





# Drainability and Fillability of Polyominoes in Diverse Models of Global Control

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## Abstract

Tilt models offer intuitive and clean definitions of complex systems in which particles are influenced by global control commands. Despite a wide range of applications, there has been almost no theoretical investigation into the associated issues of filling and draining geometric environments. This is partly because a globally controlled system (i.e., passive matter) exhibits highly complex behavior that cannot be locally restricted. Thus, there is a strong need for theoretical studies that investigate these models both (1) in terms of relative power to each other, and (2) from a complexity theory perspective. In this work, we provide (1) general tools for comparing and contrasting different models of global control, and (2) both complexity and algorithmic results on filling and draining.

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## 1 Introduction

The targeted use of global control mechanisms, applied synchronously and uniformly<sup>1</sup> to small-particle matter (i.e., *passive matter*), is both of great practical relevance and highest (theoretical) complexity. A fundamental type of global control mechanism is that of uniform movement or translation, which finds application in a variety of manufacturing processes,

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<sup>1</sup> Due to this intrinsic connection, we use the qualifiers “uniform” and “global” synonymously when applied to models of motion.



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such as filling polyhedra with a liquid for gravity casting [19, 20, 31] or the intact removal of cast objects from their molds [17, 18]. Global forces like electromagnetic fields and gravity play a key role in a range of further applications, such as amorphous computing and smart materials like smart paint [2], autonomous monitoring and repair systems as well as minimally invasive surgeries [11], medication [14], and biological robots [16].

Inspired by tilt games [1, 4, 24], theoretical research commonly uses discrete, grid-based models to study the manipulation of passive matter by global control signals. In this context, a distinction is made between single step and full tilt models [11, 23], i.e., the movement of all particles by either one step or by a maximum distance in a uniform direction.

One of the most challenging problems remains the question of filling geometric shapes using global control: Given a *board*, defined as a subset of the square grid as well as a number of “infill points” (*sources*), the question is whether, and if so how, the entire board can be filled by adding particles through the set of sources, with all particles moving in the same direction as determined by global control mechanisms. Crucially, the inherently discrete nature of particle models introduces new complications that have not appeared in the continuous frameworks in [19, 20], such as particles forming stacks by blocking each other instead of spreading like liquids.

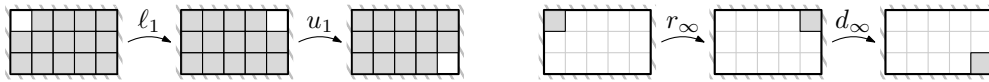
Naturally, there exist boards that cannot be filled with a given number of sources. We investigate the (therefore immediately arising) question of minimal necessary changes to the board in order to achieve fillability. Even in simplified models, globally controlled particles and targeted changes to the board create dynamic systems that are extremely complex to analyze and control: The smallest manipulation of a board can lead to far-reaching and not locally restrictable changes in terms of fillability, see for example Figures 2(b) and 2(c).

Furthermore, it is unclear how the different models relate to each other in terms of the contrasting objectives of filling or draining. In this paper, we provide a generalized, comprehensive framework for the comparative study of tilt models and formally investigate what makes a given model *more powerful* than another, i.e., allows more polyominoes to be filled or drained. This includes the special case of single step and full tilt movement.

## 1.1 Our Contributions

We build upon established models of particle movement using global control signals, namely the full tilt and single step models, examine them in the context of generalized and more powerful models, and investigate how to achieve drainability and fillability of polyominoes in these models by placing obstacles. Our main contributions are twofold.

- (1) We provide general tools to compare and contrast various models of particle movement, offering a more unified perspective and surprising new insights concerning the duality of different models (see Figure 1). In particular, we prove the following:
  - (1.1) Equivalence between drainability in the full tilt model and fillability in the single step model (Corollary 30).
  - (2.2) Limited relaxation of the restriction to global control signals does not tangibly affect drainability (Theorem 29), i.e., does not increase model power in this regard.
- (2) With regards to gaining drainability via obstacles in the full tilt model, we provide:
  - (2.1) A reduction from a 3SAT variant showing that it is NP-hard to decide whether a given number of obstacles suffices, even when restricted to thin polyominoes, i.e., those containing no  $2 \times 2$  squares (Theorem 5).
  - (2.2) A  $c$ -approximation algorithm for  $k$ -scaled boards of  $c = 4$  for  $k = 3$  and of  $c = 6$  for  $k > 3$  (Theorem 14 and Corollary 15).



■ **Figure 1** Movement of a single bubble in the single step model (left) is identical to the movement of a single particle in the full tilt model (right).

For ease of exposition, we first present algorithmic results for drainability in the full tilt model in Sections 2–4, before generalizing to fillability and other models in Section 5. Due to limited space, all proofs and full details for statements marked by ( $\star$ ) can only be found in the full version of our paper [25].

## 1.2 Related Work

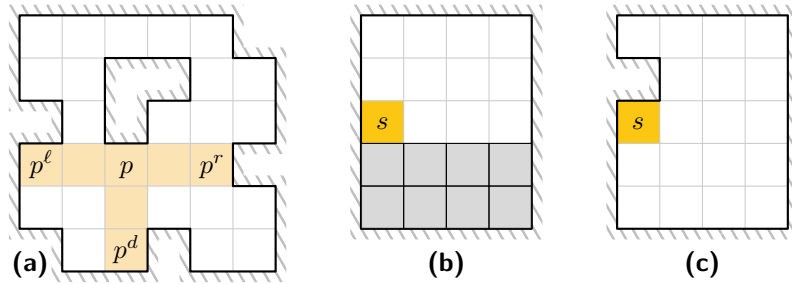
The drainability and fillability of polygonal two-dimensional shapes by uniform tilt movement has previously been studied by Aloupis et al. [6], who gave an exact polynomial-time algorithm for the minimum number of sinks (i.e., exits) necessary to drain a polygon or polyhedron. We study the discrete “tilt” model first introduced by Becker et al. [15, 23] in 2013. Particles on a grid-based *board* are moved uniformly, in either the single step or full tilt model:

The *reconfiguration* problem asks whether a given arrangement of particles can be reconfigured into another specific arrangement, given a fixed set of obstacles. The minimization variant, i.e., finding a shortest sequence of moves, was shown to be PSPACE-complete in the full tilt model by Becker et al. [11, 12]. A natural subproblem is the *relocation* problem, which asks for just one specific particle to be moved to a target position. In this variant, even deciding existence of any movement sequence is NP-hard in the full tilt model [10, 11]. Recent complexity results for single step tilt moves by Caballero et al. demonstrate that, when restricted to two or three cardinal directions, deciding the existence of a relocation sequence is NP-complete [21], as well as in the absence of obstacles (i.e., a rectangular board) and just two movement directions [23]. The more general *occupancy* problem asks whether an arrangement of particles can be modified to move *any* particle to a specific target position. Even with this relaxation, finding a shortest sequence of moves, or deciding that no such sequence exists, is PSPACE-complete [8, 11]. Caballero et al. [22] show that deciding occupancy remains PSPACE-complete in the single step model when  $2 \times 1$  dominoes are considered in addition to  $1 \times 1$  particles. However, it is easy to see that the problem is in P if only unit-size particles are considered.

A myriad of related problems have been studied in the tilt model, such as *gathering* particles into a connected configuration [14, 27], or the design of special-purpose boards for efficient reconfiguration: In particular, Balanza-Martinez et al. [9] studied the design of universal shape constructors, i.e., boards that can be used to create large classes of particle configurations. Further results exist on the reordering of labeled rectangular arrangements, which can be achieved either in linear time using quadratic area [11], or in quadratic time and linear area [32]. In addition to “workspace”-based tilt reconfiguration settings, a variety of other models exist, e.g., moving only particles at maximal coordinates [3, 5], rather than all particles. A number of questions remain on the classification and complexity of deciding which configurations can be constructed [13, 26] by sequentially introducing particles into an unobstructed system and “gluing” them onto an existing configuration using tilt movement.

## 2 Preliminaries

Particles move on a *board*, which we model as a finite subgraph of the square tiling's dual graph, i.e., a board  $B = (V, E)$  is a finite graph with  $V \subset \mathbb{Z}^2$  and  $E \subseteq \{\{u, v\} : u \in V, v \in V, \|u - v\|_1 = 1\}$ . The *boundary* of a board is the axis-aligned polygon separating the tiles in  $V$  from  $\mathbb{Z}^2 \setminus V$ . Note that the boundary uniquely determines the corresponding board. We only illustrate the special case of vertex-induced boards, which are isomorphic to sets of polyominoes and as such have boundaries whose sides only intersect at corners, even though the results in this paper hold in general. A *sub-board* is a vertex-induced subgraph of a board; a maximally connected (sub-)board is called a *region*. We refer to the elements of  $V$  as *pixels*. Every pixel is uniquely determined as the *intersection* of two *segments*, a horizontal *row segment* and a vertical *column segment*, which are maximally contiguous subsets of pixels of equal  $y$ -, or  $x$ -coordinate, respectively. A *boundary pixel* is a pixel adjacent to a side of the boundary; a *corner pixel* is a pixel adjacent to two perpendicular sides of the boundary. The set  $\{\ell, r, u, d\}$  is shorthand for the directions *left*, *right*, *up*, and *down*. For a pixel  $p$  let  $p^\ell$ ,  $p^r$ ,  $p^u$ , and  $p^d$  denote the left and right boundary pixel in its row segment, and the top and bottom boundary pixel in its column segment, respectively; see Figure 2(a).



■ **Figure 2** (a) A board with a pixel  $p$  at the intersection of its row and column segments (shaded area), along with the boundary pixels of these segments. Note that  $p = p^u$ . (b) A configuration of a board  $B$  that is minimal with respect to a sink  $s$  in the full tilt model. Particles are depicted as gray squares. (c) A drainable sub-board of  $B$ .

A configuration  $C \subseteq V$  is a subset of the pixels. We call a pixel  $p \in C$  *occupied* and a pixel  $p \in V \setminus C$  *free*. A move  $m : 2^V \rightarrow 2^V$  is a mapping between configurations; a *model* is a set of moves. Given a model  $M$  and two configurations,  $C$  and  $D$ , we say  $D$  is *reachable from  $C$  in one move*, or  $C \rightarrow_M D$ , if  $D = m(C)$  for some  $m \in M$ .  $D$  is *reachable from  $C$* , or  $C \rightarrow_M^* D$ , if there is a sequence of moves  $m_1, m_2, \dots, m_k \in M$  such that  $D = m_k \circ \dots \circ m_2 \circ m_1(C)$ , i.e.,  $\rightarrow_M^*$  is the reflexive, transitive closure of  $\rightarrow_M$ . We omit braces for singletons and write, e.g.,  $u \rightarrow_M v$  instead of  $\{u\} \rightarrow_M \{v\}$ . A pixel  $p$  is *occupiable* from a configuration  $C$  if there is a configuration  $D$  reachable from  $C$  that contains  $p$ .

The *full tilt model*,  $\text{FT} = \{u_\infty, d_\infty, \ell_\infty, r_\infty\}$ , has one move for every direction, which sees particles move maximally in that direction until they hit the boundary or are blocked by another particle. The move  $\ell_\infty$ , for example, transforms a configuration  $C$  in such a way that exactly the  $|R \cap C|$  leftmost pixels of every row segment  $R$  are occupied in  $\ell_\infty(C)$ .

A *sink* is a move associated with a pixel  $s \in V$  defined as  $C \mapsto C \setminus \{s\}$ . A configuration  $C$  is *minimal* with respect to a set of sinks  $S \subseteq V$  if there is no configuration  $D$  such that  $C \rightarrow_{\text{FTUS}}^* D$  and  $|D| < |C|$ . If  $V \rightarrow_{\text{FTUS}}^* \emptyset$ , then the board is *drainable* to  $S$ . Figure 3 illustrates particles moved to and removed at a sink.

■ **Table 1** Frequently used notation.

Notation	Meaning	Defined in
$C \rightarrow_M^* D$	$D$ is reachable from $C$ in model $M$	Section 2
$\text{FT} = \{u_\infty, d_\infty, \ell_\infty, r_\infty\}$	the full tilt model	Section 2
$G_F(B)$	the full tilt graph of a board $B$	Section 3
$G_S(B)$	the small tilt graph of a board $B$	Section 3
$B^{\uparrow k}$	the board $B$ scaled by a factor $k$	Section 4.2
$G^{\leftrightarrow S}$	the extended graph of $G$ with a set of sinks $S$	Section 4.2
$G_L(B, S)$	the large tilt graph of a board $B$ and a set of sinks $S$	Section 4.2
$\text{S1} = \{u_1, d_1, \ell_1, r_1\}$	the single step model	Section 5
$M_I$	the interval extension of model $M \in \{\text{FT}, \text{S1}\}$	Section 5
$\overline{M}$	the dual model of $M$	Section 5.2

Further notation is introduced as needed in individual sections. Table 1 provides a reference of frequently used symbols.



■ **Figure 3** Draining three particles to a sink  $s$  using full tilt moves.

### 3 Drainability in the Full Tilt Model

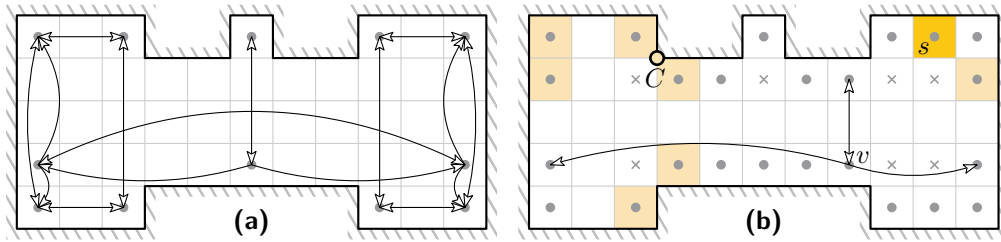
In this section, we examine an algorithmic approach to deciding drainability in the full tilt model, starting with the following theorem. This is a special case of a property that holds in more general classes of models, which is why we postpone the proof until Theorem 20 in Section 5.

► **Theorem 1.** *A board  $B = (V, E)$  is drainable to a set of sinks  $S \subseteq V$  if and only if for every  $p \in V$  there exists some  $s \in S$  such that  $p \rightarrow_{\text{FT}}^* s$ .*

The characterization in Theorem 1 suggests an approach to deciding drainability. Consider what we call the *full tilt graph*  $G_F(B) = (V_F, E_F)$  of a board  $B = (V, E)$ .  $V_F = V$  contains all pixels of  $B$  and  $E_F = \{(p, q) \in V^2 : p \neq q, q \in \{p^\ell, p^r, p^u, p^d\}\}$  connects  $p$  to  $q$  if a particle at  $p$  can reach  $q$  in a single move. Deciding drainability is equivalent to testing if there is a path in  $G_F$  from every pixel to a sink. However, this is unnecessarily inefficient. In particular, we do not need to check every pixel and require only the boundary as input.

► **Corollary 2** (\*). *A board  $B = (V, E)$  is drainable to a set of sinks  $S \subseteq V$  if and only if for every corner pixel  $c \in V$  there exists some  $s \in S$  such that  $c \rightarrow_{\text{FT}}^* s$ .*

We now describe an algorithm that, given the boundary of a board  $B = (V, E)$  and a set  $S \subseteq V$  of potential sinks, finds a minimum-cardinality subset  $S' \subseteq S$  such that  $B$  is drainable to  $S'$  – or reports that no such set exists. As a consequence, we can decide whether a board is drainable to a set  $S$  by supplying  $S$  to the algorithm and observing if it returns a subset or reports failure. The general approach is the same as the one used by the authors of [6]. However, due to the rectilinear nature of board boundaries, and our restriction to tilting in only four directions, our algorithm requires less time.



**Figure 4** (a) The small tilt graph,  $G_S(B)$ , of a board  $B$ . (b) The vertices of the large tilt graph,  $G_L(B, \{s\})$ , of  $B$  and a sink  $s$ . The lightly-shaded pixels are added due to the reflex corner  $C$  (though some are also part of the graph as corner pixels). Vertices marked with a cross are added as intersections. We only depict the edges incident on vertex  $v$  to avoid clutter.

By Corollary 2, the only pixels relevant for drainability are those reachable from corner pixels. This allows us to restrict our algorithm to the *small tilt graph*,  $G_S(B) = (V_S, E_S)$ , of  $B$ . Let  $V' \subseteq V$  be the set of corner pixels of  $B$ . Then  $V_S$  is the closure of  $V'$  under  $\rightarrow_{FT}^*$ , and  $G_S$  is the subgraph of  $G_F$  induced by  $V_S$ ; see Figure 4(a). For a boundary with  $n$  corners,  $|V_S| \in \mathcal{O}(n)$  as there are at most two additional pixels for every reflex corner of the boundary, and  $|E_S| \in \mathcal{O}(n)$  as at most three other pixels are reachable from every boundary pixel in a single move.

► **Lemma 3** ( $\star$ ).  $G_S(B)$  can be constructed from the boundary of a board  $B$  in  $\mathcal{O}(n \log n)$  time and  $\mathcal{O}(n)$  space, where  $n$  is the number of corners of the boundary.

**Proof sketch.** The main idea is to use a data structure by Sarnak and Tarjan [28] allowing us to quickly compute the boundary pixels in segments of a given pixel after preprocessing. ◀

The next step is to consider the condensation  $G_S^* = (V_S^*, E_S^*)$  of  $G_S$ , i.e., the directed graph that has as vertices the strongly-connected components of  $G_S$  and an edge from component  $A$  to component  $B$  if there is an edge from a vertex in  $A$  to a vertex in  $B$  in  $G_S$ . Then,  $B$  is drainable to every set  $S$  that contains at least one pixel from every sink  $V' \in V_S^*$ . A suitable subset of sinks, if one exists, can thus be computed in  $\mathcal{O}(|V_S|)$  time and space. We refer to the full version [25] for details regarding this approach.

► **Theorem 4.** Given the  $n$ -corner boundary of a board  $B = (V, E)$  and a set of sinks  $S \subseteq V$ , a minimum-cardinality subset  $S' \subseteq S$  such that  $V \rightarrow_{FT \cup S'}^* \emptyset$  can be found in  $\mathcal{O}(n \log n + |S|)$  time and  $\mathcal{O}(n)$  space, if one exists.

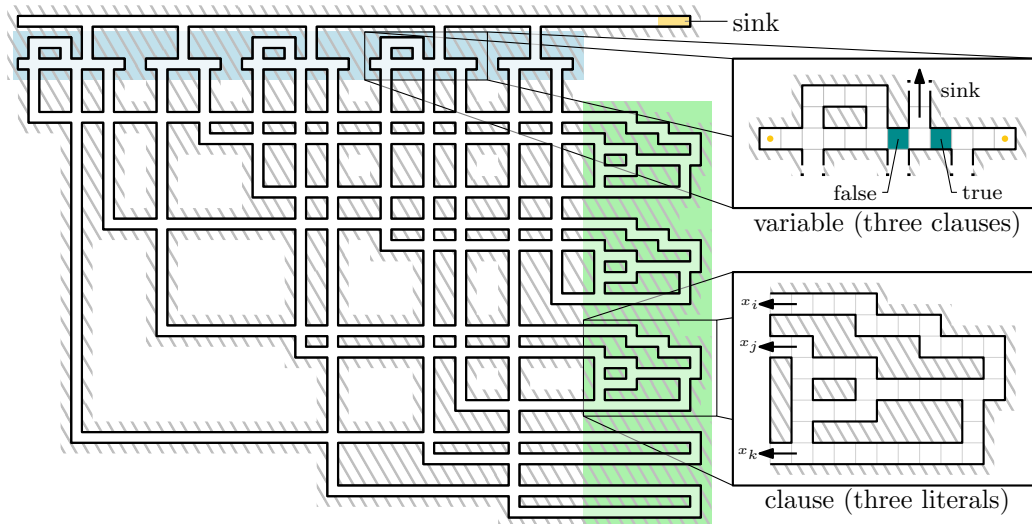
## 4 Placing Obstacles to Guarantee Drainability

Although not every board is drainable, slight changes may render a board drainable, see Figures 2(b) and 2(c). Given a board and a set  $S$  of sinks, we want to determine a maximum-size sub-board drainable to  $S$ . In other words, we want to place a minimum number of *obstacles* such that the remaining board is drainable. We refer to this as the **MAXIMUM TILT DRAINING PROBLEM**.

### 4.1 Computational Complexity

In this section, we give the high level idea of a reduction showing that the problem is NP-hard. Our reduction is from 3SAT-3 [30], and works as follows; we refer to Figure 5 for an overview. The problem 3SAT-3 is a variant of 3SAT having the additional property that every variable

appears at most 3 times; thus, we may assume that each variable occurs at least once negated, and once unnegated. Then, for every instance  $\varphi$  of 3SAT-3, we construct a polyomino  $P_\varphi$  that is an instance of the MAXIMUM TILT DRAINING PROBLEM.



■ **Figure 5** Overview of the NP-hardness reduction for the full tilt variant. The depicted instance is due to the 3SAT-3 formula  $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_4 \vee \neg x_5) \wedge (x_2 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_4) \wedge (\neg x_3 \vee x_5)$ .

For this, we add for each variable the respective variable gadget. As illustrated in Figure 5, the variable gadgets are placed in row at the top of the polyomino. For each clause, we place the respective clause gadget vertically in row at the right side of the construction. We then connect each variable with its respective clauses by thin L-shaped corridors.

It is easy to observe that we need at least one obstacle per variable to drain a variable gadget of  $P_\varphi$ . Furthermore, from every other position we can reach a variable gadget, hence, no further obstacles are needed. By carefully arguing we can show that  $P_\varphi$  is drainable if and only if  $\ell$  many obstacles are placed at very specific locations within the variable gadgets, where  $\ell$  is the number of variables in  $\varphi$ . This leads to the following theorem.

► **Theorem 5** ( $\star$ ). *It is NP-hard to decide whether placing  $k$  obstacles suffices to guarantee drainability of a thin polyomino.*

## 4.2 An Approximation Algorithm for Scaled Boards

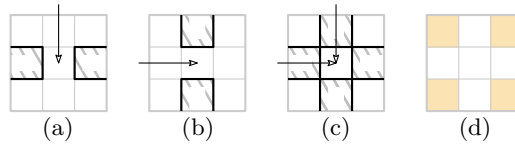
Our hardness proof relies on thin segments in the variable gadgets. To achieve positive results, we restrict the considered boards to ones that do not have thin segments by introducing scaling. In a  $k$ -scaled board  $B^{\uparrow k}$  of a board  $B$ , every tile of the underlying square tiling gets replaced by a  $k \times k$  grid of tiles. This subdivides every pixel  $p$  into  $k^2$  sub-pixels, for which  $p$  is their super-pixel. Sinks get scaled as well, which means that the move associated with a sink may remove up to  $k^2$  particles at once. By Corollary 2, only the sub-pixels reachable from corner sub-pixels are relevant in terms of drainability. We call these outer sub-pixels; the remaining sub-pixels are inner sub-pixels. Without any placed obstacles, only sub-pixels at a corner of their super-pixel can possibly be outer sub-pixels, see Figure 6(d).

The basic idea of our approach is to employ  $k$ -scaling, for  $k \geq 3$ , and use inner sub-pixels as positions for obstacles, leading to new outer sub-pixels as intermediate steps on paths from every pixel to a sink. We assume that a given board contains a sink in every region –

the only way to handle a region without a sink is to fill it with obstacles. The chief benefit we get from obstacles are new edges in the full tilt graph. We want as few new edges as possible while guaranteeing the existence of paths from every corner pixel to a sink.

For a directed graph  $G = (V, E)$  and a set of sinks  $S \subseteq V$ , we define the *extended graph*  $G^{\leftrightarrow S} = (V \cup \{r\}, E^{\leftrightarrow S})$  to be the weighted super-graph of  $G$  that contains all edges  $e \in E$  with weight zero and, for every edge  $(p, q) \in E$  with  $(q, p) \notin E$ , the *inverse edge*  $(q, p)$  with weight one, in addition to edges  $(s, r)$  with weight zero from every sink  $s$  to a newly added vertex  $r$ , called the *root*. Note that the extended graph contains an arborescence converging to  $r$ , as long as there is a sink in every undirected component of the original graph.

Conceptually, we place *turn gadgets* at sub-pixels of the heads of inverse edges in a minimum-weight arborescence of the full tilt graph. For scaling factor 3, a turn gadget consists of two obstacles placed perpendicular to the direction of the inverse edge in the middle of the pixel, see Figures 6(a) to (c). The central sub-pixel in Figure 6(c) gets disconnected by the obstacles from the rest of the board and is not considered an outer sub-pixel, even though it is a corner sub-pixel; it will be handled separately in the proof of Theorem 10.



■ **Figure 6** Turn gadgets and outer sub-pixels for scaling factor 3: (a) Horizontal turn gadget determined by a vertical inverse edge. (b) Vertical turn gadget determined by a horizontal inverse edge. (c) Horizontal and vertical turn gadgets at the same pixel determined by two inverse edges. (d) Potential outer sub-pixels.

### The Large Tilt Graph

Before we go into the details of the algorithm, we have to tackle one issue. As was the case when deciding drainability in Section 3, computing the full tilt graph from the boundary of a board is prohibitively expensive. However, the small tilt graph developed for that purpose is insufficient in this case, since it may contain undirected components without a sink. To find a compromise that allows computing a feasible solution using a reasonable amount of resources, we introduce the *large tilt graph*  $G_L(B, S) = (V_L, E_L)$  of a board  $B$  with set of sinks  $S$ , which is another subgraph of the full tilt graph and a super-graph of  $G_S$ . In addition to the vertices of  $V_S$ ,  $V_L$  contains all sinks  $s \in S$  and the pixels  $\{s^\ell, s^r, s^u, s^d\}$  for every sink  $s$ . Furthermore, for every reflex corner of the boundary between two segments, the leftmost and rightmost (for row segments) or topmost and bottommost (for column segments) pixels in those segments are included, as well as all pixels on intersections of row segments and column segments containing an included pixel. See Figure 4(b) for an example.  $G_L$  is the subgraph of  $G_F$  induced by  $V_L$ .

► **Lemma 6** (\*).  $G_L$  can be constructed from the boundary of  $B$  and a set of sinks  $S$  in  $\mathcal{O}(n \log n + K)$  time and  $\mathcal{O}(n + K)$  space, where  $n$  is the sum of the number of corners on the boundary and the number of sinks, and  $K$  is the number of vertices arising from intersections.

**Proof sketch.** We use the algorithm by Balaban [7] to find segment intersections. ◀

► **Observation 7** (\*). The large tilt graph of a region is weakly connected.

■ **Algorithm 1** Computing a drainable sub-board of  $B^{\uparrow k}$ , for  $k \geq 3$ .

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**Input:** The boundary of a board  $B = (V, E)$  and a set  $S \subseteq V$  such that every region of  $B$  contains an  $s \in S$ .

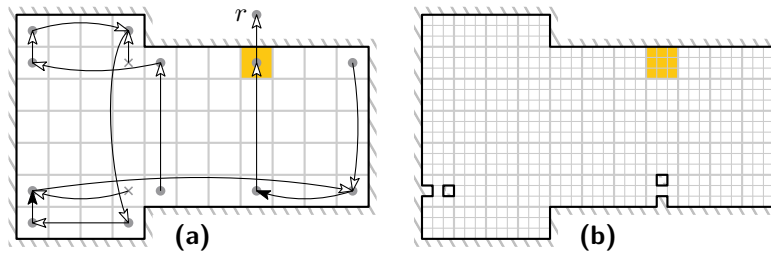
**Output:** A set of sub-pixels  $O$  such that  $B^{\uparrow k}$  without  $O$  is drainable to  $S$ .

- 1 Construct  $G_L$  using Lemma 6.
- 2 Give all edges of  $E_L$  a weight of zero and add inverse edges  $(q, p)$  of weight one whenever  $(p, q) \in E_L$  and  $(q, p) \notin E_L$ .
- 3 Add root  $r$  and edges  $(s, r)$  of weight zero for all  $s \in S$ .
- 4 Compute a minimum-weight arborescence  $T$  converging to  $r$  in the resulting  $G_L^{\rightarrow S}$ .
- 5  $O := \emptyset$
- 6 **foreach** inverse edge  $(q, p)$  in  $T$  **do**
- 7 | Add the obstacles of the turn gadget determined by  $(q, p)$  to  $O$ .
- 8 **end**
- 9 **return**  $O$

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### Analysis of the Algorithm

We provide a listing of the steps in Algorithm 1 and proceed to analyze its properties and performance. Figure 7 illustrates an arborescence in the extended large tilt graph of a board  $B$  with a single sink and the resulting obstacle placement in  $B^{\uparrow 3}$ .



■ **Figure 7** (a) An arborescence in the extended large tilt graph of a board  $B$  with a single sink. Inverse edges are drawn with a black arrowhead. (b) The resulting obstacle placement in  $B^{\uparrow 3}$ .

Before we prove that Algorithm 1 produces a drainable sub-board, it is important to observe that certain (inverse) edges of the extended large tilt graph cannot appear together in a minimum-weight arborescence of the large tilt graph, or only under special circumstances. Lemmas 8 and 9 not only ensure the correctness of a solution but also help bound the number of placed obstacles.

► **Lemma 8** ( $\star$ ). *For two distinct inverse edges  $(p_1, q_1)$  and  $(p_2, q_2)$  in a minimum-weight arborescence of the extended large tilt graph with their heads  $q_1$  and  $q_2$  in the same row segment  $R$ , at most one of their tails  $p_1$  and  $p_2$  can be in  $R$ .*

► **Lemma 9** ( $\star$ ). *If there are two distinct inverse edges  $(p_1, q)$  and  $(p_2, q)$  in a minimum-weight arborescence of the extended large tilt graph, then  $q$  is a sink.*

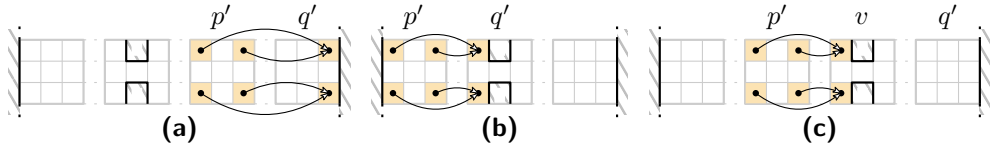
► **Theorem 10.** *Algorithm 1 produces a drainable sub-board of  $B^{\uparrow k}$ , using  $\mathcal{O}((n + K) \log n)$  time and  $\mathcal{O}(n + K)$  space, where  $n$  is the sum of the number of corners of the boundary and the number of sinks, and  $K$  is the number of vertices of  $G_L$  arising from intersections.*

**Proof.** We first show that a sub-board produced by Algorithm 1 is drainable. Note that Algorithm 1 assumes that every region of the board contains a sink, which ensures that the extended large tilt graph of  $B$  contains a path from every pixel to a sink, by Observation 7. By Lemma 9, the only pixels with two perpendicular turn gadgets are sinks. Therefore, particles at all of their sub-pixels can be removed, including the one in the middle that gets disconnected from the rest of the board by the turn gadgets, see Figure 6(c). The way the turn gadgets are constructed guarantees that this is the only possible instance of a corner sub-pixel that is not also an outer sub-pixel. By Corollary 2, it now suffices to show that for every outer sub-pixel  $p$  there is a sub-pixel  $s$  of a sink such that  $p \rightarrow_{\mathbb{F}_T}^* s$ , which we prove by strong induction on the distance  $d$  from the super-pixel  $p'$  of  $p$  to a sink in the arborescence of the extended large tilt graph.

For  $d = 0$ , we can choose  $s = p$ , since  $p$  is already a sub-pixel of a sink.

Now, assume  $d > 0$  and that every outer sub-pixel of a pixel closer than  $d$  to a sink in the arborescence has a sub-pixel of a sink reachable from it. Let  $q'$  be the unique pixel such that  $(p', q')$  is an edge in the arborescence. Without loss of generality, assume  $p'$  and  $q'$  are in the same row segment  $P$  with  $p'$  left of  $q'$ . By Lemma 8,  $P$  contains at most one vertical turn gadget. We distinguish three cases.

1. There is no vertical turn gadget in  $P$  or the only one is to the left of  $p$ . Then,  $q' = (p')^r$  and the move  $r_\infty$  moves a particle at  $p$  to a rightmost outer sub-pixel of  $q'$ , see Figure 8(a).
2. A vertical turn gadget is positioned at  $q'$ . Then, the move  $r_\infty$  moves a particle at  $p$  to a leftmost outer sub-pixel of  $q'$ , see Figure 8(b).
3. A vertical turn gadget is positioned at a pixel  $v$  strictly between  $p'$  and  $q'$ . Then, the move  $r_\infty$  moves a particle at  $p$  to a leftmost outer sub-pixel of  $v$ , see Figure 8(c). The inverse edge  $e$  determining the turn gadget at  $v$  is either  $((p')^\ell, v)$  or  $((p')^r, v)$ . Either way, the path from the other boundary pixel to a sink in the arborescence must include  $e$ . Otherwise,  $e$  could be replaced with an edge between the boundary pixels for a lower weight arborescence. Because the edge from  $p'$  is to  $q' = (p')^r$ ,  $v$  lies on the path from  $p'$  to a sink.



■ **Figure 8** The three cases in the proof of Theorem 10 for scaling factor 3.

In all three cases, we reach an outer sub-pixel with a super-pixel whose distance to a sink is strictly less than  $d$ . Thus, we can apply the induction hypothesis to reach a sink sub-pixel.

Both remaining options, namely a vertical turn gadget at  $p'$  or one to the right of  $q'$ , lead to contradictions. The first would imply an inverse edge  $((p')^\ell, p')$ , which could be replaced with  $((p')^\ell, (p')^r)$  for a lower weight arborescence. In the second,  $q'$  would not be a boundary pixel and there would need to be another vertical turn gadget at  $q'$ , contradicting Lemma 8.

As to the complexity, the initial steps can be completed in  $\mathcal{O}(n \log n + K)$  time and  $\mathcal{O}(n + K)$  space, by Lemma 6.  $G_L$  has  $\mathcal{O}(n + K)$  vertices and edges, with  $K \in \mathcal{O}(n^2)$ , so the minimum-weight arborescence can be computed in  $\mathcal{O}((n + K) \log n)$  time and  $\mathcal{O}(n + K)$  space using Tarjan’s algorithm [29], which dominates the requirements of the remaining steps. ◀

► **Lemma 11.** *For a board  $B = (V, E)$ , a set of sinks  $S \subseteq V$ , and a set of obstacles  $O \subseteq V$  such that  $B$  without  $O$  is drainable to  $S$ , there is an arborescence in the extended full tilt graph of  $B$  of total weight at most  $2|O|$ .*

**Proof.** Consider the full tilt graph of  $B$  and add, for every obstacle at a pixel  $p \in O$ , the inverse edges  $(p^\ell, p - (1, 0)^T)$ ,  $(p^\ell, p + (1, 0)^T)$ ,  $(p^d, p - (0, 1)^T)$  and  $(p^d, p + (0, 1)^T)$ . Call the resulting graph  $H$ . Now, for every two pixels  $p, q$ , with  $(p, q)$  in the full tilt graph of  $B$  without  $O$ , there is a path from  $p$  to  $q$  in  $H$ : If  $q$  is not next to an obstacle in  $O$ , then  $(p, q)$  was in  $G_F$  to begin with. Otherwise, there is a path that first uses an edge in the initial full tilt graph from  $p$  to a boundary pixel  $v$ , followed by one of the added inverse edges from  $v$  to  $q$ . Thus,  $H$  is a subgraph of  $G_F^{\leftrightarrow S}$  that contains a path from every pixel to a sink, i.e., a minimum-weight arborescence in  $G_F^{\leftrightarrow S}$  has no larger weight than an arborescence in  $H$ . At most one outgoing inverse edge can be included per vertex, for a total weight of at most  $2|O|$  for any arborescence in  $H$ . ◀

Although we do all our calculations on the large tilt graph of the given board, bounding the approximation ratio requires the weight of a minimum arborescence of the extended full tilt graph of the scaled board. Lemmas 12 and 13 together show that such an arborescence cannot have smaller weight than the one we compute.

► **Lemma 12** ( $\star$ ). *If there is an arborescence of weight  $w$  in  $G_F^{\leftrightarrow S}(B^{\uparrow k})$ , for any  $k \geq 1$ , then there is an arborescence of weight at most  $w$  in  $G_F^{\leftrightarrow S}(B)$ .*

► **Lemma 13** ( $\star$ ). *If there is an arborescence of weight  $w$  in  $G_F^{\leftrightarrow S}(B)$ , then there is an arborescence of weight at most  $w$  in  $G_L^{\leftrightarrow S}(B, S)$ .*

► **Theorem 14.** *When applied with scaling factor 3, Algorithm 1 places at most 4 times as many obstacles as used in an optimum solution for  $B^{\uparrow 3}$ .*

**Proof.** Let  $w$  be the number of inverse edges in a minimum-weight arborescence of the extended large tilt graph of  $B$  and  $S$ . Then, the number of obstacles placed by Algorithm 1 is  $|\text{ALG}| = 2w$  because two obstacles are placed per inverse edge, see Figure 6. Let  $|T(G)|$  denote the weight of a minimum-weight arborescence in a graph  $G$ . The optimum number  $|\text{OPT}|$  of obstacles can be bounded in the following way.

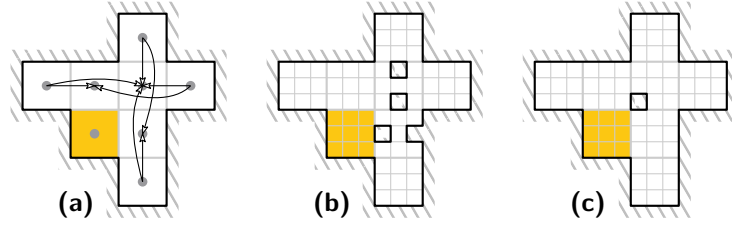
$$\begin{aligned} 2|\text{OPT}| &\geq |T(G_F^{\leftrightarrow S}(B^{\uparrow 3}))| && \text{(Lemma 11)} \\ &\geq |T(G_F^{\leftrightarrow S}(B))| && \text{(Lemma 12)} \\ &\geq |T(G_L^{\leftrightarrow S}(B, S))| = w && \text{(Lemma 13)} \end{aligned}$$

Therefore,  $\frac{|\text{ALG}|}{|\text{OPT}|} \leq 4$ . ◀

This is tight as there are 3-scaled regions for which Algorithm 1 places exactly four times as many obstacles as required, see Figure 9. For larger scaling factors we obtain the following:

► **Corollary 15** ( $\star$ ). *When applied with scaling factor  $k > 3$ , Algorithm 1 places at most 6 times as many obstacles as used in an optimum solution for  $B^{\uparrow k}$ .*

We leave it as an open question whether our approach can be adapted to 2-scaling. Using the same ideas, we can ensure drainability of 2-scaled regions by placing thin walls instead of obstacles, i.e., by removing edges of the underlying graph instead of vertices. It is not clear how obstacles can be placed on  $2 \times 2$  super-pixels to redirect particles coming from opposite sides. An approximation may still be viable, but it will require a more sophisticated argument that will certainly lose clarity.



■ **Figure 9** (a) The inverse edges in  $G_L^{\uparrow S}(B)$  of a board  $B$  with a single sink  $s \in S$ . (b) Obstacles placed by Algorithm 1 for  $B^{\uparrow 3}$ . (c) An optimum obstacle placement for the same board and sink.

Furthermore, note how two perpendicular turn gadgets as placed in Figure 6(c) split a connected board into two regions that need to be drained individually. We can easily maintain connectivity by placing another obstacle at the intersection of the turn gadgets – at the cost of increasing the approximation ratio by 1. A more elaborate question is whether we can maintain the genus of the board, i.e., do not increase the number of holes, while achieving the same approximation ratio. We leave this as an open problem as well.

## 5 Drainability and Fillability in Generalized Models

We now step away from the full tilt model and consider arbitrary models. For a model  $M$  we define the model  $M^*$  to contain all compositions of moves from  $M$ , i.e.,  $m \in M^*$  if there are  $m_1, m_2, \dots, m_k \in M$  such that  $m = m_k \circ \dots \circ m_2 \circ m_1$ , for any  $k \geq 0$ . We generalize the concepts of minimality and drainability in the obvious way. A move  $m$  is *monotone* if  $C \subseteq D$  implies  $m(C) \subseteq m(D)$ ; it is *volume-preserving* if  $|m(C)| = |C|$  for all  $C$ . Note that the moves associated with sinks are monotone but obviously not volume-preserving. Models are called monotone or volume-preserving if all of their moves have the respective property.

The most well-studied model, apart from the full tilt model, is the *single step model*,  $S1 = \{u_1, d_1, l_1, r_1\}$ , which has particles move to an adjacent pixel in one of the four directions, unless they are blocked by the boundary or another particle. Formally, call a pixel *left-blocked* in a configuration  $C$  if it and every pixel left of it in its row segment are occupied in  $C$ . Then  $p \in \ell_1(C)$  if and only if  $p$  is left-blocked in  $C$  or  $p + (1, 0)^T \in C$ . The other moves are defined analogously with respect to the other directions. Additionally, we introduce an extension to the full tilt and single step models that allows movement to be restricted to a subset of the segments parallel to the direction of movement. Let  $M \in \{FT, S1\}$ . Then the *interval extension*  $M_I$  has moves  $m^{[i,j]} \in M \times \mathbb{Z}^2$  that apply the move  $m \in M$  to those segments whose  $y$ -coordinate (in the case of row segments and horizontal moves) or  $x$ -coordinate (for column segments and vertical moves) lies in the interval  $[i, j]$ , and leave all other segments of the affected type unchanged. See the bottom half of Figure 12 for exemplary moves in  $FT_I$ . We start with an easily verified observation.

► **Observation 16.** *The models FT, S1, FT<sub>I</sub>, and S1<sub>I</sub> are monotone and volume-preserving.*

► **Proposition 17** ( $\star$ ). *A board is drainable in a monotone model if and only if  $\emptyset$  is the only minimal configuration.*

► **Lemma 18.** *A configuration  $C$  is minimal with respect to a set of sinks  $S$  in a volume-preserving model if and only if no  $s \in S$  is occupiable from  $C$ .*

**Proof.** Let  $M$  be a volume-preserving model,  $S$  a set of sinks and  $C$  a configuration. First, assume there is a sink  $s \in S$  and a configuration  $D$  reachable from  $C$  such that  $s \in D$ . Then  $|s(D)| < |D| = |C|$ , i.e.,  $C$  is not minimal. Now, assume  $C$  is not minimal. Then there

is a shortest sequence of moves  $m_1, m_2, \dots, m_k \in M \cup S$  such that  $D = m_k \circ \dots \circ m_2 \circ m_1(C)$  and  $|D| < |C|$ . Since the sequence is shortest, and all  $m \in M$  are volume-preserving,  $m_k$  must be associated with a sink  $s \in S$  and  $s \in D$ . ◀

Although the characterization of minimality in Lemma 18 applies to all volume-preserving models, its consequences vary. In the full tilt model it implies that deciding minimality is PSPACE-complete, whereas it is trivial in the single step model, as we show in Proposition 19 and Observation 21, respectively. In contrast to this, the characterization of drainability in Theorem 20 leads to a polynomial-time decision procedure for the full tilt model, as we saw when we investigated this special case in Section 3.

► **Proposition 19.** *Given a configuration  $C$  of a board  $B$  and a set  $S$  of sinks, it is PSPACE-complete to decide whether  $C$  is minimal with respect to  $S$  in the full tilt model.*

**Proof.** This is a consequence of Lemma 18 and Theorem 5.1 in [8], which states that the occupancy problem is PSPACE-complete in the full tilt model. ◀

► **Theorem 20.** *A board  $B = (V, E)$  is drainable to a set of sinks  $S$  in a monotone, volume-preserving model  $M$ , if and only if, for every  $p \in V$  there is  $s \in S$  such that  $p \rightarrow_M^* s$ .*

**Proof.** First, assume  $B$  is drainable. Then, by Proposition 17, no configuration  $\{p\}$  is minimal, for any  $p \in V$ . By Lemma 18, and since  $M$  is volume-preserving, this means there is  $s \in S$  such that  $p \rightarrow_M^* s$ .

Now, assume that for every  $p \in V$  there is  $s \in S$  with  $p \rightarrow_M^* s$ . Assume, for sake of contradiction, that there is a minimal configuration  $C \neq \emptyset$ . Let  $p \in C$  and  $m \in M^*$  such that  $m(\{p\}) = \{s\}$  for a sink  $s \in S$ . Then  $m(C) \supseteq m(\{p\}) = \{s\}$ , due to monotonicity. Thus,  $C$  is not minimal by Lemma 18 – a contradiction. Therefore,  $\emptyset$  is the only minimal configuration, which by Proposition 17 implies that  $B$  is drainable. ◀

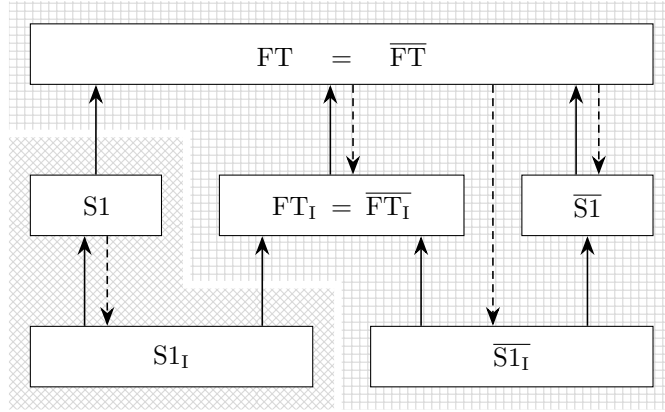
► **Observation 21.** *Every board is drainable in the single step model, as long as every one of its regions contains a sink.*

**Proof.** As every region contains a sink, there is a path on the board from every pixel  $p$  to a sink  $s$ . This path entails a sequence of single step moves transforming  $\{p\}$  into  $\{s\}$ . Thus, the board is drainable, by Observation 16 and Theorem 20. ◀

## 5.1 Relative Power of Various Models

Clearly, some models are more powerful than others, in the sense that they allow us to reach more configurations. We make this notion precise by saying that a model  $M_2$  *simulates* a model  $M_1$  if for every  $m \in M_1$  there is  $m' \in M_2^*$  such that  $m(C) = m'(C)$  for every configuration  $C$ . It is easy to see that S1 simulates FT (for any  $m \in \{u, d, \ell, r\}$ ,  $m_\infty$  can be simulated using  $D$  repetitions of  $m_1$ , where  $D$  is the maximum diameter among all regions of the board) and  $M_I$  simulates  $M$ , for  $M \in \{\text{FT}, \text{S1}\}$  (choose the interval to encompass the whole board); see the left half of Figure 10.

For the purpose of draining a board, it is useful to compare models with respect to their moves acting on single particles. To this end, we say that model  $M_2$  *simulates*  $M_1$  *on singletons* if, for every pixel  $p$  and move  $m \in M_1$ , there is  $m' \in M_2^*$  such that  $m(\{p\}) = m'(\{p\})$ , i.e., if  $p \rightarrow_{M_1} q$  implies  $p \rightarrow_{M_2}^* q$ , for all pixels  $q$ . Two models that simulate each other on singletons are called *equivalent on singletons*. Note that general simulation entails simulation on singletons and that S1 and FT simulate their respective interval extensions on singletons.



■ **Figure 10** Overview of the studied models and their relationships. A solid arrow from model  $A$  to model  $B$  indicates that  $A$  simulates  $B$ ; a dashed arrow indicates that  $A$  simulates  $B$  on singletons. Note that solid arrows include dashed arrows and arrows arising due to transitivity have been omitted. All models in the upper part (with the axis-parallel crosshatch pattern) are equivalent on singletons, as are the ones in the lower part (with the diagonal crosshatch pattern). Arrows on the left-hand side are derived in Section 5.1; those on the right-hand side and the equalities come from Section 5.2.

We now define a class of models with the property that all its members are equivalent on singletons to  $FT$ . It trivially includes  $FT$  and  $FT_1$ . Intuitively, these are the models that move at least some particles maximally, and allow doing so in all four directions.

► **Definition 22.** *A monotone and volume-preserving model  $M$  is tilt-compatible if the following conditions are satisfied for all configurations  $C$  and all occupied pixels  $p \in C$ .*

1. For all  $m \in M$ ,  $\{p^\ell, p^r, p^u, p^d, p\} \cap m(C) \neq \emptyset$ .
2. For all  $x \in \{\ell, r, u, d\}$ , there is  $m \in M$  such that  $p^x \in m(C)$ .

► **Proposition 23.** *Every tilt-compatible model is equivalent on singletons to  $FT$ .*

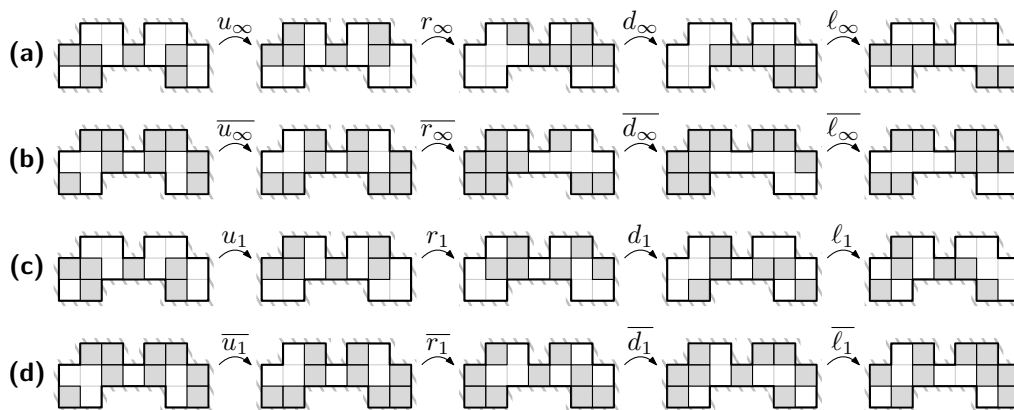
**Proof.** Let  $M$  be a tilt-compatible model and  $p$  a pixel. First, observe that  $x_\infty(\{p\}) = \{p^x\}$ , for all directions  $x \in \{u, d, \ell, r\}$ . Thus, by Condition 2 of Definition 22 and the fact that  $M$  is volume-preserving, there is  $m \in M$  with  $m(\{p\}) = x_\infty(\{p\})$ . Now, consider any  $m \in M$ . By Condition 1, and since  $m$  is volume-preserving,  $m(\{p\})$  is one of  $\{p\}$ ,  $\{p^\ell\}$ ,  $\{p^r\}$ ,  $\{p^u\}$ , or  $\{p^d\}$ . In the first case, the empty sequence  $\varepsilon \in FT^*$  satisfies  $\varepsilon(\{p\}) = \{p\}$ ; in the latter cases  $x_\infty(\{p\}) = \{p^x\} = m(\{p\})$ , for all directions  $x \in \{u, d, \ell, r\}$ . ◀

► **Corollary 24.** *For every tilt-compatible model  $M$ , a board is drainable to a set of sinks  $S$  in  $M$  if and only if it is drainable to  $S$  in  $FT$ .*

## 5.2 Duality of Various Models

It can prove enlightening to imagine free pixels to instead be occupied by a different kind of particle, called *bubbles*, and analyze their behavior under a sequence of moves. This leads to the concept of a *dual move* for a move  $m$ , defined as  $\bar{m}(C) = V \setminus m(V \setminus C)$ . In the *dual model*  $\bar{M} = \{\bar{m} : m \in M\}$  of a model  $M$ , particles move as bubbles do in  $M$ , and vice versa; examples are depicted in Figure 11.

► **Proposition 25.**  $FT = \bar{FT}$  and  $FT_1 = \bar{FT}_1$ .



■ **Figure 11** Particle movement in (a) FT, (b)  $\overline{\text{FT}}$ , (c) S1, and (d)  $\overline{\text{S1}}$ . Note that every move in  $\overline{\text{FT}}$  corresponds to a move in FT, whereas no such correspondence exists between  $\overline{\text{S1}}$  and S1.

**Proof.** We merely show  $\ell = \bar{r}$ , which easily extends to  $\ell^{[i,j]} = \bar{r}^{[i,j]}$ ; a very similar argument applies to other pairs of opposite directions. Let  $C$  be a configuration,  $R$  a row segment,  $n = |R|$ , and  $k = |R \cap C|$ . Then exactly the rightmost  $n - k$  pixels of  $R$  are occupied in  $r(V \setminus C)$ , and exactly the leftmost  $k$  are occupied in  $V \setminus r(V \setminus C) = \bar{r}(C)$  – the same ones that are occupied in  $\ell(C)$ . ◀

► **Proposition 26** (★). *If a model  $M_2$  simulates a model  $M_1$ , then  $\overline{M_2}$  simulates  $\overline{M_1}$ .*

We now come to the reason why we introduced the concept of tilt-compatible models in the first place. It allows us to connect the full tilt model via duality to the single step model and its interval extension. An overview of the resulting relationships between the models in depicted in Figure 10.

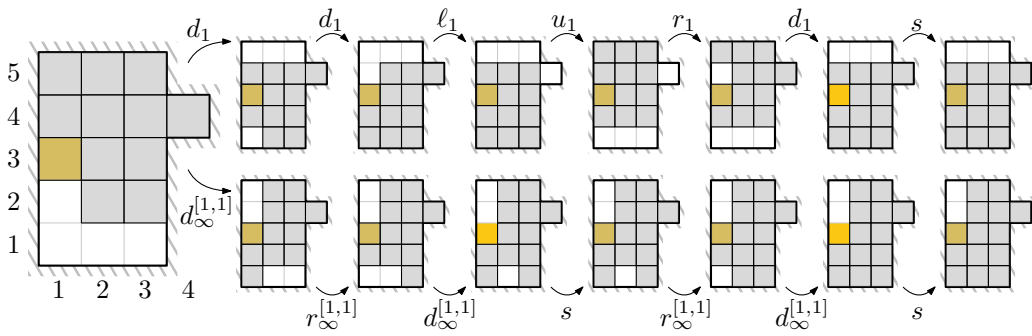
► **Proposition 27.** *The models  $\overline{\text{FT}}$ ,  $\overline{\text{FT}}_I$ ,  $\overline{\text{S1}}$ , and  $\overline{\text{S1}}_I$  are tilt-compatible.*

**Proof.** The claim is easy to see for  $\overline{\text{FT}}$  and  $\overline{\text{FT}}_I$  because they are dual to themselves. For  $\overline{\text{S1}}$  observe the movement of bubbles in S1. Let  $p$  be a free pixel in a configuration  $C$  and consider for every direction  $x \in \{u, d, \ell, r\}$  the opposite direction  $y$  ( $d, u, r$ , and  $\ell$ , respectively). Then  $p^y \notin x_1(C)$ , i.e.,  $p^y \in \bar{x}_1(V \setminus C)$ . This satisfies condition 2 of Definition 22. Since  $\bar{u}_1, \bar{d}_1, \bar{\ell}_1$ , and  $\bar{r}_1$  are the only moves of  $\overline{\text{S1}}$ , condition 1 holds as well.  $\overline{\text{S1}}_I$  simulates  $\overline{\text{S1}}$ , so condition 2 is easily satisfied. For condition 1, observe that particles move either as in  $\overline{\text{S1}}$  or not at all. ◀

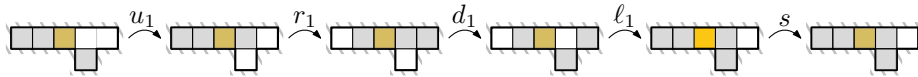
### 5.3 Fillability Instead of Drainability

Now, instead of removing as many particles as possible from a configuration, we aim to insert as many as possible. A *source* is a move associated with a pixel  $s \in V$  defined as  $C \mapsto V \cup \{s\}$ . Note that the dual move of a source at  $s$  is a sink at  $s$ . Analogously to the notions of minimality and drainability, a configuration  $C$  is *maximal* in a model  $M$  with respect to a set of sources  $S$  if there is no  $D$  reachable from  $C$  in  $M \cup S$  with  $|D| > |C|$ , and a board  $B = (V, E)$  is *fillable* from  $S$  if  $\emptyset \rightarrow_{M \cup S}^* V$ .

Due to  $\text{FT}_I$  and S1 simulating FT, it is clear that both are at least as powerful as FT when it comes to reaching large maximal configurations. Figure 12 shows that they are both strictly more powerful. In fact,  $\text{S1}_I$  is more powerful still and could, in this example, insert one more particle than  $\text{FT}_I$ . This example would seem to indicate that  $\text{FT}_I$  is more powerful than S1 but there are examples where the situation is reversed, as illustrated in Figure 13.



■ **Figure 12** A configuration that is maximal with respect to a source  $s$  in the full tilt model, and sequences of moves that lead to maximal configurations in S1 (top row) and FT<sub>1</sub> (bottom row). A sequence of moves leading to the initial configuration is  $m_1, r_\infty, m_1, r_\infty, m_1, m_2, \ell_\infty, m_2, \ell_\infty, m_2, u_\infty$ , where  $m_1$  is  $s, d_\infty, s, d_\infty, s$ , and  $m_2$  is  $u_\infty, r_\infty, d_\infty, s$ .



■ **Figure 13** A configuration that is maximal with respect to a source  $s$  in FT<sub>1</sub>, and a sequence of moves that leads to a maximal configuration in S1.

► **Observation 28.** *A configuration  $C$  is maximal in a model  $M$  with respect to  $S$  if and only if  $V \setminus C$  is minimal in  $\bar{M}$  with respect to  $S$ . Therefore, a board is fillable from  $S$  in  $M$  if and only if it is drainable to  $S$  in  $\bar{M}$ .*

Finally, we come to the point when all the groundwork laid in the previous subsections pays off. While drainability is trivial in S1, fillability is not as obvious. Duality allows us to answer the question by considering drainability in  $\bar{S1}$ , which we have connected to FT.

► **Theorem 29.** *If the dual  $\bar{M}$  of a model  $M$  is tilt-compatible, then a board  $B$  is fillable from a set  $S$  in  $M$  if and only if  $B$  is drainable to  $S$  in FT.*

**Proof.** Follows from Corollary 24 and Observation 28. ◀

Consequently, the models FT, FT<sub>1</sub>, S1, and S1<sub>I</sub> are all equivalent with respect to fillability. Most importantly, the results regarding drainability in the full tilt model presented in this paper, including hardness and approximation for obstacle placement, apply to fillability in all these models. We conclude by highlighting the most intriguing special case, which is quite amusing when taken out of the context of the previous subsections.

► **Corollary 30.** *A board is fillable from a set  $S$  in the single step model if and only if it is drainable to  $S$  in the full tilt model.*

## 6 Conclusions and Future Work

In this paper, we have analyzed ways to make a board fillable or drainable using particles moving as instructed by several models of global control signals. We have shown that placing a minimum number of obstacles to achieve fillability is NP-hard in all considered models. However, a constant-factor approximation is possible for scaled boards. The most apparent open question concerns the complexity of the obstacle placement problem for scaled boards. Is it still NP-hard or can we do better than the approximation and solve it optimally?

The next step in our future work is to investigate how to actually fill a board, once we know it is fillable. An immediate approach would be to maintain a configuration in memory and successively move bubbles to a source until the board is full. However, this may require an amount of memory that is not polynomial in the size of an appropriate encoding of the boundary. Can we efficiently compute (short) filling sequences? Of particular interest is the worst case analysis of the length of filling sequences.

It would be interesting to better understand the geometric properties of fillable regions. In the related problem of assembly there is a hierarchy of constructable shapes [8]. These shapes derive from the *external* movement of particles, whereas ours derive from *internal* movement. Are these classifications related?

Our goal so far was to completely fill a region. Applications may impose restricted areas, i.e., positions that may never be occupied by particles, while requiring other areas to be filled. How can this constraint be handled?

Although we introduced duality of models as a tool to connect fillability to drainability, it leads to interesting new questions. Maximality in a model is strongly related to the occupancy problem in the dual model. However, occupiability in the dual model has a natural interpretation in the original model, leading to what we dub the *vacancy problem*: Given a configuration  $C$  and a pixel  $p \in C$ , is there a configuration  $D$  reachable from  $C$  that does not contain  $p$ ? This problem is PSPACE-complete in the full tilt model because so is the occupancy problem and FT is dual to itself. To the best of our knowledge, this natural problem has not been examined in the single step model.

► **Conjecture 31.** *The vacancy problem is PSPACE-complete in the single step model.*

A promising approach to prove the conjecture is to follow the ideas Caballero et al. [22] used to prove the relocation problem hard for the single step model. They built on the techniques of [8] and used an observation that can be seen as a precursor to our notion of duality, implying that the same methods are likely applicable in this case.

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