb{X}(b,d] = (\mathbb{X}_{-\infty}^d, \mathbb{X}_{-\infty}^b),$$
$$\mathbb{X}[b,d) = (\mathbb{X}_b^\infty, \mathbb{X}_d^\infty), \qquad \mathbb{X}(b,d) = (\mathbb{X}, \mathbb{X}_{-\infty}^b \cup \mathbb{X}_d^\infty).$$

Let $a_1 < \ldots < a_n$ be the critical values of the function $f$. Let $s_i$ be regular values interleaved with the critical values: $s_0 < a_1 < s_1 < a_2 < \ldots < s_{n-1} < a_n < s_n$. The following constructions play an important role in the theory of persistent homology:

- Extended persistence (EP):

$$0 \to \mathsf{H}(\mathbb{X}[-\infty, s_0]) \to \ldots \to \mathsf{H}(\mathbb{X}[-\infty, s_n]) = \mathsf{H}(\mathbb{X})$$
$$\to \mathsf{H}(\mathbb{X}[-\infty, s_n)) \to \ldots \to \mathsf{H}(\mathbb{X}[-\infty, s_1)) \to \mathsf{H}(\mathbb{X}[-\infty, s_0)) = 0.$$

- Levelset zigzag (LZZ):

$$0 \leftarrow \mathsf{H}(\mathbb{X}[s_0, s_0]) \to \mathsf{H}(\mathbb{X}[s_0, s_1]) \leftarrow \mathsf{H}(\mathbb{X}[s_1, s_1]) \to \ldots \to \mathsf{H}(\mathbb{X}[s_{n-1}, s_n]) \leftarrow \mathsf{H}(\mathbb{X}[s_n, s_n]) \to 0.$$

Carlsson et al. [5] showed that these two sequences contain the same information by arranging the four types of spaces into a *Mayer–Vietoris Pyramid*, see Figure 2(left). Once homology is applied, the pyramid unrolls into an infinite Mayer–Vietoris strip of homology groups, where every diamond belongs to the Mayer–Vietoris long exact sequence, making it *exact* in the terminology of [3], see Figure 2(right). This allows one to translate the decomposition between any two paths that differ by a single diamond (and therefore between any pair of paths via composition).

Bendich et al. [2] further showed that the translation rules in [5] hold at the level of the basis elements of the individual paths, which means the entire pyramid decomposes as a direct sum of (indecomposable) pointwise 1-dimensional diamond summands, flush with its boundaries, shown in blue in Figure 2(right). The decomposition of the levelset zigzag and extended persistence are simply slices through these flush diamond summands. Bauer et al. [1] study the Mayer–Vietoris Pyramid in the cohomological setting.

We call the bottom-most space in the support of a diamond its *apex*. The homology class assigned to the diamond in that space, an *apex class*, and any cycle that belongs to such a class, an *apex representative*. We note that just like the choice of the basis for

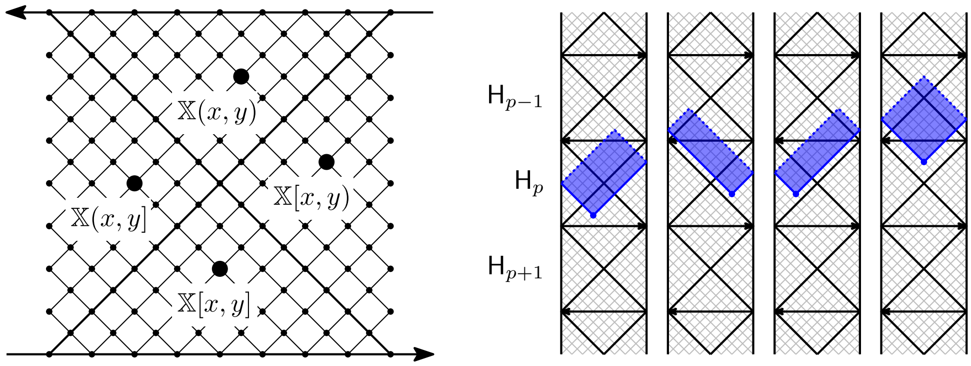



**Figure 2** Left: Mayer–Vietoris Pyramid arranges spaces $\mathbb{X}[b,d], \mathbb{X}(b,d], \mathbb{X}[b,d), \mathbb{X}(b,d)$ in a way where every diamond in the pyramid belongs to the Mayer–Vietoris long exact sequence. Right: Four types of flush diamond indecomposables in the infinite strip of homology groups; the apex of each diamond is marked.

ordinary persistence is not unique, neither is the choice of the apex classes. An apex class is characterized by lying outside the image of the maps into the apex; specifically, for the four types of apexes their classes satisfy:

$$\begin{aligned}
&[z] \in \mathsf{H}(\mathbb{X}[b,d]) \text{ s.t. } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}[b+\varepsilon,d]) \to \mathsf{H}(\mathbb{X}[b,d])) \text{ and } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}[b,d-\varepsilon]) \to \mathsf{H}(\mathbb{X}[b,d]))\\
&[z] \in \mathsf{H}(\mathbb{X}(b,d]) \text{ s.t. } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}(b-\varepsilon,d]) \to \mathsf{H}(\mathbb{X}(b,d])) \text{ and } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}(b,d-\varepsilon]) \to \mathsf{H}(\mathbb{X}(b,d]))\\
&[z] \in \mathsf{H}(\mathbb{X}[b,d)) \text{ s.t. } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}[b+\varepsilon,d)) \to \mathsf{H}(\mathbb{X}[b,d))) \text{ and } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}[b,d+\varepsilon)) \to \mathsf{H}(\mathbb{X}[b,d)))\\
&[z] \in \mathsf{H}(\mathbb{X}(b,d)) \text{ s.t. } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}(b+\varepsilon,d)) \to \mathsf{H}(\mathbb{X}(b,d))) \text{ and } [z] \notin \mathrm{im}(\mathsf{H}(\mathbb{X}(b,d-\varepsilon)) \to \mathsf{H}(\mathbb{X}(b,d)))
\end{aligned}$$

Because there is a map from the apex to every space in the diamond, we can recover a representative in any space by simply mapping an apex representative forward. All the maps in the pyramid are inclusions, except for the boundary homomorphism connecting two consecutive dimensions of homology in the strip. We say more about this translation in Section 7, but the overarching point is that given an apex representative, it is straightforward to recover every other representative, and levelset zigzag representatives in particular.

## 3 Setup

**Prism.** Given a zigzag in Equation (1), we define a larger cell complex $\overline{K}$, called a *prism*:

$$\overline{K} = \{\sigma \times [i,i+1] \mid [i,i+1] \subseteq T(\sigma), i \in \mathbb{Z}\} \,\cup\, \{\sigma \times i \mid i \in T(\sigma), i \in \mathbb{Z}\}\,, \tag{3}$$

together with a function $\overline{f} : |\overline{K}| \to \mathbb{R}$ on its underlying space, defined by the projection onto the second component, $\overline{f}(x \times t) = t$. See Figure 3. We observe that the prism consists of two types of cells: *horizontal cells* $\sigma \times [i,i+1]$ and *vertical cells* $\sigma \times i$.

We define an integer interlevel set, for $i,j \in \mathbb{Z}$, $\overline{K}_i^j = \{\sigma \times T \in \overline{K} \mid T \subseteq [i,j]\}$. We note that its underlying space $|\overline{K}_i^j| = \overline{f}^{-1}[i,j]$. Then for any pair of integers $b,d$, we get four types of pairs of subspaces,

$$\overline{K}[b,d] = \overline{K}_b^d, \quad \overline{K}(b,d] = (\overline{K}_{-1}^d, \overline{K}_{-1}^b), \quad \overline{K}[b,d) = (\overline{K}_b^{n+1}, \overline{K}_d^{n+1}), \quad \overline{K}(b,d) = (\overline{K}, \overline{K}_{-1}^b \cup \overline{K}_d^{n+1}).$$

Applying homology to each pair, we get relative homology groups that appear in the four quadrants of the Mayer–Vietoris pyramid: $\mathsf{H}(\overline{K}[b,d]), \mathsf{H}(\overline{K}(b,d])), \mathsf{H}(\overline{K}[b,d)), \mathsf{H}(\overline{K}(b,d))$.

Because even-indexed spaces in Equation (1) include into odd-indexed spaces, $K_{2i-1} \leftarrow K_{2i} \to K_{2i+1}$, we have an isomorphism of homology groups $\mathsf{H}(\overline{K}[2i,2i+1]) \simeq \mathsf{H}(\overline{K}[2i+1,2i+1]) \simeq \mathsf{H}(\overline{K}[2i+1,2i+2])$, induced by deformation retractions. It follows that the

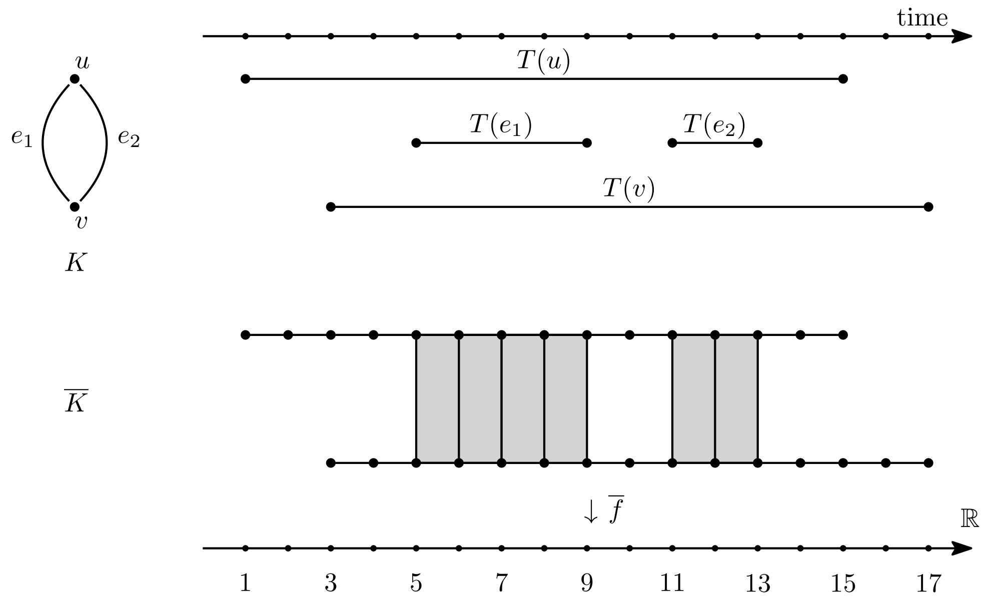



**Figure 3** A $\Delta$-complex $K$, with the times $T(\sigma)$ of its simplices illustrated, together with the corresponding prism $\overline{K}$ and its projection $\overline{f}$ onto the second coordinate.

levelset zigzag [5] of function $\overline{f}$,

$$\underbrace{\mathsf{H}(\overline{K}[0,0])}_{\mathsf{H}(K_0)} \to \underbrace{\mathsf{H}(\overline{K}[0,1]) \overset{\simeq}{\leftleftarrows} \mathsf{H}(\overline{K}[1,1]) \overset{\simeq}{\rightrightarrows} \mathsf{H}(\overline{K}[1,2])}_{\mathsf{H}(K_1)} \leftarrow \underbrace{\mathsf{H}(\overline{K}[2,2])}_{\mathsf{H}(K_2)} \to \ldots \leftarrow \underbrace{\mathsf{H}(\overline{K}[n,n])}_{\mathsf{H}(K_n)}$$

is isomorphic to our starting zigzag in Equation (2). Explicitly, the isomorphism is given by the following bijection between the four types of intervals:

$$\begin{array}{llll}
(\mathsf{H}(K_{2i}), \mathsf{H}(K_{2j})) & \leftrightarrow & (\mathsf{H}(\overline{K}[2i,2i]), \mathsf{H}(\overline{K}[2j,2j])) & \text{(open-open)} \\
(\mathsf{H}(K_{2i}), \mathsf{H}(K_{2j+1})) & \leftrightarrow & (\mathsf{H}(\overline{K}[2i,2i]), \mathsf{H}(\overline{K}[2j+1,2j+2])) & \text{(open-closed)} \\
(\mathsf{H}(K_{2i+1}), \mathsf{H}(K_{2j})) & \leftrightarrow & (\mathsf{H}(\overline{K}[2i,2i+1]), \mathsf{H}(\overline{K}[2j,2j])) & \text{(closed-open)} \\
(\mathsf{H}(K_{2i+1}), \mathsf{H}(K_{2j+1})) & \leftrightarrow & (\mathsf{H}(\overline{K}[2i,2i+1]), \mathsf{H}(\overline{K}[2j+1,2j+2])) & \text{(closed-closed)}
\end{array}$$

**Intervals.** Because we perform computation on the input complex $K$, rather than the prism, we need a notation for different intervals and pairs of intervals. The connection between these and the corresponding intervals in $\overline{K}$ will be made clear in Section 6. We denote the sub- and super-level sets:

$$K_{-1}^{i} = \{\sigma \in K \mid \min(\sigma) \leq i\}, \qquad K_{j}^{n+1} = \{\sigma \in K \mid \max(\sigma) \geq j\},$$

and define four pairs of spaces:

$$\begin{array}{ll}
K[b,d] = K_b^{n+1} \cap K_{-1}^{d}, & K(b,d] = (K_{-1}^{d}, K_{-1}^{b}), \\
K[b,d) = (K_b^{n+1}, K_d^{n+1}), & K(b,d) = (K, K_{-1}^{b} \cup K_d^{n+1}).
\end{array}$$

Because every step of the input zigzag is a simplicial complex, it follows that for any $\sigma, \tau \in K$,

$$\sigma \in \partial\tau \Rightarrow \min(\sigma) < \min(\tau) < \max(\tau) < \max(\sigma). \tag{4}$$

The following relationships between sub- and super-levelsets follow immediately:

$$\alpha \subseteq K^x_{-1} \Rightarrow \partial\alpha \subseteq K^x_{-1} \tag{5}$$

$$\alpha \subseteq K - K^{n+1}_x \Rightarrow \alpha \subseteq K^x_{-1} \tag{6}$$

**Cone.** To compute the persistence decomposition of the zigzag, we follow the algorithm of Dey and Hou [7], who use a construction similar to extended persistence [6] on the cone $\hat{K} = \omega * K$ over the input complex $K$, where $*$ indicates a join with the cone vertex $\omega$. Every simplex $\sigma$ in the base space $K$ gets value $\min(\sigma)$ from the input zigzag. Every simplex $\hat{\sigma}$ in the cone $\hat{K} - K$ gets value $\max(\sigma)$ from the input zigzag. We then define a filtration, where the base space simplices come first, ordered by increasing value, and the cone simplices come second, ordered by decreasing value. Specifically,

$$0 \to \mathsf{H}(K^1_{-1}) \to \ldots \to \mathsf{H}(K^n_{-1}) = \mathsf{H}(K) \to \mathsf{H}(K, K^{n+1}_n) \to \ldots \to \mathsf{H}(K, K^{n+1}_1) \to \mathsf{H}(K, K) = 0,$$

which in reduced homology is isomorphic to

$$0 \to \mathsf{H}(K^1_{-1}) \to \ldots \to \mathsf{H}(K) \to \mathsf{H}(K \cup \omega * K^{n+1}_n) \to \ldots \to \mathsf{H}(K \cup \omega * K^{n+1}_1) \to \mathsf{H}(\hat{K}) = 0. \tag{7}$$

The latter is an ordinary filtration and we compute its persistence via the $R = DV$ decomposition of the boundary map of the cone $\hat{K}$. Crucially, the resulting bars are in one-to-one correspondence with the bars in the decomposition of the input zigzag in Equation (2).

## 4 Lifting

Because prism $\overline{K}$ has so many cells, it is expensive to process directly. We want to perform computation on the cone $\hat{K}$ instead. To recover the apex representatives for the prism, we need to lift cone cycles in $\hat{K}$ to prism cycles in $\overline{K}$. In Section 6, we explain what exact cycles to lift to get different apex representatives. Meanwhile, we first describe an algorithm that given a (relative) cycle $z$ in $KI$ produces a cycle $\overline{z}$ in $\overline{K}I$, where $I$ is one of the four types of intervals, $(b, d]$, $[b, d)$, $(b, d)$, $[b, d]$. Algorithm 1 in this section is a reinterpretation of an inefficient, but easier-to-understand Lift-Cycle-easy algorithm in the full version of this paper [8]. We encourage the reader to go through through that algorithm first.

▶ Remark. A $p$-chain $\overline{z}$ in $\overline{K}$ consists of two types of $p$-cells: $\tau \times t$, where $\tau$ is a $p$-simplex, and $\sigma \times [t_1, t_2]$, where $\sigma$ is a $(p-1)$-simplex.

▶ Remark. We use an abridged notation, $c \cdot \sigma \times [i, j]$, to represent the chain $\sum_{k \in [i,j-1]} c \cdot \sigma \times [k, k+1]$. This choice is guided by computational efficiency: the former requires constant space vs. the $O(j - i)$ space required for the latter.

Algorithm 1 takes a $p$-cycle $z$, a direction expressed as a pair of values $(s, f)$, and an initial $(p-1)$-cycle $w_{\text{init}}$. We call the pair $(s, f)$ a direction because the order of the values specifies whether we process the cycle in increasing ($s < f$) or decreasing ($f < s$) order. The algorithm stretches the cycle from $\overline{K}[s, s]$ to $\overline{K}[f, f]$, covering $\overline{K}I$; see Figure 1.

**Correctness.** The correctness of Algorithm 1 follows from the following claim about the boundary structure, which will be important in Section 6. We let $z^y_x$ be the restriction of cycle $z$ to the simplices $\tau$ whose times $t_\tau$ lie in the interval $[x, y]$, i.e., $z^y_x = \sum_{\tau \in z, t_\tau \in [x,y]} \langle \tau, z \rangle \cdot \tau$.

▷ Claim 5. Given input cycle $z \in K$, the boundary of the lifted cycle $\overline{z}$ satisfies $\partial\overline{z} = w_{\text{init}} \times s + w_{\text{final}} \times f$, where $w_{\text{final}} = w_{\text{init}} + \partial z^d_b$.

**Algorithm 1** LIFT-CYCLE($z$, $(s, f)$, $w_{\text{init}}$.)

**Input:**
- relative $p$-cycle $z$ in $H(KI)$, where $I$ is one of $(b, d]$, $[b, d)$, $(b, d)$, $[b, d]$;
- direction from start to finish, $s, f$ $(= b, d$ or $d, b)$;
- initial boundary $w_{\text{init}} = -\partial z \cap K[s, s] \subseteq K[s, s]$

**Output:**
- relative $p$-cycle $\overline{z}$ in $H(\overline{K}I)$

1: **for** *each* $\tau \in z$ **do**
2: $\quad t_\tau \leftarrow$ a value in $T(\tau) \cap [b, d]$ $\quad$ ▷ if $T(\tau) \cap [b, d] = \emptyset$, ignore $\tau$
3: $\quad \overline{z} \leftarrow \overline{z} + \langle \tau, z \rangle \cdot (\tau \times t_\tau)$
4: **for** *each* $\sigma$ s.t. $\exists \tau \in z, \sigma \in \partial \tau$ **do** $\quad$ ▷ each face of some simplex in $z$
5: $\quad c \leftarrow \langle \sigma, w_{\text{init}} \rangle$
6: $\quad l \leftarrow s$
7: $\quad$ **for** $(t_\tau, \tau) \in \{(t_\tau, \tau) \mid \tau \in \delta\sigma\}$ in order of $t_\tau$ from $s$ to $f$ **do**
8: $\quad\quad$ **if** $l \neq t_\tau$ **then**
9: $\quad\quad\quad \overline{z} \leftarrow \overline{z} + c \cdot (\sigma \times [l, t_\tau])$
10: $\quad\quad c \leftarrow c + \langle \tau, z \rangle \cdot \langle \sigma, \partial \tau \rangle$
11: $\quad\quad l \leftarrow t_\tau$
12: $\quad$ **if** $l \neq f$ **then**
13: $\quad\quad \overline{z} \leftarrow \overline{z} + c \cdot (\sigma \times [l, f])$

*Proof.* Simplex $\sigma$ generates a cell $\sigma \times [s, l] \in \overline{z}$ iff $\langle \sigma, w_{\text{init}} \rangle \neq 0$ (Line 5). This cell contributes $\sigma \times s$ to the boundary of $\overline{z}$. Simplex $\sigma$ generates a cell $\sigma \times [l', f] \in \overline{z}$ iff $\langle \sigma, w_{\text{init}} \rangle + \sum_{\tau \in \delta\sigma \cap z_b^d} \langle \sigma, \tau \rangle \neq 0$. By definition this is the case only for simplices in $w_{\text{final}} = w_{\text{init}} + \partial z_b^d$. ◁

It follows that if $w_{\text{init}} \subseteq K[s, s]$, then $\overline{z}$ is a cycle in $\overline{K}I$. What is not immediate is why the chains $\sigma \times [l, t_\tau]$ and $\sigma \times [l, f]$ in Lines 9 and 13 are in the prism $\overline{K}$.

▷ Claim 6. Chains $\sigma \times [l, t_\tau]$ and $\sigma \times [l, f]$ added in Lines 9 and 13 of Algorithm 1 are present in $\overline{K}$.

*Proof.* Without loss of generality, assume $s < f$. There are three types of chains:

1. $\sigma \times [s, t_{\tau_1}]$. $\sigma$ has a non-zero coefficient on this interval iff $\langle \sigma, w_{\text{init}} \rangle \neq 0$ (Line 5), in which case $\sigma \in K[s, s]$. Existence of coface $\tau_1$ implies $\min(\sigma) \leq s \leq t_{\tau_1} < \max(\sigma)$.
2. $\sigma \times [t_{\tau_i}, t_{\tau_{i+1}}]$. From Equation (4), $\min(\sigma) < t_{\tau_i} \leq t_{\tau_{i+1}} < \max(\sigma)$.
3. $\sigma \times [t_{\tau_i}, f]$. Suppose that this chain is not in $\overline{K}$. This means $x = \max(\sigma) \in [t_{\tau_i}, f)$. Equation (4) implies that $\sigma$ has no cofaces in $[x, f]$. Therefore, $\sigma \in w_{\text{init}} + \partial z_s^x$ and $\sigma \notin \partial z_x^f$. Therefore, $\sigma \in \partial z$. Since we assumed $z$ is a relative cycle in $KI$, it cannot have any boundary inside the interval itself – a contradiction. ◁

▶ Remark. It may seem strange that we state nothing about the relationship between $z$ and $\overline{z}$. This is because we exploit additional properties of the algorithm in Section 6.

**Running time.** Let $m$ be the size of the input $p$-cycle $z$. Algorithm 1 requires $O(p \cdot m \log m)$ time. It goes through all simplices $\sigma$ that are faces of some simplex $\tau \in z$; there are at most $(p + 1) \cdot m$ such $(p - 1)$-simplices. For each one, the algorithm sorts its cofaces by their times $t_\tau$. Although there is no bound on the number of cofaces of one simplex, the total number of the cofaces to sort is again at most $(p+1)\cdot m$. Individual operations in the update take constant time, so the overall running time of Algorithm 1 is $O(p \cdot m \log m)$. The size of the lifted cycle is $O(p \cdot m)$.

## 5 Apex Representatives

We give a characterization of apex representatives in the prism.

▶ **Lemma 7** (Closed endpoint).

1. *If a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d])$*, contains* $\tau \times d$ *with* $\min \tau = d$*, then*

$$[\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}(b,d-1]) \to \mathsf{H}(\overline{K}(b,d])\right).$$

2. *If a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}[b,d))$*, contains* $\tau \times b$ *with* $\max \tau = b$*, then*

$$[\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}[b+1,d)) \to \mathsf{H}(\overline{K}[b,d))\right).$$

3. *If an absolute cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}[b,d])$*, contains* $\sigma \times b$ *and* $\tau \times d$ *with* $\max \sigma = b$ *and* $\min \tau = d$*, then*

$$[\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}[b+1,d]) \to \mathsf{H}(\overline{K}[b,d])\right) \quad \text{and} \quad [\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}[b,d-1]) \to \mathsf{H}(\overline{K}[b,d])\right).$$

**Proof.** All the statements follow the same proof, except for the endpoints of the intervals, so we only show the first. Suppose there is some class $[\alpha] \in \mathsf{H}(\overline{K}(b,d-1])$ that maps to $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d])$. Then $\overline{z} = \alpha + \partial\beta$ for some chain $\beta$. Since $\tau \in \overline{z}$ and $\tau \notin \alpha$ (since $\min \tau = d > d-1$), $\tau$ must be in $\partial\beta$. But from Equation (4), $\tau$ has no cofaces in $K^{d}_{-1}$ – a contradiction. ◀

▶ **Lemma 8** (Open endpoint).

1. *If the boundary* $\partial\overline{z}$ *of a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d])$*, contains* $\sigma \times b$ *with* $\min \sigma = b$*, then* $[\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}(b-1,d]) \to \mathsf{H}(\overline{K}(b,d])\right)$.
2. *If the boundary* $\partial\overline{z}$ *of a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}[b,d))$ *contains* $\sigma \times d$ *with* $\max \sigma = d$*, then* $[\overline{z}] \notin \operatorname{im}\left(\mathsf{H}(\overline{K}[b,d+1)) \to \mathsf{H}(\overline{K}[b,d))\right)$.
3. *If the boundary* $\partial\overline{z}$ *of a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d))$*, contains* $\sigma \times b$ *with* $\min \sigma = b$ *and* $\tau \times d$ *with* $\max \tau = d$*, then*

$$\overline{z} \notin \operatorname{im}\left(\mathsf{H}(\overline{K}(b-1,d)) \to \mathsf{H}(\overline{K}(b,d))\right) \quad \text{and} \quad \overline{z} \notin \operatorname{im}\left(\mathsf{H}(\overline{K}(b,d+1)) \to \mathsf{H}(\overline{K}(b,d))\right).$$

**Proof.** All the statements follow the same proof, except for the endpoints of the intervals, so we only show the first. From the long exact sequence of the triple [14, p. 118], $K^{b-1}_{-1} \subseteq K^{b}_{-1} \subseteq K^{d}_{-1}$, we know that the image in the claim is equal to the kernel of the map induced by the boundary,

$$\operatorname{im}\left(\mathsf{H}_*(\overline{K}(b-1,d]) \to \mathsf{H}_*(\overline{K}(b,d])\right) = \ker\left(\mathsf{H}_*(\overline{K}(b,d]) \xrightarrow{\partial^*} \mathsf{H}_{*-1}(\overline{K}(b-1,b])\right).$$

For $[\overline{z}]$ to be in the kernel, $\partial\overline{z}$ needs to be a relative boundary in $\overline{K}(b-1,b]$, i.e., $\partial\overline{z} = \partial\beta + \gamma$ for some $\beta \subseteq K^{b}_{-1}$ and $\gamma \subseteq K^{b-1}_{-1}$. Since $\sigma \in \partial\overline{z}$ and $\sigma \notin \gamma$, $\sigma$ must be in $\partial\beta$. But from Equation (4), $\sigma$ has no cofaces in $K^{b}_{-1}$ – a contradiction. ◀

Putting Lemmas 7 and 8 together, we get the following theorem.

▶ **Theorem 9.**

1. *If a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d])$*, contains* $\tau \times d$ *with* $\min \tau = d$*, and its boundary* $\partial\overline{z}$ *contains* $\sigma \times b$ *with* $\min \sigma = b$*, then* $\overline{z}$ *is an apex representative.*

2. *If a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}[b,d))$*, contains* $\tau \times b$ *with* $\max \tau = b$*, and its boundary* $\partial\overline{z}$ *contains* $\sigma \times d$ *with* $\max \sigma = d$*, then* $\overline{z}$ *is an apex representative.*
3. *If an absolute cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}[b,d])$*, contains* $\sigma \times b$ *and* $\tau \times d$ *with* $\max \sigma = b$ *and* $\min \tau = d$*, then* $\overline{z}$ *is an apex representative.*
4. *If the boundary* $\partial\overline{z}$ *of a relative cycle* $\overline{z}$*, with* $[\overline{z}] \in \mathsf{H}(\overline{K}(b,d))$*, contains* $\sigma \times b$ *with* $\min \sigma = b$ *and* $\tau \times d$ *with* $\max \tau = d$*, then* $\overline{z}$ *is an apex representative.*

# 6 Four Intervals

We assume $R = DV$ is the decomposition of the boundary matrix $D$ of the cone $\hat{K}$ filtration in Equation (7). We use $\sigma, \tau$ to refer to simplices in the base space $K$, and $\hat{\sigma}, \hat{\tau}$ for the simplices in the cone $\hat{K} - K$.

**Summary.** The main content of this section is summarized in Table 1: lifting the stated cycles $z$ in $K$, derived from the lazy reduction, with the given arguments to the algorithm Lift-Cycle, produces apex representatives in $\overline{K}$. The results of the section are more general, but also more verbose; they derive the expression for the cycles for any reduction.

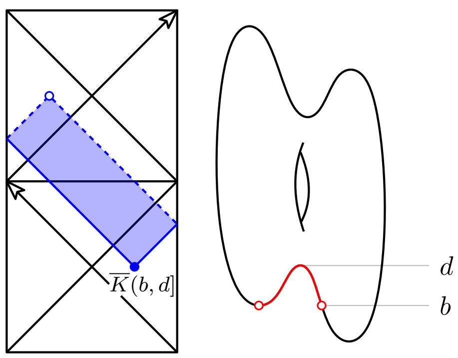
**Table 1** Summary of the chains and arguments used in different cases of the Lift-Cycle algorithm, derived from the lazy reduction.

| EP type | LZZ type | Apex | $z$ | $(s,f)$ | $w_{\text{init}}$ |
|---|---|---|---|---|---|
| Ordinary | closed-open | $K(b,d]$ | $V[\tau]$ | $(d,b)$ | 0 |
| Relative | open-closed | $K[b,d)$ | $R[\hat{\tau}] \cap K$ | $(b,d)$ | 0 |
| Extended $(d,b)$ | open-open | $K[b,d]$ | $R[\hat{\tau}]$ | $(d,b)$ | 0 |
| Extended $(b,d)$ | closed-closed | $K(b,d)$ | $V[\hat{\tau}] \cap K$ | $(b,d)$ | $-R[\hat{\tau}]$ |

## 6.1 Ordinary (closed-open)

Throughout this subsection, we assume the following **setting**:

a pair $(\sigma, \tau)$ in the cone filtration, born at $b$ and dying at $d$, i.e., $\min(\sigma) = b < d = \min(\tau)$.




▷ Claim 10. Let $z = V[\tau]$. Then $[z] \in \mathsf{H}(K(b,d])$. Furthermore, $\tau \in z$ and $\sigma \in \partial z = R[\tau]$.

Proof. Since $V$ is upper-triangular with all diagonal entries non-zero, $\tau$ is the latest simplex in $V[\tau]$, i.e., $\max\{\min(\tau') \mid \tau' \in V[\tau]\} = \min(\tau) = d$. Therefore, $V[\tau] \subseteq K^d_{-1}$. By definition, the boundary $\partial V[\tau] = R[\tau]$. $\sigma$ is its latest simplex, i.e., $\max\{\min(\sigma') \mid \sigma' \in R[\tau]\} = \min(\sigma) = b$. Therefore, $\partial V[\tau] \subseteq K^b_{-1}$. It follows that if $z = V[\tau]$, then $[z] \in \mathsf{H}(K^d_{-1}, K^b_{-1}) = \mathsf{H}(K(b,d])$. ◁

▷ Claim 11. Relative cycle $\overline{z}$ produced by the call Lift-Cycle($V[\tau], (d,b), 0$) is an apex representative in $\mathsf{H}(\overline{K}(b,d])$.

Proof. Let $z = V[\tau]$.

1. $\sigma \times b \in \partial\overline{z}$, $\min(\sigma) = b$.
   Because $\sigma \in \partial z$ (Claim 10) and $\min(\sigma) = b$, its coboundary $\delta\sigma \subseteq K^{n+1}_b$. Meanwhile, $z \subseteq K^d_{-1}$. Therefore, all cofaces of $\sigma$ in $z$ intersect $[b,d]$. Therefore, $\sigma \in \partial\,(z \cap [b,d])$. It follows from Claim 5 that

$$\sigma \times b \in \partial\overline{z} = w_{\text{final}} \times b = \partial\,(z \cap [b,d]) \times b.$$

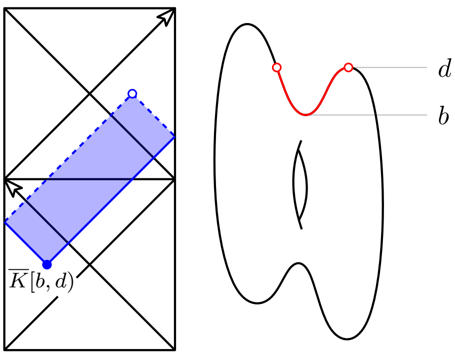
2. $\tau \times d \in \overline{z}$, $\min(\tau) = d$.
   This follows immediately since $\tau \in z$ (Claim 10) and $T(\tau) \cap [b,d] = \{d\}$.

It follows from Theorem 9 that $\overline{z}$ is an apex representative in $\mathsf{H}(\overline{K}(b,d])$. ◁

## 6.2 Relative (open-closed)

Throughout this subsection, we assume the following **setting**:

a pair $(\hat\sigma, \hat\tau)$ in the cone filtration, born at $d$ and dying at $b$, i.e., $\max(\sigma) = d > b = \max(\tau)$.




▷ Claim 12. Let chain $z$ consist of the boundary of the cone simplices in $V[\hat\tau]$ restricted to the base space,

$$z = \partial\left(V[\hat\tau] \cap (\hat K - K)\right) \cap K.$$

Then $[z] \in \mathsf{H}(K[b,d))$. Furthermore, $\tau \in z$ and $\sigma \in \partial z$.

Proof. Let $\gamma = V[\hat\tau] \cap (\hat K - K)$, i.e., $z = (\partial\gamma) \cap K$.

1. $z \subseteq K^{n+1}_b$, $\tau \in z$.
   Because $\gamma$ consists of only cone simplices, every simplex $\tau' \in z$ is in the boundary of some cone simplex $\hat\tau'$. Because $\hat\tau$ is the latest simplex in $V[\hat\tau]$, we have that $\hat\tau'$ is added before $\hat\tau$. Therefore, $\max(\tau') \geq \max(\tau) = b$. Because $\hat\tau$ is the only simplex in $\gamma$ with $\tau$ in its boundary, $\tau \in \partial\gamma \cap K = z$.

2. $\partial z \subseteq K_d^{n+1}$.
   Let
   $$\alpha = \partial z = -\partial\left(\partial\gamma \cap (\hat{K} - K)\right)$$
   be the shared boundary between the base space and the cone parts of $\partial\gamma$. We note that because cone simplices can only be in the boundary of cone simplices, $(\partial V[\hat{\tau}]) \cap (\hat{K} - K) = \partial\gamma \cap (\hat{K} - K)$. Therefore, all the cone simplices $\hat{\sigma}' \in \partial\gamma \cap (\hat{K} - K))$ are such that $\max(\sigma') \geq \max(\sigma) = d$. Therefore, any simplex in the shared boundary $\alpha$ is in $K_d^{n+1}$.
   Putting the first two sub-claims together, $[z] \in \mathsf{H}(K_b^{n+1}, K_d^{n+1}) = \mathsf{H}(K[b,d])$.
3. $\sigma \in \partial z$.
   $(\hat{\sigma}, \hat{\tau})$ is a persistence pair, therefore, $\hat{\sigma} \in \partial V[\hat{\tau}]$. Because cone simplices can only be in the boundaries of cone simplices, $\hat{\sigma} \in \partial\gamma$. Because $\hat{\sigma}$ is the only simplex in $(\partial\gamma) \cap (\hat{K} - K)$ that has $\sigma$ in its boundary, it follows that
   $$\sigma \in \partial z = \partial\left((\partial\gamma) \cap K\right) = \partial\left((\partial\gamma) \cap (\hat{K} - K)\right). \qquad \triangleleft$$

▶ Remark 13 (Lazy reduction). If $R = DV$ is obtained by lazy reduction, it follows from Corollary 3 that $V[\hat{\tau}]$ consists only of cone simplices. In other words, the chain in the prior claim simplifies to
$$z = (\partial V[\hat{\tau}]) \cap K = R[\hat{\tau}] \cap K.$$

▷ Claim 14. Relative cycle $\overline{z}$ produced by the call Lift-Cycle$(R[\hat{\tau}] \cap K, (b,d), 0)$ is an apex representative in $\mathsf{H}(\overline{K}[b,d])$.

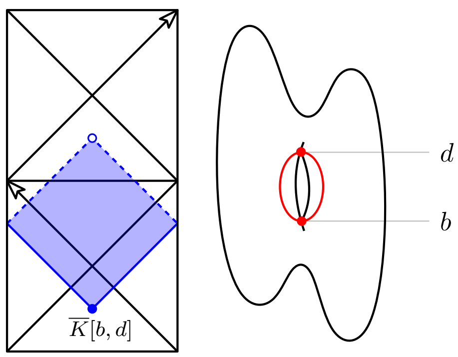
Proof. Let $z = R[\hat{\tau}] \cap K$.
1. $\sigma \times d \in \partial\overline{z}$, $\max(\sigma) = d$.
   Because $\sigma \in \partial z$ (Claim 12) and $\max(\sigma) = d$, its coboundary $\delta\sigma \subseteq K_{-1}^{d}$. Meanwhile, $z \subseteq K_b^{n+1}$. Therefore, all cofaces of $\sigma$ in $z$ intersect $[b,d]$. Therefore, $\sigma \in \partial\left(z \cap [b,d]\right)$. It follows from Claim 5 that
   $$\sigma \times d \in \partial\overline{z} = w_{\text{final}} \times d = \partial\left(z \cap [b,d]\right) \times d.$$
2. $\tau \times b \in \overline{z}$, $\max(\tau) = b$.
   This follows immediately since $\tau \in z$ (Claim 12) and $T(\tau) \cap [b,d] = \{b\}$.

It follows from Theorem 9 that $\overline{z}$ is an apex representative in $\mathsf{H}(\overline{K}[b,d])$. $\triangleleft$

## 6.3 Extended (open-open)

Throughout this subsection, we assume the following **setting**:

a pair $(\sigma, \hat{\tau})$ in the cone filtration born at $d$, dying at $b$, i.e., $\min(\sigma) = d > b = \max(\tau)$.



▷ Claim 15. Let chain $c$ consist of those simplices in $V[\hat{\tau}]$ that are either in the cone, or in $K_b^{n+1}$, i.e.,

$$c = V[\hat{\tau}] \cap (\hat{K} - K) + V[\hat{\tau}] \cap K_b^{n+1}.$$

Let $z = \partial c$; then $[z] \in \mathsf{H}(K[b,d])$. Furthermore, $\sigma, \tau \in z$.

Proof. We split the chain $V[\hat{\tau}]$ into three parts:

$$\alpha = V[\hat{\tau}] \cap (K - K_b^{n+1}), \qquad \beta = V[\hat{\tau}] \cap K_b^{n+1}, \qquad \gamma = V[\hat{\tau}] \cap (\hat{K} - K),$$

and re-write its boundary accordingly: $R[\hat{\tau}] = \partial V[\hat{\tau}] = \partial\alpha + \partial\beta + \partial\gamma$. $c = (\beta + \gamma) = V[\hat{\tau}] - \alpha$.

1. $z = \partial c \subseteq K_{-1}^d$, $\sigma \in z$.
   $\partial V[\hat{\tau}] \subseteq K_{-1}^d$ because $\sigma$ is the latest simplex in $R[\hat{\tau}]$. Putting together Equations (5) and (6), $\alpha \subseteq K - K_b^{n+1}$ implies $\partial\alpha \subseteq K_{-1}^b \subseteq K_{-1}^d$. Therefore, $z = \partial V[\hat{\tau}] - \partial\alpha \subseteq K_{-1}^d$. Because $\sigma \in \partial V[\hat{\tau}]$, but $\sigma \notin \partial\alpha$, $\sigma \in z$.
2. $z \subseteq K_b^{n+1}$, $\tau \in z$.
   Since chain $\beta \subseteq K_b^{n+1}$, its boundary $\partial\beta \subseteq K_b^{n+1}$. Since boundaries $\partial V[\hat{\tau}], \partial\alpha, \partial\beta \subseteq K$, so is $\partial\gamma$. Every simplex in $\gamma$ has the form $\omega * \tau$, where $\tau \in K_b^{n+1}$. Therefore, $\partial\gamma \subseteq K_b^{n+1}$. It follows that $z = \partial\beta + \partial\gamma \subseteq K_b^{n+1}$.
   Because $\max(\tau) = b$, it has no cofaces in $K_b^{n+1}$. Therefore, $\tau \notin \partial\beta$. Because $\hat{\tau}$ is the only simplex in $\gamma$ that has $\tau$ in its boundary, $\tau \in \partial\gamma$. Therefore, $\tau \in z = \partial\beta + \partial\gamma$.

   Therefore, $[z] \in \mathsf{H}(K_b^{n+1} \cap K_{-1}^d) = \mathsf{H}(K[b,d])$. ◁

▶ Remark 16 (Lazy reduction). Furthermore, if $R = DV$ comes from the lazy reduction, then the next claim proves that in the previous claim, chain $c = V[\hat{\tau}]$. In other words, we can use $z = R[\hat{\tau}] = \partial V[\hat{\tau}]$ directly.

▷ Claim 17 (Lazy reduction). If $R = DV$ is obtained from the lazy reduction, then all the base space simplices in $V[\hat{\tau}]$ are in $K_b^{n+1}$, i.e., $V[\hat{\tau}] \cap K \subseteq K_b^{n+1}$.

Proof. Let $\tau'$ be a simplex in $V[\hat{\tau}] \cap K$. In this case, for the lazy reduction, because $V[\tau', \hat{\tau}] \neq 0$, from Corollary 3, taking $\sigma_j = \sigma$ and $\tau_i = \tau'$, we get

$$\max(\tau') > \min(\tau') > \min(\sigma) = d > b. \qquad ◁$$

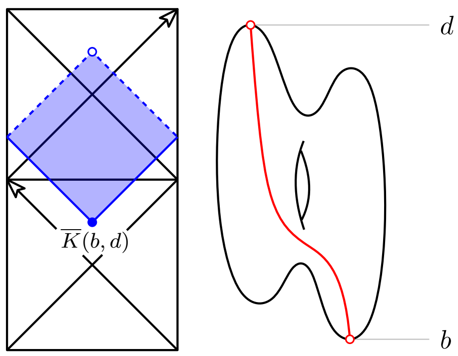
▷ Claim 18. Absolute cycle $\overline{z}$ produced by the call Lift-Cycle$(R[\hat{\tau}], (d, b), 0)$ is an apex representative in $\mathsf{H}(\overline{K}[b,d])$.

Proof. Let $z = R[\hat{\tau}]$. Because $\sigma, \tau \in z$ (Claim 15) and $\min(\sigma) = d$ and $\max(\tau) = b$, both simplices $\sigma \times d$ and $\tau \times b$ are in the lifted cycle $\overline{z}$. It follows from Theorem 9 that $\overline{z}$ is an apex representative in $\mathsf{H}(\overline{K}[b,d])$. ◁

## 6.4 Extended (closed-closed)

Throughout this subsection, we assume the following **setting**:

a pair $(\sigma, \hat{\tau})$ in the cone filtration, born at $b$, dying at $d$, i.e., $\min(\sigma) = b < d = \max(\tau)$.




▷ Claim 19. Let $z = V[\hat{\tau}] \cap K_b^{n+1}$. Then $[z] \in \mathsf{H}(K(b,d))$. Furthermore,

$$\sigma \in w_{\text{init}} = -\partial\left(V[\hat{\tau}] \cap (\hat{K} - K) + V[\hat{\tau}] \cap K_b^{n+1}\right) \subseteq K[b,b]$$

and $\tau \in (w_{\text{init}} + \partial z) = -\partial\left(V[\hat{\tau}] \cap (\hat{K} - K)\right)$.

Proof. We split the chain $V[\hat{\tau}]$ into three parts:

$$\alpha = V[\hat{\tau}] \cap (K - K_b^{n+1}), \qquad z = \beta = V[\hat{\tau}] \cap K_b^{n+1}, \qquad \gamma = V[\hat{\tau}] \cap (\hat{K} - K),$$

and re-write its boundary accordingly: $R[\hat{\tau}] = \partial V[\hat{\tau}] = \partial\alpha + \partial\beta + \partial\gamma$.

1. $\partial z \subseteq K_{-1}^b \cup K_d^{n+1}$.
   Putting together Equations (5) and (6), $\alpha \subseteq K - K_b^{n+1}$ implies $\partial\alpha \subseteq K_{-1}^b$. Because $\sigma$ is the latest simplex in $R[\hat{\tau}]$ and $\min(\sigma) = b$, $R[\hat{\tau}] \subseteq K_{-1}^b$. It follows that $\partial(\beta+\gamma) = R[\hat{\tau}] - \partial\alpha \subseteq K_{-1}^b$. Since $\partial\gamma \subseteq K_d^{n+1}$, we have $\partial z = \partial(\beta+\gamma) - \partial\gamma \subseteq K_{-1}^b \cup K_d^{n+1}$. Therefore, $[z] \in \mathsf{H}(K(b,d))$.
2. $\sigma \in w_{\text{init}} = \partial(\beta+\gamma) \subseteq K[b,b]$.
   We already saw that $\partial(\beta+\gamma) \subseteq K_{-1}^b$. To show that it is also in $K_b^{n+1}$, we note that since $\beta \subseteq K_b^{n+1}$, its boundary $\partial\beta \subseteq K_b^{n+1}$. $\partial\gamma \subseteq K_d^{n+1} \subseteq K_b^{n+1}$. Therefore, $\partial(\beta+\gamma) \subseteq K_b^{n+1}$. Because $\min(\sigma) = b$, Equation (4) implies that $\sigma$ cannot be in the boundary of any simplex $\tau \in \alpha$ since $\max(\tau) < b$. Since $\sigma \in R[\hat{\tau}]$ and $w_{\text{init}} = \partial(\beta+\gamma) = R[\hat{\tau}] - \partial\alpha$, it follows that $\sigma \in w_{\text{init}}$.
3. $\tau \in -\partial\gamma$.
   Because $\hat{\tau}$ is the only simplex in $\gamma$ that has $\tau$ in its boundary, $\tau \in -\partial\gamma$. ◁

▷ Claim 20 (Lazy reduction). If $R = DV$ is obtained from the lazy reduction, then $z = V[\hat{\tau}] \cap K$ is in $K(b,d)$, $\sigma \in w_{\text{init}} = -R[\hat{\tau}] = -\partial V[\hat{\tau}] \subseteq K[b,b]$, and $\tau \in (w_{\text{init}} - \partial z)$.

Proof. We only need to show that, in the case of the lazy reduction, $\alpha = V[\hat{\tau}] \cap (K - K_b^{n+1}) = 0$, i.e., $V[\hat{\tau}] \cap K = V[\hat{\tau}] \cap K_b^{n+1}$. Then the claim follows from the previous Claim 19.

The proof is analogous to the proof of Claim 17. Let $\tau'$ be a simplex in $V[\hat{\tau}] \cap K$. In this case, for the lazy reduction, because $V[\tau', \hat{\tau}] \neq 0$, from Corollary 3, taking $\sigma_j = \sigma$ and $\tau_i = \tau'$, we get $\max(\tau') > \min(\tau') > \min(\sigma) = b$. ◁

▶ Remark 21. $V[\hat{\tau}] \cap K$ can be empty. For example, if $K$ is a single vertex that appears at $b$ and then disappears at $d$. This is the reason why we need to bootstrap Algorithm 1 with an initial cycle $w_{\text{init}}$.

▷ Claim 22. Relative cycle $\overline{z}$ produced by the call Lift-Cycle$(V[\hat{\tau}] \cap K, (b,d), -R[\hat{\tau}])$ is an apex representative in $\mathsf{H}(\overline{K}(b,d))$.

Proof.

1. $\sigma \times b \in \partial\overline{z}$, $\min(\sigma) = b$.
   Simplex $\sigma$, with $\min(\sigma) = b$, is in the initial cycle $w_{\text{init}}$ (Claim 19). Therefore, $\sigma \times b$ is in the boundry $\partial\overline{z}$ of the lifted cycle.
2. $\tau \times d \in \partial\overline{z}$, $\max(\tau) = d$.
   Simplex $\tau$, with $\max(\tau) = d$, is in $w_{\text{init}} + \partial z$ (Claim 19). Its coboundary $\delta\tau \subseteq K^d_{-1}$. Meanwhile, $z \subseteq K^{n+1}_b$ (Claim 19). Therefore, all cofaces of $\tau$ in $z$ intersect $[b, d]$. Therefore, $\tau \in w_{\text{init}} + \partial\,(z \cap [b, d]) = w_{\text{final}}$. It follows from Claim 5 that

$$\tau \times d \in w_{\text{final}} \times d \subseteq \partial\overline{z}.$$

It follows from Theorem 9 that $\overline{z}$ is an apex representative in $\mathsf{H}(\overline{K}(b, d))$. ◁

## 7 Zigzag Representatives

To solve our original problem – to find zigzag representatives – all that remains is to map an apex representative into an appropriate space in the zigzag. Because we assumed in Remark 4 that no simplex is supported on a single space, the death and the birth for any given interval are distinct. Suppose we want to map an apex representative $\overline{z}$, with $[\overline{z}] \in \mathsf{H}(\overline{K}I)$, into a zigzag representative in $\mathsf{H}(\overline{K}[i, i]) \simeq \mathsf{H}(K_i)$.

If $i$ is odd, then the $p$-cycle $\overline{z}$ may contain a vertical cell $\tau \times i$, which does not map into $(p-1)$-cells in the levelset. To sidestep this complication, we perturb $i$. Let $x = i \pm \varepsilon \in I$ be a real value near $i$ that lies inside our birth-death interval. We assume $x = i + \varepsilon$; the other case is symmetric. We implicitly subdivide $\overline{K}$ at $x$ and extend our previous notation to allow for the interlevel sets ending at $x$, e.g., $\overline{K}^x_i$. We note that $\overline{K}^x_i$ deformation retracts onto $\overline{K}^i_i$, with the homotopy following the second (time) coordinate, and $\overline{K}^x_x$ includes into $\overline{K}^i_i$ once the cell times are shifted from $x$ to $i$. We want to compose three maps:

$$\mathsf{H}_p(\overline{K}I) \to \mathsf{H}_p(\overline{K}(i, x)) \xrightarrow{\partial^*} \mathsf{H}_{p-1}(\overline{K}[x, x]) \to \mathsf{H}_{p-1}(K_i).$$

Because $\overline{K}I$ is an apex of a diamond, and $\overline{K}(i, x)$ is a space in the diamond, the first map is an inclusion (of pairs), so $\overline{z}$ remains a relative cycle in $\overline{K}(i, x)$. Let

$$\overline{z} = \sum \alpha_\tau \cdot (\tau \times t_\tau) + \sum \alpha_\sigma \cdot (\sigma \times [t^1_\sigma, t^2_\sigma]),$$

then the second (boundary) map takes $\overline{z}$ to

$$\overline{z}[x, x] = \sum_{\substack{(\sigma \times [t^1_\sigma, t^2_\sigma]) \in \overline{z} \\ x \in [t^1_\sigma, t^2_\sigma]}} \alpha_\sigma \cdot (\sigma \times x),$$

which maps into $K_i$ by dropping the second coordinate,

$$z(i) = \sum_{(\sigma \times x) \in \overline{z}[x,x]} \alpha_\sigma \cdot \sigma.$$

By storing the apex cycle $\overline{z}$ in an interval tree [11, 15], we can retrieve any zigzag representative of size $C$ in time $O(\log m + C)$.

## References


1 Ulrich Bauer, Magnus Bakke Botnan, and Benedikt Fluhr. Structure and interleavings of relative interlevel set cohomology. *arXiv [math.AT]*, August 2021. arXiv:2108.09298.

2 Paul Bendich, Herbert Edelsbrunner, Dmitriy Morozov, and Amit Patel. Homology and robustness of level and interlevel sets. *Homology, Homotopy and Applications*, 15(1):51–72, 2013.

3 Gunnar Carlsson and Vin de Silva. Zigzag persistence. *Foundations of computational mathematics*, 10(4):367–405, August 2010. doi:10.1007/s10208-010-9066-0.

4 Gunnar Carlsson, Vin de Silva, Sara Kališnik, and Dmitriy Morozov. Parametrized homology via zigzag persistence. *Algebraic & Geometric Topology*, 19(2):657–700, March 2019. doi:10.2140/agt.2019.19.657.

5 Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov. Zigzag persistent homology and real-valued functions. In *Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry*, SCG '09, pages 247–256, New York, NY, USA, 2009. ACM. doi:10.1145/1542362.1542408.

6 David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. Extending persistence using Poincaré and Lefschetz duality. *Foundations of computational mathematics*, 9(1):79–103, February 2009. doi:10.1007/s10208-008-9027-z.

7 Tamal K. Dey and Tao Hou. Fast Computation of Zigzag Persistence. In Shiri Chechik, Gonzalo Navarro, Eva Rotenberg, and Grzegorz Herman, editors, *30th Annual European Symposium on Algorithms (ESA 2022)*, volume 244 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 43:1–43:15, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ESA.2022.43.

8 Tamal K Dey, Tao Hou, and Dmitriy Morozov. Apex representatives. *arXiv [cs.CG]*, February 2025. arXiv:2502.17704.

9 Tamal K. Dey, Tao Hou, and Dmitriy Morozov. A fast algorithm for computing zigzag representatives. In Yossi Azar and Debmalya Panigrahi, editors, *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2025, New Orleans, LA, USA, January 12-15, 2025*, pages 3530–3546. SIAM, 2025. doi:10.1137/1.9781611978322.116.

10 Herber Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. *Discrete & computational geometry*, 28(4):511–533, November 2002. doi:10.1007/s00454-002-2885-2.

11 Herbert Edelsbrunner. Dynamic data structures for orthogonal intersection queries. Technical Report F59, Inst. Information Process., Graz University of Technology, Austria, 1980. URL: https://pub.ista.ac.at/~edels/Papers/1980-01-OrthogonalIntersectionQueries.pdf.

12 Herbert Edelsbrunner and John Harer. *Computational Topology: An Introduction*. American Mathematical Soc., 2010.

13 Herbert Edelsbrunner and Dmitriy Morozov. Persistent homology. In Jacob E Goodman, Joseph O'Rourke, and Csaba D Tóth, editors, *Handbook of Discrete and Computational Geometry*. CRC Press, 2017. URL: https://www.csun.edu/~ctoth/Handbook/chap24.pdf.

14 Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

15 Edward M. McCreight. Efficient algorithms for enumerating intersecting intervals and rectangles. Technical Report CSL-80-9, Xeox Palo Alto Research Center, 1980.

16 Dmitriy Morozov and Primoz Skraba. Persistent (co)homology in matrix multiplication time. *arXiv [math.AT]*, December 2024. doi:10.48550/arXiv.2412.02591.

17 Arnur Nigmetov and Dmitriy Morozov. Topological optimization with big steps. *Discrete & computational geometry*, 72(1):310–344, July 2024. doi:10.1007/s00454-023-00613-x.