Super-Polynomial Growth of the Generalized Persistence Diagram

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──── **Abstract** ────

The Generalized Persistence Diagram (GPD) for multi-parameter persistence naturally extends the classical notion of persistence diagram for one-parameter persistence. However, unlike its classical counterpart, computing the GPD remains a significant challenge. The main hurdle is that, while the GPD is defined as the Möbius inversion of the Generalized Rank Invariant (GRI), computing the GRI is intractable due to the formidable size of its domain, i.e., the set of all connected and convex subsets in a finite grid in $\mathbb{R}^d$ with $d \geq 2$. This computational intractability suggests seeking alternative approaches to computing the GPD.

In order to study the complexity associated to computing the GPD, it is useful to consider its classical one-parameter counterpart, where for a filtration of a simplicial complex with n simplices, its persistence diagram contains at most n points. This observation leads to the question: *Given a d-parameter simplicial filtration, could the cardinality of its GPD (specifically, the support of the GPD) also be bounded by a polynomial in the number of simplices in the filtration?* This is the case for $d = 1$, where we compute the persistence diagram directly at the simplicial filtration level. If this were also the case for $d \geq 2$, it might be possible to compute the GPD directly and much more efficiently without relying on the GRI.

We show that the answer to the question above is negative, demonstrating the inherent difficulty of computing the GPD. More specifically, we construct a sequence of d-parameter simplicial filtrations where the cardinalities of their GPDs are not bounded by any polynomial in the number of simplices. Furthermore, we show that several commonly used methods for constructing multi-parameter filtrations can give rise to such "wild" filtrations.

1 Introduction

(Multi-parameter) Persistent Homology. Persistent homology is a central concept in Topological Data Analysis (TDA), which is useful for studying the multi-scale topological features of datasets [16, 17, 27]. In the classical one-parameter setting, topological features in datasets are summarized in the form of persistence diagrams or barcodes [23, 65], which serve as complete, discrete invariants of the associated persistence modules.
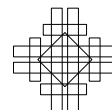
Multi-parameter persistent homology generalizes persistent homology. In the d-parameter setting, topological features of datasets are indexed by the poset $\mathbb{R}^d$ or a subposet of $\mathbb{R}^d$ [11, 18]. This generalization allows for a finer extraction of topological features, enabling a more detailed analysis of complex datasets [19, 24, 46, 47, 59, 63, 64]. However, the algebraic structure of multi-parameter persistence modules, i.e. functors from the poset $\mathbb{R}^d$ ($d \geq 2$) to the category $\mathbf{vec}_{\mathbb{F}}$ of finite dimensional vector spaces over a field $\mathbb{F}$, is significantly more complex, and in fact no complete discrete invariant exists for multi-parameter persistence modules [18].

Generalized Persistence Diagram (GPD). Many incomplete, but potentially useful invariants for multi-parameter persistence modules have been studied; e.g. [3, 4, 8, 12, 21, 34, 35, 44, 45, 51, 54, 58]. The Generalized Persistence Diagram (GPD) is one such invariant and is a natural extension of the persistence diagram for one-parameter persistence [39, 55]. The GPD is defined as the Möbius inversion of the Generalized Rank Invariant (GRI)[1], which captures "persistence" in multi-parameter persistence modules or, more generally, any linear representations of posets [40]. Many aspects of the GPD and GRI–such as stability, discriminating power, computation, connections to other invariants, and generalizations–have been studied; e.g. [5, 12, 22, 25, 26, 31, 41, 42]. Also, a vectorization method for the (restricted) GRI has recently been proposed and utilized in a machine learning context [53, 64]. There are also other works on invariants of multi-parameter persistence modules that are closely related to the GPD and GRI; e.g. [3, 4, 5, 8, 15, 34, 36].

Challenges in Computing the GPD. While many properties of the GPD have been clarified in the aforementioned works, computing the GPD, unlike its classical counterpart, remains a significant challenge. The main hurdle is that, while the GPD is defined as the Möbius inversion of the GRI, computing the GRI is intractable mainly due to the formidable size of its domain. For instance, the domain of the GRI of a persistence module over a finite grid $G \subset \mathbb{R}^d$ is the set of all *intervals* in G (cf. Definitions 2 and 5). The cardinality of the domain is huge relative to the cardinality $|G|$ of G even when $d = 2$; cf. [2, Theorem 31]. This computational intractability suggests seeking alternative approaches to computing the GPD.

In order to study the complexity associated to computing the GPD, it is useful to consider its classical one-parameter counterpart, where for a filtration of a simplicial complex with N simplices, its persistence diagram contains at most N points. This observation leads to the question: *Given a d-parameter simplicial filtration, could the cardinality of its GPD (more precisely, the support of the GPD) also be bounded by a polynomial in the number of simplices in the filtration?* In what follows, we make this question more precise.

Size of the GPD. Let K be an abstract simplicial complex, and let ΔK be the set of subcomplexes of K ordered by inclusion. A monotone map $\mathcal{F} : \mathbb{R}^d \to \Delta K$ such that there exists a $p \in \mathbb{R}^d$ with $\mathcal{F}_p := \mathcal{F}(p) = K$ is called a (***d*-parameter simplicial) filtration (of K**). The number of simplices in the filtration $\mathcal{F}$ is defined as the number of simplices in K.

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[1] The term "generalized persistence diagram" sometimes refers to different concepts; e.g. [14, 52, 56]. Also, we note that the notion of the GRI stems from the concept of the rank invariant [18].

▶ **Definition 1** ([45]). *We call the filtration $\mathcal{F}$ **finite**, if K is finite, and in $\mathcal{F}$, every simplex $\sigma \in K$ is born at finitely many points of $\mathbb{R}^d$, i.e. there exist finitely many points $q_1, \ldots, q_{t(\sigma)} \in \mathbb{R}^d$ s.t. $\sigma \in \mathcal{F}_p$ for $p \in \mathbb{R}^d$ if and only if $q_i \leq p$ in $\mathbb{R}^d$ for some $i = 1, \ldots, t(\sigma)$. We call those points $q_1, \ldots, q_{t(\sigma)}$ the **birth indices** of σ. We also say that σ is **born** at $q_1, \ldots, q_{t(\sigma)} \in \mathbb{R}^d$.*

If $\mathcal{F}$ is finite, then for each homology degree $m \in \mathbb{Z}_{\geq 0}$, the persistence module $\mathrm{H}_m(\mathcal{F}; \mathbb{F}) : \mathbb{R}^d \to \mathbf{vec}_{\mathbb{F}}$ is finitely presentable (cf. Definition 3 and Remark 4). This guarantees that the **m-th GPD** of any finite d-parameter filtration is well-defined as an integer-valued function on a *finite* subset of intervals of $\mathbb{R}^d$ (cf. Definition 5 and Remark 7), denoted by $\mathrm{dgm}_m(\mathcal{F})$. By the **size** of $\mathrm{dgm}_m(\mathcal{F})$ we mean the cardinality of the *support* of $\mathrm{dgm}_m(\mathcal{F})$, i.e. $|\{I \in \mathrm{Int}(\mathbb{R}^d) : \mathrm{dgm}_m(\mathcal{F})(I) \neq 0\}|$ (cf. Definition 2). Since the notion of the GPD reduces to that of the persistence diagram when $d = 1$ [22, Section 3], in this case, the size of the GPD becomes the number of points in the persistence diagram. Our question is as follows.

▶ **Question.** Does there exist $k \in \mathbb{Z}_{\geq 0}$ such that for *any* homology degree $m \geq 0$, the size of the m-th GPD of *any* finite simplicial filtration $\mathcal{F}$ over $\mathbb{R}^d$ is $O(N^k)$ where N stands for the number of simplicies in $\mathcal{F}$?

Note that when $d = 1$, the answer to the question is *yes*, and k can be taken to be 1, in which case we compute the persistence diagram directly at the simplicial filtration level [28]. If the answer is also *yes* when $d \geq 2$, it might be possible to compute the GPD of multi-parameter filtrations directly and much more efficiently without relying on the GRI of the persistence module $\mathrm{H}_m(\mathcal{F}; \mathbb{F}) : \mathbb{R}^d \to \mathbf{vec}_{\mathbb{F}}$.

Our contributions

1. We show that the answer to **Question** above is negative, demonstrating the inherent difficulty of computing the GPD (Theorem 13). More specifically, we construct a sequence $(\mathcal{F}_n)$ of finite d-parameter simplicial filtrations where the sizes of their GPDs are not bounded by any polynomial in the number of simplices in $\mathcal{F}_n$.
2. We also show that several well-known methods for constructing multi-parameter filtrations – Sublevel-Rips, Sublevel-Čech, Degree-Rips, and Degree-Čech – can give rise to such "wild" filtrations (Theorems 24 and 25).
3. We find that computing the value of the GPD for a 2-parameter filtration containing $O(n^s)$ simplices, for some $s \in \mathbb{N}$, via the Möbius inversion of the GRI can result in the problem of summing $\Theta(2^n)$ integers, which is an EXPTIME problem (Corollary 26). This implies that, even with the fully computed GRI, computing the GPD through Möbius inversion of the GRI remains computationally challenging.

It is noteworthy that in the proofs of some of our results, we employ techniques inspired by some recent works on the GRI [12, 22] as well as Rota's Galois connections [33]; see the proof of Theorem 13 for the case of $d > 2$ and that of Lemma 21.

Other related works. Botnan, Oppermann, Oudot investigated the computational complexity of the minimal rank decomposition of the standard rank invariant as part of their results [12, Remark 5.3]. Also, Botnan, Oppermann, Oudot, Scoccola found an upper bound on the size of the rank *exact* decomposition of a persistence module, which is a polynomial in the size of the (usual) multigraded Betti numbers of the module [13].

Clause, Kim, Mémoli identified a *tame* persistence module M over $\mathbb{Z}^2$ (a concept introduced by Miller [51]) whose GRI does not admit a Möbius inversion [22, Theorem B]. Our construction of persistence modules in Section 3 is reminiscent of that of M, making it natural to interpret M as a limit object of the types of persistence modules we construct.

Morozov and Patel elucidated a connection between 1- and 2-parameter persistence settings in order to find an output-sensitive algorithm for computing the Möbius inversion of the birth-death function [52]. We also note that there are many recent examples of utilizing Möbius inversion to extract information from (multi-parameter) persistent (co)homology: see e.g. [7, 33, 34, 49, 50, 52, 54, 56, 62].

Alonso, Kerber, Skraba showed that many multi-parameter filtration constructions, such as density- or degree-Rips bifiltrations, and across a general category of point samples in Euclidean space, the probability of the homology-induced persistence module decomposing into interval modules approaches zero as the sample size tends to infinity [1].

Organization.   In Section 2, we review basic terminology and well-known theorems related to persistence modules, as well as the GPD and GRI. In Sections 3 and 4, we establish our main results, as mentioned above. Finally, in Section 5, we discuss future research directions.

2   Preliminaries

Persistence modules and barcodes.   Throughout this paper, $P = (P, \leq)$ is a poset, regarded as the category whose objects are the elements of P, and for any pair $p, q \in P$, there exists a unique morphism $p \to q$ if and only if $p \leq q$. All vector spaces in this paper are over a fixed field $\mathbb{F}$. Let $\mathbf{vec}_{\mathbb{F}}$ denote the category of finite-dimensional vector spaces and linear maps over $\mathbb{F}$. A functor $P \to \mathbf{vec}_{\mathbb{F}}$ will be referred to as a **persistence module over P** or simply a **P-module**. A morphism between P-modules is a natural transformation between the P-modules. For any P-modules M and N, their direct sum $M \oplus N$ is defined pointwisely. A P-module M is **trivial** if $M(p) = 0$ for all $p \in P$, and we write $M = 0$. A nontrivial P-module M is **indecomposable** if the assumption $M \cong M' \oplus M''$ for some P-modules M' and M'' implies that either $M' = 0$ or $M'' = 0$. By Krull-Remak-Schmidt-Azumaya's theorem [6, 9], any P-module is isomorphic to a direct sum of indecomposable P-modules, and this decomposition is unique up to isomorphism and permutation of summands.

▶ **Definition 2.** *An **interval** I of P is a subset $I \subseteq P$ such that:*
   (i) *I is nonempty.*
   (ii) *If $p, q \in I$ and $p \leq r \leq q$, then $r \in I$.*
   (iii) *I is **connected**, i.e. for any $p, q \in I$, there is a sequence $p = p_0, p_1, \cdots, p_\ell = q$ of elements of I with p_i and p_{i+1} comparable for $0 \leq i \leq \ell - 1$.*
By $\mathrm{Int}(P)$ we denote the set of all intervals of P.

For any $p \leq q$ in P, the set $[p, q] := \{r \in P : p \leq r \leq q\}$ is called a **segment**. The collection of all segments (resp. intervals) in P will be denoted by $\mathrm{Seg}(P)$ (rep. $\mathrm{Int}(P)$). It is not difficult to see that $\mathrm{Seg}(P) \subset \mathrm{Int}(P)$.

Given any $I \in \mathrm{Int}(P)$, the **interval module** $\mathbb{F}_I$ is the P-module, with

$$\mathbb{F}_I(p) := \begin{cases} \mathbb{F} & \text{if } p \in I \\ 0 & \text{otherwise.} \end{cases}, \qquad \mathbb{F}_I(p \leq q) := \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } p \leq q \in I \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Every interval module is indecomposable [10, Proposition 2.2]. A P-module M is called **interval-decomposable** if it is isomorphic to a direct sum of interval modules. The **barcode** of an interval decomposable P-module $M \cong \bigoplus_{j \in J} \mathbb{F}_{I_j}$ is defined as the multiset $\mathrm{barc}(M) := \{I_j : j \in J\}$.

For $p \in P$, let $p^\uparrow$ (resp. $p^\downarrow$) denote the set of points $q \in P$ such that $p \leq q$ (resp. $q \leq p$). Clearly, both $p^\uparrow$ and $p^\downarrow$ belong to $\mathrm{Int}(P)$.

▶ **Definition 3.** *A P-module is called **finitely presentable** if it is isomorphic to the cokernel of a morphism $\bigoplus_{b \in B} \mathbb{F}_{p_b^\uparrow} \to \bigoplus_{a \in A} \mathbb{F}_{p_a^\uparrow}$, where $\{p_a : a \in A\}$ and $\{p_b : b \in B\}$ are finite multisets of elements of P.*

▶ **Remark 4.** It is well known that, for any finite d-parameter simplicial filtration $\mathcal{F}$ (Definition 1), the induced $\mathbb{R}^d$-module $\mathrm{H}_m(\mathcal{F}; \mathbb{F})$, for each $m \in \mathbb{Z}_{\geq 0}$, is finitely presentable. Namely, $\mathrm{H}_m(\mathcal{F}; \mathbb{F})$ is isomorphic to the cokernel of a morphism $\bigoplus_{j \in J} \mathbb{F}_{p_j^\uparrow} \to \bigoplus_{i \in I} \mathbb{F}_{p_i^\uparrow}$, where $\{p_j : J \in J\} \cup \{p_i : i \in I\}$ is a subset of the smallest finite d-d grid in $\mathbb{R}^d$ that contains all the birth indices of m- and $(m+1)$-simplices in $\mathcal{F}$. See, e.g. [45].

Generalized Rank invariant and Generalized Persistence Diagram. Any P-module M admits both a **limit** and a **colimit** [48, Chapter V]: A limit of M, denoted by $\varprojlim M$, consists of a vector space L together with a collection of linear maps $\{\pi_p : L \to M(p)\}_{p \in P}$ such that

$$M(p \leq q) \circ \pi_p = \pi_q \text{ for every } p \leq q \text{ in } P. \tag{2}$$

A colimit of M, denoted by $\varinjlim M$, consists of a vector space C together with a collection of linear maps $\{i_p : M(p) \to C\}_{p \in P}$ such that

$$i_q \circ M(p \leq q) = i_p \text{ for every } p \leq q \text{ in } P. \tag{3}$$

Both $\varprojlim M$ and $\varinjlim M$ satisfy certain universal properties, making them unique up to isomorphism.

Let us assume that P is connected. The connectedness of P alongside the equalities given in Equations (2) and (3) imply that $i_p \circ \pi_p = i_q \circ \pi_q : L \to C$ for any $p, q \in P$. This fact ensures that the **canonical limit-to-colimit map** $\psi_M : \varprojlim M \longrightarrow \varinjlim M$ given by $i_p \circ \pi_p$ for any $p \in P$ is well-defined. The **(generalized) rank** of M is defined to be[2] $\mathrm{rank}(M) := \mathrm{rank}(\psi_M)$, which is finite as $\mathrm{rank}(M) = \mathrm{rank}(i_p \circ \pi_p) \leq \dim(M(p)) < \infty$ for *any* $p \in P$. The rank of M is a count of the "persistent features" in M that span the entire indexing poset P [40].

Now, we refine the rank of a P-module, which is a single integer, into an integer-valued function. Options for the domain of the function include $\mathrm{Seg}(P)$ or the larger set $\mathrm{Int}(P)$.

▶ **Definition 5** ([22, 39]). *Let M be a P-module.*
(i) *The **generalized rank invariant (GRI)** is the map $\mathrm{rk}_M : \mathrm{Int}(P) \to \mathbb{Z}_{\geq 0}$ given by $I \mapsto \mathrm{rank}(M|_I)$ where $M|_I$ is the restriction of M to I.*
(ii) *The **generalized persistence diagram (GPD) of M** is defined as the function $\mathrm{dgm}_M : \mathrm{Int}(P) \to \mathbb{Z}$ that satisfies the equality[3]*

$$\forall I \in \mathrm{Int}(P), \quad \mathrm{rk}_M(I) = \sum_{\substack{J \in \mathrm{Int}(P) \\ J \supseteq I}} \mathrm{dgm}_M(J), \tag{4}$$

where the right-hand side of the equation includes only finitely many nonzero summands.

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[2] This construction was considered in the study of quiver representations [43].
[3] This equality generalizes *the fundamental lemma of persistent homology* [29].

We also remark that, in Definition 5, by replacing all instances of $\text{Int}(P)$ by $\text{Seg}(P)$, we obtain the notions of the *rank invariant* and its *signed barcode* [12, 13, 18].

▶ **Theorem 6** (Existence and Uniqueness of the GPD)**.** *Let M be a P-module.*
  (i) *If P is finite, then the GPD of M exists.*
  (ii) *If $P = \mathbb{R}^d$ ($d = 1, 2, \ldots$), and M is finitely presentable, then the GPD of M exists. Furthermore, dgm_M is finitely supported, i.e. there exist only finite many $I \in \text{Int}(\mathbb{R}^d)$ such that $\text{dgm}_M(I) \neq 0$.*
  (iii) *In each of the previous two cases, the GPD is unique.*

Proof. (i): If P is finite, then $\text{Int}(P)$ is finite. Thus, the claim directly follows from the Möbius inversion formula [57]. (ii): This statement is precisely that of [22, Theorem C(iii)]. (iii): In the setting of (i), the uniqueness is also a direct consequence of the Möbius inversion formula. In the setting of (ii), the uniqueness is proved in [22, Proposition 3.2].   ◀

▶ **Remark 7** ([22, Theorem C (iii)])**.** For any finitely presentable $\mathbb{R}^d$-module M, assume that M is the cokernel of a morphism $\bigoplus_{b \in B} \mathbb{F}_{p_b^\uparrow} \to \bigoplus_{a \in A} \mathbb{F}_{p_a^\uparrow}$, where A and B are finite index sets. Let $\mathcal{C} := \{p_a : a \in A\} \cup \{p_b : b \in B\} \subset \mathbb{R}^d$. Consider the finite full subposet $P := \prod_{i=1}^d \pi_i(\mathcal{C}) \subset \mathbb{R}^d$, where $\pi_i : \mathbb{R}^d \to \mathbb{R}$ is the canonical projection to the i-th coordinate. Then, the GPD of M is directly obtained from the GPD of the restriction $M|_P$, as follows.

We extend $\mathbb{R}^d$ to $\mathbb{R}^d \cup \{-\infty\}$ by declaring that $-\infty \leq x$ for all $x \in \mathbb{R}^d$. Let $\lfloor - \rfloor_P : \mathbb{R}^d \to P \cup \{-\infty\}$ be the map sending each $x \in \mathbb{R}^d$ to the maximal point $p =: \lfloor x \rfloor_P$ in $P \cup \{-\infty\}$ such that $p \leq x$ in $\mathbb{R}^d \cup \{-\infty\}$. For any $I \subseteq \mathbb{R}^d$, let $\lfloor I \rfloor_P$ denote the set of the points $\lfloor x \rfloor_P$ for $x \in I$. Then, the GPD of M, i.e. $\text{dgm}_M : \text{Int}(\mathbb{R}^d) \to \mathbb{Z}$, is equal to

$$
I \mapsto \begin{cases} \text{dgm}_{M|_P}(\lfloor I \rfloor_P), & \text{if } I = \lfloor - \rfloor_P^{-1}(J) \text{ for some } J \in \text{Int}(P) \\ 0, & \text{otherwise.} \end{cases}
$$

This implies that **there exists a bijection between the supports of dgm_M and $\text{dgm}_{M|_P}$.**

We also remark that if P is a finite poset, the condition given in Equation (4) holds if and only if

$$
\forall I \in \text{Int}(P), \quad \text{dgm}_M(I) = \sum_{\substack{J \in \text{Int}(P) \\ J \supseteq I}} \mu(J, I) \cdot \text{rk}_M(J), \tag{5}
$$

where μ is the *Möbius function* of the poset $(\text{Int}(P), \supseteq)$; see [61, Section 3.7] for a general reference on Möbius inversion, and [40] for a discussion of Möbius inversion in this specific context.

We define the **size** of rk_M, denoted by $\|\text{rk}_M\|$, as the cardinality of its support. Likewise, the size of dgm_M is defined as the cardinality of its support and is denoted by $\|\text{dgm}_M\|$.

▶ Remark 8. For any P-module M, the following holds.
  (i) (Monotonicity) $\text{rk}_M(I) \leq \text{rk}_M(J)$ for any pair $I \supseteq J$ in $\text{Int}(P)$ [39, Proposition 3.8].
  (ii) For any $I \in \text{Int}(P)$, if $\text{rk}_M(I) = 0$, then $\text{dgm}_M(I) = 0$. Hence, we have $\|\text{dgm}_M\| \leq \|\text{rk}_M\|$ [40, Remark 8].
  (iii) If M is interval decomposable and $I \in \text{Int}(P)$, then $\text{dgm}_M(I)$ equals the multiplicity of I in $\text{barc}(M)$ [25, Theorem 2.10].

In establishing results in later sections, we will utilize the following well-known concrete formulation for the limit of a P-module M (see, for instance, [39, Appendix E]):

▶ **Convention 9.** The limit of a P-module M is the pair $(L, (\pi_p)_{p \in P})$ described as:

$$L := \left\{ (\ell_p)_{p \in P} \in \prod_{p \in P} M(p) : \; \forall p \leq q \in P, M(p \leq q)(\ell_p) = \ell_q \right\}$$

where for each $p \in P$, the map $\pi_p : L \to M(p)$ is the canonical projection. Elements of L are called **sections** of M.

On 2-d grid lattices. A **join** (a.k.a. least upper bound) of $S \subseteq P$ is an element $q_0 \in P$ such that (i) $s \leq q_0$, for all $s \in S$, and (ii) for any $q \in P$, if $s \leq q$ for all $s \in S$, then $q_0 \leq q$. A **meet** (a.k.a. greatest lower bound) of $S \subseteq P$ is an element $r_0 \in P$ such that (i) $r_0 \leq s$ for all $s \in S$, and (ii) for any $r \in P$, if $r \leq s$ for all $s \in S$, then $r \leq r_0$. If a join and a meet of S exist, then they are unique. Hence, whenever they exist, we refer to them as *the* join (denoted by $\bigvee S$) and *the* meet (denoted by $\bigwedge S$), respectively. When S consists of exactly two points s_1 and s_2, we also use $s_1 \vee s_2$ and $s_1 \wedge s_2$ instead of $\bigvee S$ and $\bigwedge S$, respectively. A **lattice** is a poset such that for any pair of elements, their meet and join exist. For any $n \in \mathbb{Z}_{\geq 0}$, let $[n] := \{0 < 1 < \cdots < n\}$. By a **finite d-d grid lattice**, we mean a poset isomorphic to the lattice $[n_1] \times [n_2] \times \cdots \times [n_d]$ for some $n_1, n_2, \cdots, n_d \in \mathbb{Z}_{\geq 0}$.

Let p and q be any two points in a poset P. We say that q **covers** p if $p < q$ and there is no $r \in P$ such that $p < r < q$. By $\mathrm{Cov}(p)$, we denote the set of all points $q \in P$ that cover p.

▶ **Lemma 10** ([4, Theorem 5.3]). *Let μ be the Möbius function of the poset $(\mathrm{Int}([n]^2), \supseteq)$. Then, for all $J, I \in \mathrm{Int}([n]^2)$ with $J \supseteq I$,*

$$\mu(J, I) = \begin{cases} 1, & \text{if } I = J, \\ \displaystyle\sum_{\substack{J = \bigwedge S \\ \emptyset \neq S \subseteq \mathrm{Cov}(I)}} (-1)^{|S|}, & \text{otherwise,} \end{cases}$$

where $|S|$ denotes the cardinality of S. Note that when $I \neq J$ and there is no nonempty $S \subseteq \mathrm{Cov}(I)$ such that $J = \bigwedge S$, the sum above is empty, and thus $\mu(J, I) = 0$. We also remark that $\bigwedge S$ represents the smallest interval in $[n]^2$ that contains all the intervals in S.

A subposet $L \subset P$ (resp. $U \subset P$) is called a **lower (resp. upper) fence** of P if L is connected, and for any $q \in P$, the intersection $L \cap q^{\downarrow}$ (resp. $U \cap q^{\uparrow}$) is nonempty and connected [25, Definition 3.1].

▶ **Lemma 11** ([25, Proposition 3.2]). *Let L and U be a lower and an upper fence of a connected poset P, respectively. Given any P-module M, we have*

$$\varprojlim M \cong \varprojlim M|_L \quad \text{and} \quad \varinjlim M \cong \varinjlim M|_U.$$

Let I be any subset of P. By $\min(I)$ and $\max(I)$, we denote the collections of minimal and maximal elements of I, respectively. A **zigzag poset** of n points is $\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \ldots \bullet_{n-1} \leftrightarrow \bullet_n$ where $\leftrightarrow$ stands for either $\leq$ or $\geq$.

▶ **Definition 12** ([25, Definition 3.5]). *For $I \in \mathrm{Int}([n]^2)$ with $\min(I) = \{p_0, p_1, \ldots, p_k\}$, and $\max(I) = \{q_0, q_1, \ldots, q_l\}$, we define the following two zigzag posets*

$$\min_{ZZ}(I) := \{p_0 < (p_0 \vee p_1) > p_1 < (p_1 \vee p_2) > \cdots < (p_{k-1} \vee p_k) > p_k\}$$

$$= \min(I) \cup \{p_i \vee p_{i+1} : i = 0, \ldots, k-1\},$$

$$\max_{ZZ}(I) := \{q_0 > (q_0 \wedge q_1) < q_1 > (q_1 \wedge q_2) < \cdots > (q_{l-1} \wedge q_l) > q_l\}$$

$$= \max(I) \cup \{q_i \wedge q_{i+1} : i = 0, \ldots, l-1\},$$

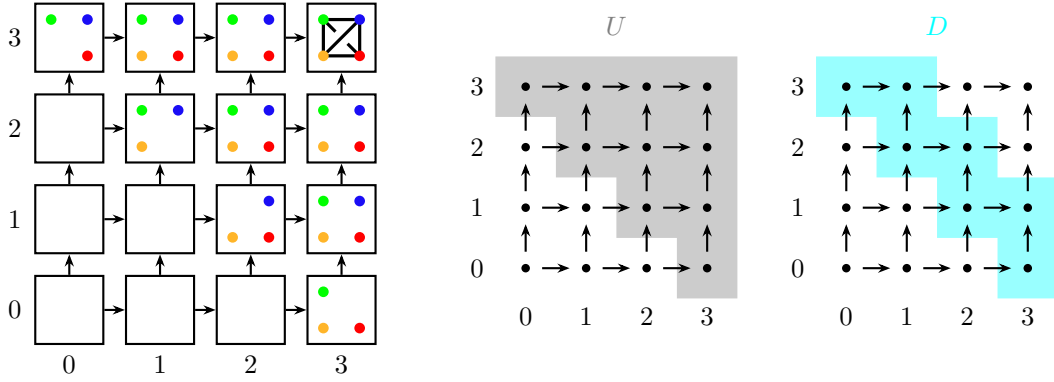*which are lower and upper fences of I, respectively [25, Section 3.2].*

Figure 1 For $n = 3$, the filtration over $[n]^2$ defined in Equation (6), along with the intervals U and D of $[n]^2$, as described in Equation (9).

3   Super-Polynomial Growth of the GPD

The goal of this section is to establish the following theorem.

▶ **Theorem 13** (Super-polynomial Growth of the GPD). *Let $d, m \in \mathbb{Z}_{\geq 0}$ with $d \geq 2$. There does* not *exist $k \in \mathbb{Z}_{\geq 0}$ such that, for* all *finite filtrations $\mathcal{F}$ over $\mathbb{R}^d$ containing N simplices, $\|\mathrm{dgm}_m(\mathcal{F})\|$ belongs to $O(N^k)$.*

Let K be an abstract simplicial complex and let $m \in \mathbb{Z}_{\geq 0}$. Let $C_m(K; \mathbb{F})$ and $Z_m(K; \mathbb{F})$ be the group of m-chains and that of m-cycles of K, respectively. Let K^m be the m-skeleton of K.

To prove the statement, we construct a sequence $(\mathcal{F}_n)$ of filtrations where $\mathcal{F}_n$ contains $O(n^t)$ simplices for some $t \in \mathbb{Z}_{\geq 0}$, but $\|\mathrm{dgm}_m(\mathcal{F}_n)\| \notin O(n^k)$ for *any* $k \in \mathbb{Z}_{\geq 0}$. By Remarks 4 and 7, it suffices to construct filtrations $\mathcal{F}_n$ over the discrete posets $[n]^d$.

▶ Remark 14. In [45], the *size* of a finite d-parameter filtration of a simplicial complex K is defined as the total number of birth indices across all $\sigma \in K$. Even if we adopt this quantity as N in the above theorem, the statement remains true.

We also remark that some techniques used in the following proof are similar to those employed in the proof of [22, Theorem B]. We first prove the statement for the case of $d = 2$.

Proof of Theorem 13 when $d = 2$. We consider the cases of $m = 0$ and $m \geq 1$ separately.

Case 1 ($m = 0$). Fix $n \in \mathbb{Z}_{\geq 0}$. Let $K := K_n$ be the simplicial complex on the vertex set $K^0 = \{z_0, z_1, \dots, z_n\}$ that consists of the 1-simplices $\{z_i, z_j\}$ ($0 \leq i < j \leq n$) and their faces. We define a filtration $\mathcal{F} := \mathcal{F}_n$ of K over $[n]^2$ as follows. For $(i, j) \in [n]^2$, let (see Figure 1):

$$
\mathcal{F}_{(i,j)} := \begin{cases} \emptyset, & \text{if } i + j < n \\ K^0 - \{\{z_i\}\}, & \text{if } i + j = n \\ K^0, & \text{if } i + j > n \text{ and } (i, j) \neq (n, n) \\ K, & \text{if } (i, j) = (n, n). \end{cases} \tag{6}
$$

Let $M := \mathrm{H}_0(\mathcal{F}; \mathbb{F}) : [n]^2 \to \mathbf{vec}_\mathbb{F}$. Then, we have:

$$M(i,j) \cong \begin{cases} 0, & \text{if } i+j < n \\ \mathbb{F}^n, & \text{if } i+j = n \\ \mathbb{F}^{n+1}, & \text{if } i+j > n \text{ and } (i,j) \neq (n,n) \\ \mathbb{F}, & \text{if } (i,j) = (n,n). \end{cases} \tag{7}$$

Let $i \in [n]$. Then, for every $(x,y) \in [n]^2$ that covers $(i, n-i)$, we have that

$$M((i, n-i) \leq (x,y)) \quad = \quad \iota_i^n : \qquad\qquad \mathbb{F}^n \to \mathbb{F}^{n+1}$$
$$(v_1, \dots, v_n) \mapsto (v_1, \dots, v_i, 0, v_{i+1}, \dots, v_n). \tag{8}$$

In what follows, we use ι_i in place of ι_i^n. Now, we consider the following two intervals of $[n]^2$ (see Figure 1):

$$U := \{(i,j) \in [n]^2 : \ n \leq i+j\} \ \text{ and } \ D := \{(i,j) \in [n]^2 : n \leq i+j \leq n+1\}. \tag{9}$$

By proving the following claims, we complete the proof for the case of $m = 0$.

$\triangleright$ **Claim 15.** If $J \in \text{Int}([n]^2)$ is not a proper subset of U, then $\text{rk}_M(J) = 0$.

Proof. First, assume that there exists a point $p \in J - U$. Then, by the construction of M, $\dim(M(p)) = 0$ and thus $\text{rk}_M(J) = 0$ by monotonicity of rk_M (Remark 8 (i)). Next, assume that $J = U$. Since $\text{rk}_M(U) = \text{rank}(\varprojlim M|_U \to \varinjlim M|_U)$, it suffices to show $\varprojlim M|_U = 0$. Again, since $D = \min_{\text{ZZ}}(U)$, by Lemma 11, it suffices to show that $\varprojlim M|_D = 0$. By Equation (8), the diagram $M|_D$ is given by

$$\mathbb{F}^n \xrightarrow{\iota_0} \mathbb{F}^{n+1} \xleftarrow{\iota_1} \mathbb{F}^n \xrightarrow{\iota_1} \mathbb{F}^{n+1} \xleftarrow{\iota_2} \mathbb{F}^n \xrightarrow{\iota_2} \mathbb{F}^{n+1} \xleftarrow{\iota_3} \mathbb{F}^n \xrightarrow{\iota_3} \cdots \xleftarrow{\iota_n} \mathbb{F}^n.$$

From this, it is not difficult to see that $\varprojlim M|_D = 0$: For $p \in D$, let $\pi_p : \varprojlim M|_D \to M(p)$ be the canonical projection. For any $i = 0, \dots, n$ and for any $v \in \varprojlim M|_D$, let $\pi_{(i,n-i)}(v) =: (v_1^i, v_2^i, \dots, v_n^i) \in \mathbb{F}^n = M(i, n-i)$. Then, Equations (2) and (8) imply the following pair of equalities for each $i = 1, \dots, n$:

$$\begin{aligned} \pi_{(i,n-i+1)}(v) &= \iota_{i-1} \circ \pi_{(i-1,n-i+1)}(v) &= (v_1^{i-1}, \dots, v_{i-1}^{i-1}, 0, v_i^{i-1}, v_{i+1}^{i-1}, \dots, v_n^{i-1}), \\ \pi_{(i,n-i+1)}(v) &= \iota_i \circ \pi_{(i,n-i)}(v) &= (v_1^i, \dots, v_{i-1}^i, v_i^i, 0, v_{i+1}^{i-1}, \dots, v_n^i), \end{aligned}$$

and thus $v_i^i = v_i^{i-1} = 0$ for $i = 1, \dots n$. From this, we deduce that $v_j^i = 0$ for all $i = 0, 1, \dots, n$ and $j = 1, 2, \dots, n$. Therefore, $\pi_{(i,n-i)}(v) = 0$ for all $i = 0, 1, \dots, n$ and in turn $v = 0$. Since $v \in \varprojlim M|_D$ was arbitrary, we have $\varprojlim M|_D = 0$. $\triangleleft$

Consider the following intervals of $[n]^2$ (see Figure 2):

$$U_i := U - \{(i, n-i)\} \ \text{ for each } i \in [n]. \tag{10}$$

$\triangleright$ **Claim 16.** Let $J \in \text{Int}([n]^2)$ such that $\bigcap_{i=0}^n U_i \subseteq J \subsetneq U$. Then, $\text{rk}_M(J) = 1$.

Proof. Since $J \subsetneq U$, there exists $i \in [n]$ such that $J \subseteq U_i$. Fix such i. Since $\max(J) = \{(n,n)\}$, by Lemma 11, we have the isomorphism $\varinjlim M|_J \overset{(*)}{\cong} M(n,n)$. Observe that for every $p \in J \subseteq U_i$, $\{z_i\} \in \mathcal{F}_p$. Hence, by Convention 9 and the definition of M, we have that

$$([z_i])_{p \in J} \in \varprojlim M|_J \subset \bigoplus_{p \in J} \text{H}_0(\mathcal{F}_p; \mathbb{F}).$$

Clearly, this tuple $([z_i])_{p \in J}$ maps to $[z_i] \in \text{H}_0(\mathcal{F}_{(n,n)}; \mathbb{F}) = M(n,n)$ via the limit-to-colimit map of M over J, followed by the isomorphism $(*)$. Since $[z_i] \in M(n,n)$ is nonzero and $\dim M(n,n) = 1$ (cf. Equation (7)), we have that $\text{rk}_M(J) = 1$, as desired. $\triangleleft$
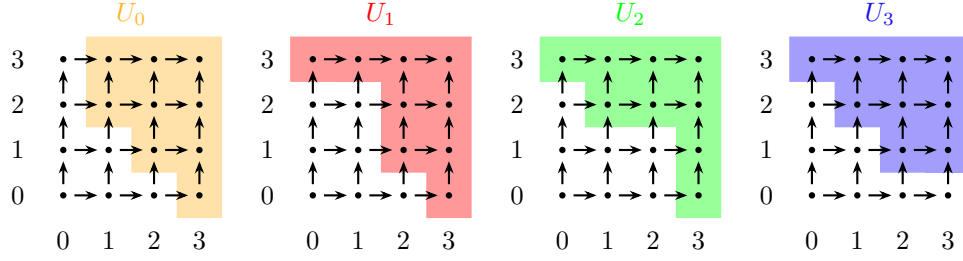
Figure 2 For $n = 3$, the intervals U_0, U_1, U_2, U_3 of $[n]^2$ that are defined in Equation (10).

▷ **Claim 17.** $\mathrm{dgm}_M(U_i) = 1$ for every $i \in [n]$.

Proof. We have:

$$
\begin{aligned}
1 &= \mathrm{rk}_M(U_i) && \text{by Claim 16} \\
&= \sum_{\substack{I \in \mathrm{Int}([n]^2) \\ I \supseteq U_i}} \mathrm{dgm}_M(I) && \text{by Equation (4)} \\
&= \mathrm{dgm}_M(U_i) && \text{by Claim 15 and Remark 8 (ii).} \quad \lhd
\end{aligned}
$$

For each nonempty $S \subset [n]$, we define $U_S := \bigcap_{i \in S} U_i$, which is an interval of $[n]^2$.

▷ **Claim 18.** For each nonempty $S \subset [n]$, we have that $\mathrm{dgm}_M(U_S) = (-1)^{|S|+1}$.

Proof. Let $I := U_S$. When $|S| = 1$, there exists $i \in [n]$ such that $I = U_i$. We already proved that $\mathrm{dgm}_M(U_i) = (-1)^{1+1} = 1$. If $|S| > 1$, then

$$
\begin{aligned}
\mathrm{dgm}_M(I) &= \sum_{\substack{J \in \mathrm{Int}(P) \\ J \supseteq I}} \mu(J, I) \cdot \mathrm{rk}_M(J) && \text{by Equation (5)} \\
&= \sum_{\substack{J \in \mathrm{Int}(P) \\ U \supsetneq J \supseteq I}} \mu(J, I) \cdot \mathrm{rk}_M(J) && \text{by Claim 15} \\
&= \sum_{\substack{J \in \mathrm{Int}(P) \\ U \supsetneq J \supseteq I}} (-1)^{|J-I|} \cdot \mathrm{rk}_M(J) && \text{by Lemma 10} \\
&= \sum_{S' \subsetneq S} (-1)^{|S'|} && \text{by Claim 16, letting } S' = J - I \\
&= (1-1)^{|S|} - (-1)^{|S|} = (-1)^{|S|+1} && \text{by the binomial theorem}
\end{aligned}
$$

as desired. The number of nonempty subsets $S \subset [n]$ is $2^{n+1} - 1$ and each S corresponds to the interval U_S such that $\mathrm{dgm}_M(U_S) \neq 0$. Hence, $\|\mathrm{dgm}_0(\mathcal{F})\| = \|\mathrm{dgm}_M\| \geq 2^{n+1} - 1$. In particular, there is no $k \in \mathbb{Z}_{\geq 0}$ such that $2^{n+1} - 1 \in O(N^k)$ for $N := (n^2 + n)/2$, the number of simplices in the filtration $\mathcal{F}$, completing the proof. $\lhd$

Case 2 ($m \geq 1$). Let $K' := K_{n,m}$ be the smallest simplicial complex on the vertex set $\{x_0, \cdots, x_n, y_0, \cdots, y_m\}$, containing all m-simplices τ, τ_{ij}, and $(m+1)$-simplices σ_{ij} that are the underlying sets of the following oriented simplices (see Figure 3a):

$$
\begin{aligned}
\tau^+ &:= [y_0, \cdots, y_m] \\
\tau_{ij}^+ &:= [x_i, y_0, \cdots, y_{j-1}, y_{j+1}, \cdots, y_m], && \text{for } 0 \leq i \leq n, \, 0 \leq j \leq m, \\
\sigma_{ij}^+ &:= [x_{i-1}, x_i, y_0, \cdots, y_{j-1}, y_{j+1}, \cdots, y_m], && \text{for } 1 \leq i \leq n, \, 0 \leq j \leq m.
\end{aligned}
$$

For $0 \le k \le n$, let L_k be the m-dimensional subcomplex of K', whose m-simplices are exactly τ and $\{\tau_{kj} : 0 \le j \le m\}$ (see Figure 3a).

We define the filtration $\mathcal{F}' := \mathcal{F}'_{n,m} : [n]^2 \to \Delta K' (= \Delta K_{n,m})$ as (see Figure 3b):

$$
\mathcal{F}'_{(i,j)} := \begin{cases}
\emptyset, & \text{if } i+j < n \\
\bigcup_{k \in [n]-\{i\}} L_k, & \text{if } i+j = n \\
\bigcup_{k \in [n]} L_k, & \text{if } i+j > n \text{ and } (i,j) \ne (n,n) \\
K', & \text{if } (i,j) = (n,n).
\end{cases}
\tag{11}
$$

Note that, for each $0 \le i \le n$, the m-chain $c_i = \tau^+ - \sum_{j=0}^{m}(-1)^j \tau_{ij}^+$ in $C_m(K'; \mathbb{F})$ satisfies $\partial c_i = 0$, i.e. c_i is an m-cycle. Moreover, $\mathrm{H}_m(L_i; \mathbb{F})$, which is isomorphic to $Z_m(L_i; \mathbb{F})$, is the 1-dimensional vector space generated by the homology class $[c_i]$.

(a) For $m = 1$, the simplicial complex $K' = K_{n,m}$ given in **Case 2**.

(b) For $m = 1$ and $n = 3$, the filtration $\mathcal{F}' = \mathcal{F}'_{n,m}$ given in **Case 2**.

Figure 3 Illustration of the constructions described in **Case 2**.

Let $N := \mathrm{H}_m(\mathcal{F}'; \mathbb{F}) : [n]^2 \to \mathbf{vec}_{\mathbb{F}}$. We claim that N is isomorphic to M, given in Equation (7). Indeed,

$$
N(i, j) \cong
\begin{cases}
0, & \text{if } i + j < n \\
\langle [c_0], \cdots, [c_{i-1}], [c_{i+1}], \cdots, [c_n] \rangle \cong \mathbb{F}^n, & \text{if } i + j = n \\
\langle [c_0], \cdots, [c_n] \rangle \cong \mathbb{F}^{n+1}, & \text{if } i + j > n \text{ and } (i, j) \neq (n, n) \\
\mathbb{F}, & \text{if } (i, j) = (n, n)
\end{cases}
$$

and for each pair $p \leq q$ in $[n]^2$, $N(p \leq q) \cong M(p \leq q)$. Therefore, there is no $k \in \mathbb{Z}_{\geq 0}$ such that $\|\mathrm{dgm}_m(\mathcal{F}')\| = \|\mathrm{dgm}_N\| = \|\mathrm{dgm}_M\| = 2^{n+1} - 1 \notin O(N^k)$ for N, the number of simplices in the filtration $\mathcal{F}'$, which belongs to $O(n^{m+2})$. ◀

In order to prove Theorem 13 for the case where $d > 2$, we utilize a result from [12, Section 3.2] and techniques present in [22, Section 3]. See the full version [38] for details.

▶ **Remark 19.** Notably, the supports of the GPDs we have considered contain many overlapping intervals with opposite-signed values. This makes it difficult to interpret the information the GPDs convey, unlike persistence diagrams for one-parameter persistence modules.

4 Super-Polynomial Growth of the GPDs Induced by Finite Metric Spaces

In this section, we show that the sizes of the 1-st GPDs of the sublevel-Rips, sublevel-Čech, degree-Rips, and degree-Čech bifiltrations are *not* bounded by *any* polynomial in the number of points in the input metric space (Theorems 24 and 25). We prove these results by constructing specific point sets in $\mathbb{R}^3$ that induce persistence modules isomorphic, or structurally similar to, those appearing in the proof of Theorem 13. Also, we make use of Rota's Galois connection.

A **Galois connection** between two posets P and Q is a pair of monotone maps $g_1 : P \rightleftarrows Q : g_2$ satisfying $g_1(p) \leq q$ iff $p \leq g_2(q)$ for any $p \in P$ and $q \in Q$. Let $f : P \to Q$ be any function. For any $h : P \to \mathbb{Z}$, the **pushforward** of h along f is the function $f_\sharp h : Q \to \mathbb{Z}$ given by $f_\sharp h(q) = \sum_{p \in f^{-1}(q)} h(p)$. For any $\ell : Q \to \mathbb{Z}$, the **pullback** of ℓ along f is the function $f^\sharp \ell : P \to \mathbb{Z}$ defined by $f^\sharp \ell(p) = (\ell \circ f)(p)$. For any poset P, let ∂_P denote the Möbius inversion operator over P. To say that the GPD of a P-module M exists is equivalent to stating that $\partial_{\mathrm{Int}(P)} \mathrm{rk}_M$ is well-defined and equals dgm_M.

▶ **Lemma 20** ([33, Theorem 3.1]). *Let P be a finite poset, Q be any poset, and $g_1 : P \rightleftarrows Q : g_2$ be a Galois connection. Then, $\partial_Q \circ g_2^\sharp = g_{1\sharp} \circ \partial_P$.*

Lemma 20 is crucial for establishing the following lemma, whose second item is essential for the proofs of both Theorem 24 and Theorem 25. See the full version [38] for the proof.

▶ **Lemma 21.** *Let P be a finite poset, M be a P-module, and $I \subset P$ be an interval of P. Then,*

(i) *for every $J \in \mathrm{Int}(I)$, we have $\mathrm{dgm}_{M|_I}(J) = \sum_{\substack{J' \in \mathrm{Int}(P) \\ J \in \Pi(J' \cap I)}} \mathrm{dgm}_M(J')$, where $\Pi(J' \cap I)$ denotes the coarsest partition of $J \cap I$ into disjoint intervals of I.*

(ii) *Furthermore, if P and I are finite 2-d grid, then $\|\mathrm{dgm}_M\| \geq \|\mathrm{dgm}_{M|_I}\|$.*

▶ **Remark 22.** Lemma 21 (ii) does not generally hold in higher dimensions, i.e. for $d \geq 3$, there exist d-d grids P and I with $I \subseteq P$ and a P-module M such that $\|\mathrm{dgm}_M\| < \|\mathrm{dgm}_{M|_I}\|$.

4.1   Sublevel-Rips and Sublevel-Čech Bifiltrations

For $r \in \mathbb{R}_{\geq 0}$, the **Vietoris-Rips complex** of a metric space (X, d_X) at scale r is defined as

$$\mathrm{Rips}((X, \mathsf{d}_X))_r = \left\{ \text{finite } \sigma \subset X : \max_{x,y \in \sigma} \mathsf{d}_X(x, y) \leq r \right\}.$$

Now assume that $X \subset \mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with a metric d, and $\mathsf{d}_X = \mathsf{d}|_{X \times X}$. The **Čech complex of (X, d_X) in $(\mathbb{R}^d, \mathsf{d})$ at scale r** is

$$\check{\mathrm{Cech}}((X, \mathsf{d}_X))_r = \left\{ \text{finite } \sigma \subset X : \bigcap_{x \in \sigma} B_r(x) \neq \emptyset \right\},$$

where $B_r(x) = \{ y \in \mathbb{R}^d : \mathsf{d}(x, y) \leq r \}$.

▶ **Lemma 23** ([32, Lemma VII]). *Let $(\mathbb{R}^d, \rho)$ be the Euclidean space with the supremum metric ρ. For any finite $X \subset \mathbb{R}^d$ and any $r \in \mathbb{R}_{\geq 0}$, we have $\check{\mathrm{Cech}}((X, \rho|_{X \times X}))_r = \mathrm{Rips}((X, \rho|_{X \times X}))_{2r}$.*

Let $\gamma : X \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$ and any $r \in \mathbb{R}_{\geq 0}$ (see e.g. [18]),
 (i) the **sublevel-Rips complex at scale r and threshold a** is $\mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma))_{a,r} := \mathrm{Rips}(\gamma^{-1}((-\infty, a]))_r$. The **sublevel-Rips bifiltration** of (X, d, γ) is the simplicial filtration $\mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma)) := \{\mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma))_{a,r}\}_{a \in \mathbb{R}, r \in \mathbb{R}_{\geq 0}}$ over $\mathbb{R} \times \mathbb{R}_{\geq 0}$.
 (ii) the **sublevel-Čech complex at scale r and threshold a** is $\check{\mathrm{Cech}}^{\downarrow}((X, \mathsf{d}, \gamma))_{a,r} := \check{\mathrm{Cech}}(\gamma^{-1}((-\infty, a]))_r$. The **sublevel-Čech bifiltration** of (X, d, γ) is the simplicial filtration $\check{\mathrm{Cech}}^{\downarrow}((X, \mathsf{d}, \gamma)) := \{\check{\mathrm{Cech}}^{\downarrow}((X, \mathsf{d}, \gamma))_{a,r}\}_{a \in \mathbb{R}, r \in \mathbb{R}_{\geq 0}}$ over $\mathbb{R} \times \mathbb{R}_{\geq 0}$.

Let

$$\Gamma_X := \mathrm{im}(\gamma) \subset \mathbb{R} \quad \text{and} \quad T_X := \mathrm{im}(\mathsf{d}) \subset \mathbb{R}_{\geq 0}. \tag{12}$$

By Remarks 4 and 7, for each $m \in \mathbb{Z}_{\geq 0}$, the size of the GPD of $M := \mathrm{H}_m(\mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma)), \mathbb{F}) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbf{vec}_{\mathbb{F}}$ equals $\|\mathrm{dgm}_{M|_{\Gamma_X \times T_X}}\|$.

▶ **Theorem 24** (Super-polynomial GPD of Sublevel-Rips and Sublevel-Čech filtrations). *There does* not *exist $k \in \mathbb{Z}_{\geq 0}$ such that for* every *metric space (X, d) and* every *function $\gamma : X \to \mathbb{R}$, $\|\mathrm{dgm}_1(\mathcal{F}_{(X, \mathsf{d}, \gamma)})\|$ belongs to $O(|X|^k)$, where $\mathcal{F}_{(X, \mathsf{d}, \gamma)} \in \{\mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma)), \check{\mathrm{Cech}}^{\downarrow}((X, \mathsf{d}, \gamma))\}$.*

Given any set $A \subset \mathbb{R}^d$ and $r \in \mathbb{R}$, let $rA := \{ra \in \mathbb{R}^d : a \in A\}$. When $d \geq 2$, let $\pi : \mathbb{R}^d \to \mathbb{R}^2$ be the projection $(x_1, \ldots, x_d) \mapsto (x_1, x_2)$.

Proof Sketch. By Lemma 23, it suffices to show the statement when $\mathcal{F}_{(X, \mathsf{d}, \gamma)} = \mathrm{Rips}^{\downarrow}((X, \mathsf{d}, \gamma))$. Let $n \geq 1$. We construct an $X_n \subset \mathbb{R}^3$, which consists of $4n(n + 2)$ points and inherits the metric d_n from the supremum metric on $\mathbb{R}^3$. Our goal is to show that $\|\mathrm{dgm}_1(\mathrm{Rips}^{\downarrow}((X_n, \mathsf{d}_n, \gamma_n)))\|$ does not belong to $O(n^k)$ for any k, which then does not belong to $O(|X_n|^k)$ for any k. Let $\epsilon_n := 1/n^2$. Let $X_n = \left(\bigcup_{i=0}^{n} Y_i \right) \cup \left(\bigcup_{i=0}^{n-1} Z_i \right)$ (see Figure 4), where, for $i \in [n]$, Y_i is the subset of the plane $z = 2in$ in $\mathbb{R}^3$ whose projected image πY_i onto $\mathbb{R}^2$ is described below, and Z_i is the subset of the plane $z = (2i + 1)n$ in $\mathbb{R}^3$ whose projected image πZ_i onto $\mathbb{R}^2$ is described below. Let $A := \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$, $b_j := n - j\epsilon_n$ for $j \in [n]$, $\pi Y_i := \bigcup_{\substack{0 \leq j \leq n \\ j \neq i}} b_j A$, and $\pi Z_i := b_0 A$. Note that $|X_n| = \left| \bigcup_{i=0}^{n} Y_i \right| + \left| \bigcup_{i=0}^{n-1} Z_i \right| = $
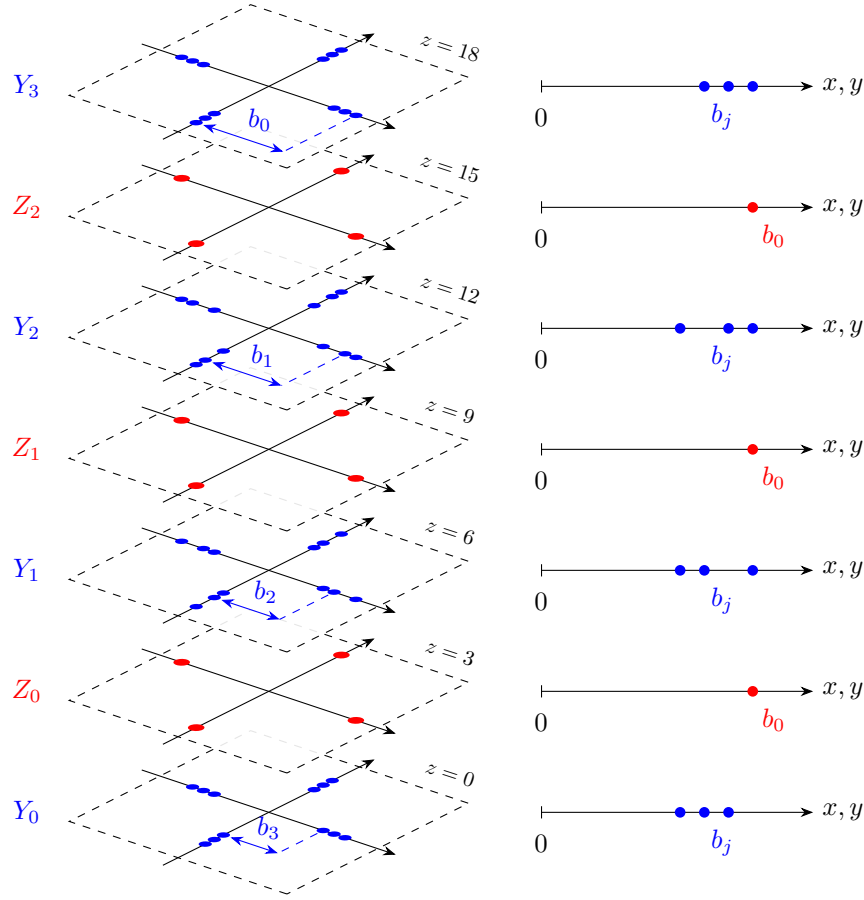
Figure 4 For $n = 3$, the set X_n that is defined in the proof of Theorem 24.

$4n(n + 1) + 4n = 4n(n + 2)$. Let $\gamma_n : X_n \to \mathbb{R}$ be

$$
\gamma_n(w) = \begin{cases} j, & \text{if } w \in \bigcup\limits_{i=0}^{n} Y_i \text{ and } \pi(w) \in b_j A \text{ for some } 0 \le j \le n, \\ n, & \text{if } w \in \bigcup\limits_{i=0}^{n-1} Z_i \end{cases}
$$

so that $\Gamma_{X_n} = \mathrm{im}(\gamma_n) = \{0 < 1 < \ldots < n\} \subset \mathbb{R}$. In the full version [38], we show that for $\mathcal{F}_n := \mathrm{Rips}^{\downarrow}((X_n, \mathsf{d}_n, \gamma_n))$ on $\Gamma_{X_n} \times T_{X_n}$ (Figure 5), $\|\mathrm{dgm}_1(\mathcal{F}_n)\|$ does not belong to $O(|X_n|^k)$ for any k. In particular, we identify an interval of $\Gamma_{X_n} \times T_{X_n}$ on which the restriction of $M_n := \mathrm{H}_1(\mathcal{F}_n; \mathbb{F})$ is isomorphic to M from Equation (7), and then apply Lemma 21 (ii). ◄

4.2   Degree-Rips and Degree-Čech Bifiltrations

Let (X, d) be a metric space. For $d \in \mathbb{Z}$ and $r \in \mathbb{R}_{\geq 0}$,
   (i) the **Degree-Rips complex at scale r and degree d** [45], denoted $\mathrm{DRips}((X, \mathsf{d}))_{d,r}$, is the maximal subcomplex of $\mathrm{Rips}((X, \mathsf{d}))_r$ whose vertices have degree at least $d - 1$ in the 1-skeleton of $\mathrm{Rips}((X, \mathsf{d}))_r$. The **Degree-Rips bifiltration** of (X, d) is the simplicial filtration $\mathrm{DRips}((X, \mathsf{d})) := \{\mathrm{DRips}((X, \mathsf{d}))_{d,r}\}_{d \in \mathbb{Z}^{\mathrm{op}}, r \in \mathbb{R}_{\geq 0}}$ over $\mathbb{Z}^{\mathrm{op}} \times \mathbb{R}_{\geq 0}$.
   (ii) The **Degree-Čech complex at scale r and degree d** [30], denoted $\mathrm{D\check{C}ech}((X, \mathsf{d}))_{d,r}$, is the maximal subcomplex of $\check{\mathrm{C}}\mathrm{ech}((X, \mathsf{d}))_r$ whose vertices have degree at least $d - 1$ in the 1-skeleton of $\check{\mathrm{C}}\mathrm{ech}((X, \mathsf{d}))_r$. The **Degree-Čech bifiltration** of (X, d) is the simplicial filtration $\mathrm{D\check{C}ech}((X, \mathsf{d})) := \{\mathrm{D\check{C}ech}((X, \mathsf{d}))_{d,r}\}_{d \in \mathbb{Z}^{\mathrm{op}}, r \in \mathbb{R}_{\geq 0}}$ over $\mathbb{Z}^{\mathrm{op}} \times \mathbb{R}_{\geq 0}$.
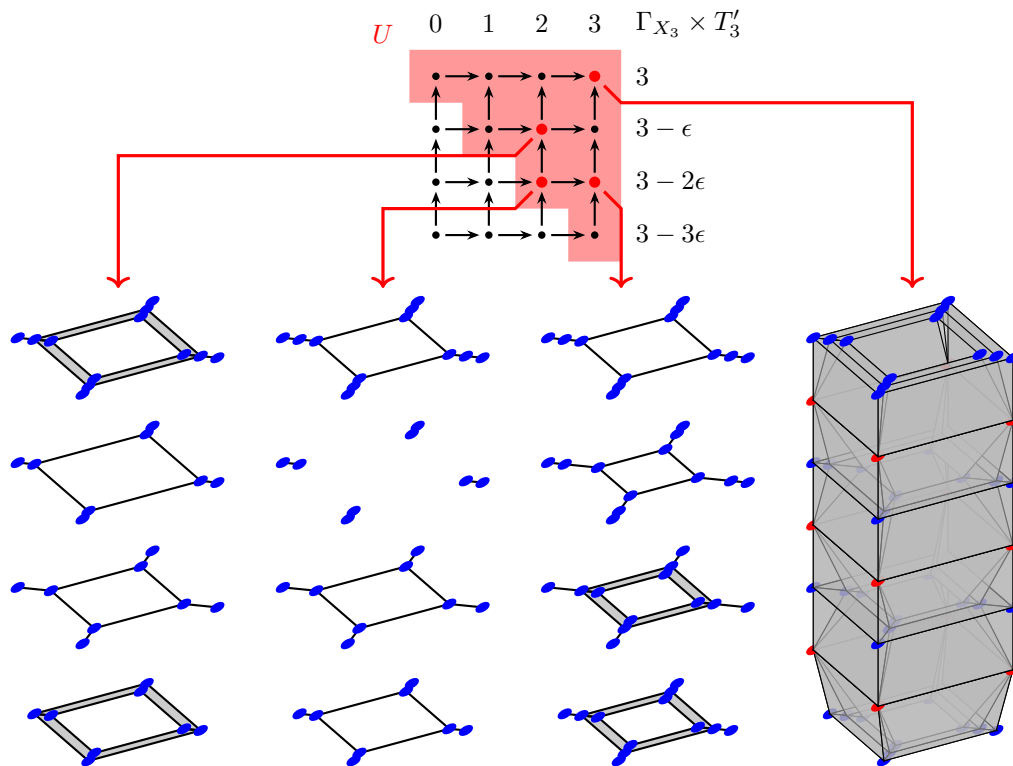
Figure 5 Partial illustration of the filtration $\mathcal{F}_3$ on $\Gamma_{X_3} \times T'_3$. The body of $\mathcal{F}_3(3,3)$ is homotopy equivalent to the cylinder $S^1 \times [0,1]$. The 1-cycles in $\mathcal{F}_3(2, 3 - \epsilon_3)$, $\mathcal{F}_3(2, 3 - 2\epsilon_3)$, and $\mathcal{F}_3(3, 3 - 2\epsilon_3)$ become homologous in $\mathcal{F}_3(3,3)$.

We consider the finite subposets

$$J_X := \{|X| - 1 < |X| - 2 < \ldots < 1\} \subset \mathbb{Z}^{\text{op}} \quad \text{and} \quad T_X := \text{im}(\mathsf{d}) \subset \mathbb{R}_{\geq 0}.$$
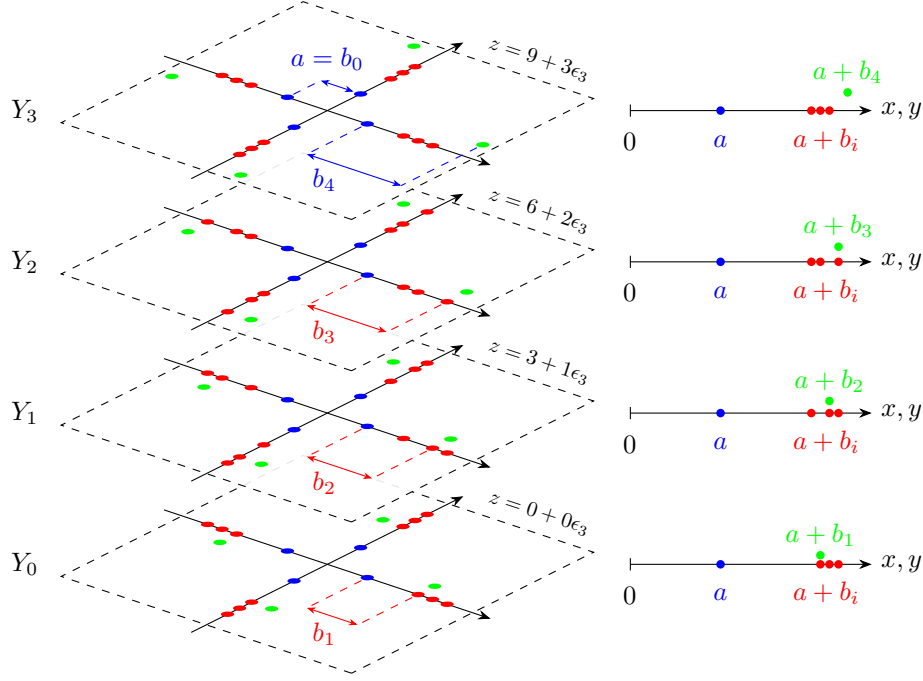
By Remarks 4 and 7, for each $m \in \mathbb{Z}_{\geq 0}$, the size of the GPD of $M := \text{H}_m(\text{DRips}((X, \mathsf{d})), \mathbb{F})$: $\mathbb{Z}^{\text{op}} \times \mathbb{R}_{\geq 0} \to \textbf{vec}_{\mathbb{F}}$ equals $\|\text{dgm}_{M|_{J_X \times T_X}}\|$.

▶ **Theorem 25** (Super-polynomial GPD of Degree-Rips and Degree-Čech filtrations)**.** *There does* not *exist* $k \in \mathbb{Z}_{\geq 0}$ *such that for* every *finite metric space* (X, d), $\|\text{dgm}_1(\mathcal{F}_{(X,\mathsf{d})})\|$ *belongs to* $O(|X|^k)$, *where* $\mathcal{F}_{(X,\mathsf{d})} \in \{\text{DRips}((X, \mathsf{d})), \text{D\v{C}ech}((X, \mathsf{d}))\}$.

Proof Sketch. By Lemma 23, it suffices to show the statement when $\mathcal{F}_{(X,\mathsf{d})} = \text{DRips}((X, \mathsf{d}))$.

Let $n \geq 1$. We construct an $X_n \subset \mathbb{R}^3$, which consists of $4(n + 1)^2$ points and inherits the metric d_n from the supremum metric on $\mathbb{R}^3$. Our goal is to show that $\|\text{dgm}_1(\text{DRips}((X_n, \mathsf{d}_n)))\|$ does not belong to $O(n^k)$ for any k, which then does not belong to $O(|X_n|^k)$ for any k. Let $\epsilon_n := 1/n^2$. Let $X_n := \bigcup_{i=0}^{n} Y_i$ (see Figure 6), where, for $i \in [n]$, Y_i is the subset of the plane $z = (n+\epsilon_n)i$ in $\mathbb{R}^3$ whose projected image πY_i onto $\mathbb{R}^2$ is described below. Let $A := \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$, $B := \{(1, \epsilon_n), (-\epsilon_n, 1), (-1, -\epsilon_n), (\epsilon_n, -1)\}$, and $a := n - n\epsilon_n$. For $j \in [n + 1]$, let $b_j := a + j\epsilon_n = n + (j - n)\epsilon_n$. Let

$$\pi Y_i := \textcolor{blue}{aA} \cup \Big(\bigcup_{\substack{0 \leq j \leq n \\ j \neq i}} \textcolor{red}{(a + b_i)A} \Big) \cup \textcolor{green}{(a + b_{i+1})B}.$$

Figure 6 For $n = 3$, the set X_n that is defined in the proof of Theorem 25.

In the full version [38], we show that for the filtration $\mathcal{F}_n := \text{DRips}((X_n, \mathsf{d}_n))$, $\|\text{dgm}_1(\mathcal{F}_n)\|$ does not belong to $O(|X_n|^k)$ for any k. For this, we identify an interval of $\Gamma_{X_n} \times T_{X_n}$ on which the restriction of $M_n := \text{H}_1(\mathcal{F}_n; \mathbb{F})$ is similar to M from Equation (7). Then, we prove claims analogous to Claim 15 through 18, and apply Lemma 21 (ii). ◄

It is known that, for $m \in \mathbb{Z}_{\geq 0}$, adding m integers has time complexity $O(m)$ [37]. Hence, as a direct corollary of Theorems 13, 24, and 25, we obtain:

▶ **Corollary 26.** *Let $n, d \in \mathbb{Z}_{\geq 0}$, $d \geq 2$, and $I \in \text{Int}([n]^d)$. For an arbitrary $[n]^d$-module M, computing $\text{dgm}_M(I)$ via the Möbius inversion of the GRI of M requires at least $\Theta(2^n)$ in time. In particular, M can arise as the homology of a simplicial filtration over $[n]^d$, with the number of simplices being polynomial in n.*

Proof. Let M be the persistence module defined in the proof of Theorem 13 for $d = 2$ and $m = 0$. Let $I := U_{[n]}$, as described immediately above Claim 18 in that proof. By Lemma 10, we have $\text{dgm}_M(I) = \sum_{S' \subsetneq [n]} (-1)^{|S'|} = (-1)^{n+2}$. This sum contains $2^{n+1} - 1$ nonzero summands, requiring $\Theta(2^n)$ time for computation. ◄

5 Conclusion

We showed that the size of the GPD can be super-polynomial in the number of simplices in a given multi-parameter filtration, and such sizes can arise even from well-known constructions. Furthermore, we noted that the GRI can be "locally densely supported" (see Claim 18), which suggests that using the Möbius inversion of the GRI to compute the GPD can be intractable. These findings highlight the need to compute the GPD (or any invariant approximating it) directly, without relying on the GRI.

From a different perspective, developing an output-sensitive algorithm for computing the GPD might be a more practical approach, potentially extending or adapting ideas from [52]. Moreover, combining optimization techniques to address challenges in computing the GPD appears to be a promising research direction. In line with this, [20] proposes a method for "sparsifying" the GPD via gradient descent.

It also remains an interesting question whether special types of multi-parameter filtrations not explored in this work can also give rise to GPDs of super-polynomial size. Examples include multicover and barycentric (subdivision) bifiltrations [60].

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