# Tiling Random Regular Graphs Efficiently

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## Abstract

We show that for every $\epsilon > 0$ there exists a sufficiently large $d_0 \in \mathbb{N}$ such that for every $d \geq d_0$, **whp** the random $d$-regular graph $G(n,d)$ contains a $T$-factor for every tree $T$ on at most $(1-\epsilon)d/\log d$ vertices. This is best possible since, for large enough integer $d$, **whp** $G(n,d)$ does not contain a $\frac{(1+\epsilon)d}{\log d}$-star-factor. Our method gives a randomised algorithm which **whp** finds said $T$-factor and whose expected running time is $O(n^{1+o(1)})$, as well as an efficient deterministic counterpart.

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## 1 Introduction

Let $G$ be an $n$-vertex graph and $H$ be an $s$-vertex graph. An *$H$-factor* in $G$ is a union of $\lfloor \frac{n}{s} \rfloor$ vertex-disjoint isomorphic copies of $H$ in $G$.

There has been an extensive study into the threshold of appearance of $H$-factors in the binomial random graph $G(n,p)$. The case where $H = K_2$ corresponds to finding a perfect matching in $G(n,p)$. The sharp threshold for appearence of a perfect matching was established by Erdős and Rényi [10]. Early results for general $H$ were obtained by Alon and Yuster [2] and Ruciński [26]; they determined the threshold up to a constant factor for a specific family of graphs and gave bounds for the general case. For the case where $H$ is a tree, Łuczak and Ruciński [19] characterised "pendant" structures, and proved that in the random graph process (that is, when edges are added one after the other uniformly at random), the hitting time of the appearance of an $H$-factor is the same as the hitting time of the disappearance of these forbidden "pendant" structures. In particular, one is able to infer the precise threshold for the appearance of an $H$-factor in this case. In 2008, Johansson, Kahn and Vu [16] determined the threshold (up to a multiplicative constant) for the existence of an $H$-factor for every strictly-1-balanced[1] graph $H$ and determined the threshold up to a logarithmic factor for an arbitrary graph $H$. In the case of cliques

[1] A graph $H$ is strictly-1-balanced if $\frac{|E(H)|}{|V(H)|-1} > \frac{|E(J)|}{|V(J)|-1}$ for every proper subgraph $J \subsetneq H$ with $|V(J)| \geq 2$.


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$K_s$, Heckel (for $s = 3$) [12] and Riordan (for $s \geq 4$) [24] determined the *sharp* threshold for the appearance of a $K_s$-factor. Recently, a hitting time result for the appearance of a $K_s$-factor was proved [13], and the sharp threshold for the appearance of an $H$-factor for every strictly-1-balanced graph $H$ was determined [6].

Much less is known in the case of *random d-regular graphs*. The random $d$-regular graph $G(n, d)$ is a graph chosen uniformly at random among all simple $d$-regular graphs on the vertex set $[\![n]\!] := \{1, \ldots, n\}$ (throughout the paper, we treat $d$ as fixed and consider the asymptotics in $n$). Since, for every pair of integers $d \geq 2$ and $k \geq 3$, the number of cycles of length $k$ in $G(n, d)$ is asymptotically distributed as a Poisson random variable with mean $(d-1)^k/(2k)$ (see [28]), **whp**[2] there are $o(n)$ cycles of length $k$ in $G(n, d)$. Thus, we may (and will) restrict our attention to tree factors.

For the case of $H = K_2$, Bollobás [4] proved that whp there exists an $H$-factor (that is, a perfect matching) in $G(n, d)$ for every $d \geq 3$. There has been some research on the more general case of stars. Naturally, one cannot hope for a $K_{1,t}$-factor for $t > d$, since the graph is $d$-regular. For $d \geq 3$, using a first moment argument, one can show that **whp** $G(n, d)$ does not contain a $K_{1,d}$-factor (see [3, Corollary 2]). Robinson and Wormald [25] showed that for $d \geq 3$, **whp** $G(n, d)$ contains a Hamilton cycle, and thus a $K_{1,2}$-factor. In a subsequent work, Assiyatun and Wormald [3] showed that for $d \geq 4$, **whp** $G(n, d)$ contains a $K_{1,3}$-factor. One may then suspect that for any $d \geq 3$, typically $G(n, d)$ contains a $K_{1,d-1}$-factor. However, using first moment calculations, one can show that this is not the case for $d \geq 5$.

There are then two natural avenues to venture into: first, for sufficiently large $d$, to determine all $k$ such that **whp** $G(n, d)$ contains a factor of stars on $k$ vertices; second, more ambitiously, one could try to find all trees $T$ for which **whp** $G(n, d)$ contains a $T$-factor.

Considering a related but slightly different problem, Alon and Wormald [1] showed that for any $d$-regular graph $G$, there exists an absolute constant $c'$, such that $G$ contains a star-factor, in which every star has at least $c'd/\log d$ vertices (not necessarily all of the same size). They further noted that this is optimal up the choice of the constant $c' > 0$. Indeed, the existence of a factor of stars on at least $k$ vertices implies the existence of a dominating set of size at most $\frac{n}{k-1}$, and for any $\epsilon > 0$ and sufficiently large $d$, **whp** the smallest dominating set in $G(n, d)$ is of size at least $\frac{(1-\epsilon)n \log d}{d}$ (see [1, page 3]). Let us note here that if one only assumes that $G$ is $d$-regular, then one cannot hope to obtain a factor of stars of size exactly $k$ for any $3 \leq k = O(d/\log d)$. Indeed, consider for example a $d$-regular graph $G$ formed by a collection of vertex disjoint copies of $K_{d+1}$ and vertex disjoint copies of complete bipartite graphs $K_{d,d}$. Then, since $\gcd(d + 1, 2d) \in \{1, 2\}$, for any choice of $k > 2$, one cannot find a factor of stars of size $k$.

Our first main result shows that typically a random $d$-regular graph $G$ contains a star-factor with the asymptotically *optimal* possible size. In fact, we extend this result to factors of *any* tree (not necessarily a star).

▶ **Theorem 1.** *For every constant $0 < \epsilon < 1$, there exists a sufficiently large integer $d_0$ such that the following holds for any $d \geq d_0$. **Whp**, for every tree $T$ on at most $\frac{(1-\epsilon)d}{\log d}$ vertices, there exists a $T$-factor in $G(n, d)$.*

We note that throughout the paper, we will assume that $|V(G)|$ is divisible by $|V(T)|$, to avoid unnecessary technical details, however all proofs can be directly extended to the general case. Further, we note that we may fix the tree $T$ and show that **whp** there is a $T$-factor in $G(n, d)$; since there are at most $d^2 \cdot 4^d$ such trees (see [22]) and $d$ is fixed, by the union bound the statement then holds for every tree $T$.

[2] With high probability, that is, with probability tending to one as $n$ tends to infinity.

A detailed sketch of the proof of Theorem 1 is presented in Section 2. Let us briefly recap the main strategy here. We show that **whp**, there exists a balanced partition of $|V(G)|$ into $|V(T)|$ parts so that, for every pair of parts $V_i, V_j$ where $\{i,j\} \in E(T)$, every vertex in $V_i \cup V_j$ has the number of neighbours in the other part concentrated around the mean $d/|V(T)|$. We say that such a partition is *nice*. We obtain this nice partition through four different applications of the algorithmic version of the Lovász Local Lemma, due to Moser and Tardos [21]. In particular, this allows us to find such a nice partition which is *close* to a random partition; further, this gives us a randomised algorithm to find this partition whose average running time is $\tilde{O}(n)$ **whp** (see Theorem 4 and Corollary 5). In fact, we show that, in *any* $d$-regular graph, which does not have short cycles close to each other, a fraction of the possible partitions are close to nice partitions. Utilising a description of the distribution of edges in random graphs with specified degree sequences ([11], see also [20, Theorem 2.2]), we conclude that **whp** almost all partitions of a random regular graph induce multipartite graphs with good expansion properties. This allows us to find a partition with such expansion properties which is also close to a nice partition, ensuring the existence of a perfect matching between pairs of sets. Using an algorithm as in [7], we can find these perfect matchings in time $n^{1+o(1)}$. This gives us our second main result.

▶ **Theorem 2.** *For every constant $0 < \epsilon < 1$, there exists a sufficiently large integer $d_0$ such that the following holds for any $d \geq d_0$. There is a randomised algorithm that **whp** finds a $T$-factor in $G(n,d)$ in expected time $n^{1+o(1)}$, for every tree $T$ on at most $\frac{(1-\epsilon)d}{\log d}$ vertices.*

Since the events that we consider in our applications of the algorithmic version of the Lovász Local Lemma are determined by poly$(d)$ random variables over domains of size at most $d$, [21, Theorem 1.4] shows that there exists a *deterministic* algorithm that **whp** finds these $T$-factors in polynomial in $n$ time in $G(n,d)$.

Let us make some additional remarks.

- It is not hard to verify that our proof follows through for a uniformly chosen graph on $n$ vertices with a given degree sequence, whose degrees lie in the interval $[d,(1+\delta)d]$ for some small $\delta > 0$. We believe slight modifications of our technique, specifically in Section 5, should allow us to obtain the same result for such graphs whose degrees are between $d$ and $O(d)$.
- We stress that in order to show the existence of a perfect matching between relevant sets in the partition, we need our partition to be close to a random partition, and thus the application of the algorithmic version of the Lovász Local Lemma is crucial, even if we do not aim to get Theorem 2.

A possible simplification, which allows using a non-constructive version of the Lovász Local Lemma, is applying "sprinkling" due to the contiguity result from [15] instead of applying the direct estimation of probabilities in $G(n,d)$. As soon as a nice partition is obtained, we add independently edges of $G(n,\epsilon' d)$, where $\epsilon' \ll \epsilon$. Although it simplifies the proof, it does not allow deriving Theorem 2 and the generalisation to non-regular random graphs with specified degree sequences. Moreover, this does not allow obtaining any probability bounds, in contrast to our approach. Indeed, our proof gives that the probability a random $d$-regular graph has a $T$-factor (for any tree $T$ with $|V(T)| \leq (1-\epsilon)d/\log d$) is at least $1 - n^{-\Theta_d(1)}$. In fact, the latter probability bound is tight. Indeed, consider the vertices $\{1, ..., 10d\} \in [\![n]\!]$, say. The probability they form a connected component without a $T$-factor in $G(n,d)$ is at least $n^{-100d^2}$. We thus obtain the following corollary.

▶ **Corollary 3.** *For every constant $0 < \epsilon < 1$, there exists a sufficiently large integer $d_0$ such that the following holds for any $d \geq d_0$. For any tree $T$ on at most $\frac{(1-\epsilon)d}{\log d}$ vertices, the probability that $G(n,d)$ contains a $T$-factor is $1 - n^{-\Theta_d(1)}$.*

One key complication that arises when using any variant of the Lovász Local Lemma to prove Theorem 1 is that it is impossible to directly apply it, as every "bad" event has too many dependencies. A similar issue was addressed independently in the paper by Draganić and Krivelevich [9] on connected dominating sets, where they proposed a (significantly different and shorter) proof strategy to show that a $d$-regular graph without short close cycles has a nice partition. Notably, their method requires $\Theta(n)$ applications of the Lovász Local Lemma (and consequently $O(n^2)$ resamples in the algorithmic version), which precludes a linear time reduction to finding perfect matchings. In contrast, our approach applies the Lovász Local Lemma only a constant number of times, enabling such a reduction.

Let us finish this section with several avenues for future research. While in this general setting Theorem 1 is asymptotically best possible, there are specific cases (such as when $T$ is a path [25], or in fact any $T$ of bounded degree [23, 14]) where one can **whp** obtain a $T$-factor for trees of size significantly larger than $\frac{d}{\log d}$ in $G(n,d)$. It would be interesting to try characterising for every value of $k = k(n)$ families of trees $T$ on $k$ vertices for which one can **whp** obtain a $T$-factor in $G(n,d)$. Another possible direction would be to consider $(n,d,\lambda)$-graphs, with $\lambda \ll d$, instead of $G(n,d)$ – see [17] for background, and [18, 23, 14] for related results on spanning subgraphs in such graphs that, in particular, imply the existence of $T$-factors for trees $T$ of bounded maximum degree.

## 1.1 Organisation

In Section 1.2 we set out some notation which will be of use throughout the paper. We then discuss the proof's structure and strategy in Section 2. In Section 3 we set out how we will utilise the algorithmic version of Lovász Local Lemma. Section 4 is devoted to the key proposition (Proposition 6), and is perhaps the most involved and novel part of the paper. Finally, in Section 5 we present two typical properties of $G(n,d)$ and show how to deduce Theorem 1 from these properties and Proposition 6. All missing proofs appear in the full version on ArXiv [8].

## 1.2 Notation

Given a graph $H$, a vertex $v \in V(H)$, and a set $A \subseteq V(H)$, we denote by $d_H(v)$ the degree of $v$ and by $d_H(v,A)$ the number of neighbours of $v$ in $A$ (in $H$). When the graph in question is clear we may omit the subscript. We write $d(A) = \sum_{v\in A} d(v)$. Given $A, B \subseteq V(H)$, we denote by $e(A,B)$ the number of edges with one endpoint in $A$ and the other endpoint in $B$, and by $e(A)$ the number of edges with both endpoints in $A$ (each edge counted only once). We denote by $N(A,B)$ the neighbourhood of $A$ in $B$, that is, the set of vertices in $B$ which are adjacent to some vertex in $A$. All logarithms are with the natural base. Moreover, for every positive integer $n$, define $[\![n]\!] \coloneqq \{1, 2, \dots, n\}$. We use the fairly standard notation that given sequences $a = (a_n)$ and $b = (b_n) \geq 0$, $a = o(b)$ if, for every $\varepsilon > 0$, $|a_n| \leq \varepsilon b_n$ for all large enough $n$. Given sequences $a' = (a'_d = a'_d(n))$ and $b' = (b'_d = b'_d(n)) \geq 0$, we sometimes also use $a = o_d(b)$ to say that, for every $\varepsilon > 0$, $|a'_d| \leq \varepsilon b'_d$ for all large enough $d$ and $n$. We systematically ignore rounding signs when it does not affect computations.

# 2 Proof outline

Unsurprisingly, finding a tree factor is much harder when the size of the tree is close to the optimal size (that is, $d/\log d$). In this section, we will present the proof outline for Theorem 1 in the case when $k \geq \frac{d}{10\log d}$. We will further point out the steps where the proof becomes simpler for trees of smaller size.

Let $T$ be a tree on $k$ vertices and let us label these vertices by $V(T) := [\![k]\!]$. The overall strategy for finding a $T$-factor in $G \sim G(n,d)$ is quite intuitive. We will find $k$ disjoint sets $V_1, \dots, V_k \subseteq V(G)$ of equal size and show that **whp** there exists a perfect matching between every $V_i$ and $V_j$ such that $\{i,j\} \in E(T)$. To do so, our proof proceeds in two main steps. In the first step, we find "good" sets $V_1, \dots, V_k$ (in fact, we show such a partition typically exists in any $d$-regular graph $G$ without two short cycles close to each other). In the second step, we show the typical existence of a perfect matching between every relevant pair of these sets. The properties achieved in the first step facilitate the execution of the second step.

The first step of the proof, presented in Section 4, is perhaps the most involved and novel part. In this step, we establish key properties of the sets $V_1, \dots, V_k$ which will be crucial in verifying the typical existence of perfect matchings in the second step. First, we show that for every $\{i,j\} \in E(T)$, the degree of every $v \in V_i$ into $V_j$ is around $d/k$. Notice that, this property alone does not suffice to establish the existence of a perfect matching between $V_i$ and $V_j$. To that end, we will also make sure that the sets $V_1, \dots, V_k$ are "close" to uniformly chosen sets. More specifically, we will require that a large proportion of each set satisfies a few typical properties.

In the second step (Section 5), we show that **whp**, for every $\{i,j\} \in E(T)$, there exists a perfect matching between $V_i$ and $V_j$ by showing that Hall's condition is satisfied. That is, we will show that **whp**, for every $W \subseteq V_i$, we have $|N(W,V_j)| \geq |W|$. To that end, we utilise a useful bound given by Gao and Ohapkin (see [11, Corollary 8], and also [20]). First, we will show that **whp** for every two "small" sets $U, W \subseteq V(G)$ with $|U| = |W|$, there are not too many edges going from $U$ to $W$. Moreover, since the sets $V_1, \dots, V_k$ were constructed in the first step in a way such that the degree of every vertex $v \in V_i$ to appropriate $V_j$'s is not too small, we obtain a lower bound on $e(W,V_j)$ for every $W \subseteq V_i$. In particular, if $|N(W,V_j)| < |W|$, we will get a contradiction for small sets $W \subseteq V_i$. In the same spirit, [11, Corollary 8] together with the properties of the sets $V_1, \dots, V_k$ allows us to bound $|N(W,V_j)|$ for every "large" $W \subseteq V_i$ *if* the sets $V_1, \dots, V_k$ were chosen uniformly at random. Indeed, given randomly chosen disjoint sets $A$ and $B$ (that is, sets formed without first exposing $G(n,d)$), the graph $G[A,B]$, given its degree sequence, has a uniform distribution. Our first step ensures that these sets behave similarly to uniformly chosen sets.

Let us return to the first step of the proof and describe the strategy of showing the typical existence of the "good" sets $V_1, \dots, V_k$. We begin with a random partition of the vertices into $k$ parts, $S_1, \dots, S_k$. A key tool in establishing the existence of such sets $V_1, \dots, V_k$ is the Lovász Local Lemma. Since we want our sets to be close to the initial random sets $S_1, \dots, S_k$ (so that we may later be able to apply [11, Corollary 8], we will in fact utilise the algorithmic version of the Lovász Local Lemma, due to Moser and Tardos (see Theorem 4 in Section 3). Utilising the algorithmic version of the Lovász Local Lemma, we can show that in each step of the algorithm we resample a small number of random variables assigned to vertices, and thus the initial random sets $S_1, \dots, S_k$ will not be "far" from $V_1, \dots, V_k$. Let us note here that when applying the algorithmic version of the Lovász Local Lemma, in the initial step of the algorithm one evaluates all "bad" events (requires $\tilde{O}(n)$ time), and then, at each "resampling" step, one re-evaluates only those $O(1) = O_d(1)$ events that depend on the resampled random variables. Since the expected number of steps of the algorithm of Moser and Tardos is $O(n)$, this gives the overall expected running time $\tilde{O}(n)$.

Now, for every vertex $v \in V(G)$, denote by $X_v$ a uniform random variable on $[\![k]\!]$. For every $i \in [\![k]\!]$, set $S_i := \{v \in V(G) \colon X_v = i\}$. Moreover, for every vertex $v \in V(G)$, denote by $B_v$ the event that there exists $\{i,j\} \in E(T)$ such that $v \in S_i$ and $d(v,S_j) \notin [\delta d/k, Cd/k]$ where $\delta > 0$ is a sufficiently small constant and $C > 0$ is a sufficiently large constant. Notice that if $\neg B_v$ occurs for every $v \in V(G)$, then we get the desired bounds on the degrees which is the first key point in the first step.

Note that, for every vertex $v \in V(G)$ and $j \in [\![k]\!]$, we have $d(v, S_j) \sim \text{Bin}(d, 1/k)$. This distribution is the heart of the obstacle concerning "large" trees. The reason for it is that whenever $k \leq \frac{d}{10 \log d}$, then the probability that the degree of $v$ into $S_j$, for some $j \in [\![k]\!]$, is not in the interval $[\delta d/k, Cd/k]$ is at most $d^{-8}$. Furthermore, the event $B_v$ is determined by $d+1$ random variables $X_u$, and $B_v$ is independent of all but at most $d^2$ other events $B_u$. Therefore, we can apply the Lovász Local Lemma. It is worth noting here that if $k \leq \frac{d}{10 \log d}$, we may omit the requirement that $\{i, j\} \in E(T)$ in the definition of $B_v$. Then, if for every vertex $v \in V(G)$ we have that $\neg B_v$ holds, then $d(v, S_j) \in [\delta d/k, Cd/k]$ for every vertex $v \in V(G)$ and index $j \in [\![k]\!]$.

However, as $k$ gets closer to $\frac{(1-\epsilon)d}{\log d}$, the probability that $\text{Bin}(d, 1/k) < \delta d/k$ is not smaller than $d^{-1-\epsilon'}$, for some $\epsilon' > 0$ tending to zero as $\epsilon$ tends to zero. Thus, the treatment of this case is much more delicate and involves several rounds of applications of the algorithmic version of the Lovász Local Lemma, in order to refine the initial random partition. In the rest of this section, we describe these rounds.

Notice that, for every $i \in [\![k]\!]$, the event $B_v$ conditioned on $v \in S_i$ is more likely to occur as the degree of the $i$-th vertex in $T$ gets larger. For this reason, we will treat vertices of small degree and vertices of large degree in $T$ differently (this treatment is in Section 4.1). Assume that $[\![h]\!]$ is the set of vertices of $T$ with "large" degrees. We slightly decrease the probability that $X_v = i$, for every $i \in [\![h]\!]$. Then, after one application of the Lovász Local Lemma we will be able to get rid of vertices which have more than $Cd/k$ neighbours into $S_j$, for some $j \in [\![h]\!]$. Next, we consider the neighbourhood of the vertices which have degree less than $\delta d/k$ into $S_j$, for some $j \in [\![h]\!]$. We "resample" the vertices in this neighbourhood outside of $S_1, \ldots, S_h$ into $S_1, \ldots, S_h$, that is, we move them into one of the sets $S_1, \ldots, S_h$ uniformly at random. In this way, once again using the Lovász Local Lemma, we will have that $d(v, S_j) \in [\delta d/k, Cd/k]$ for every $v \in V(G)$ and $j \in [\![h]\!]$.

Next, in Section 4.2, we partition the remaining vertices among $S_{h+1}, \ldots, S_k$. In this third application of the Lovász Local Lemma, we will ensure that after the resampling we have the property that, for every vertex $v \in V(G)$ and for all but at most $\epsilon^{-2}$ indices $j \in [\![k]\!]$, we have $d(v, S_j) \in [\delta d/k, Cd/k]$. At this point, there will still be vertices $v \in S_i$ which have less than $\delta d/k$ neighbours into some $S_j$ where $\{i, j\} \in E(T)$. After the fourth application of the Lovász Local Lemma, we will be able to obtain a partition with no such vertices (that is, $d(v, S_j) \in [\delta d/k, Cd/k]$ for every $v \in S_i$ and $\{i, j\} \in E(T)$).

Finally, in Section 4.4, we adjust the sets $S_1, \ldots, S_k$ to be of size $n/k$ each. This is the purpose of the fifth and final round of the Lovász Local Lemma. In this round, we will move vertices from sets of size bigger than $\frac{n}{k}$ to sets of size smaller than $\frac{n}{k}$ in a random manner. We will do so while ensuring the vertices $v$ we move satisfy that $d(v, S_j) \in [\delta d/k, Cd/k]$ for every $j \in [\![k]\!]$. After this round, while keeping the bounds over the degrees of the vertices, we will be able to make each set $S_i$ to be close to $n/k$ up to and additive $n/d^{100}$ error term. Finally, to make the sets exactly of size $\frac{n}{k}$, we introduce a deterministic argument adjusting the sets $S_1, \ldots, S_k$ while changing the degree of every vertex to every set $S_i$ by at most one. We thus obtain the required sets $V_1, \ldots, V_k$.

## 3 Algorithmic version of Lovász Local Lemma

We will make extensive use of the algorithmic version of the Lovász Local Lemma, due to Moser and Tardos [21].

▶ **Theorem 4** (Theorem 1.2 of [21], rephrased). *Let $U$ be a finite set. Let $X = (\xi_u)_{u \in U}$ be a tuple of mutually independent random variables. Let $\mathcal{F}$ be a finite set of events determined by $X$. Suppose that there exists $q$ such that for every event $F \in \mathcal{F}$, $\mathbb{P}_X(F) \leq q$. Moreover,* *suppose that every $F \in \mathcal{F}$ depends on at most $\Delta$ other events $F' \in \mathcal{F}$. Suppose that $\beta \in (0,1)$ satisfies $q \le \beta(1-\beta)^{\Delta}$. Then, there exists an evaluation of $X$ which does not satisfy any event in $\mathcal{F}$.*

*Furthermore, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of mutually independent copies of $X$, that is, for every $n \in \mathbb{N}$, $X_n \sim X$. Initially, we sample $X_0$ and let $Z_0 \coloneqq X_0$. At step $t \ge 1$, we pick one $F \in \mathcal{F}$ satisfied by $Z_{t-1}$ (if one exists) in an arbitrary manner. Then, we consider all $u \in U$ such that this witness $F$ depends on the $u$-th coordinate of $Z_{t-1}$, and set $(Z_t)_u = (X_t)_u$ for all such $u$ and $(Z_t)_u = (Z_{t-1})_u$ for all other $u$. If no such $F \in \mathcal{F}$ exists, the process halts and we set $\tau \coloneqq t$. Then, $\mathbb{E}[\tau] \le |\mathcal{F}|\frac{\beta}{1-\beta}$.*

In fact, we will utilise the following corollary, whose proof appears in [8].

▶ **Corollary 5.** *Let $U$ be a finite set. Let $m \in \mathbb{N}$. Let $X = (\xi_u)_{u\in U}$ be a set of mutually independent random variables, supported on $[\![m]\!]$. Let $S_1,\dots,S_m$ be a partition of $U$ satisfying $S_i = \{u \in U \colon \xi_u = i\}$ for every $i \in [\![m]\!]$. Let $\mathcal{F}$ be a finite set of events determined by $S_1,\dots,S_m$. Suppose that there exists $q$ such that $\mathbb{P}_X(F) \le q$ for every event $F \in \mathcal{F}$. Moreover, suppose that every $F \in \mathcal{F}$ is determined by at most $\Delta_1$ random variables $\xi_u$, and depends on at most $\Delta_2$ other events $F' \in \mathcal{F}$. Furthermore, suppose that $\beta \in (0,1)$ satisfies that $q \le \beta(1-\beta)^{\Delta_2}$. Then, the probability (under the measure of $X$) that there exists a partition of $U$ into $U_1,\dots,U_m$ which does not satisfy any event in $\mathcal{F}$ and $\left|S_i \triangle U_i\right| \le 2\Delta_1|\mathcal{F}|\frac{\beta}{1-\beta}$ for every $i \in [\![m]\!]$, is at least $\frac{1}{2}$.*

## 4 Planting the seeds

As mentioned in Section 2, the proof of Theorem 1 consists of two main steps. In order to find a $T$-factor in $G(n,d)$, we will partition the vertices of $G(n,d)$ into $|V(T)|$ sets of the same size, each set represents a different vertex in the tree $T$, and find a perfect matching between the $i$-th set and the $j$-th set for every $\{i,j\} \in E(T)$. In this section, we prove the first step of the proof. That is, we find a partition of the vertices into $|V(T)|$ sets which satisfies two crucial properties, which in turn will allow us to find the desired perfect matchings in the second step in Section 5.

For every pair of integers $d$ and $n$, let $\mathcal{G}_d$ be the family of all $d$-regular graphs on $n$ vertices, such that there are no two cycles of length at most 10 at distance less than 10 from each other. Recall that we treat $d$ as fixed, and consider the asymptotic as $n \to \infty$. Note that **whp** $G(n,d) \in \mathcal{G}_d$ (see, for example, [27]).

The main result of this section, which is the first step in the proof of Theorem 1, is the following.

▶ **Proposition 6.** *For every $\epsilon > 0$, there exist a sufficiently small constant $\delta \coloneqq \delta(\epsilon) > 0$, a sufficiently large constant $C \coloneqq C(\epsilon) > 0$, and a sufficiently large integer $d_0$ such that the following holds for any $d \ge d_0$.*

*Let $T$ be a tree on $k \le (1-\epsilon)\frac{d}{\log d}$ vertices. Let $G \in \mathcal{G}_d$ and suppose that $n$ is divisible by $k$. Let $S_1,\dots,S_k$ be a uniformly random partition of $V(G)$: $\mathbb{P}(v \in S_i) = \frac{1}{k}$ for every $i \in [\![k]\!]$, $v \in V(G)$ and the random choice of $i \in [\![k]\!]$ for $v \in V(G)$ is performed independently of all the other vertices. Then, with probability bounded away from zero, there are disjoint sets $V_1,\dots,V_k \subseteq V(G)$, each of size $\frac{n}{k}$, with the following properties:*

**(A)** $|S_i \triangle V_i| = o_d(n/k)$ *for every* $i \in [\![k]\!]$.

**(B)** $d(v,V_j) \in \left[\frac{\delta d}{k}, \frac{Cd}{k}\right]$ *for every* $\{i,j\} \in E(T)$ *and* $v \in V_i$.

The proof of Proposition 6 is composed of four steps. Let us present here the overview of the proof and the organisation of this section.

In the first step of the proof of Proposition 6, appearing in Section 4.1, we construct the sets in the partition of $V(G)$ which correspond to vertices of high degree in $T$, where our treatment of vertices of high degree in $T$ will be separate from those of low degree.

In the second step of the proof of Proposition 6, appearing in Section 4.2, we construct the remaining sets in the partition of $V(G)$ (that is, the sets corresponding to vertices of low degree in $T$). After this step, for every $\{i,j\} \in E(T)$, we will no longer have vertices in the $i$-th set in the partition whose degree into the $j$-th set is greater than $Cd/k$. However, we may still have a small amount of vertices with degree less than $\delta d/k$. In the third step of the proof of Proposition 6, appearing in Section 4.3, we get rid of all such vertices. Lastly, in the final step of the proof of Proposition 6, appearing in Section 4.4, we will balance the sets of the partition to be of size exactly $n/k$ while we ensure that the requested properties are kept.

As discussed in Section 2, the proof is much simpler whenever one assumes $k \le \frac{d}{10\log d}$. Indeed, then some of the above steps may be skipped. We focus on the case where $k \ge \frac{d}{10\log d}$. In Sections 4.1–4.4, we let $T$ be a tree on $[\![k]\!]$ and assume that $k \ge \frac{d}{10\log d}$.

## 4.1 Vertices of high degree in the tree

In this section, we build the sets in the partition of $V(G)$ that correspond to vertices of high degree in the tree. Let $\beta := \beta(\epsilon) > 0$ be a sufficiently small constant. Denote by $H_{deg}(T)$ the set of vertices of $T$ whose degree is at least $d^{1-\beta}$, and let $h := |H_{deg}(T)|$, noting that $h < d^{\beta}$, since $k = |V(T)| < d$. Assume WLOG that $[\![h]\!] \subseteq V(T) = [\![k]\!]$ is exactly the set $H_{deg}(T)$. We construct the first $h$ sets of the partition, ensuring that every $v \in V(G)$ will have degree between $\delta d/k$ and $Cd/k$ into each one of these sets.

A key tool here is the algorithmic version of the Lovász Local Lemma. We build the first $h$ sets in two rounds. First, we construct random $h$ sets by assigning each vertex into each one of them with probability $(1-\alpha)/k$ for a suitable choice of $\alpha$. After this sample, **whp** we will not have vertices with degree larger than $Cd/k$ into any of the sets. However, we will have a small amount of vertices of degree smaller than $\delta d/k$ into some of the sets. We denote this set of vertices by $B$. In the next round, we will resample the neighbourhood of $B$ outside of the first $h$ sets in the partition, and put each vertex into each one of the first $h$ sets with an appropriate probability, ensuring the expected size of the sets is $n/k$. As we will see, this probability will be at least $900/k$. This, in turn, will allow us to get rid of vertices with less than $\delta d/k$ neighbours into any of the first $h$ sets in the partition.

The next lemma determines the value of $\alpha$ which should be considered.

▶ **Lemma 7.** *There exists $c \in [\epsilon/4, 5]$ such that $\alpha = d^{-c}$ satisfies*

$$\mathbb{P}\left(Bin\left(d, \frac{1-\alpha}{k}\right) \le \delta \log d\right) = \frac{\alpha^2}{d \cdot h}.$$

In relation to Theorem 2, we note that for the proof to follow, it is sufficient to $n^{-3}$-approximate $\alpha$. As we know that $c \in [\epsilon/4, 5]$, the bisection method (see, for example, [5]) gives us an algorithm of finding an $n^{-3}$-approximation of $\alpha$ within $O_\epsilon(\log n)$ steps.

Throughout the rest of the section, we let $\alpha$ be as in the statement of Lemma 7. For every vertex $v \in V(G)$, define the random variable $X_v$ with the following distribution.

$$\mathbb{P}(X_v = i) = \begin{cases} \frac{1-\alpha}{k}, & i \in [\![h]\!] \\ 1 - \frac{(1-\alpha)h}{k}, & i = 0. \end{cases}$$

For every $i \in [\![h]\!]$, let $S_i \coloneqq \{v \in V(G) \colon X_v = i\}$. Given vertex disjoint sets $U_1, \ldots, U_h$, let

$$B(U_1, \ldots, U_h) \coloneqq \{v \in V(G) \colon \exists i \in [\![h]\!], d(v, U_i) \le \delta \log d\},$$
$$W(U_1, \ldots, U_h) \coloneqq \{v \in V(G) \colon \exists i \in [\![h]\!], d(v, U_i) \ge C \log d\}.$$

Set $B_1 \coloneqq B(S_1, \ldots, S_h)$ and $W_1 \coloneqq W(S_1, \ldots, S_h)$.

Let us first estimate several probabilities (which are proved in [8]) that will be useful for us in the proof.

▶ **Lemma 8.** *For every $v \in V(G)$, we have the following.*

1. $\mathbb{P}(v \in W_1) \le d^{-100}$.
2. $\mathbb{P}(v \in B_1) \le \frac{\alpha^2}{d}$.
3. $\mathbb{P}\left(v \in N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i\right) \le 2\left(1 - \frac{h(1-\alpha)}{k}\right)\alpha^2$.

We are now ready for the first (out of several) key step in the proof.

▶ **Lemma 9.** *With probability at least $\frac{1}{2} - o(1)$, there exist disjoint sets $A_1^{(1)}, \ldots, A_h^{(1)} \subseteq V(G)$ which satisfy the following. Let $B_2 = B\left(A_1^{(1)}, \ldots, A_h^{(1)}\right)$ and let $W_2 = W\left(A_1^{(1)}, \ldots, A_h^{(1)}\right)$. Then,*

1. $\left|S_i \triangle A_i^{(1)}\right| \le \frac{n}{d^{50}}$ *and* $\left||A_i^{(1)}| - \frac{(1-\alpha)n}{k}\right| \le \frac{n}{d^{50}}$ *for every* $i \in [\![h]\!]$.
2. $W_2 = \varnothing$.
3. $\left|N(B_2) \setminus \bigcup_{i\in[\![h]\!]} A_i^{(1)}\right| \le 3\alpha^2 n$.

**Proof.** For every $v \in V(G)$, let $F_v$ be the event that $v \in W_1$. Let $\mathcal{F} \coloneqq \{F_v\}_{v\in V(G)}$. By Lemma 8, for every $v \in V(G)$, we have $\mathbb{P}(F_v) \le d^{-100} =: q$. Observe that every $F_v$ is determined by $\Delta_1 \coloneqq d$ random variables (its neighbours). Furthermore, every $F_v$ depends on at most $\Delta_2 \coloneqq d^2$ other events (revealing whether a vertex $v$ satisfies $F_v$ may only affect the probability that $u$ satisfies $F_u$ for $u$ which is in the second neighbourhood of $v$). Furthermore, note that $\beta \coloneqq 4d^{-100}$ satisfies

$$\beta(1-\beta)^{\Delta_2} = 4d^{-100}(1 - 4d^{-100})^{d^2} \ge 4d^{-100}e^{-d^{-90}} \ge d^{-100} = q. \tag{1}$$

By Corollary 5, we obtain that with probability at least $\frac{1}{2}$, there exist sets $A_1^{(1)}, \ldots, A_h^{(1)}$ such that $W_2 = \varnothing$ and $\left|S_i \triangle A_i^{(1)}\right| \le 2 \cdot d \cdot n \cdot \frac{4d^{-100}}{1-4d^{-100}} \le \frac{n}{d^{51}}$ for every $i \in [\![h]\!]$. Let $A_1^{(1)}, \ldots, A_h^{(1)}$ be these sets (if they do not exist, set $A_i^{(1)} = S_i$ for every $i \in [\![h]\!]$). By the Chernoff bound, **whp** $\left||S_i| - \frac{(1-\alpha)n}{k}\right| \le n^{2/3}$ for every $i \in [\![h]\!]$, and thus we obtain the first and second items of the lemma.

As for the third item, by Lemma 8,

$$\mathbb{E}\left[\left|N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i\right|\right] \le 2\left(1 - \frac{h(1-\alpha)}{k}\right)\alpha^2 n.$$

Moreover, the event that $v \in N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i$ depends on at most $d^4$ other events $u \in N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i$. Thus,

$$\begin{aligned}\mathrm{Var}\left(\left|N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i\right|\right) &= \sum_{u,v} \mathrm{Cov}\left(v \in N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i, u \in N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i\right) \\ &\le nd^4.\end{aligned}$$

Thus, by Chebyshev's inequality, **whp**

$$\left|N(B_1) \setminus \bigcup_{i\in[\![h]\!]} S_i\right| \leq 2\left(1 - \frac{h(1-\alpha)}{k}\right)\alpha^2 n + n^{2/3}. \tag{2}$$

Furthermore, whenever we change the location of a vertex (that is, move it from $S_i$ to $S_j$), we change the size of $B_1$ by at most $d$. Hence,

$$\big||B_2| - |B_1|\big| \leq d \cdot \sum_{i\in[\![h]\!]} \big|S_i \triangle A_i^{(1)}\big| \leq d \cdot h \cdot \frac{n}{d^{51}},$$

and thus $||N(B_2)| - |N(B_1)|| < \frac{n}{d^{40}}$. Together with (2), we obtain that

$$\left|N(B_2) \setminus \bigcup_{i\in[\![h]\!]} A_i^{(1)}\right| \leq 3\alpha^2 n,$$

where we used that $\alpha \geq d^{-5}$. ◀

Recall that, in order to prove Proposition 6, we need to show that the sets $V_1, \ldots, V_k$ exist with positive probability (bounded away from zero). We will actually prove that such sets exist with probability $1/4 - o(1)$. In particular, by Lemma 9, the sets $A_1^{(1)}, \ldots, A_h^{(1)}$ (satisfying the statement of the lemma) exist with probability $1/2 - o(1)$. For every possible tuple of disjoint sets $(S_1, \ldots, S_h)$, if there exists a tuple of sets $(A_1^{(1)}, \ldots, A_h^{(1)})$, satisfying the conclusion of Lemma 9, we fix such a tuple. Otherwise, we let $A_i^{(1)} = S_i$ for all $i \in [\![h]\!]$. Further in this section, we assume that the event from Lemma 9, that has probability at least $1/2 - o(1)$, actually occurs; we call the tuple $(S_1, \ldots, S_h)$ *nice* in this case. Note that, under this assumption, the sets $A_i^{(1)}$ satisfy that no vertex has more than $C \log d$ neighbours in each one of them.

We turn to show that with probability bounded away from zero, there exist sets $A_1^{(2)}, \ldots, A_h^{(2)}$ that are "not far" from $S_1, \ldots, S_h$, and every vertex has between $\delta \log d$ and $2C \log d$ neighbours in each one of $A_1^{(2)}, \ldots, A_h^{(2)}$. Let $U$ be a set of size $\frac{\alpha n}{1000}$ which contains $N(B_2) \setminus \bigcup_{i\in[\![h]\!]} A_i^{(1)}$ (note that by Lemma 9, $\left|N(B_2) \setminus \bigcup_{i\in[\![h]\!]} A_i^{(1)}\right| \leq 3\alpha^2 n < \frac{\alpha n}{1000}$). We will make use of the following lemma.

▶ **Lemma 10.** *There exist $\frac{900}{k} \leq p_1, \ldots, p_h \leq \frac{1100}{k}$ such that, for every $i \in [\![h]\!]$,*

$$|A_i^{(1)}| + p_i|U| = \frac{n}{k}.$$

Let $p_1, \ldots, p_h$ be the probabilities from the statement of Lemma 10. Let us note that $\sum_{i\in[\![h]\!]} p_i \leq \frac{1100h}{k} < 1$, since $h < d^\beta$ and $k \geq \frac{d}{10 \log d}$. Further, for every $v \in U$ we define a random variable $X_v$ such that $\mathbb{P}(X_v = i) = p_i$ for every $i \in [\![h]\!]$. Similarly to the proof of Lemma 9, these random variables define a partition, for which we will apply the algorithmic version of Lovász Local Lemma (Corollary 5). Somewhat more explicitly, we will show that given this partition, the probability that a vertex has some set in which it has less than $\delta \log d$ neighbours or more than $2C \log d$ neighbours is sufficiently small, so that a rather simple application of Corollary 5 obtains the required sets.

▶ **Lemma 11.** *With probability at least $1/2 - o(1)$, there exist disjoint subsets $A_1^{(2)}, \ldots, A_h^{(2)} \subseteq V(G)$ which satisfy the following.*

1. $\left|S_i \triangle A_i^{(2)}\right| = o_d(n/k)$ *for every $i \in [\![h]\!]$.*

2. $\left|A_i^{(2)} - \frac{n}{k}\right| = O(n/d^{50})$ *for every* $i \in [\![h]\!]$.
3. $d(v, A_i^{(2)}) \in [\delta \log d, 2C \log d]$ *for every* $i \in [\![h]\!]$ *and* $v \in V(G)$.

The complete proof of the above lemma appears in [8].

## 4.2 Vertices of low degree in the tree

We have proved that, if $(S_1, \ldots, S_h)$ is nice (this happens with probability at least $1/2 - o(1)$), then there exist sets $A_1^{(2)}, \ldots, A_h^{(2)}$ satisfying the properties as in the statement of Lemma 11. Throughout this section, we fix a nice tuple $(S_1, \ldots, S_h)$ and a tuple $(A_1^{(2)}, \ldots, A_h^{(2)})$, satisfying the conclusion of Lemma 11.

Recall that the sets $A_1^{(2)}, \ldots, A_h^{(2)}$ correspond to the high-degree vertices in $T$, and every vertex has between $\delta \log d$ and $2C \log d$ neighbours in each of these sets. As described in the beginning of Section 4, in this section we aim to establish similar sets for low-degree vertices.

Let

$$U \coloneqq V(G) \setminus \bigcup_{i \in [\![h]\!]} A_i^{(2)}. \tag{3}$$

For every $v \in U$, let $X_v$ be the random variable such that $\mathbb{P}(X_v = i) = \frac{1}{k-h}$, for every index $i \in \{h+1, \ldots, k\}$. All $X_v$ are independent. For every $i \in \{h+1, \ldots, k\}$, set $S_i \coloneqq \{v \in U \colon X_v = i\}$. This defines the random variable that we will consider, when applying Corollary 5. For convenience, let us also set $A_i^{(2)} \coloneqq S_i$ for every $i \in \{h+1, \ldots, k\}$ (recall that $A_i^{(2)}$ is already defined for every $i \in [\![h]\!]$).

Given a partition $U_1, \ldots, U_k$ of $V(G)$, let $B(U_1, \ldots, U_k)$ be the set of vertices $v \in V(G)$ satisfying the following. There exist $i \in [\![k]\!]$ and $j \in \{h+1, \ldots, k\}$ such that $\{i, j\} \in E(T)$, $v \in U_i$ and $d(v, U_j) \leq \delta \log d$. Further, let $W(U_1, \ldots, U_k)$ be the set of vertices $v \in V(G)$ that satisfy at least one of the following.

1. There exists $i \in [\![k]\!]$ such that $d(v, U_i) \geq 2C \log d$.
2. There exist more than $1/\epsilon^2$ indices $i \in \{h+1, \ldots, k\}$ such that $d(v, U_i) < \delta \log d$.
3. There exists $i \in [\![k]\!]$ such that $d(v, U_i \cap B(U_1, \ldots, U_k)) > \log \log d$.

In the following lemma, we show that there exists a "good" partition $A_1^{(3)}, \ldots, A_k^{(3)}$, which is not far from $S_1, \ldots, S_k$. Its complete proof is in [8]. The key element of the proof lies in yet another application of the algorithmic version of Lovász Local Lemma (Corollary 5), this time on the random partition $A_1^{(2)}, \ldots, A_k^{(2)}$.

▶ **Lemma 12.** *With probability at least* $1/2 - o(1)$*, there exists a partition of* $V(G)$ *into* $A_1^{(3)}, \ldots, A_k^{(3)}$ *which satisfies the following. Let*

$$B_4 = B\left(A_1^{(3)}, \ldots, A_k^{(3)}\right) \quad \text{and} \quad W_4 = W\left(A_1^{(3)}, \ldots, A_k^{(3)}\right).$$

*Then,*

1. $\left|A_i^{(3)} \triangle S_i\right| = o_d(n/k)$ *for every* $i \in [\![k]\!]$.
2. $\left||A_i^{(3)}| - \frac{n}{k}\right| = O(n/d^{49})$ *for every* $i \in [\![k]\!]$.
3. $W_4 = \varnothing$.
4. $|B_4| \leq n d^{-1-\epsilon/5}$.
5. *For every* $i \in [\![k]\!]$*, there are at least* $\frac{n}{4k}$ *vertices* $v \in A_i^{(3)}$ *which satisfy that for every* $j \in [\![k]\!]$*,* $d(v, A_j^{(3)}) \geq \delta \log d$.
6. $d(v, A_i^{(3)}) \in [\delta \log d, 2C \log d]$ *for every* $i \in [\![h]\!]$ *and* $v \in V(G)$.

## 4.3 Eliminating bad vertices

A tuple $(S_1, \ldots, S_k)$ is *nice* if $(S_1, \ldots, S_h)$ is nice and there exist sets $A_i^{(3)}$, $i \in [\![k]\!]$, satisfying the conclusion of Lemma 12. Due to Lemmas 9 and 12, with probability at least $1/4 - o(1)$, the considered random tuple of sets $(S_1, \ldots, S_k)$ is nice. As in the previous section, for every nice tuple $(S_1, \ldots, S_k)$, we fix the corresponding sets $A_i^{(3)}$, $i \in [\![k]\!]$, satisfying the conclusion of Lemma 12. If $(S_1, \ldots, S_k)$ is not nice, then we simply set $A_i^{(3)} = S_i$ for all $i \in [\![k]\!]$.

We assume that the event from Lemma 12 (that has probability at least $1/4 - o(1)$ due to Lemma 9), actually occurs; i.e. the tuple $(S_1, \ldots, S_k)$ is *nice*. We recall that $A_1^{(3)}, \ldots, A_k^{(3)}$ satisfy that for every vertex $v \in V(G)$ and an index $i \in [\![h]\!]$, we have $d(v, A_i^{(3)}) \in [\delta \log d, 2C \log d]$. Further, for every vertex $v \in V(G)$ and an index $i \in [\![k]\!]$, we have $d(v, A_i^{(3)}) \leq 2C \log d$. For every vertex $v \in B_5$, denote by $I_v \subseteq \{h+1, \ldots, k\}$ the set of indices $j \in \{h+1, \ldots, k\}$ such that $d(v, A_j^{(3)}) \leq \delta \log d$. Set $\Gamma_v \coloneqq \{h+1, \ldots, k\} \setminus (I_v \cup N_T(I_v))$. For every vertex $v \in B_5$, denote by $X_v$ the uniform random variable over the set of integers $\Gamma_v$. This allows us to define a random partition, to which we apply the algorithmic version of Lovász Local Lemma and obtain the following sets.

▶ **Lemma 13.** *There exist sets $A_1^{(4)}, \ldots, A_k^{(4)}$ such that the following holds:*

1. $\left|A_i^{(4)} \triangle A_i^{(3)}\right| \leq nd^{-1-\epsilon/5}$ *for every $i \in [\![k]\!]$.*
2. $d(v, A_j^{(4)}) > \frac{\delta \log d}{2}$ *for every $\{i, j\} \in E(T)$ and for every $v \in A_i^{(4)}$.*
3. $d(v, A_j^{(4)}) < 3C \log d$ *for every vertex $v \in V(G)$ and every $j \in [\![k]\!]$.*
4. *For every $i \in [\![k]\!]$, there are at least $\frac{n}{5k}$ vertices $v \in A_i^{(4)}$ which satisfy that for every $j \in [\![k]\!]$, $d(v, A_j^{(4)}) \geq \frac{\delta \log d}{2}$.*

The complete proof of this lemma is in [8].

## 4.4 Balancing the sets

We have showed that, if $(S_1, \ldots, S_k)$ is nice (this happens with probability at least $1/4 - o(1)$), then there exist sets $A_1^{(4)}, \ldots, A_k^{(4)}$ satisfying the properties as in the statement of Lemma 13. Throughout this section, we fix a nice tuple $(S_1, \ldots, S_k)$ and a tuple $(A_1^{(4)}, \ldots, A_k^{(4)})$, satisfying the conclusion of Lemma 13.

Recall that the sets $A_1^{(4)}, \ldots, A_k^{(4)}$ have good degree distribution in between them, yet their size could be up to $nd^{-1-\epsilon/5}$-far from $n/k$. We now turn to show that there exist sets $A_1^{(5)}, \ldots, A_k^{(5)}$, all "close" to $A_1^{(4)}, \ldots, A_k^{(4)}$, and all of size $\frac{n}{k} \pm O(nd^{-50})$, which still satisfy the "good degrees" assumption. This will, in turn, allow us to complete the balancing of the sets deterministically, and obtain sets of size exactly $\frac{n}{k}$ which satisfy the "good degrees" assumption.

To that end, let us reorder the sets such that $A_1^{(4)}, \ldots, A_m^{(4)}$ are of size at least $\frac{n}{k}$, and $A_{m+1}^{(4)}, \ldots, A_k^{(4)}$ are of size less than $\frac{n}{k}$, for some $m \in [\![k]\!]$. Further, for every $i \in [\![k]\!]$, let $\Delta_i = \left||A_i^{(4)}| - \frac{n}{k}\right|$, noting that by the first item in Lemma 13 and by the second item in Lemma 12, $\Delta_i \leq nd^{-1-\epsilon/6}$. We have that for every $i \in [\![k]\!]$, there are at least $\frac{n}{5k}$ vertices $v \in A_i^{(4)}$ such that $d(v, A_j^{(4)}) \in \left[\frac{\delta \log d}{2}, 3C \log d\right]$ for every $j \in [\![k]\!]$. For every $i \in [\![m]\!]$, let $Q_i \subseteq A_i(4)$ be a set of exactly $\frac{n}{5k}$ such vertices, and set $Q \coloneqq \bigcup_{i \in [\![m]\!]} Q_i$.

For every $i \in [\![m]\!]$ and $v \in Q_i$, set $M_v \sim \text{Bernoulli}(p_i)$ where $p_i = \frac{\Delta_i}{n/5k}$. This Bernoulli random variable represents whether the vertex $v$ is moved to the $j$-th set, for some $j \in \{m+1, \ldots, k\}$, or not. In addition, let $Z_v$ be the random variable over the set $\{m+1, \ldots, k\}$ defined by $\mathbb{P}(Z_v = j) = \frac{\Delta_j}{\Delta_1 + \ldots + \Delta_m}$ for every $j \in \{m+1, \ldots, k\}$. Note that $\sum_{j \in \{m+1, \ldots, k\}} \mathbb{P}(Z_v =$

$j) = 1$ since $\Delta_1 + \ldots + \Delta_m = \Delta_{m+1} + \ldots + \Delta_k$. The random variable $Z_v$ represents the index $j \in \{m+1, \ldots, k\}$ for which the vertex $v$ may move (it will indeed move to $A_j^{(4)}$ if and only if $M_v = 1$). We stress that for every $v \in Q$, $M_v$ and $Z_v$ are independent, and are also independent over different $v$. Let

$$\tilde{A}_i^{(4)} := \begin{cases} A_i^{(4)} \setminus \{v \in Q_i \colon M_v = 1\}, & i \in \llbracket m \rrbracket \\ A_i^{(4)} \cup \{v \in Q \colon M_v = 1 \text{ and } Z_v = i\}, & i \in \{m+1, \ldots, k\}. \end{cases}$$

Note that by the above construction, if $A_i^{(4)}$ is of size smaller than $\frac{n}{k}$, then we may only move vertices into it, whereas when $A_i^{(4)}$ is of size larger than $\frac{n}{k}$ we may only move vertices outside of it. Further, if the set $A_i^{(4)}$ is of size exactly $\frac{n}{k}$, then $\Delta_i = 0$, and thus the set will remain unchanged.

The proof of the following lemma then follows from some typical properties of the sets $\tilde{A}_1^{(4)}, \ldots, \tilde{A}_k^{(4)}$, an application of Corollary 5, and bounding the probability a vertex $v$ satisfies one of the following:

- There exists $\{i, j\} \in E(T)$ such that $v \in \tilde{A}_i^{(4)}$ and $d(v, \tilde{A}_j^{(4)}) \notin \left[\frac{\delta \log d}{3}, 4C \log d\right]$.
- $d(v, A_i^{(4)}) > \delta \log d/2$ for every $i \in \llbracket k \rrbracket$; further, there is some $i \in \llbracket k \rrbracket$ such that $d(v, \tilde{A}_i^{(4)}) \leq \delta \log d/3$.

See [8] for the full proof.

▶ **Lemma 14.** *With probability at least $\frac{1}{2} - o(1)$ (in the product measure induced by $M_v$ and $Z_v$, $v \in Q$), there exist disjoint sets $A_1^{(5)}, \ldots, A_k^{(5)}$ such that the following holds.*

1. *For every $\{i, j\} \in E(T)$ and for every $v \in A_i^{(5)}$, we have that $d(v, A_j^{(5)}) \in \left[\frac{\delta \log d}{3}, 4C \log d\right]$.*
2. *For every $i \in \llbracket k \rrbracket$, we have $\left|A_i^{(5)} \triangle \tilde{A}_i^{(4)}\right| = O\left(\frac{n}{d^{50}}\right)$.*
3. *For every $i \in \llbracket k \rrbracket$, there are at least $\frac{n}{6k}$ vertices $v \in A_i^{(5)}$ which satisfy that $d(v, A_j^{(5)}) \in \left[\frac{\delta \log d}{3}, 4C \log d\right]$ for every $j \in \llbracket k \rrbracket$.*

Fix sets $A_1^{(5)}, \ldots, A_k^{(5)}$ satisfying the properties as in the statement of Lemma 14. We are now ready to complete the proof of Proposition 6. To that end, we first show we can move the vertices between the sets $A_1^{(5)}, \ldots, A_k^{(5)}$ to obtain sets $V_1, \ldots, V_k$, each with exactly $\frac{n}{k}$ vertices, while maintaining the degree distribution between the sets.

▶ **Lemma 15.** *There exists sets $V_1, \ldots, V_k$ such that the following holds.*

1. *For every $\{i, j\} \in E(T)$ and for every $v \in V_i$, we have that $d(v, V_j) \in \left[\frac{\delta \log d}{4}, 5C \log d\right]$.*
2. *$\left|V_i \triangle A_i^{(5)}\right| = o_d(n/k)$ for every $i \in \llbracket k \rrbracket$.*
3. *$|V_i| = \frac{n}{k}$ for every $i \in \llbracket k \rrbracket$.*

**Proof.** It suffices to show that we can move vertices from sets of size larger than $\frac{n}{k}$ to sets of smaller size, without changing the degree of any vertex into any of the sets by more than one. Consider the following procedure. We start with the sets $A_1^{(5)}, \ldots, A_k^{(5)}$.

Recall that at least $\frac{n}{6k}$ vertices $v \in A_i^{(5)}$ satisfy $d(v, A_j^{(5)}) \in \left[\frac{\delta \log d}{3}, 4C \log d\right]$ for every $i, j \in \llbracket k \rrbracket$. Furthermore, by the second item in Lemma 14 together with typical properties of $A_1^{(4)}, \ldots, A_k^{(4)}$, $\left||A_i^{(5)}| - \frac{n}{k}\right| = O(n/d^{50})$, for every $i \in \llbracket k \rrbracket$.

We proceed inductively. Suppose we have already moved vertices $v_1, \ldots, v_t$ and that there still exists a set $A_i^{(5)}$ of size larger than $n/k$. In particular, $t \leq k \cdot O(n/d^{50}) < n/d^{45}$. Note that $\frac{n}{6k} - d^2 t \geq \frac{n}{7k} > 0$, and thus there exists a vertex $v \in A_i^{(5)}$ which satisfies the following two properties:

- $v$ is not in the second neighbourhood of any $v_1, \dots, v_t$, and,
- for every $j \in [\![k]\!]$, the number of neighbours of $v$ in $A_j^{(5)}$ lies in the interval $\left[\frac{\delta \log d}{3}, 5C \log d\right]$.

We move the vertex $v$ to an arbitrary set of size smaller than $\frac{n}{k}$. Observe that the above two properties guarantee that throughout the entire process, the degree of any vertex into any set will change by at most one. ◀

We are now ready to prove Proposition 6.

**Proof of Proposition 6.** We have showed that, with probability $1/4 - o(1)$ (in the measure induced by $(S_1, \dots, S_k)$), there exist sets $V_1, \dots, V_k$ that satisfy item 1 and item 3 from Lemma 15 and that $|V_i \triangle S_i| = o_d(n/k)$ for every $i \in [\![k]\!]$, due to Lemmas 12(1), 13(1), 14(2), and 15(2).

Now, let $\tilde{S}_1, \dots, \tilde{S}_k$ be a uniformly random partition of $V(G)$: every $v \in V(G)$ is assigned to $\tilde{S}_i$ for an index $i \in [\![k]\!]$ chosen uniformly at random, independently of all the other vertices. All that is left then is to observe that there is a coupling $(S_i', \tilde{S}_i)$ such that $(S_1', \dots, S_k') \stackrel{d}{=} (S_1, \dots, S_k)$ and $|S_i' \triangle \tilde{S}_i| = o_d(n/k)$ for every $i \in [\![k]\!]$ **whp**.

Indeed, consider the following coupling. Initially, for every $i \in [\![k]\!]$, we set $S_i' = \tilde{S}_i$. We then keep every vertex $v \in S_i'$, for every $i \in [\![h]\!]$, with probability $1 - \alpha$, and with probability $\alpha$ we remove $v$ from $S_i$. Let $N_1$ be the set of removed vertices. Recall that the choice of $\alpha$ is according to Lemma 7, thus we removed at most $n/d^{1+\epsilon/5}$ vertices from every set $S_i'$ **whp**. Observe that $(S_1', \dots, S_h') \stackrel{d}{=} (S_1, \dots, S_h)$ and **whp** $|S_i' \triangle \tilde{S}_i| = o_d(n/k)$. However, sets $S_{h+1}', \dots, S_k'$ should be still perturbed since the set $U$ from (3) is obtained after two resamples due to Corollary 5. Thus, we now consider the first two applications of the algorithmic Lovász Local Lemma. By Lemmas 9 and 11 and since we defined $A_i^{(1)} := S_i$ for partitions $(S_1, \dots, S_h)$ that do not satisfy the event from the statement of Lemma 9, for every $i \in [\![h]\!]$, the number of vertices which are moved inside/outside of $S_i$ is $o_d(n/k)$. Denote by $N_2^+$ and $N_2^-$ the sets of vertices which were moved in these first two applications of the algorithmic Lovász Local Lemma from outside of $S_1' \cup \ldots \cup S_h'$ to $A_1^{(2)} \cup \ldots \cup A_h^{(2)}$ and from $S_1' \cup \ldots \cup S_h'$ outside of $A_1^{(2)} \cup \ldots \cup A_h^{(2)}$, respectively. We have that $|N_2^-| = o_d(n)$ and $|N_2^+| = o_d(n)$. We then partition the set $N_1 \cup N_2^- \setminus N_2^+$ that has size $o_d(n)$ **whp** uniformly at random into $S_{h+1}'', \dots, S_k''$. Letting $S_i' := S_i' \cup S_i''$, for every $i \in [\![k]\!] \setminus [\![h]\!]$, and recalling that $k - h = (1 - o_d(1))k$, we get that $(S_1', \dots, S_k') \stackrel{d}{=} (S_1, \dots, S_k)$ and that $|S_i' \triangle \tilde{S}_i| = o_d(n/k)$ for every $i \in [\![k]\!]$ **whp**, completing the proof. ◀

## 5 Growing the trees

In this section, we show how to construct the tree factor, given the vertex partition guaranteed by Proposition 6. Recall that **whp**, $G(n, d) \in \mathcal{G}_d$ (see, for example, [27]). This enables us to apply Proposition 6.

The following two claims, whose proofs appear in [8], show that typically there are not "too many" edges between any two small sets of equal size, and bound the neighbourhoods of large sets as well.

▷ Claim 16. For every $\epsilon, \delta > 0$, there exists $\eta > 0$ such that, for sufficiently large $d$, **whp** the following holds in $G \sim G(n, d)$. Let $k \le \frac{(1-\epsilon)d}{\log d}$ be an integer. Then, for every two disjoint sets $A, B \subseteq V(G)$ satisfying $|A| = |B| < \eta \cdot \frac{n}{k}$, we have $e(A, B) < |A| \cdot \delta \cdot \frac{d}{k}$.

▷ Claim 17. For every $\epsilon, \eta > 0$, there exist $\epsilon_1, \epsilon_2 > 0$ such that, for sufficiently large $d$, the following holds in $G \sim G(n, d)$. Let $2 \le k \le \frac{(1-\epsilon)d}{\log d}$ be an integer and let $S_1, \dots, S_k$ be a uniformly random partition of $V(G)$. Then, **whp**, for every $i \ne j \in [\![k]\!]$ and $A \subset S_i$ satisfying $\eta \frac{n}{k} \le |A| \le 0.5(1 + \epsilon_1)\frac{n}{k}$, we have $|N(A, S_j)| \ge (1 + \epsilon_2)|A|$.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Let $\epsilon > 0$ be a constant and let $d$ be a sufficiently large integer. Let $T$ be a tree on $k \le \frac{(1-\epsilon)d}{\log d}$ vertices. We will show that **whp** $G \sim G(n,d)$ contains a $T$-factor. Let $\delta = \delta(\epsilon) > 0$ and $C = C(\epsilon) > 0$ be the constants guaranteed by Proposition 6. In addition, let $\eta = \eta(\epsilon, \delta)$ be the constant guaranteed by Claim 16 and let $\epsilon_1 = \epsilon_1(\eta^2, \epsilon)$ and $\epsilon_2 = \epsilon_2(\eta^2, \epsilon)$ be the constants guaranteed by Claim 17.

Now, let $S_1, \dots, S_k$ be such that every $v \in V(G)$ is assigned to $S_i$ for an index $i \in [\![k]\!]$ chosen uniformly at random, independently from all the other vertices. Note that **whp** $G(n,d) \in \mathcal{G}_d$ and the statements of Claims 16 and 17 are satisfied. We then fix a deterministic $G \in \mathcal{G}_d$ that satisfies conclusions of both claims.

Let $\Sigma$ be the set of all partitions of $V(G)$ into $k$ ordered sets. Let $\Sigma' \subset \Sigma$ be the set of all *nice* $(S_1, \dots, S_k)$, i.e. those that satisfy the conclusion of Proposition 6. Due to Proposition 6, there exists a constant $\gamma > 0$ such that $|\Sigma'|/|\Sigma| \ge \gamma$. On the other hand, let $\Sigma'' \subset \Sigma$ be the set of all good $(S_1, \dots, S_k)$, i.e. those that satisfy the conclusion of Claim 17. We know that $|\Sigma''|/|\Sigma| = 1 - o(1)$. We immediately get that there exists a tuple $(S_1, \dots, S_k)$ which is simultaneously nice and good. Since this tuple is nice, there exist sets $V_1, \dots, V_k$ which satisfy all the desired requirements. Under these assumptions, we will be able to show deterministically that there exists a perfect matching between $V_i$ and $V_j$ for every $\{i,j\} \in E(T)$ which implies the existence of a $T$-factor. One way to show the latter implication is, for example, by induction on $k$. Assume without loss of generality that $k \in V(T)$ is a leaf and that, by induction assumption, we have a $T'$-factor in $\cup_{i=1}^{k-1} V_i$ where $T' = T \setminus \{k\}$. We may then complete $T'$ to a $T$-factor via the perfect matching between $V_k$ and $V_i$ where $i$ is the only neighbour of $k$ in $T$.

Fix $\{i,j\} \in E(T)$. We will show that Hall's condition is satisfied between $V_i$ and $V_j$ in $G$. Let $W \subseteq V_i$. We will prove that $|N(W, V_j)| \ge |W|$. By Proposition 6, for every $v \in V_i$, we have $d(v, V_j) \in \left[\delta \cdot \frac{d}{k}, C \cdot \frac{d}{k}\right]$. Hence, $e(W, V_j) \ge |W| \cdot \delta \cdot \frac{d}{k}$. We split the proof into three parts depending on the size of $|W|$.

First of all, we show that if $|W| < \eta \cdot \frac{n}{k}$, then $|N(W, V_j)| > |W|$. Assume towards contradiction that this is false. Then, there exists a set $B \subseteq V_j$ satisfying $N(W, V_j) \subseteq B$ and $|B| = |W|$. By Claim 16, we have $e(W, B) = e(W, V_j) < |W| \cdot \delta \cdot \frac{d}{k}$, a contradiction.

Next, assume that $\eta \cdot \frac{n}{k} \le |W| \le 0.5(1 + \epsilon_1)\frac{n}{k}$. We have

$$|W \cap S_i| \ge |W| - |V_i \setminus S_i| \ge \eta \cdot \frac{n}{k} - |V_i \setminus S_i| \ge \eta^2 \cdot \frac{n}{k},$$

where the last inequality is true since $|V_i \setminus S_i| = o_d(n/k)$ by Proposition 6. Thus,

$$\begin{aligned} |N_{V_j}(W)| &\ge |N_{S_j}(W \cap S_i) \cap V_j| \\ &\ge (1 + \epsilon_2)|W \cap S_i| - |V_j \setminus S_j| \\ &\ge (1 + \epsilon_2)(|W| - |V_i \setminus S_i|) - |V_j \setminus S_j| > |W|, \end{aligned}$$

where the second inequality is true by Claim 17 and the last inequality is true since $|V_i \setminus S_i|, |V_j \setminus S_j| = o_d(n/k)$ and $|W| = \Omega(n/k)$.

Finally, assume that $0.5(1 + \epsilon_1)\frac{n}{k} < |W| \le \frac{n}{k}$. Assume towards contradiction that $|N(W, V_j)| < |W|$. Let $B \subseteq V_j \setminus N(W, V_j)$ be an arbitrary set of size $|V_i \setminus W|$. Notice that $N(B, V_i) \subseteq V_i \setminus W$, otherwise there exists $v \in B$ which is adjacent to $u \in W$. This in turn implies that $v \in N(W, V_j)$ and, in particular, $v \notin B$ – a contradiction. Moreover,

$$|B| = |V_i| - |W| < |V_i| - 0.5(1 + \epsilon_1)\frac{n}{k} = 0.5(1 - \epsilon_1)\frac{n}{k}.$$

Therefore, by the previous argument (with $i$ and $j$ reversed), $|N(B, V_i)| > |B|$. However, since $N(B, V_i) \subseteq V_i \setminus W$, we have $|N(B, V_i)| \leq |V_i \setminus W| = |B|$ – contradiction. ◀

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