

Parameterized Algorithms for Matching Integer Programs with Additional Rows and Columns

Alexandra Lassota 

Eindhoven University of Technology, The Netherlands

Koen Ligthart 

Eindhoven University of Technology, The Netherlands

Abstract

We study integer linear programs (ILP) of the form $\min\{c^\top x \mid Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}$ and analyze their parameterized complexity with respect to their distance to the generalized matching problem, following the well-established approach of capturing the hardness of a problem by the distance to triviality. The generalized matching problem is an ILP where each column of the constraint matrix has 1-norm of at most 2. It captures several well-known polynomial time solvable problems such as matching and flow problems. We parameterize by the size of variable and constraint backdoors, which measure the least number of columns or rows that must be deleted to obtain a generalized matching ILP. This extends generalized matching problems by allowing a parameterized number of additional arbitrary variables or constraints, yielding a novel parameter.

We present the following results: (i) a fixed-parameter tractable (FPT) algorithm for ILPs parameterized by the size p of a minimum variable backdoor to generalized matching; (ii) a randomized slice-wise polynomial (XP) time algorithm for ILPs parameterized by the size h of a minimum constraint backdoor to generalized matching as long as c and A are encoded in unary; (iii) we complement (ii) by proving that solving an ILP is $W[1]$ -hard when parameterized by h even when c, A, l, u have coefficients of constant size. To obtain (i), we prove a variant of lattice-convexity of the degree sequences of weighted b -matchings, which we study in the light of SBO jump M -convex functions. This allows us to model the matching part as a polyhedral constraint on the integer backdoor variables. The resulting ILP is solved in FPT time using an integer programming algorithm. For (ii), the randomized XP time algorithm is obtained by pseudo-polynomially reducing the problem to the exact matching problem. To prevent an exponential blowup in terms of the encoding length of b , we bound the Graver complexity of the constraint matrix and employ a Graver augmentation local search framework. The hardness result (iii) is obtained through a parameterized reduction from ILP with h constraints and coefficients encoded in unary.

2012 ACM Subject Classification Mathematics of computing \rightarrow Combinatorial optimization; Theory of computation \rightarrow Complexity classes

Keywords and phrases Integer Programming, fixed-parameter Tractability, polyhedral Optimization, Matchings

Digital Object Identifier 10.4230/LIPIcs.ICALP.2025.112

Category Track A: Algorithms, Complexity and Games

Related Version *Full Version*: <https://arxiv.org/abs/2503.05548v1> [36]

1 Introduction

We study integer linear programs (ILPs) and analyze their parameterized complexity with respect to their distance to the generalized matching problem. In general, an ILP is of the form

$$\min\{c^\top x \mid Ax = b, l \leq x \leq u, x \in \mathbb{Z}^n\}, \quad (\text{G})$$



© Alexandra Lassota and Koen Ligthart;

licensed under Creative Commons License CC-BY 4.0

52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025).

Editors: Keren Censor-Hillel, Fabrizio Grandoni, Joël Ouaknine, and Gabriele Puppis

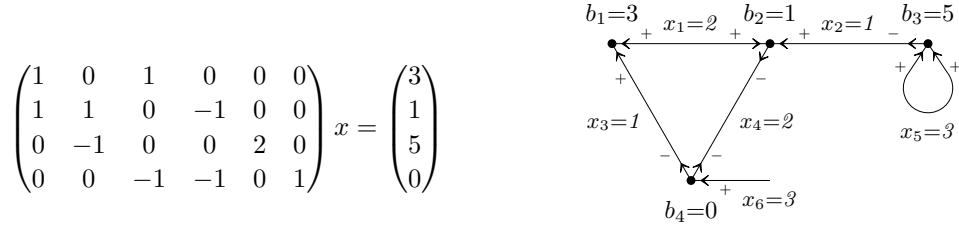
Article No. 112; pp. 112:1–112:18



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany





(a) Example constraints $Mx = b$ with $\|M\|_1 = 2$ of a generalized matching instance.

(b) Graphic interpretation of a solution to the system of Figure 1a.

■ **Figure 1** An example instance of the generalized matching problem and a corresponding solution, disregarding potential variable bounds.

where $l \in (\mathbb{Z} \cup \{-\infty\})^n$, $u \in (\mathbb{Z} \cup \{\infty\})^n$, $c \in \mathbb{Z}^n$ and $A \in \mathbb{Z}^{m \times n}$. The underlying integer linear programming problem is to either decide that the system is infeasible, there exists an optimal feasible solution, or it is unbounded and provide a solution with an unbounded direction of improvement.

Integer linear programming is a powerful language that has been applied to many key problems such as graph problems [20, 21], scheduling and bin packing [25, 32], multichoice optimization [19] and computational social choice [5, 34], among others. Unfortunately, solving ILPs is, in general, NP-hard.

However, most ILP formulations of problems are naturally well-structured. See e.g. [20, 25, 26, 32, 34] and the references therein. Motivated by this insight and their daunting general hardness, classes of ILPs with highly-structured constraint matrices and parameterizations have been intensively and successfully studied to obtain polynomial and FPT time algorithms [20, 25, 26, 32, 34]. Arguably most famous is Lenstra's 1983 algorithm who presented the first FPT time algorithm for constraint matrices with few columns [33]. The body of literature for the over three-decades-long research and the many structures, parameters, and applications are too vast to cover here, we thus refer to [24] for an overview.

Central to this paper is the polynomial time solvable class of the generalized matching problem, which is the ILP restricted to coefficient matrices satisfying $\|A\|_1 \leq 2$, i.e., all columns have 1-norm at most 2.

► **Theorem 1** (Theorem 36.1 in [45]). *When $\|A\|_1 \leq 2$, $ILP(G)$ can be solved in strongly polynomial time.*

The generalized matching problem captures a variety of well-known problems in P such as minimum cost flow, minimum cost (b)-matching and certain graph factor problems [45]. This connection becomes apparent by interpreting the variables as values assigned to edges of a graph whose vertices correspond to the constraints of the ILP¹. In this way, a constraint matrix A with $\|A\|_1 \leq 2$ can be interpreted as an incidence matrix of a generalized bidirected graph where all endpoints of edges are given signs and where degenerate self-loops and half-edges may be present [14]. An example of a graphic interpretation of an instance of the generalized matching problem is shown in Figure 1. For instance, a problem such as the minimum cost perfect matching problem on a graph $G = (V, E)$ can be cast as a generalized matching ILP. In this scenario, $A \in \{0, 1\}^{V \times E}$ is the incidence matrix of G and $l = \mathbf{0}$, $u = \mathbf{1}$, $b = \mathbf{1}$. That is, $A_{ij} = 1$ if and only if vertex i is incident to edge j . Here, $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-ones vector respectively.

¹ Note that, due to the conventions in ILP, the vertices of the graph are the numbers $[m] := \{1, \dots, m\}$ and the edges are identified with the numbers $[n]$.

This paper studies the degree to which we can extend the generalized matching problem while maintaining tractability. For this purpose, we study constraint matrices A which are similar to the constraint matrix of a generalized matching problem. Such matrix A consists of a small corner block $C \in \mathbb{Z}^{h \times p}$, wide block $W \in \mathbb{Z}^{h \times n}$, tall block $T \in \mathbb{Z}^{m \times p}$ and matching-like block $M \in \mathbb{Z}^{m \times n}$ with $\|M\|_1 \leq 2$ and is of the form

$$A = \begin{pmatrix} C & W \\ T & M \end{pmatrix}.$$

We consider the cases where either of $h = 0$ or $p = 0$, yielding the ILPs

$$\min\{a^\top y + c^\top x \mid Ty + Mx = b, e \leq y \leq g, l \leq x \leq u, (y, x) \in \mathbb{Z}^{p+n}\}, \quad (\text{T})$$

which describes a generalized matching ILP admitting additional variables, and

$$\min\{c^\top x \mid Wx = d, Mx = b, l \leq x \leq u, x \in \mathbb{Z}^n\}, \quad (\text{W})$$

which describes a generalized matching ILP admitting additional constraints. The ILPs (T) and (W) represent generalizations of well-known flow and matching problems. See [3] for an application of bipartite matching with additional variables to a scheduling problem.

In the context of the fixed-parameter tractability of ILPs, graphs and corresponding structural parameters associated with the coefficient matrix of an ILP are usually studied. However, in our case, the coefficient matrix of the perfect matching problem may have arbitrary associated primal or dual graphs, unlike the case for two-stage stochastic and n -fold ILP, which have associated coefficient matrix graphs with limited tree-depth. See [17] for an overview. In addition, the incidence matrix of an undirected graph may have arbitrarily large subdeterminants, which makes it unsuited for methods such as the one discussed in [22].

The goal of this paper is to study the complexity of the above integer linear programs (T) and (W), i.e., ILPs where the constraint matrices are *nearly* generalized matching constraint matrices. For this purpose, we parameterize by the number of variables p and constraints h that must be deleted from ILP (G) to obtain a generalized matching problem. Such parameter choice follows the classical approach of studying parameters measuring the *distance to triviality*, also called *(deletion) backdoors to triviality*—a concept that was first proposed by Niedermeier [42]. Roughly speaking, this approach introduces a parameter that measures the distance of the given instance from an instance that is solvable in polynomial time. This approach was already used for many different problems such as clique, set cover, power dominating set, longest common subsequence, packing problems, and the satisfiability problem [42, 4, 25, 23].

For some problems, such as satisfiability, having obtained a variable backdoor immediately leads to an FPT algorithm parameterized by the size of the backdoor. For ILP, however, this is not as straightforward as variable domains may be arbitrarily large, which invalidates a brute-force approach to obtain FPT results in terms of variable backdoor size. In fact, solving ILPs in FPT time parameterized by the number of variables is already highly nontrivial [33, 44].

Despite such difficulties, a significant number of results have been obtained in the context of backdoors in ILPs. The complexity of ILPs which become totally unimodular after removing a constant number of columns and rows is actively being researched [2, 41]. In addition, a large line of research has established efficient algorithms for solving ILPs given a variable or constraint backdoor to a remaining ILP which consists of many isolated ILPs with small coefficients [12, 13, 17]. Two-stage stochastic and n -fold ILP, which are two important

special cases of such block-structured ILPs, may be viewed as ILPs of the form shown in respectively (T) and (W) where M is a block-diagonal matrix with small nonzero blocks and small coefficients. To relate these results with backdoors and backdoor identification, Dvořák et al. [13] define fracture numbers, which describe the number of variable and/or constraint removals needed to decouple the ILP into small blocks. They give an overview of the parameterized complexity of ILPs parameterized by various backdoor sizes and coefficient sizes of the constraint matrix. Their work and the recent result of Cslovjcek et al. [12] shows that the complexity of ILPs parameterized by backdoor size is subtle, as ILPs may already become NP-hard when there are only 3 complicating global variables connecting constant dimension, otherwise independent ILPs with unary coefficient size [13]. On the other hand, when additionally parameterizing by the coefficient size of the small blocks, the problem becomes FPT [12]. With this paper, we aim to reveal the parameterized complexity with respect to backdoor sizes to another class of well-known efficiently solvable ILPs, that of the generalized matching problem.

We show that ILP parameterized by p , the number of variables that, upon deletion, give a general matching problem, is FPT by designing a corresponding algorithm. Further, we give a randomized, XP time algorithm with respect to h , and prove the corresponding W[1]-hardness result for this case.

We note that studying the complexity of ILP with respect to p is also motivated by the hardness of the general factor problem with a gap size of 2 [37]. Lovász's reduction from edge coloring on cubic graphs shows that, even when T consists solely of columns with only one nonzero coefficient with value 3, ILP remains NP-hard. This, together with the tractability of generalized matching and our FPT result, shows that if one were to judge the complexity of an ILP by independently and individually labeling each column as "hard" or "easy", the easy columns are precisely those with 1-norm at most 2.

Finally, given an arbitrary constraint matrix A of ILP (G), a permutation of the columns or rows to obtain the form ILP (T) or (W) with minimum p or h respectively can be obtained efficiently². That is, one can identify a minimum cardinality variable or constraint deletion backdoor to the generalized matching ILP. To obtain the form ILP (T), one may greedily move all columns with 1-norm greater than 2 to the left. The problem of minimizing h in ILP (W) is equivalent to the 3-uniform hitting set problem, which is NP-hard but FPT in h , cf. [23]. For this reason, we will assume that the given ILPs are of the form ILP (T) or (W) throughout the rest of the paper.

1.1 Contributions

In this paper, we show that solving integer linear programs is fixed-parameter tractable parameterized by the number of columns of A with 1-norm greater than 2.

► **Theorem 2.** *ILP (T) is FPT parameterized by p .*

This reveals an entirely new class of ILPs that is FPT and adds to the story of parameterized ILPs and distance to triviality paradigm. To obtain this algorithm, we use a remainder guessing strategy introduced by Cslovjcek et al. [12]. Central in this approach is modeling non-backdoor variables as polyhedral constraints on the backdoor variables. In [12], these polyhedral constraints may be exponentially complex, whereas we exploit the structure of

² Through the use of elementary row operations, a system may be modified to a row-equivalent system admitting smaller backdoors. See [9] for such study on block-structured ILP. We leave this problem open in the context of the parameterizations discussed in this paper.

matching problems and employ an efficient description of a global polyhedral constraint. To accomplish this, we study the convexity of degree sequences for which subgraphs with matching degrees exist. These systems have been studied previously, see [1], and are a well-known example of discrete systems known as jump systems [8, 39, 40, 35]. For our application, we prove a new result involving the lattice-convexity of a particular class of jump M-convex functions on the shifted lattice $2\mathbb{Z}^m + r$. See Section 2 for a technical overview of this algorithm.

As for the constraint backdoors, we show that matching-like ILPs are solvable in polynomial time with a randomized algorithm for a fixed number of additional complicating constraints if the constraints and the objective are encoded in unary. As ILP (W) can encode the NP-hard subset sum problem for $h = 1$, we will need to assume that the coefficients of the constraint matrix are small. That is, we assume that they are bounded by Δ in absolute value. The given randomized XP time algorithm unifies the known tractability in terms of membership in RP of multiple classes of constrained problems where the edges of a graph correspond to variables.

► **Theorem 3.** *ILP (W) can be solved with a randomized XP time algorithm in terms of h . More specifically, it can be solved in time $\tilde{O}(\|c\|_\infty \log^2 \|c\|_\infty \cdot \Delta^h \log^5 \Delta) \cdot n^{10+h} \cdot \mathcal{O}(\Delta h m)^{8h+h^2}$.*

In this paper, the running time is measured in the number of arithmetic operations on numbers with encoding length polynomial in the encoding length of the instance. The \tilde{O} hides polylogarithmic factors of the encoding length of the instance.

Camerini, Galbiati and Maffioli [10] already observed in 1992 that one can solve constrained flow, circulation and matching problems in randomized, pseudo-polynomial time. Theorem 3 differs in that it reveals the tractability of a generalized class of problems by removing the exponential dependency on the encoding length of b that would appear in a purely pseudo-polynomial algorithm.

The randomized XP time algorithm is obtained through the use of a Graver augmentation framework, which has been effective in obtaining FPT algorithms for two-stage stochastic and n -fold ILP with small coefficients [17]. For those ILPs, the Graver complexity is bounded by a function of the parameters, resulting in FPT algorithms, whereas the XP time dependence on h of Theorem 3 is the result of ILP (W) having a large Graver complexity and proximity. We provide upper and lower bounds for these respective quantities. See Sections 2.2 and 4. The latter section also reveals that ILP (T) has a similarly large Graver complexity even for binary matrices T , which motivates the use of the different technique.

We provide a $W[1]$ -hardness result in terms of the number of additional constraints h to match the randomized XP time algorithm in a complexity theoretic sense. The hardness persists even for binary constrained perfect matching on a restricted class of graphs. Theorem 4 additionally shows that parameterizing with Δ in addition to the number of complicating constraints h is unlikely to result in an FPT algorithm.

► **Theorem 4.** *Solving ILP (W) encoded in unary is $W[1]$ -hard parameterized by h . This hardness persists when restricting to M being the incidence matrix of a graph which is the disjoint union of even length cycles and $W \in \{0, 1\}^{h \times n}$, $c = \mathbf{0}$, $l = \mathbf{0}$, $u = \mathbf{1}$, $b = \mathbf{1}$.*

We like to conclude the introduction with noting that recent methodology used by Eisenbrand and Rothvoss [18] can be used to derive a similar results to Theorem 2 in a different way. This work was carried out independently to ours.

2 Technical Overview

This section gives a technical overview of our results. It aims at presenting the main ideas and new concepts. Therefore, details are omitted.

We first preprocess an ILP instance in Section 2.1, after which both algorithms are covered in Sections 2.2 and 2.3. The $W[1]$ -hardness of Theorem 4 is additionally discussed in the latter section. Section 4 provides lower bounds on the Graver complexity and proximity.

2.1 Reduction to perfect b -matching

Before presenting the key components of the algorithms that solve the generalized matching problem with additional variables or constraints, we first discuss a reduction that modifies ILPs (T) and (W) in a favorable way. By doing so, it suffices to give algorithms for such restricted instances, see Sections 2.2 and 2.3.

In particular, we show that ILPs (T) and (W) can, in polynomial time, be transformed to equivalent ILPs where the generalized matching submatrix is transformed to the constraint matrix of the perfect b -matching problem. Such matrix M is a binary matrix with unique columns that each have precisely two nonzero entries, and is the incidence matrix of a simple graph $G(M)$, the *constraint graph*, with vertices $[m]$. Since M is the incidence matrix of $G(M)$, we may identify a variable $i \in [n]$ with the edge of $G(M)$ connecting the vertices j_1, j_2 such that $M_{i,j_1} = M_{i,j_2} = 1$. Now, a perfect b -matching x of $G(M)$ is a solution to the generalized matching ILP with constraint matrix M , nonnegative integral variables $x \in \mathbb{Z}_{\geq 0}^n$, and right-hand-side b . That is, x assigns a value to every edge such that for every vertex i the sum of x_j over the incident edges j is exactly b_i .

The used reduction is an extension of the reduction in [45] from generalized matching to b -matching. However, we show in Lemma 5 that we can also keep track of the additional general variables and constraints.

► **Lemma 5.** *An ILP of the form*

$$\min \left\{ a^\top y + c^\top x \mid \begin{pmatrix} C & W \\ T & M \end{pmatrix} (y, x) = \begin{pmatrix} d \\ b \end{pmatrix}, e \leq y \leq g, l \leq x \leq u, (y, x) \in \mathbb{Z}^{p+n} \right\},$$

for which $\|M\|_1 \leq 2$ and the variable bounds are finite, can be reduced in input+output-linear time to an instance of the same form that additionally satisfies

- $M' \in \{0, 1\}^{m' \times n'}$ is the incidence matrix of a simple graph,
- $c' \geq 0, W' \in \mathbb{Z}_{\geq 0}^{h' \times n'}$,
- $e', l' = 0, \|g'\|_1 = \mathcal{O}(\|g - e\|_1), u' = \infty$,
- $n', m' = \mathcal{O}(n + m)$,
- $h' = h, p' = p$,
- $\|d'\|_1 = \mathcal{O}(\|d\|_1 + \|A\|_1(\|e\|_1 + \|l\|_1))$,
- $\|b'\|_1 = \mathcal{O}(\|u - l\|_1)$ if $p = 0$ and $\|b'\|_1 = \mathcal{O}(\|b\|_1 + \|g - e\|_1 + \|u - l\|_1 + \|A\|_1(\|e\|_1 + \|l\|_1))$ otherwise,
- a', c', C', T', W' contain the same collections of entries as a, c, C, T, W up to sign changes, the insertion of zeros and the insertion of at most one binary row to T .

Here, parameters with primes are used to denote the parameters of the new instance and $u = \infty$ denotes the absence of upper bounds on x . The proof transforms the constraint matrix and variable bounds to the intended form by creating additional constraints and variables. It consists of steps that individually treat columns with negative entries, columns

with 1-norm strictly less than 2 and finite variable upper bounds. Most of these obstacles are circumvented by conceptually subdividing edges or adding redundant constraints. See [36] for the proof.

By virtue of Lemma 5, we can now restrict ourselves to finding algorithms for ILPs (T) and (W) which represent perfect b -matching problems with additional variables or constraints.

2.2 Few arbitrary variables

We now focus on solving ILP (T). We first motivate the main idea underlying the algorithm and present the algorithm itself, before we give an outline of remaining parts. The key idea is to treat the backdoor variables y and the matching variables x separately. Let us first consider the problem of determining the feasibility of ILP (T). In essence, we replace the constraint $Ty + Mx = b$ and matching variables x with a polyhedral constraint of the form $b - Ty \in P$ for some polyhedron P that models that there must exist a $x \in \mathbb{Z}_{\geq 0}^n$ such that $Mx = b - Ty$, in other words, there must exist a perfect $(b - Ty)$ -matching in $G(M)$. To construct such polyhedral constraint on y , the first idea may be to use the convex hull of all points $z \in \mathbb{Z}^m$ such that there exists a perfect z -matching—unfortunately, a naive execution of this strategy fails as when $G(M) = K_3$, the complete graph on 3 vertices, we have that $z = (0, 0, 0)$, $z = (2, 2, 2)$ and the convex combination $z = (1, 1, 1)$ are in the hull, but the latter does not admit a perfect z -matching. Hence, we attempt to model a discrete set that is not lattice-convex on \mathbb{Z}^3 with a convex constraint, which is impossible.

To contrast this, the work by Cslovjcek et al. [12] shows that, even for general matrices M , one can work around this issue by restricting y to a fixed remainder $r \in \mathbb{Z}^m$ modulo some large integer B . That is, one can construct a polyhedron P_r such that for all $y \in B\mathbb{Z}^m + r$, we have that $\{Mx = b - Ty, x \in \mathbb{Z}_{\geq 0}^n\}$ is feasible if and only if $b - Ty \in P_r$. Note that through an affine transformation, this may equivalently be posed as a polyhedral constraint directly on y . Their choice of B may grow with the dimensions of M , which calls for a more problem specific approach to solve ILP (T) in FPT time. When M is the incidence matrix of a simple graph, we show that we may restrict the modulus B to be the constant 2, which paves the way for an FPT algorithm. In addition, we show that one may optimize over a linear objective function by encoding the objective contribution of the matching part of the ILP in an additional dimension of P_r . The required convexity property to make this work translates to a lattice-convexity property, see Lemma 7, of the function $f_{c,M}$ from Definition 6.

► **Definition 6.** Let $f_{c,M} : \mathbb{Z}^m \rightarrow \mathbb{Z} \cup \{\infty\}$ be defined by

$$f_{c,M}(z) = \min\{c^\top x \mid Mx = z, x \in \mathbb{Z}_{\geq 0}^n\}.$$

That is, $f_{c,M}(z)$ is the cost of a minimum c -cost perfect z -matching of $G(M)$ or ∞ if $G(M)$ has no perfect z -matching.

► **Lemma 7.** Let $z^{(1)}, z^{(2)}, \dots, z^{(\ell)} \equiv r \pmod{2}$ be given and $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)} \geq 0$ be real convex multipliers, i.e., $\sum_{k \in [\ell]} \lambda^{(k)} = 1$, such that $z := \sum_{k \in [\ell]} \lambda^{(k)} z^{(k)} \equiv r \pmod{2}$. Then $f_{c,M}(z) \leq \sum_{k \in [\ell]} \lambda^{(k)} f_{c,M}(z^{(k)})$.

We use modular congruence on vectors to denote component-wise modular congruence. Next, we show in which way Lemma 7 allows us to reformulate ILP (T). First, we perform a binary search on the objective to obtain the problem of finding a feasible point of ILP (T) subject to an additional linear constraint $a^\top y + c^\top x \leq \omega^*$. We guess the remainder vector t of y in an optimal solution modulo 2. It then suffices to search for an integer point y restricted to $y \equiv t \pmod{2}$ in the box $e \leq y \leq g$ subject to the constraint that there is a

perfect $(b - Ty)$ -matching with cost at most $\omega^* - a^\top y$. This is equivalent to finding a point in the set

$$\{y \in \mathbb{Z}^p : y \equiv t \pmod{2}, e \leq y \leq g, f_{c,M}(b - Ty) \leq \omega^* - a^\top y\}.$$

In particular, as the parity of $b - Ty \equiv r \pmod{2}$ is constant for $y \equiv t \pmod{2}$, Lemma 7 shows that there exists a convex set P_r such that the constraint $f_{c,M}(b - Ty) \leq \omega^* - a^\top y$ may be replaced with $(\omega^* - a^\top y, b - Ty) \in P_r$. When we obtain a good representation of P_r , we use the algorithm by Reis and Rothvoss [44] to solve the integer program in FPT time. After guessing all 2^p parity vectors t , an optimal solution to ILP (T) is found.

We now discuss the missing ingredients needed to make the previously discussed algorithm work. First, we explain how Lemma 7 is derived. To do so, we consider jump systems, where degree sequences of graphs and functions such as $f_{c,M}$ have been studied exhaustively [8, 39, 40, 35]. Murota [40] observed that $f_{c,M}$ describes a jump M-convex function, which is a valuated generalization of jump systems. In fact, recent work on the general factor problem [35] reveals that a similar jump M-convex function defined for weighted graph factorizations has additional exploitable properties. Informally, small steps, as in Definition 8, connecting points in the effective domain of such function may be rearranged in any order. For this reason, Kobayashi defines strongly base orderable (SBO) jump systems and a valuated extension. Our function $f_{c,M}$ defined over weighted uncapacitated perfect b -matchings can be seen to also satisfy the properties of Definition 9. As Lemma 7 holds for general SBO jump M-convex functions, we discuss our proof in terms of this abstract property. We use \sqsubseteq to denote the conformal partial order defined by $x \sqsubseteq y$ if and only if $|x_i| \leq |y_i|$ and $x_i y_i \geq 0$ for all i .

► **Definition 8.** A 2-step decomposition of a vector $d \in \mathbb{Z}^m$ is a multiset of steps $p^{(1)}, \dots, p^{(\ell)} \in \mathbb{Z}^m$, satisfying $\|p^{(k)}\|_1 = 2$ and $p^{(k)} \sqsubseteq d$ for all $k \in [\ell]$, such that $d = \sum_{k \in [\ell]} p^{(k)}$.

► **Definition 9.** A function $f: \mathbb{Z}^m \rightarrow \mathbb{Q} \cup \{\infty\}$ is SBO jump M-convex if for every two points $z^{(1)}, z^{(2)}$ in the domain $\{z \in \mathbb{Z}^m : f(z) < \infty\}$ there exist a 2-step decomposition $p^{(1)}, \dots, p^{(\ell)}$ of $z^{(2)} - z^{(1)}$ and real numbers $g^{(1)}, \dots, g^{(\ell)}$ such that

- $f(z^{(2)}) = f(z^{(1)}) + \sum_{i \in [\ell]} g^{(i)}$,
- for all $I \subseteq [\ell]$ it holds that $f(z^{(1)} + \sum_{k \in I} p^{(k)}) \leq f(z^{(1)}) + \sum_{k \in I} g^{(k)}$.

The alternating path argument employed by Kobayashi [35] to show that 1-capacitated perfect b -matchings satisfy Definition 9 can be adapted to show the SBO jump M-convexity of $f_{c,M}$. For this, we use a known gadget construction that relates b -matchings to perfect matchings [45]. See [36] for the full proof.

Lemma 7 is derived by first showing that a 2-step decomposition of a difference $z^{(2)} - z^{(1)}$ can be used to construct a small even step $\sum_{k \in I} p^{(k)} \in \{-2, 0, 2\}^m$ by combining some of the steps. Such an even, small step can be used to modify a pair of points $z^{(2)}, z^{(1)}$ closer to each other while remaining on the lattice $2\mathbb{Z}^m + r$ and without increasing the sum of the function values on these points. Such pairwise modifications may then be exhaustively performed to move the points appearing in a convex combination towards the target z as in Lemma 7 and obtain the required result. The proof of Lemma 7 is provided in Section 3.

The final ingredient needed to implement the FPT algorithm for ILP (T) is a good characterization of a suitable polyhedron P_r that models the matching part of the ILP. We obtain this by letting P_r be the convex hull of the vectors $(\omega, z) \in \mathbb{R} \times (2\mathbb{Z}^m + r)$ for which $f_{c,M}(z) \leq \omega$ and z is in some bounding box. That is, P_r is the epigraph of a natural convex

extension of $f_{c,M}$ restricted to a bounding box. Since proximity arguments allow us to assume that e and g are finite, it is possible to bound z , see [11]. We obtain a polynomial time separation oracle for P_r through the ellipsoid method [27] and by noting that P_r can efficiently be optimized over³. This follows from the fact that weights associated with a vertex i may be transferred to the edge weights of the incident edges, which shows that optimizing over $(\omega, z) \in P_r$ corresponds to solving a minimum cost parity constrained b -matching problem. The parity constraints can straightforwardly be translated into constraints compatible with the polynomially solvable generalized matching problem [45, 15].

2.3 Few arbitrary constraints

The randomized XP time algorithm for solving ILP (W) is obtained through a series of reductions. As a key intermediate step, we will employ a pseudo-polynomial reduction in [45] that transforms the minimum cost perfect b -matching to the minimum cost perfect matching problem on a graph with a total of $\|b\|_1$ vertices. In order to prevent an exponential blow-up in terms of the encoding size of b , we split the solution process up into parts in which b may be bounded through the use of the Graver augmentation framework [17]. This is an exact local search framework which iteratively improves a primal solution to an ILP by taking smaller steps solving a similar, but slightly easier ILP. In these easier ILPs, the variable domains may be bounded by a function of the Graver complexity of the constraint matrix of the ILP. By Lemma 5, bounds on the variable domains translate into bounds on the right-hand side b . Before providing the required bounds, we first introduce the Graver augmentation framework and relevant definitions.

► **Definition 10.** *The Graver basis $\mathcal{G}(A) \subseteq \mathbb{Z}^n \setminus \{0\}$ of an integer matrix $A \in \mathbb{Z}^{m \times n}$ is the set of conformally minimal nonzero integral kernel elements of A . That is, $g \in \mathcal{G}(A)$ if and only if $g \neq 0$, $Ag = 0$ and there is no $g' \notin \{0, g\}$ such that $Ag' = 0$ and $g' \sqsubseteq g$.*

The Graver complexity of A refers to norm bounds on $\mathcal{G}(A)$, and we are interested in $g_\infty(A) = \max\{\|g\|_\infty \mid g \in \mathcal{G}(A)\}$ and $g_1(A) = \max\{\|g\|_1 \mid g \in \mathcal{G}(A)\}$, referring to the ∞ -norm and 1-norm respectively. An oracle that can compute a single small improving step, used in the local search framework, is called a Graver-best oracle.

► **Definition 11.** *A Graver-best oracle is an oracle that, given a constraint matrix A , objective c , variable bounds l and u , finds a Graver-best step. That is, it finds an integral solution x to $\{Ax = 0, l \leq x \leq u\}$ such that*

$$c^\top x \leq \min \{c^\top g \mid l \leq g \leq u, g \in \mathcal{G}(A)\}.$$

Such an oracle can be implemented by solving an ILP over variable domains of size at most $\mathcal{O}(g_\infty(A))$, which we show to be polynomially bounded for fixed h for ILP (W). In this case, we may employ Lemma 5 to find that implementing a Graver-best oracle can be done by finding a minimum cost constrained b -matching where $\|b\|_1 \leq n\|b\|_\infty = \mathcal{O}(ng_\infty(A))$ is polynomially bounded. Theorem 12 concludes with how this Graver-best oracle can be used to optimize an ILP.

► **Theorem 12** (Lemma 12 in [17], Lemma 5 in [16]). *Given a feasible solution x to ILP (G) with finite l and u , an optimal solution to the ILP can be found with $\mathcal{O}(n \log(\|l - u\|_\infty) \log(c^\top x - c^\top x^*))$ queries of a Graver-best oracle. Here, $c^\top x^*$ is the value of an optimal solution.*

³ We note that it may be possible to derive a combinatorial separation algorithm, but this is not needed to establish the fixed-parameter tractability of ILP (T). For a closely related separation problem, see [46].

By constructing an auxiliary ILP with a trivial feasible solution to an ILP of the form (W) as done in [30], we can use Theorem 12 to find a feasible solution as well as optimize such solution. For this reason, we will assume that we are given a feasible solution to ILP (W).

The polynomial bound on $g_\infty(A)$ for fixed h follows from literature: Berndt, Mnich and Stamm [7] show that the Graver basis elements of a matrix M with $\|M\|_1 \leq 2$ are small.

► **Theorem 13** (Theorem 13 in [7]). *Let $M \in \mathbb{Z}^{m \times n}$ be an integer matrix with $\|M\|_1 \leq 2$. Then $g_\infty(M) \leq 2$ and $g_1(M) \leq 2m + 1$.*

Note that we may assume that $m = \mathcal{O}(n)$ after preprocessing. Lemma 3 in [16], which proves a Graver complexity bound for n -fold ILPs, reveals that Theorem 13 suffices to yield a polynomial bound on the Graver complexity of the composite matrix appearing in ILP (W).

► **Corollary 14.**

$$g_\infty \left(\begin{pmatrix} W \\ M \end{pmatrix} \right) \leq 2(2\Delta h(2m + 1) + 1)^h = \mathcal{O}(\Delta h m)^h.$$

The number of Graver-best steps that need to be computed in Theorem 12 can be polynomially bounded by computing a solution to the LP relaxation of ILP (W) and using Theorem 3.14 in [29], which shows that the proximity of an ILP is bounded by n times the obtained Graver bound from Corollary 14.

► **Corollary 15.** *Let y^* be an optimal solution to the LP relaxation of ILP (W). If the ILP is feasible, then there exists an optimal ILP solution x^* with $\|y^* - x^*\|_\infty \leq n \cdot \mathcal{O}(\Delta h m)^h$.*

Thus, we may restrict the search of an optimal solution to ILP (W) to a polynomially bounded domain, which results in the needed bound on $c^\top x - c^\top x^*$ to apply Theorem 12. Now, after applying the pseudo-polynomial reduction to a perfect matching problem [45], which is compatible with additional constraints as noted in [10], we find that solving ILP (W) can be polynomially reduced to finding a minimum cost constrained perfect matching in a graph with polynomially many vertices. As all variables in this resulting problem are binary, we can condense all h constraints into a single constraint by encoding the constraints in base- B for sufficiently large B . This generalizes the reduction steps used in [6, 43] and only increases the coefficient size of the constraint matrix polynomially for fixed h . The resulting constrained perfect matching problem can then be solved with a randomized pseudo-polynomial algorithm such as the one by Mulmuley, Vazirani and Vazirani [38]. Combining all steps and binary searching on the values of the variables in an optimal solution yields the randomized XP time algorithm from Theorem 3.

To the best of our knowledge, the complexity of constrained matching where the objective is encoded in binary is still open. It is worth noting that the reduction chain shows that ILP (W) is polynomially equivalent to the exact matching problem, which aims to find a perfect matching in a graph with exactly a given target weight. Finding a deterministic pseudo-polynomial algorithm for this problem has been a central and intensively studied open problem for over four decades.

Finally, to complement the XP time complexity of the algorithm from Theorem 3, we reveal the W[1]-hardness of solving ILP (W) parameterized by h in Theorem 4. This result is obtained through a parameterized reduction from the ILP problem, which is strongly W[1]-hard when parameterized by the number of constraints [13].

► **Theorem 16** (Theorem 22 in [13]). *Determining the feasibility of the system*

$$\{Wx = d : x \in \mathbb{Z}_{\geq 0}^n\},$$

where $W \in \mathbb{Z}_{\geq 0}^{h \times n}$ is encoded in unary and $d \in \{0, 1\}^h$, is W[1]-hard parameterized by h . This hardness persists when restricting x to $0 \leq x \leq u$ where u is encoded in unary.

The essence of the reduction is to split every variable x_i into u_i many binary variables. The resulting binary variables can then be duplicated and linked through graph cycles to reduce the unary sized coefficients in W and obtain a binary constraint matrix.

3 Lattice convexity of SBO jump M-convex functions on $2\mathbb{Z}^m + r$

As the lattice-convexity of $f_{c,M}$ plays a key role in the correctness of the algorithm discussed in Section 2.2, we focus on the explicit proof of the lattice-convexity of SBO jump M-convex functions in this extended abstract. The proof is split into two steps. Lemma 17 shows how two points in a convex combination can be pairwise modified without increasing the overall sum of function values. This operation is then exhaustively employed in the proof of the final lattice-convexity result.

► **Lemma 17.** *Let f be an SBO jump M-convex function. Let $z^{(1)}, z^{(2)} \equiv r \pmod{2}$ and let $i^* \in [m]$ be so that $z_{i^*}^{(1)} - z_{i^*}^{(2)} \geq 2$. Then, there exist $z^{(1)'}$ and $z^{(2)'}$ such that*

- $z^{(1)'}, z^{(2)'} \equiv r \pmod{2}$,
- $z^{(1)'} + z^{(2)'} = z^{(1)} + z^{(2)}$,
- $z_{i^*}^{(1)'} = z_{i^*}^{(1)} - 2, z_{i^*}^{(2)'} = z_{i^*}^{(2)} + 2$,
- $z_i^{(1)} = z_i^{(2)} \implies z_i^{(1)} = z_i^{(1)'} = z_i^{(2)'} = z_i^{(2)}$ for all $i \in [m]$,
- $f(z^{(1)'}) + f(z^{(2)'}) \leq f(z^{(1)}) + f(z^{(2)})$.

Proof. W.l.o.g., consider the case where $z^{(1)} \geq z^{(2)}$. The general case is equivalent up to changing signs. Consider a 2-step decomposition $p^{(1)}, \dots, p^{(\ell)}$ of $z^{(2)} - z^{(1)}$ and $g \in \mathbb{R}^\ell$ from Definition 9. We create a multigraph representation G of this decomposition with vertex set $[m]$. For every step $p^{(k)}$ that has a -1 component at coordinates i_1 and i_2 , add an edge that connects i_1 and i_2 . For a step $p^{(k)}$ for which $p_i^{(k)} = -2$, add a self-loop on i . Since each vertex in G has an even degree and i^* is not an isolated vertex by assumption, we can pick an arbitrary cycle that contains i^* . Let $I \subseteq [\ell]$ be the indices of the steps corresponding to the edges of the cycle, which defines an even step from $z^{(1)}$ to $z^{(2)}$. We define $z^{(1)'} = z^{(1)} + \sum_{k \in I} p^{(k)}$ and $z^{(2)'} = z^{(2)} - \sum_{k \in I} p^{(k)}$. Note that the step $\sum_{k \in I} p^{(k)}$ is in $\{-2, 0\}^m$ because the degree of each vertex in the cycle is 0 or 2. In particular, the degree of i^* is exactly 2. We conclude that the first, second and third properties are satisfied. The satisfaction of the fourth property follows from that $p_i^{(k)} = 0$ when $z_i^{(1)} = z_i^{(2)}$. Finally, observe that $f(z^{(1)'}) + f(z^{(2)'}) \leq f(z^{(1)}) + \sum_{k \in I} g^{(k)} + f(z^{(1)}) + \sum_{k \in [\ell] \setminus I} g^{(k)} = f(z^{(1)}) + f(z^{(2)})$, showing that the fifth requirement is satisfied. ◀

We can now exhaustively employ Lemma 17 pairwise on vectors of a given convex combination to derive the lattice-convexity of f . Lemma 18 generalizes Lemma 7.

► **Lemma 18.** *Let f be an SBO jump M-convex function. Let $z^{(1)}, z^{(2)}, \dots, z^{(\ell)} \equiv r \pmod{2}$ be given and $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)} \geq 0$ be real convex multipliers, i.e., $\sum_{k \in [\ell]} \lambda^{(k)} = 1$, such that $z := \sum_{k \in [\ell]} \lambda^{(k)} z^{(k)} \equiv r \pmod{2}$. Then $f(z) \leq \sum_{k \in [\ell]} \lambda^{(k)} f(z^{(k)})$.*

Proof. Since we may assume that λ is rational, we may additionally assume that $\lambda_k = 1/\ell$ after duplicating vectors of the convex combination a suitable number of times.

For every $i^* \in [m]$ we exhaustively employ the following procedure: if there exist $z^{(k_1)}, z^{(k_2)}$ with $z_{i^*}^{(k_1)} - z_{i^*}^{(k_2)} \leq -2$ and $z_{i^*}^{(k_2)} - z_{i^*}^{(k_1)} \geq 2$, apply Lemma 17 to replace $z^{(k_1)}$ and $z^{(k_2)}$ with the resulting $z^{(k_1)'}$ and $z^{(k_2)'}$ from this lemma. This preserves the total sum $\sum_{k \in [\ell]} z^{(k)} = \ell z$ and guarantees that $\sum_{k \in [\ell]} f(z^{(k)})$ does not increase. Observe that if such $z^{(k_1)}$ and $z^{(k_2)}$ do

not exist, w.l.o.g. all $z_{i^*}^{(k)} \geq z_{i^*}$. As z_{i^*} is the average of $z_{i^*}^{(k)}$, it must hold that $z_{i^*} = z_{i^*}^{(k)}$. Since the application of Lemma 17 does not modify any $z_i^{(k)}$ at i for which $z_i^{(k)} = z_i$ for all $k \in [\ell]$, exhaustive application will eventually result in $z^{(k)'} = z$.

It is left to show that Lemma 17 can be exhaustively applied. To see this, we employ $\sum_{k \in [\ell]} |z_{i^*} - z_{i^*}^{(k)}|$ as progress measure. With respect to the progress made after applying Lemma 17, it holds that

$$\begin{aligned} & \sum_{k \in [\ell]} |z_{i^*} - z_{i^*}^{(k)'}| - \sum_{k \in [\ell]} |z_{i^*} - z_{i^*}^{(k)}| \\ &= |z_{i^*} - z_{i^*}^{(k_1)'}| + |z_{i^*} - z_{i^*}^{(k_2)'}| - |z_{i^*} - z_{i^*}^{(k_1)}| + |z_{i^*} - z_{i^*}^{(k_2)}| \\ &< 0, \end{aligned}$$

which follows from Lemma 17 that guarantees that $z_{i^*}^{(k_1)}$ and $z_{i^*}^{(k_2)}$ both move strictly closer to z_{i^*} . After applying Lemma 17 exhaustively for all $i^* \in [m]$, we end up with a situation where $z^{(k)'} = z$ for all k and that $\ell f(z) = \sum_{k \in [\ell]} f(z^{(k)'}) \leq \sum_{k \in [\ell]} f(z^{(k)})$. ◀

We note that the lattice-convexity of the domain $\{z \in \mathbb{Z}^m : f_{c,M}(z) < \infty\}$ on $2\mathbb{Z}^m + r$ immediately follows from a generalization of Tutte's characterization of graphs with perfect matchings, see Corollary 31.1a in [45]. Alternatively, it is a special case of Theorem 1.2 in [1].

4 Lower bounds

In this section, we provide lower bounds for the Graver complexity of the coefficient matrices of ILPs (T) and (W) and a lower bound for the proximity of ILP (W). These results are obtained through explicit constructions of constraint matrices.

First, we observe that it is unlikely that a Graver augmentation procedure along the lines of Theorem 12 will be effective at solving ILP (T), even in the case that the coefficients in the block T are bounded by a constant.

► **Proposition 19.** *For any fixed positive integer p , there exists an infinite family of binary constraint matrices $(T \ M) \in \{0, 1\}^{m \times (p+n)}$ of ILP (T) with Graver basis elements that have an ∞ -norm of order $\Omega(n)^p$.*

Proof. Consider the matrix

$$(T \ M) = \left(\begin{array}{cccccc|cccccc|cccc} 1 & 0 & 0 & \cdots & 0 & 0 & I & O & O & \cdots & O & O & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & O & I & O & \cdots & O & O & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & O & O & I & \cdots & O & O & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & O & O & O & \cdots & I & O & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & O & O & O & \cdots & O & I & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \cdots & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \cdots & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \cdots & \mathbf{0}^\top & \mathbf{0}^\top & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \cdots & \mathbf{1}^\top & \mathbf{0}^\top & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \cdots & \mathbf{0}^\top & \mathbf{1}^\top & 1 \end{array} \right)$$

where the vectors and submatrices that comprise the matrix have dimension k for a given $k \in \mathbb{Z}_{\geq 1}$. Here \mathbf{O} denotes the all-zero matrix and I the identity matrix. By construction, any nonzero kernel element must have that the absolute value of the last entry is k^p times the absolute value of the first entry. Such nonzero kernel element also exists. Therefore, there must be Graver basis elements with ∞ -norm at least k^p . We conclude with $n = pk + 1$, i.e., $k = (n - 1)/p$. ◀

We now turn to the Graver complexity of the constraint matrix of ILP (W). In this context, Proposition 20 exhibits a construction showing that Corollary 14 is asymptotically tight when h is constant.

► **Proposition 20.** *For any fixed positive integer h , there exists an infinite family of constraint matrices $\begin{pmatrix} W \\ M \end{pmatrix} \in \{0, 1, \Delta\}^{(h+m) \times n}$ of ILP (W) with Graver basis elements that have an ∞ -norm of order $\Omega(\Delta m)^h$.*

Proof. We construct a matrix that essentially matches the lower bound example

$$\begin{pmatrix} \Delta' & -1 & & & \\ & \Delta' & & & \\ & & \ddots & -1 & \\ & & & \Delta' & -1 \end{pmatrix}$$

in [31] for some large Δ' by using smaller coefficients of size $\Delta \leq \Delta' = \Delta k$ in the matrix W for some fixed $k \in \mathbb{Z}_{\geq 1}$. First, define the auxiliary $k \times k$ block U_z for $z \in \{0, 1\}$ by

$$U_z = \begin{pmatrix} -z & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & \ddots & \\ & & & \ddots & \ddots \\ & & & & 1 & -1 \end{pmatrix} \quad (1)$$

which will ensure that the variables associated with this block have equal value. Then construct the constraint matrix

$$\begin{pmatrix} W \\ M \end{pmatrix} = \begin{pmatrix} \Delta \mathbf{1}^\top & -1 & & & \\ & \Delta \mathbf{1}^\top & -1 & & \\ & & \ddots & \ddots & \\ & & & \Delta \mathbf{1}^\top & -1 \\ \hline U_0 & & & & \\ & e_1 & U_1 & & \\ & & & e_1 & \ddots \\ & & & & \ddots & \ddots \\ & & & & & e_1 & U_1 \end{pmatrix}, \quad (2)$$

where e_1 denotes the unit vector $(1, 0, \dots, 0) \in \mathbb{Z}^k$.

Again, all kernel elements have last entry with absolute value equal to $(\Delta k)^h$ times the first entry. A nonzero kernel element with first entry 1 exists. Since $m = hk$, we obtain $k = m/h$ and the stated lower bound. Note that the constructed matrices use elements from $\{-1, 0, 1, \Delta\}$. However, they can be adapted to use only elements from $\{0, 1, \Delta\}$ using the ideas from Lemma 5, yielding the same asymptotic lower bound. ◀

A similar construction in Proposition 21 shows that Corollary 15 is tight for constant h and $n = \Theta(m)$. We note that a conjecture posed by Berndt, Mnich and Stamm involving upper bounds of the proximity of ILPs and their relation to the Graver-complexity [7] hints at that the upper bound from Corollary 14 may be able to be improved to $m \cdot \mathcal{O}(\Delta hm)^h$, at least for the case where $u = \infty$.

► **Proposition 21.** *For any fixed positive integer h , there exists an infinite family of constraint matrices $\binom{W}{M} \in \{0, 1, \Delta\}^{(h+m) \times n}$ of ILP (W) and vertex LP relaxation solutions to the corresponding ILP (W) such that the nearest integral solution is at ∞ -norm distance $\Omega(m) \cdot \Omega(\Delta m)^h$.*

Proof. To show this, we use a large part of the matrix in Equation (2) and replace one block with the incidence matrix of a gadget graph. For this let $k \in \mathbb{Z}_{\geq 1}$ and construct the b -matching instance shown in Figure 2. Write the gadget perfect b -matching problem from Figure 2 as the ILP $\tilde{M}x = \tilde{b}$ for nonnegative integral x . Denote $\tilde{M} = (\tilde{M}_1 \quad \tilde{M}_2) \in \{0, 1\}^{12k \times (-1+15k)}$ where the columns of \tilde{M}_2 correspond to the variables y_1, \dots, y_k , which may be assigned value 0 in a vertex LP relaxation solution, but are forced to take value k in an integral solution. Finally, construct the system

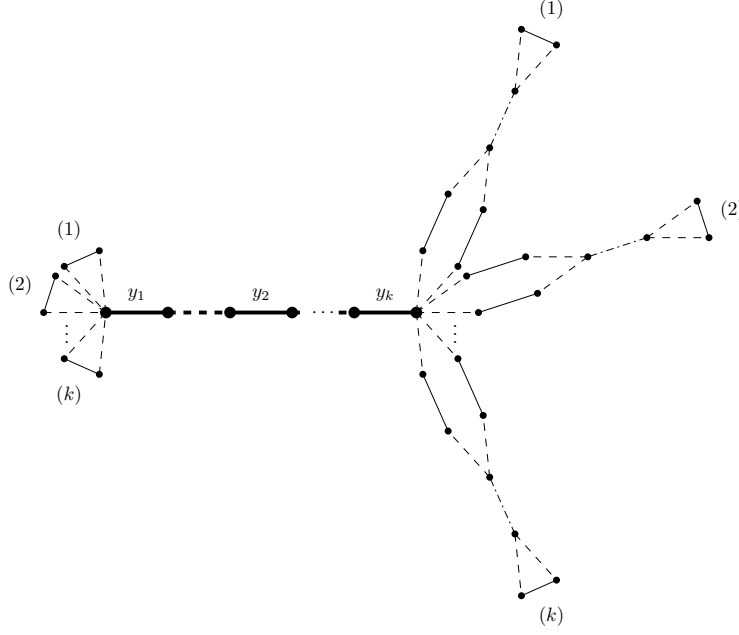
$$\left(\begin{array}{ccccccc} & \Delta \mathbf{1}^\top & -1 & & & & \\ & & & \Delta \mathbf{1}^\top & -1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \Delta \mathbf{1}^\top & -1 \\ \hline \tilde{M}_1 & \tilde{M}_2 & & & & & \\ & & e_1 & U_1 & & & \\ & & & & e_1 & \ddots & \\ & & & & & \ddots & \ddots \\ & & & & & & e_1 & U_1 \end{array} \right) x = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{b} \\ \mathbf{0} \\ \vdots \\ \vdots \\ \mathbf{0} \end{pmatrix}, \quad x \geq \mathbf{0}$$

using the $k \times k$ auxiliary block U_1 from Equation (1). A vertex fractional solution exists with the last component of x being zero, whereas the only integral solution must assign $k \cdot (\Delta k)^h$ to this variable. We have that $m = 12k + (h-1)k$, i.e., $k = \Omega(m)$. The lower bound follows. The -1 entries can be eliminated through the use of Lemma 5. ◀

We note that constant multiplicative factors of $1/p$ and $1/h$ in the lower bounds of Propositions 19–21 are hidden in the Ω .

5 Concluding notes and open problems

We study the complexity of ILP parameterized by the size of variable or constraint backdoors to the generalized matching problem, called p and h respectively. In particular, we show that solving ILPs is in FPT when parameterized by p . We study the convexity of degree sequences in the light of SBO jump M-convex functions to express the non-backdoor variables efficiently as polyhedral constraints. Additionally, we present a randomized XP time algorithm to solve ILPs for fixed h when the objective and constraint matrix are encoded in unary. This algorithm employs a Graver augmentation procedure to reduce the problem to the exact matching problem. Finally, we match this latter result by showing that ILP is W[1]-hard parameterized by h .



■ **Figure 2** A b -matching gadget showing that the 1-norm proximity for b -matching can be of order $\Omega(|V|^2)$. All vertices have $b_v = 1$ except for the vertices incident to y_j , which have $b_v = k$. The only integral solution assigns a value of zero to the dashed edges, a value of k to the edges y_j and a value of 1 to the other edges. On the other hand, a vertex fractional solution exists which assigns a value of zero to the full thick and dash-dotted edges, a value of k to the dashed thick edges, and $\frac{1}{2}$ to all other edges. The size of the gadget is given by $|V| = 12k$ and $|E| = -1 + 15k$.

Having studied the case where $p > 0, h = 0$ and where $p = 0, h > 0$, a natural open question is whether ILP can still be solved in polynomial time when it is a generalized matching problem with a fixed number of both additional variables and constraints. This can be seen as an analogue of 4-block n -fold ILP in the context of block structured ILPs, which are known to be solvable in slice-wise polynomial time [28]. Since the restricted case of $p = 0, h > 0$ in the context of generalized matching is W[1]-hard in terms of h , the question is whether $p > 0, h > 0$ also admits a randomized XP time algorithm when c, A are encoded in unary. We note that providing a polynomial Graver-complexity upper bound for ILP (T) suffices to obtain this result. In such case, the Graver-complexity of the entire constraint matrix will also be polynomially bounded and a Graver-best step can be computed by brute forcing the values of the p arbitrary variables, after which a problem of ILP (W) can be solved efficiently.

In addition, aside from the long-standing open question of whether the exact matching problem is in P, a natural open question coming from Theorem 3 is whether the exponential dependence on the encoding length of c is necessary. To answer this question positively, it suffices to restrict to 0/1-budgeted perfect matching with its objective in encoded binary. See [36].

Finally, we suspect that some intermediate steps used in obtaining Theorem 2 can be refined. That is, we suspect that Lemma 18 holds for general jump M-convex functions. In addition, it may be possible to make a combinatorial separation algorithm for $P_{r,U}$ in a more direct fashion as done in [46].

References

- 1 Richard P. Anstee and Yunsun Nam. Convexity of degree sequences. *Journal of Graph Theory*, 30(2):147–156, 1999. doi:10.1002/(SICI)1097-0118(199902)30:2<147::AID-JGT8>3E3;0.CO;2-C.
- 2 Manuel Aprile, Samuel Fiorini, Gwenaél Joret, Stefan Kober, Mihal T. Seweryn, Stefan Weltge, and Yelena Yuditsky. Integer programs with nearly totally unimodular matrices: the cographic case. In *Proceedings of the 2025 ACM-SIAM Symposium on Discrete Algorithms, SODA*, pages 2301–2312, 2025. doi:10.1137/1.9781611978322.76.
- 3 Egon Balas and William R. Pulleyblank. The perfectly matchable subgraph polytope of a bipartite graph. *Networks*, 13(4):495–516, 1983. doi:10.1002/net.3230130405.
- 4 Max Bannach, Sebastian Berndt, Marten Maack, Matthias Mnich, Alexandra Lassota, Malin Rau, and Malte Skambath. Solving packing problems with few small items using rainbow matchings. In *Proceedings of the 45th International Symposium on Mathematical Foundations of Computer Science, MFCS*, pages 11:1–11:14, 2020. doi:10.4230/LIPIcs.MFCS.2020.11.
- 5 John Bartholdi, Craig A Tovey, and Michael A Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare*, 6(2):157–165, 1989. doi:10.1007/BF00303169.
- 6 André Berger, Vincenzo Bonifaci, Fabrizio Grandoni, and Guido Schäfer. Budgeted matching and budgeted matroid intersection via the gasoline puzzle. *Mathematical Programming*, 128(1-2):355–372, 2011. doi:10.1007/s10107-009-0307-4.
- 7 Sebastian Berndt, Matthias Mnich, and Tobias Stamm. New support size bounds and proximity bounds for integer linear programming. In *Proceedings of the 49th International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM*, pages 82–95, 2024. doi:10.1007/978-3-031-52113-3_6.
- 8 André Bouchet and William H. Cunningham. Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics*, 8(1):17–32, 1995. doi:10.1137/S0895480191222926.
- 9 Marcin Brianski, Martin Koutecký, Daniel Král’, Kristýna Pekárková, and Felix Schröder. Characterization of matrices with bounded graver bases and depth parameters and applications to integer programming. *Mathematical Programming*, 208(1-2):497–531, 2024. doi:10.1007/s10107-023-02048-x.
- 10 Paolo M. Camerini, Giulia Galbiati, and Francesco Maffioli. Random pseudo-polynomial algorithms for exact matroid problems. *Journal of Algorithms*, 13(2):258–273, 1992. doi:10.1016/0196-6774(92)90018-8.
- 11 William J. Cook, Albertus M. H. Gerards, Alexander Schrijver, and Éva Tardos. Sensitivity theorems in integer linear programming. *Mathematical Programming*, 34(3):251–264, 1986. doi:10.1007/BF01582230.
- 12 Jana Cslovjcek, Martin Koutecký, Alexandra Lassota, Michal Pilipczuk, and Adam Polak. Parameterized algorithms for block-structured integer programs with large entries. In *Proceedings of the 2024 ACM-SIAM Symposium on Discrete Algorithms, SODA*, pages 740–751, 2024. doi:10.1137/1.9781611977912.29.
- 13 Pavel Dvořák, Eduard Eiben, Robert Ganian, Dusan Knop, and Sebastian Ordyniak. The complexity landscape of decompositional parameters for ILP: programs with few global variables and constraints. *Artificial Intelligence*, 300:103561, 2021. doi:10.1016/j.artint.2021.103561.
- 14 Jack Edmonds and Ellis L. Johnson. Matching: A well-solved class of integer linear programs. In *Proceedings of the Calgary International Conference on Combinatorial Structures and Their Applications*, pages 89–92, 1970. doi:10.1007/3-540-36478-1_3.
- 15 Jack Edmonds and Ellis L. Johnson. Matching, Euler tours and the Chinese postman. *Mathematical Programming*, 5(1):88–124, 1973. doi:10.1007/BF01580113.
- 16 Friedrich Eisenbrand, Christoph Hunkenschröder, and Kim-Manuel Klein. Faster algorithms for integer programs with block structure. In *Proceedings of the 45th International Colloquium*

- on Automata, Languages, and Programming, ICALP, pages 49:1–49:13, 2018. doi:10.4230/LIPIcs.ICALP.2018.49.
- 17 Friedrich Eisenbrand, Christoph Hunkenschröder, Kim-Manuel Klein, Martin Koutecký, Asaf Levin, and Shmuel Onn. An algorithmic theory of integer programming, 2019. arXiv:1904.01361.
 - 18 Friedrich Eisenbrand and Thomas Rothvoss. A parameterized linear formulation of the integer hull, 2025. doi:10.48550/arXiv.2501.02347.
 - 19 T. Ermolieva, Y Ermoliev, P. Havlik, A. Lessa-Derci-Augustynczik, N. Komendantova, T. Kahil, J. Balkovic, R. Skalsky, C. Folberth, P. S. Knopov, and G. Wang. Connections between robust statistical estimation, robust decision-making with two-stage stochastic optimization, and robust machine learning problems. *Cybernetics and Systems Analysis*, 59(3):385–397, 2023. doi:10.1007/s10559-023-00573-3.
 - 20 Michael R. Fellows, Daniel Lokshtanov, Neeldhara Misra, Frances A. Rosamond, and Saket Saurabh. Graph layout problems parameterized by vertex cover. In *Proceedings of the 19th International Symposium on Algorithms and Computation, ISAAC*, pages 294–305, 2008. doi:10.1007/978-3-540-92182-0_28.
 - 21 Jirí Fiala, Tomas Gavenciak, Dusan Knop, Martin Koutecký, and Jan Kratochvíl. Parameterized complexity of distance labeling and uniform channel assignment problems. *Discrete Applied Mathematics*, 248:46–55, 2018. doi:10.1016/j.dam.2017.02.010.
 - 22 Samuel Fiorini, Gwenaél Joret, Stefan Weltge, and Yelena Yuditsky. Integer programs with bounded subdeterminants and two nonzeros per row. In *Proceedings of the 62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS*, pages 13–24, 2021. doi:10.1109/FOCS52979.2021.00011.
 - 23 Serge Gaspers and Stefan Szeider. Backdoors to satisfaction. In *The Multivariate Algorithmic Revolution and Beyond*, pages 287–317. Springer, 2012. doi:10.1007/978-3-642-30891-8_15.
 - 24 Tomáš Gavenciak, Martin Koutecký, and Dušan Knop. Integer programming in parameterized complexity: Five miniatures. *Discrete Optimization*, 44(Part 1):100596, 2022. doi:10.1016/j.disopt.2020.100596.
 - 25 Michel X. Goemans and Thomas Rothvoss. Polynomiality for bin packing with a constant number of item types. *Journal of the ACM*, 67(6):38:1–38:21, 2020. doi:10.1145/3421750.
 - 26 Jens Gramm, Rolf Niedermeier, and Peter Rossmanith. Fixed-parameter algorithms for CLOSEST STRING and related problems. *Algorithmica*, 37(1):25–42, 2003. doi:10.1007/s00453-003-1028-3.
 - 27 Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer, 1988. doi:10.1007/978-3-642-97881-4.
 - 28 Raymond Hemmecke, Matthias Köppe, and Robert Weismantel. A polynomial-time algorithm for optimizing over N -fold 4-block decomposable integer programs. In *Proceedings of the 14th International Conference on Integer Programming and Combinatorial Optimization, IPCO*, pages 219–229, 2010. doi:10.1007/978-3-642-13036-6_17.
 - 29 Raymond Hemmecke, Matthias Köppe, and Robert Weismantel. Graver basis and proximity techniques for block-structured separable convex integer minimization problems. *Mathematical Programming*, 145(1-2):1–18, 2014. doi:10.1007/s10107-013-0638-z.
 - 30 Raymond Hemmecke, Shmuel Onn, and Lyubov Romanchuk. n -Fold integer programming in cubic time. *Mathematical Programming*, 137(1-2):325–341, 2013. doi:10.1007/s10107-011-0490-y.
 - 31 Christoph Hunkenschröder, Kim-Manuel Klein, Martin Koutecký, Alexandra Lassota, and Asaf Levin. Tight lower bounds for block-structured integer programs. In *Proceedings of the 25th International Conference on Integer Programming and Combinatorial Optimization, IPCO*, pages 224–237, 2024. doi:10.1007/978-3-031-59835-7_17.

- 32 Klaus Jansen, Kim-Manuel Klein, Marten Maack, and Malin Rau. Empowering the configuration-ip: new PTAS results for scheduling with setup times. *Math. Program.*, 195(1):367–401, 2022. doi:10.1007/s10107-021-01694-3.
- 33 Hendrik W. Lenstra Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983. doi:10.1287/moor.8.4.538.
- 34 Dusan Knop, Martin Koutecký, and Matthias Mnich. Voting and bribing in single-exponential time. *ACM Transactions on Economics and Computation*, 8(3):12:1–12:28, 2020. doi:10.1145/3396855.
- 35 Yusuke Kobayashi. Optimal general factor problem and jump system intersection. In *Proceedings of the 24th International Conference on Integer Programming and Combinatorial Optimization, IPCO*, pages 291–305, 2023. doi:10.1007/978-3-031-32726-1_21.
- 36 Alexandra Lassota and Koen Ligthart. Parameterized algorithms for matching integer programs with additional rows and columns, 2025. doi:10.48550/arXiv.2503.05548.
- 37 László Lovász. The factorization of graphs. II. *Acta Mathematica Academiae Scientiarum Hungarica*, 23(1-2):223–246, 1972. doi:10.1007/BF01889919.
- 38 Ketan Mulmuley, Umesh V. Vazirani, and Vijay V. Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7(1):105–113, 1987. doi:10.1007/BF02579206.
- 39 Kazuo Murota. M-convex functions on jump systems: A general framework for minsquare graph factor problem. *SIAM Journal on Discrete Mathematics*, 20(1):213–226, 2006. doi:10.1137/040618710.
- 40 Kazuo Murota and Ken’ichiro Tanaka. A steepest descent algorithm for M-convex functions on jump systems. *IEICE Transactions on Fundamentals of Electronics Communications and Computer Sciences*, E89-A(5):1160–1165, 2006. doi:10.1093/ietfec/e89-a.5.1160.
- 41 Martin Nägele, Richard Santiago, and Rico Zenklusen. Congruency-constrained TU problems beyond the bimodular case. *Mathematics of Operations Research*, 49(3):1303–1348, 2024. doi:10.1287/moor.2023.1381.
- 42 Rolf Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006. doi:10.1093/ACPROF:OSO/9780198566076.001.0001.
- 43 Christos H. Papadimitriou and Mihalis Yannakakis. The complexity of restricted spanning tree problems. *Journal of the ACM*, 29(2):285–309, 1982. doi:10.1145/322307.322309.
- 44 Victor Reis and Thomas Rothvoss. The subspace flatness conjecture and faster integer programming. In *Proceedings of the 64th IEEE Annual Symposium on Foundations of Computer Science, FOCS*, pages 974–988, 2023. doi:10.1109/FOCS57990.2023.00060.
- 45 Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24 of *Algorithms and Combinatorics*. Springer, 2003. URL: <https://link.springer.com/book/9783540443896>.
- 46 Fan Zhang. A separation algorithm for b-matching degree-sequence polyhedra. *Mathematics of Operations Research*, 28(1):92–102, 2003. doi:10.1287/moor.28.1.92.14263.