Geometric Realizations of Dichotomous Ordinal Graphs

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Abstract

A *dichotomous ordinal graph* consists of an undirected graph with a partition of the edges into *short* and *long* edges. A *geometric realization* of a dichotomous ordinal graph G in a metric space X is a drawing of G in X in which every long edge is strictly longer than every short edge. We call a graph G *pandichotomous* in X if G admits a geometric realization in X for every partition of its edge set into short and long edges.

We exhibit a very close relationship between the degeneracy of a graph G and its *pandichotomic Euclidean or spherical dimension*, that is, the smallest dimension k such that G is pandichotomous in $\mathbb{R}^k$ or the sphere $\mathcal{S}^k$, respectively. First, every d-degenerate graph is pandichotomous in $\mathbb{R}^d$ and $\mathcal{S}^{d-1}$ and these bounds are tight for the sphere and for $\mathbb{R}^2$ and almost tight for $\mathbb{R}^d$, for $d \geq 3$. Second, *every* n-vertex graph that is pandichotomous in $\mathbb{R}^k$ has at most μkn edges, for some absolute constant $\mu < 7.23$. This shows that the pandichotomic Euclidean dimension of any graph is linearly tied to its degeneracy and in the special case $k \in \{1, 2\}$ resolves open problems posed by Alam, Kobourov, Pupyrev, and Toeniskoetter.

Further, we characterize which complete bipartite graphs are pandichotomous in $\mathbb{R}^2$: These are exactly the $K_{m,n}$ with $m \leq 3$ or $m = 4$ and $n \leq 6$. For general bipartite graphs, we can guarantee realizations in $\mathbb{R}^2$ if the short or the long subgraph is constrained: namely if the short subgraph is outerplanar or a subgraph of a rectangular grid, or if the long subgraph forms a caterpillar.

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1 Introduction

For an *ordinal embedding*, we are given a set of objects $x_1, \ldots, x_n$ in an abstract space, together with a set of *ordinal constraints* of the form $\operatorname{dist}(x_i, x_j) < \operatorname{dist}(x_k, x_\ell)$. The objective is to compute a set of points $p_1, \ldots, p_n$ in a metric space while preserving as many ordinal constraints as possible. Ordinal embeddings were first studied in the 1960's by Shepard [20, 21] and Kruskal [13, 14] in the context of psychometric data analysis. Recently, there have been applications in the field of Machine Learning [23]. The computation of ordinal embeddings is also known in the literature as *non-metric multi-dimensional scaling.* For an extensive literature review on ordinal embeddings refer to [24].

Of particular interest in relation to our work is the application of ordinal embeddings to the problem of recognizing multidimensional preferences [5, 7, 19] in the field of Computational Social Science. The objects are either *voters* or *options*, which, together with the ordinal constraints (i.e., the voters' preferences), naturally define a bipartite graph. However, the goal is to find an embedding in $\mathbb{R}^d$ where all constraints are satisfied rather than to seek for an approximation. Efficient algorithms exist when $d = 1$ [6, 7], while for any $d \geq 2$ the problem is as hard as the existential theory of the reals [19]. In the simplest case, the preferences form a *dichotomy*, that is, a voter may either support or reject an option [9, 19].

This setting is naturally modeled as a *dichotomous ordinal graph*, which consists of an undirected graph $G = (V, E_s \cup E_\ell)$ together with a partition of the edges into a set E_s of *short* edges and a set E_ℓ of *long* edges. A *geometric realization* of a dichotomous ordinal graph G is a drawing Γ of G in some metric space along with a threshold $\delta > 0$ such that the short edges of G are exactly those that have length at most δ in Γ. In this work we consider two natural classes of drawings in which edges are drawn as geometrically shortest paths: (1) straight-line drawings in the Euclidean space $\mathbb{R}^d$ and (2) geodesic drawings on the sphere $\mathcal{S}^d$. Figure 1 shows two straight-line drawings of the same dichotomous ordinal graph in $\mathbb{R}^2$. The drawing in (a) is a valid geometric realization, whereas the drawing in (b) is not.

A dichotomous ordinal graph may or may not be realizable in any particular space. For instance, it is easy to see that K_4 where the short edges induce a star cannot be realized in $\mathbb{R}^1$. In general, it is NP-hard to decide whether a dichotomous ordinal graph admits a geometric realization in $\mathbb{R}^2$, even if the underlying graph is a complete graph and the short edges induce a planar graph [3, Lemma 1]. According to Peters [19, Theorem 4] the problem is $\exists\mathbb{R}$-complete for bipartite graphs.[1]

(a) valid drawing of $\triangle uvw$.

(b) invalid drawing of $\triangle uvw$.

Figure 1 A dichotomous ordinal triangle. Short edges are shown blue/solid, long edges red/dashed.

In light of the motivation, it is desirable that a geometric realization always exists, no matter the preferences of the voters. We call a graph $G = (V, E)$ *pandichotomous* in a metric space X if G admits a geometric realization in X for every partition of E into short and long edges. Graphs that are pandichotomous in $\mathbb{R}^1$ are also called total-threshold-colorable [1, 2] or total weak unit interval graphs [3]. Angelini, Bekos, Gronemann, and Symvonis [4] exhibited some graph classes that are pandichotomous in $\mathbb{R}^2$, such as *double-wheels* (a cycle and two

[1] But even the full version of the paper [18] does not contain an explicit proof but only refers to the work of Kang and Müller [12, Theorem 1]. The graph constructed in this proof is not bipartite. While it is plausible that the construction could be adapted, it does not seem immediately obvious to us.

additional vertices connected to all vertices of the cycle), *2-degenerate* graphs (can be reduced to the empty graph by repeatedly removing a vertex of degree at most two), and *subcubic* graphs (vertex degree at most three). On the negative side, there exists a dichotomous ordinal graph G whose underlying graph can be obtained from a double-wheel by adding a single edge such that G does not admit a geometric realization in $\mathbb{R}^2$ [4]. Clearly, being pandichotomous is a *monotone* graph property, that is, if a graph G is pandichotomous in X, then so is every subgraph of G. For $X = \mathbb{R}^d$, being pandichotomous is also closed under taking disjoint unions, but for $X = \mathcal{S}^d$ this is not immediate.

A related question is the existence of a realization of a graph as a *unit disk graph*, where vertices are represented by unit disks, and they are connected by an edge if and only if the corresponding disks intersect. More generally, the *sphericity* [10, 15] of a graph G is the smallest d such that G can be realized as an intersection graph of unit balls in $\mathbb{R}^d$. The main difference between geometric realizations of dichotomous ordinal graphs and unit ball realizations lies in the different types of edges. In a unit disk realization there are only two types: edge and non-edge, and all of them have to be faithfully represented. In a geometric realization of dichotomous ordinal graphs there are three types of edges: long, short, and non-edges, and we have no constraints concerning the last type. Therefore, only upper bounds on the sphericity carry over to the dichotomous setting. Since every graph on n vertices has sphericity at most n [15], every graph on n vertices is also pandichotomous in $\mathbb{R}^n$. So for every finite graph G we can define its *pandichotomic Euclidean dimension* $\mathrm{ped}(G)$ to be the smallest dimension d such that G is pandichotomous in $\mathbb{R}^d$. Analogously, we define the *pandichotomic spherical dimension* $\mathrm{psd}(G)$. Trivially, $\mathrm{ped}(G) \leq \mathrm{psd}(G) + 1$.

Results. In this paper, we initiate the study of the pandichotomic dimension of graphs, with a focus on the bipartite case. In Section 3, we characterize the complete bipartite graphs that are pandichotomous in $\mathbb{R}^2$. Specifically, we show that $K_{m,n}$ is pandichotomous in $\mathbb{R}^2$ if and only if either (1) $m \leq 3$ or (2) $m = 4$ and $n \leq 6$.

A dichotomous ordinal graph has two natural induced subgraphs: The *short* subgraph, induced by the short edges, and the *long* subgraph, induced by the long edges. In Section 4, we show that if either of these subgraphs is sufficiently constrained, then a realization in $\mathbb{R}^2$ always exists. Specifically, this is the case if (1) the graph is bipartite and the short subgraph is outerplanar (Theorem 8); (2) the short subgraph forms a subgraph of a rectangular grid (Theorem 9); or (3) the long subgraph forms a caterpillar (Theorem 10). In (1) and (2), we can also ensure that the short subgraph is realized without edge crossings. However, there are bipartite dichotomous ordinal graphs that do not admit a geometric realization in $\mathbb{R}^2$, even though the short subgraph is planar (see, e.g., Lemma 6).

In Section 5, we study the pandichotomic Euclidean dimension and show that it is very closely related to the degeneracy. A graph is *k-degenerate* if it can be constructed starting from the empty graph by iteratively applying the following operation: Add a new vertex and make it adjacent to at most k existing vertices. The *degeneracy* $\mathrm{d}(G)$ of a graph G is the smallest k such that G is k-degenerate.

We show that all d-degenerate graphs are pandichotomous on $\mathcal{S}^{d-1}$ and in $\mathbb{R}^d$, for $d \geq 2$ (Theorem 11). In particular, it follows that all bipartite planar graphs are pandichotomous on $\mathcal{S}^2$ and in $\mathbb{R}^3$ (Corollary 12). Our bounds imply $\mathrm{ped}(G) \leq \mathrm{d}(G)$ and $\mathrm{psd}(G) \leq \mathrm{d}(G) - 1$, for every graph G with $\mathrm{d}(G) \geq 2$. We also show that these bounds are tight for the sphere (Corollary 14) and for $\mathbb{R}^2$ (Theorem 15) and almost tight for $\mathbb{R}^d$, for $d \geq 3$ (Theorem 13).

We also show that *every* n-vertex graph that is pandichotomous in $\mathbb{R}^d$ has at most μdn edges, for some constant $\mu < 7.23$ (Theorem 16), and this bound is optimal up to the value of μ. In the special cases $d \in \{1, 2\}$, this affirmatively answers two open problems posed explicitly by

Alam, Kobourov, Pupyrev, and Toeniskoetter [3]. Consequently, $\mathrm{d}(G)/(2\mu) \leq \mathrm{ped}(G) \leq \mathrm{d}(G)$ and $\mathrm{d}(G)/(2\mu) - 1 \leq \mathrm{psd}(G) \leq \mathrm{d}(G) - 1$, for every graph G (Corollary 19). In other words, the pandichotomic Euclidean and spherical dimensions are linearly tied to the degeneracy. As a direct consequence of these bounds (Corollary 20), we determine up to a constant factor the smallest dimension d for which every n-vertex (bipartite) dichotomous ordinal graph is realizable in $\mathbb{R}^d$ (or $\mathcal{S}^{d-1}$).

2 Preliminaries

For two points $p, q \in \mathbb{R}^d$ we denote by $\|p - q\|$ the Euclidean distance between p and q. For a point $c \in \mathbb{R}^d$ and a positive real number r, the *ball* $\mathrm{B}(c, r)$ of radius r around c is the set $\{p \in \mathbb{R}^d \colon \|p - c\| \leq r\}$ of all points that have Euclidean distance at most r to c. For a set $A \subset \mathbb{R}^d$ we denote by ∂A the *boundary* of A. The boundary of $\mathrm{B}(c, r)$ is formed by the *sphere* $\mathrm{S}(c, r) = \{p \in \mathbb{R}^d \colon \|p - c\| = r\}$. A *unit* ball or sphere has a radius of $r = 1$. The *geodesic distance* $\mathrm{d}_\gamma(p, q) \in [0, 2\pi]$ between two points $p, q \in \mathcal{S}_d$ is determined by the central angle of a shortest great circle arc between p and q.

Given a finite set X of geometric objects, such as hyperplanes or spheres, in $\mathbb{R}^d$, the *arrangement* $\mathcal{A}(X)$ of X is the partition of $\mathbb{R}^d$ induced by X into so-called cells of various dimension. The maximal connected open subsets of $\mathbb{R}^d \setminus \bigcup_{R \in X} R$ are the *d-cells* of $\mathcal{A}(X)$. And every relatively open k-dimensional intersection of two or more d-cells forms a k-cell of $\mathcal{A}(X)$. The d-cells are also called *full-dimensional cells* or even just *cells* of $\mathcal{A}(X)$. The 0-cells and 1-cells are also called *vertices* and *edges* of $\mathcal{A}(X)$, respectively.

For geometric realizations of dichotomous ordinal graphs in $\mathbb{R}^d$ we may fix a global scale and assume without loss of generality a threshold of $\delta = 1$. Furthermore, as we consider finite graphs only, we may assume that no two vertices have distance exactly one.

▶ **Observation 1.** *If a dichotomous ordinal graph admits a geometric realization in $\mathbb{R}^d$, then it admits a realization in $\mathbb{R}^d$ where no two vertices have distance exactly one.*

In contrast, such a global rescaling does not work in general for geometric realizations on $\mathcal{S}^d$. If the short subgraph of a dichotomous ordinal graph has several connected components, then we can draw these components mutually far apart. This yields the following.

▶ **Observation 2.** *A dichotomous ordinal graph admits a geometric realization in $\mathbb{R}^d$ if and only if each subgraph induced by a connected component of its short subgraph does.*

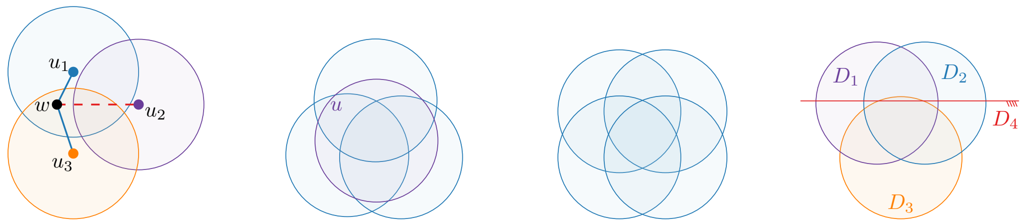
3 Complete Bipartite Graphs

In this section, we prove the following theorem. The proof is split into Lemmas 4, 6, and 7, in combination with the fact that being pandichotomous is a monotone graph property.

▶ **Theorem 3.** *The complete bipartite graph $K_{m,n}$ is pandichotomous in $\mathbb{R}^2$ if and only if either (1) $m \leq 3$ or (2) $m = 4$ and $n \leq 6$.*

A convenient way to reason about geometric realizations for bipartite graphs is in terms of arrangements of spheres. Consider a bipartite dichotomous ordinal graph $G = (U \cup W, E)$ and suppose that the vertices of U are already drawn as points in $\mathbb{R}^d$. Then, to obtain a geometric realization for G the task is to place each $w \in W$ such that for each $u \in U$ with $uw \in E$ we have $w \in \mathrm{B}(u, 1)$ if and only if the edge uw is short; see Figure 2a.

Let $U = \{u_1, \ldots, u_n\}$, let $D_i = \mathrm{B}(u_i, 1)$, let $C_i = \partial D_i$, and let $\mathcal{C}$ denote the arrangement of $C_1, \ldots, C_n$. To every vertex $w \in W$ we associate a set $V(w) \subseteq U$ such that $u \in V(w)$ if and only if uw is a short edge in G. We refer to $V(w)$ as a *singleton*, a *pair*, or a *triple*



(a) $K_{3,m}$ (b) $K_{4,m}$ all pairs (c) $K_{4,m}$ all triples (d) arbitrary radii

Figure 2 Arrangements of circles to represent one color class of a bipartite graph in $\mathbb{R}^2$.

if $|V(w)| = 1$, 2, or 3, respectively. A subset $X \subseteq U$ is *realized* by a drawing of U if there is a cell r in $\mathcal{C}$ such that $r \subseteq D_i$ if and only if $u_i \in X$. Then G admits a geometric realization if and only if there exists a drawing of U where $V(w)$ is realized, for all $w \in W$.

▶ **Lemma 4.** *The complete bipartite graph $K_{3,m}$ is pandichotomous in $\mathbb{R}^2$, for all $m \in \mathbb{N}$. The complete bipartite graph $K_{4,m}$ is pandichotomous in $\mathbb{R}^2$, for all $1 \leq m \leq 6$.*

Proof. For $K_{3,m}$ we can draw $U = \{u_1, u_2, u_3\}$ so that all eight subsets of U are realized; see Figure 2a. Therefore, any dichotomous ordinal $K_{3,m}$ admits a geometric realization. For $|U| \geq 4$ such a universal placement is not possible because an arrangement of n circles has at most $n(n-1)+2$ cells [22]. So an arrangement of four circles has at most 14 cells, whereas a four-element set has 16 subsets. However, for $|U| = 4$ and $|W| \leq 6$ we can always obtain a geometric realization as follows. Let $V(W) = \{V(w) \colon w \in W\} \subset 2^U$.

If there are at least three pairs in $V(W)$, then, given that $|V(W)| \leq |W| \leq 6$, the number of triples plus the number of singletons in $V(W)$ together is at most three. Thus, as $|U| = 4$, there exists a vertex $u \in U$ such that $\{u\} \notin V(W)$ and $U \setminus \{u\} \notin V(W)$. So we can use the drawing depicted in Figure 2b, where we assign u to the central disk. As all subsets of U other than $\{u\}$ and $U \setminus \{u\}$ are realized, this is a valid geometric realization of G.

Otherwise, there are at most two pairs in $V(W)$. We use the drawing depicted in Figure 2c, where we assign the vertices of U to the disks so that both pairs in $V(W)$ appear consecutively in the circular order of disks. (This works regardless of whether or not these pairs share a vertex.) As all subsets of U other than the two pairs that correspond to opposite circles in the drawing are realized, this is a valid geometric realization of G. ◀

The following elementary lemma about unit disks turns out helpful.

▶ **Lemma 5.** *Let D, E be unit disks in $\mathbb{R}^2$, let c be a chord of D, and let A be a closed part of D bounded by c and ∂D whose interior A° does not contain the center of D. Then $c \subset E \iff A \subset E$.*

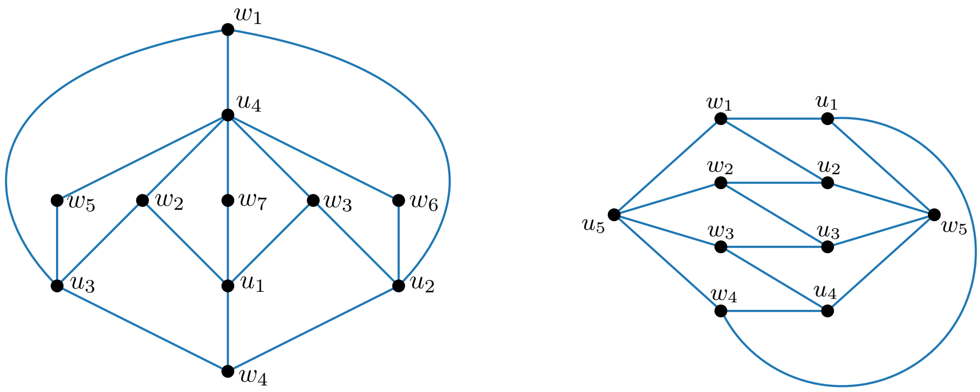
▶ **Lemma 6.** *There exists a dichotomous ordinal $K_{4,7}$ that is not realizable in $\mathbb{R}^2$.*

Proof Sketch. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, \ldots, w_7\}$ denote the vertex partition. For each w_i, we can specify an associated set $U_i \subseteq U$ (such that exactly the edges between w_i and U_i are short; see Figure 3a). We choose all four subsets of size three and the three subsets of size two that contain u_4, and distribute them among the vertices of W arbitrarily. In any geometric representation, each set U_i corresponds to a cell in the induced arrangement $\mathcal{C}$ of unit circles. Two more cells are required implicitly: The outer cell, which corresponds to $\emptyset \subset U$, and a cell that corresponds to the whole set U and is required by Helly's Theorem [11] because disks are convex and we specified all triples to be among the sets U_i.

Using these properties of $\mathcal{C}$ we can show that it cannot be realized using unit circles. It is crucial that the circles must have the same radius: If arbitrary radii are allowed, then a geometric realization exists (Figure 2d). ◀

▶ **Lemma 7.** *There exists a dichotomous ordinal $K_{5,5}$ that is not realizable in $\mathbb{R}^2$.*

Proof. First we specify a dichotomous ordinal $K_{5,5}$, see Figure 3 for an illustration.



(a) short edges of $K_{4,7}$ **(b)** short edges of $K_{5,5}$

Figure 3 A dichotomous ordinal $K_{4,7}$ and $K_{5,5}$, respectively, that does not admit a geometric realization. The drawn edges are the short edges. Edges between vertices labeled u on one hand and w on the other hand, that are not drawn, are long.

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$ denote the partition of the vertex set. To each $w_i \in W$, for $1 \leq i \leq 5$, we associate a set $\alpha(w_i) \subseteq U$ as follows:

$$\alpha(w_i) = \{u_i, u_{i\oplus 1}, u_5\}, \text{for } 1 \leq i \leq 4, \text{ and } \alpha(w_5) = U \setminus \{u_5\},$$

where $i \oplus 1 = (i \bmod 4) + 1$. Now consider an arbitrary but fixed geometric realization of this dichotomous ordinal $K_{5,5}$. In a slight abuse of notation we identify the vertices with the corresponding points in the geometric realization. Denote by D_i the unit disk centered at u_i, which represents the region of points that are in short distance to u_i. Let $W^- = \{w_1, w_2, w_3, w_4\}$. As $W^- \subset D_5$, whereas $w_5 \notin D_5$, by convexity of D_5 all points of W^- lie in an open halfplane through w_5. Let $\phi_1, \phi_2, \phi_3, \phi_4$ denote the counterclockwise order of the points from W^- around w_5 within this halfplane. By symmetry of W^- we may assume that $\phi_1 = w_1$ without loss of generality. We consider two cases.

Case 1: The set $\alpha(\phi_4)$ contains one of u_1 or u_2. Then by symmetry between u_1 and u_2 we may assume without loss of generality that $\phi_4 = w_4$. See Figure 4 (left) for illustration. By convexity of D_1, the triangle $\triangle = w_1 w_4 w_5 = \phi_1 \phi_4 w_5$ is contained in D_1. As $w_2, w_3 \notin D_1$, we have $\phi_2, \phi_3 \notin \triangle$. As $w_2, w_3, w_5 \in D_3$ but $w_1, w_4 \notin D_3$, the circle ∂D_3 crosses the line segment $w_1 w_4$ twice in D_5, and it also crosses both of the line segments $\phi_1\phi_2$ and $\phi_3\phi_4$. Similarly, as $\triangle \subset D_1$ but $w_2, w_3 \notin D_1$, the circle ∂D_1 crosses both line segments $w_5 w_2$ and $w_5 w_3$, and it also crosses both of the line segments $\phi_1\phi_2$ and $\phi_3\phi_4$. Let c denote the chord of D_3 induced by $\phi_1\phi_4$. As $D_1 \supset c$, by Lemma 5 we know that D_1 contains the part A of D_3 on the side of c that does not contain the center of D_3. This is the part that contains w_5 because the other part contains $\phi_2, \phi_3 \notin D_1$. Similarly, as $D_5 \supset c$, by Lemma 5 it follows that also $D_5 \supset A$, which is a contradiction to $w_5 \notin D_5$. Thus, this case is impossible.

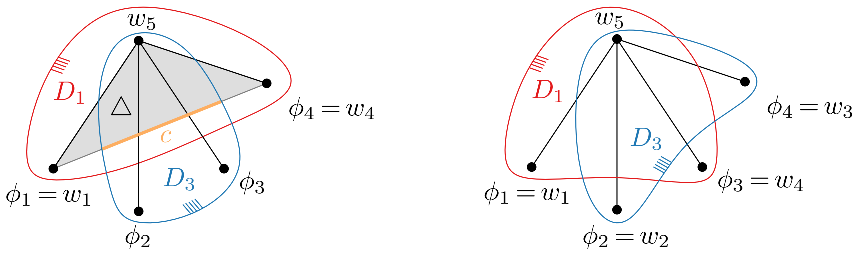


Figure 4 The two cases in the proof of Lemma 7.

Case 2: The set $\alpha(\phi_4)$ does not contain any of u_1 or u_2. Then we have $\phi_4 = w_3$ and both $|\alpha(\phi_1) \cap \alpha(\phi_3)| = 1$ and $|\alpha(\phi_2) \cap \alpha(\phi_4)| = 1$. By symmetry we may assume without loss of generality that $\phi_2 = w_2$ and $\phi_3 = w_4$. See Figure 4 (right) for illustration. Consider the two disks D_1 and D_3. They both contain w_5, and so the corresponding circles ∂D_1 and ∂D_3 intersect all of the rays $w_5\phi_i$, for $1 \leq i \leq 4$. As $\phi_1, \phi_3 \in D_1 \setminus D_3$ and $\phi_2, \phi_4 \in D_3 \setminus D_1$, the rays $w_5\phi_1$ and $w_5\phi_3$ intersect ∂D_1 before ∂D_3, whereas the rays $w_5\phi_2$ and $w_5\phi_4$ intersect ∂D_3 before ∂D_1. But then ∂D_1 and ∂D_3 cross in each of the cones $\phi_i w_5 \phi_{i+1}$, for $1 \leq i \leq 3$, which is impossible, since any two distinct circles intersect in at most two points.

As we arrived at a contradiction in both cases, we conclude that no geometric realization of this dichotomous ordinal $K_{5,5}$ exists. If, however, we allow disks with arbitrary radii, then a geometric realization exists, as depicted in Figure 5. ◀

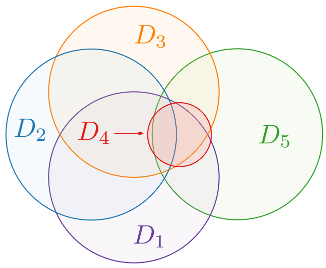


Figure 5 A realization of the dichotomous ordinal $K_{5,5}$ from Lemma 7 by disks of arbitrary radii.

4 Graphs with Constrained Short or Long Subgraphs

We show that every bipartite dichotomous ordinal graph admits a geometric realization if the short subgraph G_s is outerplanar or a subgraph of the rectangular grid or if the long subgraph G_ℓ is a caterpillar. In the first case, we construct a plane drawing of G_s in which the BFS-layers are drawn on horizontal lines (Figure 7b). In the second case, we suitably perturb the grid (Figure 6). And in the third case, we suitably place points on $\mathcal{S}^2$.

▶ **Theorem 8.** *A bipartite dichotomous ordinal graph admits a geometric realization if the subgraph induced by the short edges is outerplanar.*

Proof Sketch. Let $G = (V, E_s \cup E_\ell)$ be a bipartite dichotomous ordinal graph such that $G_s = (V, E_s)$ is outerplanar. By Observation 2, we may assume that G_s is connected. We root G_s at an arbitrary vertex r. Let V_k, $k = 0, \ldots,$ be the BFS layers of G_s rooted at r, i.e., $V_0 = \{r\}$, V_1 is the set of neighbors of r, and V_{k+1}, $k \geq 1$ is the set of neighbors of the vertices in V_k that are not already in V_{k-1}. We say that w is a *child* of v and v is a *parent* of w if vw is an edge of G_s, $v \in V_k$ and $w \in V_{k+1}$ for some k. By outerplanarity, each vertex has at most two parents. We construct a planar drawing of G_s with the following properties.

- The root r is drawn with y-coordinate $y_0 = 0$. All vertices in layer V_k, $k > 0$ are on a horizontal line ℓ_k with y-coordinate y_k strictly between $k-1$ and k.
- The distance between a vertex and its children is at most 1 while the distance between two vertices on consecutive layers is greater than 1 if they are not adjacent in G_s.
- For each vertex v there is a vertical strip S_v such that
  - v is in S_v,
  - S_w is contained in the union of the strips of w's parents,
  - S_u and S_v are internally disjoint if u and v are on the same layer.

Special care has to be taken if a vertex $w \in V_{k+1}$ has two parents u and v, i.e., if w closes an internal face. In that case, we want to draw w on line ℓ_{k+1}, on the common boundary of S_u and S_v, and with distance exactly one to both u and v. ◀

▶ **Theorem 9.** *A dichotomous ordinal graph $G = (V, E_s \cup E_\ell)$ admits a geometric realization if the set of short edges induces a subgraph of the grid.*

Proof Sketch. Extend $G_s = (V, E_s)$ by the remaining grid edges and require the new edges to be long. Use the construction in Figure 6 to place the vertices. Then the short edges are shorter than $n^2 + 1/2$, while the long edges have length at least $n^2 + 1$. Scale the drawing. ◀

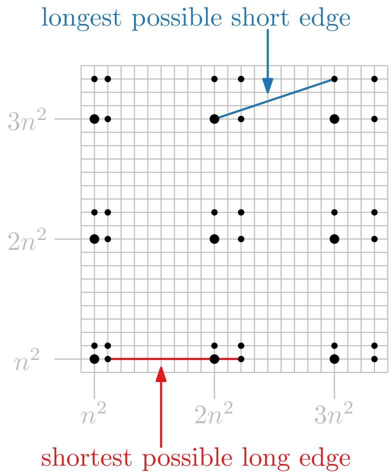
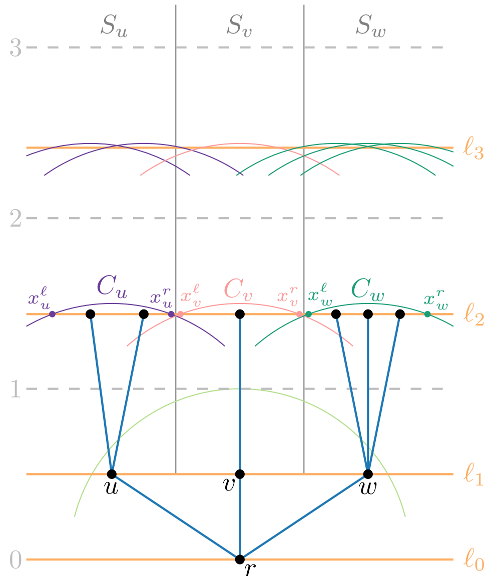


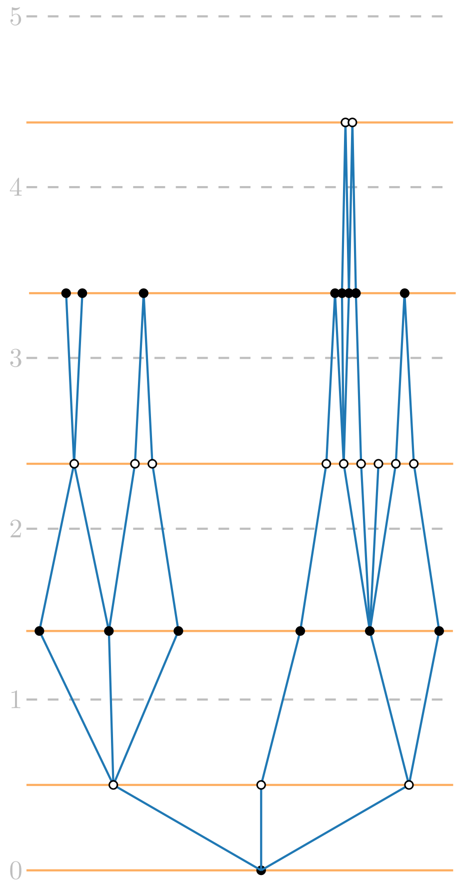
Figure 6 For each grid point (i, j), $1 \leq i \leq n$, $1 \leq j \leq n$ there are four possible points. If $i > 1$, the x-coordinate is in^2 if the edge between $(i-1, j)$ and (i, j) is short and $in^2 + i$ otherwise. If $j > 1$, the y-coordinate is jn^2 if the edge between $(i, j-1)$ and (i, j) is short and $jn^2 + j$ otherwise.

▶ **Theorem 10.** *A dichotomous ordinal graph admits a geometric realization in $\mathbb{R}^2$ and on $\mathcal{S}^2$ if each component of the long subgraph forms a caterpillar.*

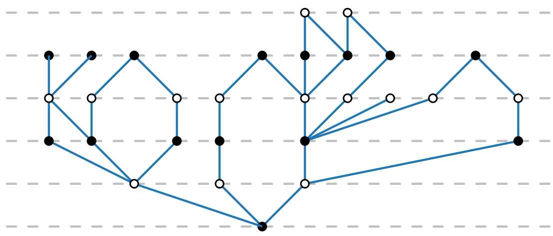
Proof Sketch. We draw the vertices on a circle C of radius $\frac{1}{2} + \epsilon$, for some small $\epsilon > 0$. For each point $p \in C$ there is a region $R(p) \subset C$ around its antipodal point $\overline{p}$ such that for a point $q \in C$ we have $||p - q|| > 1 \iff q \in R(p)$. We draw the vertices $v_1, \dots, v_k$ on the spine such that for $i = 1, \dots, k-1$ the vertex v_{i+1} lies at the counterclockwise boundary of $R(v_i)$. Then we place the leaves (if any) attached to v_i within a very short arc around the point v'_i close to $\overline{v_i}$. More precisely, we obtain v'_i by shifting $\overline{v_i}$ counterclockwise along C by a small angle that monotonically increases with i. In this way, we ensure that the leaves of v_i are far from v_i but close to the leaves of the neighbors of v_i along the spine. ◀



(a) construction for a tree



(b) geometric realization when G_s is the graph in (c)



(c) bipartite outerplanar graph

Figure 7 How to construct a geometric realization of a bipartite dichotomous ordinal graph if the short edges induce an outerplanar graph.

5 Pandichotomic Dimension and Degeneracy

In this section, we exhibit a strong connection between the degeneracy of a graph and its pandichotomic dimension. Among others, we show that the pandichotomic dimension is bounded both from above and from below by a linear function of the degeneracy, and we give bounds on the coefficients in these linear functions.

▶ **Theorem 11.** *Every d-degenerate graph, for $d \geq 2$, is pandichotomous on $\mathcal{S}^{d-1}$ and in $\mathbb{R}^d$.*

Proof. Let G be a d-degenerate dichotomous ordinal graph, for $d \geq 2$, and let $u_1, \ldots, u_n$ be a vertex ordering such that u_i has at most d neighbors in $\{u_1, \ldots, u_{i-1}\}$, for $1 \leq i \leq n$. To obtain a geometric realization of G we place the vertices on the sphere $S = \{p \in \mathbb{R}^d \colon \|p\| = \sqrt{2}/2\}$ such that for every d-tuple of vertices the corresponding vectors are linearly independent. For a vertex $p \in S$ denote by $\vec{v}_p$ the corresponding vector; the points in distance less than one to p on S form a hemisphere, which consists of all points $q \in S$ such that $\vec{v}_p\vec{v}_q > 0$, where $\vec{v}_p\vec{v}_q$ denotes the scalar product between $\vec{v}_p$ and $\vec{v}_q$.

We place the vertices $u_1, \ldots, u_n$ in this order. The first vertex u_1 is placed arbitrarily on S. Then for each u_i, for $2 \leq i \leq n$, we have a set $Q = \{q_1, \ldots, q_t\} \subset S$ of points, which correspond to the $t \leq d$ neighbors of u_i in $u_1, \ldots, u_{i-1}$. Further, for each point in Q we know whether it should be close to or far from u_i. Now we need to find a suitable point on S that satisfies these constraints. We obtain the set $Q' = \{q'_1, \ldots, q'_t\}$ from Q by setting $q'_j = q_j$, if q_j should be close to u_i and setting q'_j to the point antipodal to q_j on S if q_j should be far from u_i. As Q is linearly independent, so is Q'. Furthermore, the linear system $\vec{v}_{q'_j}\vec{x} = \vec{1}$, for $1 \leq j \leq t$, has a (unique) solution $\vec{x}_0$. Normalizing $\vec{x}_0$ we obtain a point $r \in S$, such that $\vec{v}_r\vec{v}_{q'_j} > 0$, for all $1 \leq j \leq t$. Thus, we can place u_i at r such that all constraints with respect to Q are satisfied. The points on S that lie in a k-dimensional subspace spanned by k vertices, for $k < d$, can easily be avoided when selecting r. In this way we ensure that for all d-tuples of vertices on S the corresponding vectors are linearly independent. This realization also works for geodesic distances on $\mathcal{S}^{d-1}$ if we take an arc with angle $\pi/2$ as a unit. ◀

▶ **Corollary 12.** *Every bipartite planar graph is pandichotomous on $\mathcal{S}^2$ and in $\mathbb{R}^3$.*

Proof. Let $G = (V, E)$ be a planar bipartite graph on n vertices. As a consequence of Euler's formula, we have $|E| \leq 2n - 4$. By the handshaking lemma $\sum_{v \in V} \deg(v) = 2|E| \leq 4n - 8$ and, thus, the average degree in G is strictly less than 4. Consequently, every planar bipartite graph is 3-degenerate, and the statement follows by Theorem 11. ◀

We contrast the upper bound on the pandichotomic dimension in terms of the degeneracy provided by Theorem 11 by an almost matching lower bound in the Euclidean (Theorem 13) and a matching lower bound in the spherical case (Corollary 14), even for bipartite graphs. Note that every lower bound example is not just sporadic but spans an infinite family of examples, given that being pandichotomous is a monotone graph property.

▶ **Theorem 13.** *For every $d \geq 2$, there exists a $(d+2)$-degenerate bipartite graph that is not pandichotomous in $\mathbb{R}^d$.*

Proof. We build a graph $G = (V, E)$ starting with a set V_0 of $d + 2$ isolated vertices. Then for each of the 2^{d+2} many choices of assigning short or long to the vertices of V_0 we add a new vertex and connect it by edges to V_0 accordingly. Observe that G is bipartite and $(d + 2)$-degenerate by construction. Consider a geometric realization of G and the corresponding arrangement $\mathcal{A}$ of the $d + 2$ unit spheres centered at the vertices of V_0. In order to bound the number of full-dimensional cells in $\mathcal{A}$, we consider the standard parabolic lifting map λ, which orthogonally projects a point in $\mathbb{R}^d$ to the unit paraboloid in $\mathbb{R}^{d+1}$. This map has a number of useful properties, see, for instance, Edelsbrunner's book [8, Section 1.4]. Specifically, the image $\lambda(S)$ of a sphere $S \subset \mathbb{R}^d$ lies on a hyperplane $h_S \subset \mathbb{R}^{d+1}$ such that a point $p \in \mathbb{R}^d$ lies inside S if and only if its image $\lambda(p)$ lies below h_S. It follows that we may regard $\mathcal{A}$ as an arrangement of $d + 2$ hyperplanes in $\mathbb{R}^{d+1}$. As such an arrangement has at most $\sum_{i=0}^{d+1} \binom{d+2}{i} = 2^{d+2} - 1 < 2^{d+2}$ many full-dimensional cells [16, Proposition 6.1.1], for at least one of the vertices in $V \setminus V_0$ its distance requirements with respect to V_0 are violated. Thus, there is no geometric realization of G in $\mathbb{R}^d$. ◀

▶ **Corollary 14.** *For every $d \geq 2$, there exists a $(d+1)$-degenerate bipartite dichotomous ordinal graph that is not realizable on $\mathcal{S}^{d-1}$.*

Proof. We use the same graph G as in the proof of Theorem 13, except that we start with a set V_0 of $d+1$ rather than $d+2$ isolated vertices. We consider $\mathcal{S}^{d-1}$ as a unit sphere $S \subset \mathbb{R}^d$. For a point $p \in S$, the set of points on S in distance at most some fixed unit from p can be expressed as an intersection $S \cap u_p$ where u_p is a halfspace.

Suppose there is some geometric realization of G, and let $P \subset S$ denote the set of $d+1$ points that represent the vertices of V_0 in this realization. Consider the set H of the $d+1$ bounding hyperplanes of the halfspaces u_p, for $p \in P$. The arrangement $\mathcal{A}$ of H has at most $\sum_{i=0}^{d} \binom{d+1}{i} = 2^{d+1} - 1 < 2^{d+1}$ many full-dimensional cells [16, Proposition 6.1.1]. Thus, for at least one of the vertices in $V \setminus V_0$ there is no cell that satisfies its distance requirements with respect to V_0. It follows that there is no geometric realization of G in $\mathcal{S}^{d-1}$. ◀

In the Euclidean case, we give a tight lower bound in $\mathbb{R}^2$, albeit not for bipartite graphs.

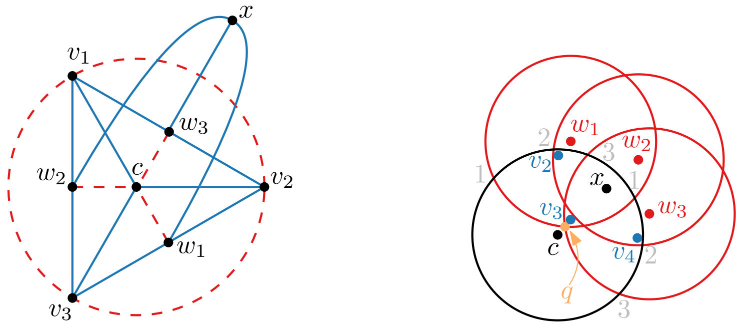


Figure 8 A 3-degenerate graph that cannot be realized in the plane along with a sketch of the proof. Long edges are dashed. The three points v_1, v_2, v_3 would have to lie in the intersection of the disk $B(c, 1)$ and a unit disk within $B(w_1, 1) \cup B(w_2, 1) \cup B(w_3, 1)$. But there is not enough space for three points with pairwise distance at least one.

▶ **Theorem 15.** *There exists a 3-degenerate graph that is not pandichotomous in $\mathbb{R}^2$.*

Proof. We build a dichotomous ordinal graph G on eight vertices as follows (Figure 8). Start with a set $V = \{v_1, v_2, v_3\}$ and a complete graph on V, all edges long. Then insert a vertex c and connect it to all vertices in V by a short edge. Next, insert a set $W = \{w_1, w_2, w_3\}$ of vertices as follows: the vertex w_i, for $i \in \{1, 2, 3\}$, is connected to c by a long edge and to each vertex in $V_i^- := V \setminus \{v_i\}$ by a short edge. Finally, we add a vertex x that is connected to all vertices in W by a short edge. Observe that G is 3-degenerate as constructed.

Consider any geometric realization Γ of G. In a slight abuse of notation we identify the vertices of G with the corresponding points that represent them in Γ. For a set $P \subset \mathbb{R}^2$ denote $I(P) = \bigcap_{p \in P} \mathrm{B}(p, 1)$ and $U(P) = \bigcup_{p \in P} \mathrm{B}(p, 1)$. As $x \in I(W)$, we have $I(W) \neq \emptyset$ and $U(W)$ is simply connected. The constraints imposed by G enforce $c \notin U(W)$ and $V \subset \mathrm{B}(c, 1) \cap U(W)$. In particular, for each $i \in \{1, 2, 3\}$, the set V_i^- is contained in the lens $L_i = I(\{c, w_i\})$. As $c \notin U(W)$, each L_i has diameter strictly smaller than $\sqrt{3}$. Any lens in $\mathbb{R}^2$ of diameter strictly smaller than $\sqrt{3}$ contains at most two points at a pairwise distance at least one. Thus, in particular, we have $v_i \notin L_i$, for all $i \in \{1, 2, 3\}$.

We analyze the interaction of the circles $S_i = \mathrm{S}(w_i, 1)$, for $i \in \{1, 2, 3\}$, with $C = \mathrm{S}(c, 1)$. As $|L_i \cap V| \geq 2$ and as any two points in V are at distance strictly larger than one, we have $|S_i \cap C| = 2$, for all $i \in \{1, 2, 3\}$. As $c \notin U(W)$ and as $U(W)$ is simply connected, some part of C is disjoint from $U(W)$. We use such a part to break up the circular sequence of intersection points between C and the S_i into a linear sequence $\mathcal{I}$ of six points (some but not all of which may coincide). On the one hand, as $|L_i \cap V| \geq 2$, for all $i \in \{1,2,3\}$, and as $|V| = 3$, we have $L_i \cap L_j \neq \emptyset$, for all $i,j \in \{1,2,3\}$. On the other hand, if $L_i \subseteq L_j$, for some $i,j \in \{1,2,3\}$ with $i \neq j$, then, given that $v_j \in V_i$ we also have $v_j \in L_j$, a contradiction. It follows that up to the indexing of W we have $\mathcal{I} = 1,2,3,1,2,3$, where we write an integer i to indicate an intersection $C \cap S_i$.

Let q denote the intersection point of $S_1 \cap S_3$ in $\mathrm{B}(c,1)$. Note that $q \notin L_2$ because $q \in L_2$ would imply $L_1 \cap L_3 \subseteq L_2$ and thus $v_2 \in L_2$, a contradiction. We continuously move a unit disk D starting from $D = \mathrm{B}(w_2,1)$ towards c such that the center d of D is on the line segment w_2c. We stop the movement as soon as $d \in \mathrm{S}(q,1)$. This process is well defined because $q \notin \mathrm{B}(w_2,1)$ and $q \in L_1 \cap L_3 \subset \mathrm{B}(c,1)$. Furthermore, during the whole movement we have $L_2 \subset D \subset U(W)$ and the disk D' at the end of the movement also contains $L_1 \cap L_3$. Thus, the lens $L = D' \cap \mathrm{B}(c,1)$ contains all of V and given that $L \subset U(W)$ it has diameter strictly smaller than $\sqrt{3}$, a contradiction. ◀

The examples discussed above demonstrate that the bound from Theorem 11 is tight in the worst case, that is, for *some* graphs. In the following, we will show that the bound is also tight for *all* graphs, up to a multiplicative constant. We first estimate the edge density of pandichotomous graphs in $\mathbb{R}^d$ and show that it is linearly bounded in d. In fact, we prove a stronger result, namely that every sufficiently dense graph G not only is not pandichotomous in $\mathbb{R}^d$, but in fact, asymptotically *almost all* partitions of the edge-set of G into short and long edges yield dichotomous ordinal graphs that are not realizable in $\mathbb{R}^d$.

▶ **Theorem 16.** *Let $d \in \mathbb{N}$, let $\varepsilon > 0$, and let $G = (V,E)$ be a graph with n vertices and m edges. Let $c < 7.182$ denote the unique positive root of the function $x - 3 - x \cdot H(1/x)$, where $H\colon (0,1) \to \mathbb{R}$ is the binary entropy function $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$.*

If $m > (c+\varepsilon)dn$, then for asymptotically almost all[2] *partitions $E = E_s \cup E_\ell$ of the edge-set of G, the dichotomous ordinal graph $(V, E_s \cup E_\ell)$ has no geometric realization in $\mathbb{R}^d$. In particular, every graph that is pandichotomous in $\mathbb{R}^d$ has at most $(c+o(1))dn$ edges. The $o(1)$ term approaches zero quickly as $dn \to \infty$. For illustration, we also show that $m < \mu dn$, for all $n \in \mathbb{N}$ and $d \geq 2$ and some explicit constant $\mu < 7.23$.*

In the proof of Theorem 16, we will use the following result by Warren [25] on the number of sign-patterns of a collection of multivariate polynomials.

▶ **Theorem 17** (Warren [25, Theorem 2])**.** *Let $q_1, \ldots, q_m$ be real polynomials in N variables, each of degree at most d. Then the number of connected components of the set*

$$\left\{ \mathbf{x} \in \mathbb{R}^N \;\middle|\; \forall i \in \{1,\ldots,m\} : q_i(\mathbf{x}) \neq 0 \right\}$$

is at most

$$2(2d)^N \sum_{k=0}^{N} 2^k \binom{m}{k}.$$

The following is an immediate consequence of the above.

[2] That is, a fraction of the partitions that tends to 1 as $dn \to \infty$.

▶ **Corollary 18.** *Let* $q_1, \ldots, q_m$ *be real polynomials in* N *variables, each of degree at most* d *and let* $\mathcal{S} := \{(\mathrm{sgn}(q_1(\mathbf{x})), \ldots, \mathrm{sgn}(q_m(\mathbf{x}))) \mid \forall i \in \{1, \ldots, m\} : q_i(\mathbf{x}) \neq 0\} \subseteq \{+1, -1\}^m$*. Then we have*

$$|\mathcal{S}| \leq 2(2d)^N \sum_{k=0}^{N} 2^k \binom{m}{k}.$$

Proof. Let $X := \{\mathbf{x} \in \mathbb{R}^N \mid \forall i \in \{1, \ldots, m\} : q_i(\mathbf{x}) \neq 0\}$. Consider the map $f : X \to \{+1, -1\}^m$, defined by $f(\mathbf{x}) = (\mathrm{sgn}(q_1(\mathbf{x})), \ldots, \mathrm{sgn}(q_m(\mathbf{x})))$. It is easy to see, using the continuity of the polynomials $q_1, \ldots, q_m$, that f is continuous. Let C be a topological component of X. The restriction of f to C is a continuous map on a connected topological space that takes on only finitely many (at most 2^m) values. Since every such map is constant, we find that f is constant on every component of X. Hence, the size of $\mathcal{S} = \mathrm{im}(f)$ is bounded by the number of components of X. The statement now follows from Theorem 17. ◀

Proof of Theorem 16. Assume w.l.o.g. that $G = (V, E)$ has vertex set $V = \{1, \ldots, n\}$. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be an n-tuple of points in $\mathbb{R}^d$, no two of which are at distance exactly one. The *distance-pattern* of $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ w.r.t. G is the map $s_G(\mathbf{x}_1, \ldots, \mathbf{x}_n) : E \to \{+1, -1\}$, where

$$s_G(\mathbf{x}_1, \ldots, \mathbf{x}_n)(ij) := \begin{cases} +1, & \text{if } \|\mathbf{x}_i - \mathbf{x}_j\| > 1, \\ -1, & \text{if } \|\mathbf{x}_i - \mathbf{x}_j\| < 1, \end{cases}$$

for every edge $ij \in E$. In other words, given a configuration of points $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ in $\mathbb{R}^d$, the distance-pattern captures the information of which edges of G in a geometric realization placing vertices $1, \ldots, n$ at points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively, are long or short.

Next, we estimate the size of the set $\mathcal{S}$ of all possible distance-patterns for G. Concretely, $\mathcal{S}$ is the set of all mappings $s_G(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ ranges over all n-tuples of points in $\mathbb{R}^d$, no two of which are at distance 1. Let us denote by M the number of partitions $E_s \cup E_\ell = E$ of the edge-set of G such that the dichotomous ordinal graph $(V, E_s \cup E_\ell)$ is realizable in $\mathbb{R}^d$. Let p denote the fraction of such partitions compared to all possible partitions of E. Note that $M = p2^m$ and that $p = 1$ if and only if G is pandichotomous.

▷ Claim 1. $|\mathcal{S}| \geq M$.

Proof of Claim 1. Consider any one of the M partitions $E_s \cup E_\ell$ of the edge-set E for which $(V, E_s \cup E_\ell)$ is realizable in $\mathbb{R}^d$. By Observation 1 we may assume a threshold of $\delta = 1$ and that we use a set of points with no unit distances. The distance-pattern of the corresponding tuple of n points in d-space then has value $+1$ for every long and -1 for every short edge.

Since any two distinct partitions of E into short and long edges result in different distance-patterns in $\mathcal{S}$ this way, it follows that $|\mathcal{S}| \geq M$, as claimed. ◁

Next, we obtain an upper bound on $|\mathcal{S}|$ using Theorem 17.

▷ Claim 2. $|\mathcal{S}| \leq 2 \cdot 4^{dn} \cdot \sum_{k=0}^{dn} 2^k \binom{m}{k}$.

Proof of Claim 2. For any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we have $\|\mathbf{x} - \mathbf{y}\| > 1$ if and only if $\|\mathbf{x} - \mathbf{y}\|^2 - 1 > 0$, and $\|\mathbf{x} - \mathbf{y}\| < 1$ if and only if $\|\mathbf{x} - \mathbf{y}\|^2 - 1 < 0$. Further, the expression

$$\|\mathbf{x} - \mathbf{y}\|^2 - 1 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_d - y_d)^2 - 1$$

is a polynomial of degree two in the coordinates of $\mathbf{x}, \mathbf{y}$. It is therefore natural to associate with every edge $ij \in E$ of G a polynomial $q_{i,j}$, defined as $q_{i,j}(\mathbf{x}_1, \dots, \mathbf{x}_n) := \|\mathbf{x}_i - \mathbf{x}_j\|^2 - 1$. The collection $\big(q_{i,j} \,\big|\, ij \in E\big)$ then forms a set of m polynomials, each of degree 2, in the $N := dn$ variables corresponding to the coordinates of $\mathbf{x}_1, \dots, \mathbf{x}_n$. For every tuple $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of points in $\mathbb{R}^d$, no two at unit distance, we have $s_G(\mathbf{x}_1, \dots, \mathbf{x}_n)(ij) = \operatorname{sgn}(q_{i,j}(\mathbf{x}_1, \dots, \mathbf{x}_n))$.

Hence, the set $\mathcal{S}$ defined before coincides with the set $\mathcal{S}$ defined in Corollary 18 for the collection $\big(q_{i,j} \,\big|\, ij \in E\big)$ of m multivariate polynomials. We therefore obtain, using the degree bound 2 for the polynomials,

$$|\mathcal{S}| \le 2 \cdot (2 \cdot 2)^N \cdot \sum_{k=0}^{N} 2^k \binom{m}{k} = 2 \cdot 4^{dn} \cdot \sum_{k=0}^{dn} 2^k \binom{m}{k}. \qquad \triangleleft$$

Using Claims 1 and 2, we obtain the inequality

$$p2^m = M \le |\mathcal{S}| \le 2 \cdot 4^{dn} \cdot \sum_{k=0}^{dn} 2^k \binom{m}{k}.$$

Since $c > 2$, we have $m \ge (c+\varepsilon)dn > 2dn$. This implies that the term $\binom{m}{k}$ is monotonically increasing for $k = 0, 1, \dots, dn$. Hence, we may bound the right hand side above by

$$2 \cdot 4^{dn} \cdot \binom{m}{dn} \cdot \sum_{k=0}^{dn} 2^k = 2 \cdot 4^{dn} \cdot \binom{m}{dn} \cdot (2^{dn+1} - 1) < 4 \cdot 8^{dn} \cdot \binom{m}{dn}.$$

Let $x := \frac{m}{dn} \ge c + \varepsilon$. Using the estimate $\binom{a}{\lambda a} \le 2^{aH(\lambda)}$, which holds for any $a \in \mathbb{N}$ and $\lambda \in (0,1)$, see e.g., [17, Lemma 9.2], we find

$$p2^{xdn} = p2^m < 4 \cdot 8^{dn} 2^{xdn \cdot H(1/x)}.$$

Taking logarithms and dividing by dn this simplifies to

$$x < \frac{2 - \log_2(p)}{dn} + 3 + x \cdot H(1/x). \tag{1}$$

Consider the function $f(x) = x - 3 - x \cdot H(1/x) = x - 3 - x \log_2 x + (x-1)\log_2(x-1)$. Observe that f is strictly monotonically increasing and continuous for $x \ge 2$ and that it has a unique positive root $c \approx 7.1815$.

It follows that $f(c+\varepsilon) > f(c) = 0$, and, since $x \ge c + \varepsilon$, Equation (1) above implies that

$$f(c+\varepsilon) \le f(x) \le \frac{2 - \log_2(p)}{dn} \implies \log_2(p) \le 2 - f(c+\varepsilon)dn.$$

Since the right hand side of the last inequality tends to $-\infty$ as $dn \to \infty$, it follows that $p \to 0$ for $dn \to \infty$. Hence, it is indeed true that asymptotically almost all partitions $E_s \cup E_\ell = E$ of the edge-set of G into short and long edges yield dichotomous ordinal graphs with no geometric representation in $\mathbb{R}^d$. This completes the proof of the first statement.

It remains to show that if G is pandichotomous ($p = 1$), then the explicit bound $m \le \mu dn$ with $\mu := 7.2240208$ holds for all $n \in \mathbb{N}$ and $d \ge 2$. Set $\phi(z) = 2/z$ and observe that $\phi \to 0$ for $z \to \infty$. We can then express Equation (1) as $f(x) < \phi(dn)$. The trivial bound $m \le \binom{n}{2}$ suffices to show that $m \le \mu dn$, unless $\mu dn < \binom{n}{2}$, i.e., unless $\lceil 4\mu + 1 \rceil \le n$. As ϕ is monotonically decreasing, it follows that $f(x) < \phi(dn) \le \phi(2n) \le \phi(2 \cdot \lceil 4\mu + 1 \rceil)$. For the value $\mu = 7.2240208$ we obtain $f(x) < \phi(60) = 1/30$, whereas $f(\mu) > 0.033333334 > 1/30$. Therefore, as f is monotonically increasing, it follows that $\frac{m}{dn} = x < \mu$, as claimed. ◀

▶ **Corollary 19.** *For some absolute constant* $\mu < 7.23$ *we have* $\frac{\mathrm{d}(G)}{2\mu} \leq \mathrm{ped}(G) \leq \mathrm{d}(G)$ *for every graph* G *and* $\frac{\mathrm{d}(G)}{2\mu} - 1 \leq \mathrm{psd}(G) \leq \mathrm{d}(G) - 1$ *for every graph* G *with* $\mathrm{d}(G) \geq 2$*.*

Proof. By Theorem 16 every graph G that is pandichotomous in $\mathbb{R}^d$ has at most μdn edges, for some absolute constant $\mu < 7.23$. Thus, the minimum degree of G is at most $\lfloor 2\mu d \rfloor$ and so G is k-degenerate, for $k = \lfloor 2\mu d \rfloor$. This implies $\mathrm{d}(G) \leq 2\mu \cdot \mathrm{ped}(G)$ and so $\frac{\mathrm{d}(G)}{2\mu} \leq \mathrm{ped}(G) \leq \mathrm{d}(G)$. Since $\mathrm{psd}(G) \geq \mathrm{ped}(G) - 1$ we obtain the lower bound for the spherical dimension. The upper bounds follow from Theorem 11. ◀

▶ **Corollary 20.** *Every* n*-vertex graph is pandichotomous in* $\mathcal{S}^{n-2}$ *and* $\mathbb{R}^{n-1}$*. But there exists an absolute constant* $\mu < 7.23$ *such that for every* $d < \frac{n-1}{4\mu}$ *there exist* n*-vertex bipartite graphs that are not pandichotomous in* $\mathbb{R}^d$ *or* $\mathcal{S}^{d-1}$*.*

Proof. Since n-vertex graphs are $(n-1)$-degenerate, the first statement is a direct consequence of Theorem 11. The complete bipartite n-vertex graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ has minimum degree $\lfloor n/2 \rfloor$. Hence, Corollary 19 implies that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is not pandichotomous in $\mathbb{R}^d$ for any dimension d such that $2\mu d < \lfloor n/2 \rfloor$, so in particular for $d < (n-1)/(4\mu)$. ◀

6 Conclusion

We study pandichotomous graphs with an emphasis on bipartite graphs on one hand and the relationship between the degeneracy and the pandichotomous dimension on the other hand. Some interesting open questions remain, such as:

- Is every planar graph, planar 3-tree, or planar bipartite graph pandichotomous in $\mathbb{R}^2$?
- Are bipartite dichotomous ordinal graphs realizable in $\mathbb{R}^2$ if the short graph is a 2-tree?
- Characterize the (complete) bipartite graphs that are pandichotomous in $\mathbb{R}^k$, for $k \geq 3$.
- Given $d \geq 3$, is there a bipartite d-degenerate graph that is not pandichotomous in $\mathbb{R}^{d-1}$? We believe that the answer to this question should be positive.

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