


Uniform Bounds on Product Sylvester-Gallai Configurations

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Abstract

In this work, we explore a non-linear extension of the classical Sylvester-Gallai configuration. Let \mathbb{K} be an algebraically closed field of characteristic zero, and let $\mathcal{F} = \{F_1, \dots, F_m\} \subset \mathbb{K}[x_1, \dots, x_N]$ denote a collection of irreducible homogeneous polynomials of degree at most d , where each F_i is not a scalar multiple of any other F_j for $i \neq j$. We define \mathcal{F} to be a product Sylvester-Gallai configuration if, for any two distinct polynomials $F_i, F_j \in \mathcal{F}$, the following condition is satisfied:

$$\prod_{k \neq i, j} F_k \in \text{rad}(F_i, F_j).$$

We prove that product Sylvester-Gallai configurations are inherently low dimensional. Specifically, we show that there exists a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$, independent of \mathbb{K} , N , and m , such that any product Sylvester-Gallai configuration must satisfy:

$$\dim(\text{span}_{\mathbb{K}}(\mathcal{F})) \leq \lambda(d).$$

This result generalizes the main theorems from (Shpilka 2019, Peleg and Shpilka 2020, Oliveira and Sengupta 2023), and gets us one step closer to a full derandomization of the polynomial identity testing problem for the class of depth 4 circuits with bounded top and bottom fan-in.

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1 Introduction

In 1893, Sylvester posed the following basic problem in extremal combinatorial geometry.

► **Problem 1** ([49]). *Suppose a finite set of real points are arranged such that any line through two points of the set contains a third point in the set. Prove that the set of points lies on a straight line.*

The first complete affirmative answer to Problem 1 was given by Melchior [35]. The same problem was independently posed in 1943 by Erdős [17], and was answered by Gallai [20]. The above statement is now known as the Sylvester-Gallai theorem.



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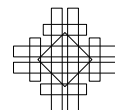
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It is important to note that the answer to Sylvester’s problem depends on the base field. For instance, it was known (quite possibly even to Sylvester) that the answer to Problem 1 is negative when the base field is \mathbb{C} , as the nine inflection points of any non-singular planar cubic form a 2-dimensional complex configuration satisfying Sylvester’s conditions. In 1966, Serre asked in [46] whether any complex configurations satisfying Sylvester’s conditions must be 2-dimensional – that is, whether the configurations arising from non-singular planar cubics are extremal. An affirmative answer to Serre’s question was given in 1983 by Hirzebruch [29], using highly non-elementary algebro-geometric tools.

Ever since the solution of Sylvester’s problem, several generalizations and variations have been studied in combinatorial geometry, theoretical computer science (TCS) and coding theory [15, 27, 42, 16, 7, 13, 39, 40] – see [9, 12] for brief surveys on these generalizations. The usual setup of such generalizations is as follows: given a *finite* collection of geometric objects (points, in the case of Sylvester’s problem) satisfying *enough local conditions* (collinearity of certain triples of points, in the case of Problem 1), one wants to know if such collection of geometric objects must be “*low dimensional*” (all points must be in one line, in the case of Problem 1). As is usual in the literature, any configuration satisfying the proposed local conditions are called *Sylvester-Gallai configurations*, and the result stating that such configurations are low dimensional is referred to as a *Sylvester-Gallai* type theorem.

One line of generalizations of Problem 1 arises from projective duality, which we now discuss. By projective duality, any point $P = (p_1, \dots, p_N)$ gives rise to a dual hyperplane, defined by the zero set of the linear form $\ell_P := p_1x_1 + \dots + p_Nx_N$, which we denote by $V(\ell_P)$. Given three points P, Q, R , the condition that R is in the line defined by P, Q is equivalent to $V(\ell_P, \ell_Q) \subset V(\ell_R)$ in the dual space. Lastly, given points P_1, \dots, P_m , the condition that they are collinear is equivalent to $\dim \text{span}_{\mathbb{R}} \{\ell_{P_1}, \dots, \ell_{P_m}\} = 2$. Thus, we can recast Sylvester’s problem as follows: given a finite set of distinct hyperplanes defined by the set of linear forms $\mathcal{L} := \{\ell_1, \dots, \ell_m\} \subset \mathbb{R}[x_1, \dots, x_N]$ such that for any $i \neq j \in [m]$, there is $k \neq i, j$ such that $V(\ell_i, \ell_j) \subset V(\ell_k)$, then it must be the case that $\dim \text{span}_{\mathbb{R}} \{\mathcal{L}\} = 2$. For simplicity, if one applies projective duality in two dimensions, the foregoing statement becomes quite natural: given a finite set of distinct lines (the set \mathcal{L}) such that for any two lines $\ell_i, \ell_j \in \mathcal{L}$, there must be a third line in \mathcal{L} which passes through the intersection of ℓ_i, ℓ_j (i.e., $V(\ell_i, \ell_j)$), then it must be the case that the set of lines forms a pencil (i.e., all lines intersect at a common point). In fact, this dual formulation was precisely the setting treated in [29].

Motivated by questions in algebraic complexity theory, Gupta [24] proposed non-linear, algebro-geometric generalizations to the above (dual) formulation of Sylvester’s problem (and their set variants, such as [15]). In this work, we make progress on Gupta’s program and fully resolve one such generalization of Sylvester’s problem. Before we describe our main result, in the following subsection we describe the connection between the Sylvester-Gallai configurations that we study and the Polynomial Identity Testing (PIT) problem, a fundamental problem in algebraic complexity theory.

1.1 Polynomial Identity Testing (PIT) and Sylvester-Gallai configurations

The Polynomial Identity Testing (PIT) problem is the task of determining whether a given algebraic circuit computes the zero polynomial. While there are simple and efficient randomized algorithms for the PIT problem, developing an efficient deterministic solution remains a significant open problem in theoretical computer science. The PIT problem is

closely tied to fundamental topics such as lower bounds for algebraic circuits [28, 2, 30] and derandomizing key problems in mathematics and computer science [3, 19, 33, 36, 26, 18]. For a deeper exploration of PIT and its applications, see [43, 48, 44].

The recent works [1, 25, 50] have shown that the PIT problem for general circuits can be reduced to solving it for low-depth circuits, such as unrestricted depth-3 or homogeneous depth-4 circuits, which has made these circuit classes a focus of recent study. Notably, progress has been made towards deterministic, polynomial-time PIT for depth-3 circuits of constant top fanin via the connection between this problem and linear Sylvester-Gallai configurations arising from depth-3 identities [14, 32, 45].

Let us now explain how Sylvester-Gallai configurations arise in the context of depth-3 and depth-4 PIT. Let $\mathbf{x} := (x_1, \dots, x_N)$ be a tuple of N variables and $S := \mathbb{C}[\mathbf{x}]$ be the polynomial ring in N variables. Suppose we are given a depth three circuit with top fan-in three (denoted $\Sigma^3\Pi\Sigma$) that computes a polynomial P . Such a polynomial has the form

$$P = \prod_{i=1}^m \ell_i(\mathbf{x}) + \prod_{j=1}^m g_j(\mathbf{x}) + \prod_{k=1}^m h_k(\mathbf{x}), \quad (1)$$

where ℓ_i, g_j, h_k are linear forms. When $P \equiv 0$, that is, when we have an identity, and moreover this representation of the identity is “efficient,”¹ we may wonder whether the linear forms in the identity must be “low dimensional” (i.e., all such linear forms must “depend on few variables”). To capture the dimensionality of the circuit given by Equation (1), [14] defined the *rank* of the circuit as being the linear span of the forms that appear in the circuit. In case of Equation (1), we have that the rank is $\dim \text{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\}_{i,j,k \in [m]}$.

Consider a linear form ℓ_i from the first gate and a second linear form g_j from the second gate. Since $P \equiv 0$, we have that for any $\mathbf{a} \in \mathbb{C}^N$ such that $\ell_i(\mathbf{a}) = g_j(\mathbf{a}) = 0$, it must be the case that $\prod h_k(\mathbf{a}) = 0$. In other words, we have that $V(\ell_i, g_j) \subset V(\prod h_k) = \bigcup_k V(h_k)$. Since the algebraic set $V(\ell_i, g_j)$ is irreducible, we must have $V(\ell_i, g_j) \subset V(h_a)$ for some $a \in [m]$. Note that this last condition (combined with the symmetry among the gates) is exactly the local constraint arising from the dual formulation of Sylvester’s problem. In [14], the authors asked whether sets $\{\ell_i\}_{i \in [m]}, \{g_j\}_{j \in [m]}, \{h_k\}_{k \in [m]}$ arising from Equation (1) which satisfy the above conditions must be low-dimensional – more precisely, whether $\dim \text{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\}_{i,j,k} = O(1)$.

As it turns out, the Sylvester-Gallai like condition in the previous paragraph can be seen as a set-version of Sylvester’s problem, which was studied by [15]. This work showed that such configurations must have constant rank. Thus, we deduce that $\text{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\} = O(1)$. This shows that any $\Sigma^3\Pi\Sigma$ identity essentially depends on constantly many variables. Combined with the results of [31], this gives a black box deterministic PIT algorithm for such circuits. While significantly extra complexity and subtleties arise when one works with $\Sigma^k\Pi\Sigma$ circuits for $k > 3$, the linear forms in “efficient” identities arising from such circuits satisfy enough local relationships to deduce that they have low rank. This approach was carried out by [32, 45], who generalized the above approach and showed an intrinsic and elegant connection between $\Sigma^k\Pi\Sigma$ identities and Sylvester-Gallai configurations (see [45, Theorem 1.4]). As a result of this connection, Sylvester-Gallai theorems imply deterministic, polynomial-time PIT algorithms for $\Sigma^k\Pi\Sigma$ circuits, when $k = O(1)$.

Motivated by these results, Gupta [24] studied depth four circuits with constant top and bottom fan-ins. This circuit family is denoted $\Sigma^k\Pi\Sigma\Pi^d$, and it computes polynomials which can be written as a sum of k products of polynomials of degree at most d .

¹ The formal definition of efficient is simple and minimal, as defined in [32].

Consider a polynomial Q computed by a $\Sigma^3\Pi\Sigma\Pi^d$ circuit. It has the form

$$Q = \prod_{i=1}^{m_1} A_i(\mathbf{x}) + \prod_{j=1}^{m_2} B_j(\mathbf{x}) + \prod_{k=1}^{m_3} C_k(\mathbf{x}),$$

where A_i, B_j, C_k are polynomials of degree at most d . If $Q \equiv 0$ and the representation is efficient, as in the previous case, we have $V(A_i, B_j) \subseteq V(\prod C_k)$.² However, as the forms are not necessarily linear, we have that $V(A_i, B_j)$ is not necessarily irreducible, and thus we no longer have the same Sylvester-Gallai type configuration, as we cannot guarantee the existence of $k \in [m]$ such that $V(A_i, B_j) \subset V(C_k)$. Nevertheless, as there are many constraints of the form $V(A_i, B_j) \subseteq V(\prod C_k)$, [24] asked whether such Sylvester-Gallai type conditions would be enough to prove that $\dim \text{span}_{\mathbb{C}} \{A_i, B_j, C_k\}_{i,j,k}$ is small enough.

Given any “efficient” $\Sigma^3\Pi\Sigma\Pi^d$ circuit (not necessarily computing the zero polynomial), [24] called it a Sylvester-Gallai circuit if the above constraints on the zerosets (i.e. $V(A_i, B_j) \subseteq V(\prod C_k)$) hold. The above discussion shows that all such circuits that compute 0 are Sylvester-Gallai circuits, but the converse is not true: for instance, multiplying a single form by a constant can change whether or not the circuit computes 0 without changing the above condition.

Non Sylvester-Gallai circuits are relatively simple compared to Sylvester-Gallai circuits. Non Sylvester-Gallai circuits do not have too many cancellations (the above discussion shows that they can never compute 0), and the circuit structure can be preserved by certain linear variable reduction maps, which is the main tool used in the works [24, 23] to handle these circuits. This allows these works to certify the non-zerosness of non-Sylvester-Gallai circuits.

The cancellations in Sylvester-Gallai circuits on the other hand are a lot more subtle, and it is not clear if they are preserved by the linear maps of [24, 23]. Similar to how linear Sylvester-Gallai configurations have constant rank, Gupta conjectured that sets of higher degree forms satisfying the local condition $V(A_i, B_j) \subset V(\prod_k C_k)$ must “depend on constantly many variables.” If this is true, then we can efficiently test if Sylvester-Gallai circuits are nonzero using the methods of [8]. A simpler form of Gupta’s main conjecture is the following generalization of the configurations studied in [15]:

► **Conjecture 2** (Gupta’s main conjecture - simple form). *Let $\{A_i\}_{i \in [m_1]} \cup \{B_j\}_{j \in [m_2]} \cup \{C_k\}_{k \in [m_3]}$ be a set of irreducible polynomials of degree $\leq d$ such that for any $i \in [m_1], j \in [m_2]$, we have that $V(A_i, B_j) \subseteq V(\prod_k C_k)$ (and the same relations hold when exchanging the roles of the polynomials A, B, C). Then there exists a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that*

$$\dim \text{span}_{\mathbb{C}} \{A_i, B_j, C_k\}_{i,j,k} \leq \lambda(d).$$

A first step towards establishing the above conjecture is to establish the Sylvester-Gallai version of the above conjecture. This motivates us to consider the Sylvester-Gallai configurations we study in this article, which we now define. By way of the algebraic-geometric dictionary, we will henceforth replace the geometric condition $V(A, B) \subset V(C)$ by the equivalent algebraic condition $\text{rad}(C) \subset \text{rad}(A, B)$.

► **Definition 3** (Product Sylvester-Gallai configurations). *A finite set $\mathcal{F} \subset \mathbb{C}[x_1, \dots, x_N]$ of irreducible forms of degree at most d is a d -product-SG-configuration if the following hold.*

- $F_i \notin (F_j)$ for any $F_i, F_j \in \mathcal{F}$. (each form encodes a different hypersurface)
- For each $F_i, F_j \in \mathcal{F}$, we have (Sylvester-Gallai condition)

$$\prod_{F \in \mathcal{F} \setminus \{F_i, F_j\}} F \in \text{rad}(F_i, F_j).$$

² By symmetry among the gates, we also have $V(A_i, C_k) \subseteq V(\prod B_j)$ and $V(B_j, C_k) \subseteq V(\prod A_i)$.

As stated above, the reader may find it strange that the product constraint no longer seems local, since the condition for each F_i, F_j involves every other form in the configuration. However, by a standard Bézout type argument it follows that the condition $\prod_{F \in \mathcal{F} \setminus \{F_i, F_j\}} F \in \text{rad}(F_i, F_j)$ is equivalent to the (local) condition that there are indices k_1, \dots, k_{d^2} different from i, j such that

$$\prod_{a=1}^{d^2} F_{k_a} \in \text{rad}(F_i, F_j).$$

Since d is a constant, each (local) condition now only depends on constantly many forms in the configuration as expected. As this equivalent condition is more cumbersome to state, we prefer to work with the above definition.

In a similar fashion to [24], we can also conjecture that such “product” configurations are “low dimensional.” This is the content of the following conjecture:

► **Conjecture 4** (Product Sylvester-Gallai Conjecture). *There is a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that if \mathcal{F} is a d -product-SG-configuration then $\dim \text{span}_{\mathbb{C}} \{\mathcal{F}\} = \lambda(d)$.*

The above conjecture was first considered by [39], where the authors proved Conjecture 4 for $d = 2$. Our main theorem, which we now state, generalizes their main result to forms of every degree, thus fully settling Conjecture 4.

► **Theorem 5** (Product Sylvester-Gallai Theorem). *There is a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that if \mathcal{F} is a d -product-SG-configuration then $\dim \text{span}_{\mathbb{C}} \{\mathcal{F}\} = \lambda(d)$.*

Our proof techniques build upon the techniques introduced in [47, 38] to study radical Sylvester-Gallai configurations, which we define in Section 1.3. In Section 1.3 we also discuss related and previous works on non-linear Sylvester-Gallai configurations and the PIT problem for depth-4 circuits. In Section 1.4, we give a high level outline of our proof, highlighting the technical difficulties that need to be overcome, as well as the new conceptual contributions of this work. We will now discuss the main contributions of this work, where we state our main technical result, which is of independent interest. Formal statements and proofs of all claims can be found in the full version of this paper.

1.2 Contributions of this paper

This paper has two main contributions. The first is the proof of Theorem 5, which states that product Sylvester-Gallai configurations of degree bounded by d have dimension uniformly upper bounded by a function which depends only on d . This result simultaneously generalizes [47, 39, 38], putting us one step closer towards a proof of Conjecture 2, and also to the main conjecture of Gupta [24]. To achieve this, we adapt the inductive approach from [38] to work with product configurations. This adaptation, as we discuss in Section 1.4, requires us to deal with several subtleties, one of them being to establish a useful property of the general quotients of [47, 38]: we prove that such general quotients behave well with respect to (a generalized notion of) absolute irreducibility. More precisely, we prove that that these quotients send *absolutely reducible* forms to *reducible* elements, and that forms which are *absolutely irreducible* over the vector space being quotiented out will *remain absolutely irreducible* after a general quotient.

The second contribution is our main technical result which relates absolute irreducibility of a polynomial P with an *effective and uniform bound* on the non-primality of certain ideals containing P . In a simplified form, it can be stated as:

► **Theorem 6** (Prime bound - informal). *Let $S := \mathbb{C}[x_1, \dots, x_N, y_1, \dots, y_n]$ and $P \in S \setminus (x_1, \dots, x_N)$ be a form of degree d that is irreducible over $\mathbb{C}(x_1, \dots, x_N)[y_1, \dots, y_n]$. The number of non-associate irreducible forms $Q \in \mathbb{C}[x_1, \dots, x_N]$ such that (P, Q) is not prime is bounded above by a function of d .*

We believe that the above theorem, and its more general version (which can be found in the full version), together with its proof, are of independent interest. We now briefly discuss the ingredients needed in the proof of the above theorem. If one only combines results from Gröbner basis theory and elimination theory with techniques from algebraic geometry, one can obtain an effective bound for the above theorem which depends on N, n and d . Since the goal of a Sylvester-Gallai theorem is to obtain bounds *independent of the number of variables*, such a bound will not be good enough for our purposes. By combining the Stillman's uniformity bounds of [5] with results from elimination theory and algebraic geometry, we are able to obtain an effective bound (depending only on n, d) on the number of non-associate irreducible forms Q such that (P, Q) is not prime.

Once the dependence on N has been removed and a finite bound on the number of (non-associate) bad forms Q has been established, we can then use Bertini's theorem and the Ananyan-Hochster principle to further drop the dependency on n , finally obtaining a bound which simply depends on the degree d . To achieve this, we give very effective bounds on the degrees of generators of Gröbner bases for ideals generated by a small number of variables of low degree, and also for certain elimination ideals of such ideals. These bounds could be of independent interest.

An important point to note is that [38] builds on the Ananyan-Hochster principle to show their transfer theorems, which we also use. However, to prove the above theorem, we also need to apply the Ananyan-Hochster principle in a different manner: combined with results from Gröbner basis theory, we use it to find low degree polynomials in elimination ideals of ideals generated by a small number of low degree forms.

1.3 Related Work

1.3.1 Sylvester-Gallai configurations

In [24], as a first step of his plan, Gupta proposed the study of a direct generalization of the linear Sylvester-Gallai configurations, which are captured by the following definition.

► **Definition 7** (Radical Sylvester-Gallai configuration). *Let $\mathcal{F} \subset \mathbb{C}[x_1, \dots, x_N]$ be a finite set of irreducible forms of degree at most d . We say that \mathcal{F} is a d -radical Sylvester-Gallai configuration if the following hold.*

- $F_i \notin (F_j)$ for any $F_i, F_j \in \mathcal{F}$.
- For every $F_i, F_j \in \mathcal{F}$, we have $F_k \in \text{rad}(F_i, F_j)$ for some $F_k \in \mathcal{F} \setminus \{F_i, F_j\}$.

The following conjecture is a strengthening of [24, Conjecture 2].

► **Conjecture 8** (Radical Sylvester-Gallai conjecture). *There is a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that if \mathcal{F} is a d -radical Sylvester-Gallai configuration then $\dim \text{span}_{\mathbb{C}} \{\mathcal{F}\} = \lambda(d)$.*

Conjecture 8 was proved by [47] in the special case when $d = 2$. Subsequently, [37] proved Conjecture 8 for $d = 3$, and [38] fully resolved the above conjecture in the affirmative. Note that the condition $F_k \in \text{rad}(F_i, F_j)$ for forms of degree d is a natural generalization of the condition $\ell_k \in (\ell_i, \ell_j)$: by the Nullstellensatz, these conditions are equivalent to $V(F_k) \supset V(F_i, F_j)$ and $V(\ell_k) \supset V(\ell_i, \ell_j)$ respectively. Therefore Conjecture 8 is a direct generalisation of the usual Sylvester-Gallai theorem for forms of higher degrees.

The first work to consider product Sylvester-Gallai configurations was the work of [39], where the authors settled Conjecture 4 for quadratic forms. In follow up work [40], the same authors settled Conjecture 2 for the case when $d = 2$, therefore proving that Gupta's PIT algorithm runs in deterministic, poly-time for $\Sigma^3\Pi\Sigma\Pi^2$ circuits. Note that the above path in the proofs of Sylvester-Gallai type results is a natural path towards resolving the PIT problem for $\Sigma^3\Pi\Sigma\Pi^d$ circuits, since Conjecture 8 follows from Conjecture 4, and it can be shown that Conjecture 2 would follow from a robust version of Conjecture 4.

Robust and higher dimensional generalizations of the radical Sylvester-Gallai theorems have also been studied by the works [41, 22, 21]. Aside from settling such interesting questions in extremal combinatorial geometry, such variations are also motivated by the PIT problem for depth-4 circuits, in the hope that these more general versions may be useful towards obtaining a (potentially simpler) proof of Gupta's conjectures.

1.3.2 PIT for depth four circuits

Depth-4 circuits with bounded top fan-in are among the “easiest” classes for which we do not have deterministic, poly-time PIT algorithms. There has thus been some work on deterministic PIT for these circuits using methods other than the Sylvester-Gallai based methods discussed above.

In [11], the authors give a quasipolynomial time PIT algorithm for depth 4 circuits of bounded top and bottom fanins, via the Jacobian method of [4]. By using the logarithmic derivative and its power series, they are able to modify the top sum gate in the circuit to a powering gate.

The breakthrough work of [34] on lower bounds for bounded depth circuits gives another approach to PIT for this model. Hardness-randomness tradeoffs have been well studied in algebraic complexity [2, 30], and the work of [10] showed that these tradeoffs also hold in the bounded depth setting. Combining these results gives a subexponential time PIT algorithm for depth four circuits. Based on the lower bounds of [34], a hitting set generator for bounded depth circuits was constructed by [6], which gives another subexponential time PIT algorithm for depth four circuits, with improved parameters.

It is important to note that neither of the above methods seem to be able to give a truly polynomial time algorithm for the PIT problem for $\Sigma^k\Pi\Sigma\Pi^d$ circuits. The only currently known method that might do so is the approach of [24].

1.4 High-level proof overview

In this subsection we provide an overview of the proof of our main result, Theorem 5. We begin by outlining the high-level approach used in previous works to bound the dimension of the span of radical Sylvester-Gallai configurations and discuss the challenges of applying this method in our context. Next, we describe the tools that we employ to address these challenges. Finally, we provide a concise summary of how these components are combined to establish our main theorem.

To illustrate the ideas in this overview, we use a special case of product Sylvester-Gallai configurations with a simplifying assumption as an example. In the final part of this subsection, we explain how this assumption can be eliminated to extend the argument to the general case.

1.4.1 High-level approach

At a high level, our proof uses a similar approach as the previous works on higher degree Sylvester-Gallai configurations [47, 39, 40, 41, 21, 38], which we now summarize.

Let $R := \mathbb{C}[x_1, \dots, x_N]$, and $S := R[y_1, \dots, y_n]$ be two polynomial rings and $\mathcal{F} \subset S$ be a radical (or product) Sylvester-Gallai configuration, where the forms in \mathcal{F} have degree at most d . Thus, we can write $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_d$, where \mathcal{F}_e are the forms in \mathcal{F} of degree e . We start with the assumption that each form only depends on constantly many variables in S . A number of key ideas can already be highlighted in this easier setting.

We want to control the highest degree forms \mathcal{F}_d in our configuration, with the goal of reducing to the case when the highest degree is $d - 1$, where we can proceed inductively.

If many ideals of the form (F_i, F_j) , where $F_i, F_j \in \mathcal{F}_d$, are radical (or prime), then the set \mathcal{F}_d is essentially a (robust) linear Sylvester-Gallai configuration. In this case, the linear Sylvester-Gallai theorems imply that constantly many forms F_1, \dots, F_a are a basis for $\text{span}_{\mathbb{C}}\{\mathcal{F}_d\}$, and if x_1, \dots, x_r is the union of the set of variables of F_1, \dots, F_a , then $\mathcal{F}_d \subset \mathbb{C}[x_1, \dots, x_r]$. This is sufficient to control the forms of \mathcal{F}_d and apply our inductive step. The interesting case is thus when there are many ideals generated by forms in \mathcal{F}_d that are not radical (or prime). In this case, the goal is to show that the forms in \mathcal{F}_d must share many variables in common. The steps listed below show how previous works have dealt with this case.

1. First, one devises a structure theorem on ideals that are not radical (or prime). In principle, such structure theorems show that if an ideal generated by two polynomials is not radical (or not prime), then the variables appearing in the polynomials must have some special “dependencies”.
2. Next, combine the structure theorem with the local conditions to show the existence of constantly many “good variables” x_1, \dots, x_b such that $\mathcal{F}_d \subset (x_1, \dots, x_b)$.
In this step, our approach differs from previous works, as in a product configuration, it may not be possible to directly achieve this high level of control on the forms in \mathcal{F}_d . We discuss this in more detail in Step 2 below.
3. Finally, we “randomly project” (i.e., apply a general quotient) the special variables x_1, \dots, x_b to a new variable z in order to reduce the degree of the forms in \mathcal{F}_d .
Also, in this step our approach will differ from previous works. Instead of reducing degree directly, we will make the forms in \mathcal{F} “factor more.”

We now elaborate on this approach, pointing out the new difficulties we face in this work, and sketch how we deal with these issues.

1.4.1.1 Step 1 – Structure theorems

Earlier works ([47, 37]) gave fairly strong classification of ideals that are not radical in the first step. In particular, they consider the possible minimal primes that a non-radical ideal of the form (P, Q) can have, and possible multiplicities that these minimal primes can have, giving a structural result for each of these cases. However this strategy is hard to generalise beyond the case of ideals generated by two cubics, since these ideals have very high degree and their primary decomposition becomes much more complex as the degree grows.

Subsequent work by [38] used a significantly weaker (albeit more general) structure theorem, and showed that this suffices to carry out the strategy above. The structure theorem in [38] is the following. If $P \in S$ is an irreducible form of degree d that is not in the ideal (x_1, \dots, x_N) , then there are at most $3d^3$ square-free forms $Q_i \in R$ such that (P, Q_i) is not radical. Informally, this statement says that if a form P depends non-trivially on some

variable(s) y_i , then (P, Q) must be radical for almost every polynomial Q that only depends on x_1, \dots, x_N .

For product Sylvester-Gallai configurations, the ideals of interest are those that are not prime. One could hope to generalize the structural result from [38] to this setting. However, a statement of the above form is no longer true here. Take the form $P = y_1^4 - x_1x_2y_2^2$. This form depends on the variables y_1, y_2 . However, for every irreducible linear form $Q \in R$, the ideal $(P, x_1x_2 - Q^2)$ is not prime, even though P and $x_1x_2 - Q^2$ are irreducible. This is our first technical difficulty.

We overcome this by showing that the above only happens because P is reducible as a form in $\overline{\mathbb{C}}(x_1, \dots, x_N)[y_1, y_2]$, since it factors as $P = (y_1^2 - \sqrt{x_1x_2}y_2)(y_1^2 + \sqrt{x_1x_2}y_2)$. This is where our first key conceptual and technical contribution comes in. We show that for any degree d form in $R[y_1, \dots, y_m]$ that is irreducible in $\overline{\mathbb{C}}(x_1, \dots, x_N)[y_1, \dots, y_n]$ (such forms are called absolutely irreducible over $R[y_1, \dots, y_m]$), there are only finitely many $Q_i \in R$ such that (P, Q_i) is not prime.

A fairly subtle issue remains in the above statement. In order to solve the product Sylvester-Gallai problem, we need quantitative bounds on the number of “bad forms” Q_i . Moreover, we need a bound which is solely a function of $\deg(P)$. If we use standard techniques from commutative algebra and algebraic geometry, we obtain bounds which depend on the number of variables (that is: N, n) and on $\deg(P)$. This is not sufficient for us, as we want to show that $\dim \text{span}_{\mathbb{C}} \{\mathcal{F}\}$ does not depend on the number of variables in the polynomial ring. To prove that the number of bad forms is in fact *independent* of N and n , we need to work a bit harder, and we need to make use of two additional tools: Bertini-type theorems (to remove the dependence on n), and the Ananyan-Hochster construction of small strong subalgebras (this will remove the dependence on N). This yields, in its basic form, Theorem 6, which we discussed in Section 1.2.

1.4.1.2 Step 2 – Finding a small set of “good common variables”

Suppose \mathcal{F} is a radical Sylvester-Gallai configuration. By the above, we can assume that \mathcal{F}_d is not a robust linear Sylvester-Gallai configuration. Our assumption implies that for many pairs F_i, F_j , the ideal (F_i, F_j) is not radical. In particular, for a small $0 < \varepsilon < 1$, we can find $3d^3$ forms $F_1, \dots, F_{3d^3} \in \mathcal{F}_d$ such that (F_i, F_j) is not radical for $i \leq 3d^3$ and $j \leq (1 - \varepsilon)|\mathcal{F}_d|$. The radical bound mentioned in Step 1 shows that there is $a \in O(1)$ and an ideal generated by linear forms (ℓ_1, \dots, ℓ_a) that contains every such F_j . Repeating this argument, we can find an ideal (ℓ_1, \dots, ℓ_b) with b constant such that $\mathcal{F}_d \subset (\ell_1, \dots, \ell_b)$.

When \mathcal{F} is a product Sylvester-Gallai configuration a number of issues arise: first, our structure theorem deals with prime ideals (instead of radical) and our assumptions are that the forms are absolutely irreducible. Hence, we will only be able to draw the following weaker conclusion: there are constantly many variables x_1, \dots, x_b such that every $F \in \mathcal{F}_d$ is either in (x_1, \dots, x_b) , or absolutely reducible in the ring $S' := \mathbb{C}(x_1, \dots, x_b)[x_{b+1}, \dots, x_N, y_1, \dots, y_n]$. Therefore, when we apply the general quotient (i.e. a random projection), we can no longer guarantee that the degree of our forms reduce. We deal with this issue by changing the way we “decompose” the configuration $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_d$. This is a key difference between our approach and previous approaches, since this issue does not arise when studying radical Sylvester-Gallai configurations. This new decomposition also allows us to solve an issue that arises when we apply the general quotient. We now state this issue, and discuss the solution.

1.4.1.3 Step 3 – General quotients & degree reduction/lowering degree

The image of an irreducible form under a projection map can be reducible. In order to apply induction, the definition of product Sylvester-Gallai configurations has to be generalised to allow the forms in \mathcal{F} to be reducible. Hence, we update our definition of product Sylvester-Gallai configurations to the following.

► **Definition 9** (Product Sylvester–Gallai configurations). *A finite set $\mathcal{F} \subset \mathbb{C}[x_1, \dots, x_N]$ of forms of degree at most d is a d -product-SG-configuration if the following hold.*

- $\gcd(F_i, F_j) = 1$ for all $i \neq j$.
- For every $F_i, F_j \in \mathcal{F}$, we have $\prod_{F \in \mathcal{F} \setminus \{F_i, F_j\}} F \in \text{rad}(F_i, F_j)$.

This generalisation already appears in [38], where reducibility is not an issue when dealing with radical ideals, since the radical bound also applies to reducible forms. The prime bound however can no longer apply, irrespective of any assumption: if P is reducible then (P, Q) can never be prime. In particular, if every form in \mathcal{F}_d is reducible, then we can draw no conclusion about \mathcal{F}_d using the above ideas.

We tackle both these issues by turning this reducibility into an advantage: instead of trying to reduce the degree as in the inductive approaches of [47, 38, 39], we will try and make the forms in our configuration “factor more”.

In order to formalise what we mean by “factor more”, we introduce the notion of factor sets of a product configuration. Given a (generalized) product Sylvester-Gallai configuration \mathcal{F} , we define the factor set \mathcal{I} of \mathcal{F} to be the set of irreducible factors of all forms in \mathcal{F} . As before, we write $\mathcal{I} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_c$, where $\mathcal{I}_j \subset S_j$.

If every form in \mathcal{F} factors a lot, then all the forms in \mathcal{I} will have low degree. We will induct on the highest integer c such that \mathcal{I}_c is nonzero. Now, the set of “common variables” we look for will not be variables common to \mathcal{F}_d , but variables common to \mathcal{I}_c . The absolute irreducibility condition remains, and we end up with variables x_1, \dots, x_b such that every form in \mathcal{I}_c is either in the ideal (x_1, \dots, x_b) or absolutely reducible. We postpone the discussion of how we do this step to the next subsection. We now show how these common variables are used in the rest of the induction step.

Step 2 lets us find the “common variables” x_1, \dots, x_b . Define a projection map φ that maps each x_i for $1 \leq i \leq b$ to a random multiple of a new variable z , and acts as identity on the other variables. If $F \in (x_1, \dots, x_b)$ then $\varphi(F) = zF'$, where $\deg(F') < \deg(F)$. We define $\varphi(\mathcal{F})$ to be the image of \mathcal{F} under φ , after factoring out powers of z . Such a projection map preserves radical and product Sylvester-Gallai structure, and also allows bounds on the dimension of $\varphi(\mathcal{F})$ to be lifted back to bounds on the dimension of \mathcal{F} .

If we have $\mathcal{F}_d \subset (x_1, \dots, x_b)$ (as in the case of radical Sylvester-Gallai configurations) then $\varphi(\mathcal{F})$ is a Sylvester-Gallai configuration with forms of degree at most $d - 1$, and induction can be applied. If on the other hand \mathcal{F} is a product Sylvester-Gallai configuration, then we have the weaker condition that every $F \in \mathcal{I}_c$ is either in (x_1, \dots, x_b) or absolutely reducible in S' . We show that the image of any F that is absolutely reducible in S' under the map φ is reducible. Thus, in either case we conclude that $\mathcal{I}'_c = \emptyset$, where \mathcal{I}' is a factor set of $\varphi(\mathcal{F})$, and we can apply induction.

In the next subsection we show how we find these “common variables” in our setting. Note that we are still under the assumption that every form in \mathcal{F} (and thus in \mathcal{I}) depends on only constantly many variables. We will also show how this assumption is removed.

1.4.2 Putting everything together

Let \mathcal{F} be a product Sylvester-Gallai configuration of forms of degree at most d . We now allow forms in \mathcal{F} to be reducible, but still require that they are square-free and are pairwise relatively prime. Let $\mathcal{I} := \mathcal{I}_1 \cup \dots \cup \mathcal{I}_d$ be the set of irreducible factors of forms in \mathcal{F} . We will induct on the largest integer c such that $\mathcal{I}_c \neq \emptyset$. We first go over the base case in our induction.

Our base case is when every form $F_i \in \mathcal{F}$ is a product of linear forms, say $F_i = \prod_a \ell_{ia}$. In this case, $\mathcal{I} = \mathcal{I}_1 = \{\ell_{ia}\}_{i,a}$. Now suppose we pick ℓ_{ia} and ℓ_{jb} with $i \neq j$. The ideal (ℓ_{ia}, ℓ_{jb}) is a minimal prime of (F_i, F_j) . Therefore, the product Sylvester-Gallai condition implies that $F_k \in (\ell_{ia}, \ell_{jb})$ for some k , which in turn implies that $\ell_{ke} \in (\ell_{ia}, \ell_{jb})$ for some choice of index e . This is a linear Sylvester-Gallai relationship among the elements of \mathcal{I}_1 . The choice of ℓ_{ia}, ℓ_{jb} was arbitrary, except the condition that $i \neq j$. Therefore, the set \mathcal{I}_1 is a robust linear Sylvester-Gallai configuration, and therefore has bounded rank. This in turn implies that the forms in \mathcal{F} have bounded rank, since they are spanned by monomials of degree at most d in a basis for $\mathcal{I} = \mathcal{I}_1$.

We now sketch the induction step. Suppose c is the largest integer such that $\mathcal{I}_c \neq \emptyset$. Our induction step is further split into two steps, as discussed in the previous subsection.

1. First we show that there are variables x_1, \dots, x_b such that every form in \mathcal{I}_c is either in the ideal (x_1, \dots, x_b) or absolutely reducible in $S' := \mathbb{C}(x_1, \dots, x_b)[x_{b+1}, \dots, x_N, y_1, \dots, y_n]$. This is a relaxed version of step 2 in the previous approach, applied to \mathcal{I}_c instead of \mathcal{F}_d .
2. We then show that projecting x_1, \dots, x_b to a new variable z results in a product Sylvester-Gallai configuration with $\mathcal{I}_e = \emptyset$ for any $e \geq c$.

We elaborate on these in the special case when $c = 2$, since this already captures all the main ideas. As before, if \mathcal{I}_2 is a robust linear Sylvester-Gallai configuration, then step 2 follows easily by picking a basis for \mathcal{I}_2 . Consider $A, B \in \mathcal{I}_2$, and suppose $A|F_i$ and $B|F_j$ for some $F_i, F_j \in \mathcal{F}$. Suppose (A, B) is prime. Then (A, B) is a minimal prime of (F_i, F_j) . The product Sylvester-Gallai configuration implies $F_k \in (A, B)$ for some k , and therefore $C \in (A, B)$ for some $C|F_k$. Further, by homogeneity, it must be that $C \in \mathcal{I}_2$. This gives us a linear Sylvester-Gallai relationship between the elements $A, B, C \in \mathcal{I}_2$. The assumption that \mathcal{I}_2 is not a robust linear Sylvester-Gallai configuration will therefore imply that (A, B) is not prime for many pairs A, B . Since the forms in \mathcal{I}_2 are irreducible (even though the forms in \mathcal{F} might not be), by a combinatorial argument and Theorem 6, we deduce the existence of variables x_1, \dots, x_b , completing step 1 above.

We now move on to second step. Suppose φ is a projection map that sends x_1, \dots, x_b to random multiples of a new variable z . The map φ sends elements in (x_1, \dots, x_b) to multiples of z . We also prove that φ sends elements of $\mathcal{I}_2 \setminus (x_1, \dots, x_b)$ that are absolutely reducible in S' to reducible forms. Since $c = 2$, the degree of each of these forms has to be 1. Therefore, if we apply φ to some $F \in \mathcal{F}$, every factor of F will be linear, even though it is possible that $\deg(F) = \deg(\varphi(F))$. This reduces the problem to the base case, and the properties of φ allow us to lift the resulting bound on the dimension of $\varphi(\mathcal{F})$ back to a bound on the dimension of \mathcal{I} .

1.4.2.1 Removing the initial assumption

We started with the assumption that every form in \mathcal{F} depended only on constantly many variables. In general, it is of course possible that the forms depend on many or even all the variables. This issue was overcome in [38], building on the seminal work of [5]. They show that forms H_1, \dots, H_a of high enough strength behave essentially like variables, in

the sense that $\mathbb{C}[H_1, \dots, H_a]$ is isomorphic to a polynomial ring. Further, the extension $\mathbb{C}[H_1, \dots, H_a] \subset S$ has many of the properties that the extension $\mathbb{C}[x_1, \dots, x_a] \subset S$ has, the most useful of them being that this extension preserves arbitrary intersection of ideals.

Motivated by this, [38] define the notion of strong vector spaces and algebras, which are vector spaces of small dimension spanned by forms of high rank, and the algebras they generate. They show that in the general case of the radical Sylvester-Gallai problem (when the forms depend on more than constantly many variables), while there might not be variables x_1, \dots, x_b such that $\mathcal{F}_d \subset (x_1, \dots, x_b)$, we can find a strong vector space V such that $\mathcal{F}_d \subset (V)$.

The above change complicates the projection step, since random restriction of higher degree forms is no longer well defined. To fix this, [38] defined generalised projection maps, and again proved that such maps have all the properties that the linear projection maps have. In particular, they preserve the radical Sylvester-Gallai structure, and they allow bounds on the dimension of the image of \mathcal{F} to be lifted back to bounds on \mathcal{F} . With these tools, they set up an inductive framework to show radical Sylvester-Gallai configurations have bounded dimension, with linear Sylvester-Gallai configurations acting as the base case.

Extending our results to the setting of strong algebras in order to apply this framework requires two technical results. First we define the notion of absolute reducibility with respect to strong vector spaces. This allows us to extend the prime bound to strong algebras. We then show that general projection maps send forms that are absolutely reducible with respect to strong vector spaces to reducible forms. These two extensions to the above framework allow us to prove our main theorem in the general case.

2 Conclusion

In this work, we prove that product Sylvester-Gallai configurations of forms have bounded dimension, generalising the result of [38], and getting one step closer towards a PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits. To achieve this, we give a novel, and very effective, sufficient condition for the primality of ideals of the form (P, Q) , where P “depends on more variables” than Q , via the transfer principle from [38]. This turns out to be far more subtle, and to require more tools from algebraic geometry, than giving a sufficient condition for the ideal to be radical.

Our work leaves two important questions, in order to bridge the gap between our results and a complete analysis for a polynomial-time, deterministic PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits. The first question is to extend this work and prove a higher codimension product Sylvester-Gallai theorem for all degrees, simultaneously generalising the main theorems of [21] and this work. The second question is to provide a tight connection between such Sylvester-Gallai problems and rank bounds for identities in $\Sigma^k\Pi\Sigma\Pi^d$, thereby generalizing the beautiful result in [45, Theorem 1.4] to circuits of depth four.

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