

Low-Temperature Sampling on Sparse Random Graphs*

Andreas Galanis ✉ 

Department of Computer Science, University of Oxford, UK

Leslie Ann Goldberg ✉ 

Department of Computer Science, University of Oxford, UK

Paulina Smolarova ✉ 

Department of Computer Science, University of Oxford, UK

Abstract

We consider sampling in the so-called low-temperature regime, which is typically characterised by non-local behaviour and strong global correlations. Canonical examples include sampling independent sets on bipartite graphs and sampling from the ferromagnetic q -state Potts model. Low-temperature sampling is computationally intractable for general graphs, but recent advances based on the polymer method have made significant progress for graph families that exhibit certain expansion properties that reinforce the correlations, including for example expanders, lattices and dense graphs.

One of the most natural graph classes that has so far escaped this algorithmic framework is the class of sparse Erdős-Rényi random graphs whose expansion only manifests for sufficiently large subsets of vertices; small sets of vertices on the other hand have vanishing expansion which makes them behave independently from the bulk of the graph and therefore weakens the correlations. At a more technical level, the expansion of small sets is crucial for establishing the Kotecky-Priess condition which underpins the applicability of the framework.

Our main contribution is to develop the polymer method in the low-temperature regime for sparse random graphs. As our running example, we use the Potts and random-cluster models on $G(n, d/n)$ for $d = \Theta(1)$, where we show a polynomial-time sampling algorithm for all sufficiently large q and d , at all temperatures. Our approach applies more generally for models that are monotone. Key to our result is a simple polymer definition that blends easily with the connectivity properties of the graph and allows us to show that polymers have size at most $O(\log n)$.

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1 Introduction

We consider the problem of sampling from high-dimensional distributions in the so-called low-temperature regime, which is typically characterised by non-local behaviour. A classical example that has been studied in this context is the problem of sampling independent sets

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on bipartite graphs where it is perhaps intuitive to expect that a typical independent set is correlated with one of the two sides of the bipartition. Another standard example is the ferromagnetic Potts model on (not necessarily proper) q -colourings weighted by the number of monochromatic edges (see below for definitions) where, at low temperatures, a typical colouring is expected to be correlated with one of the q monochromatic colourings.

Starting from [32, 34], a series of works have demonstrated that this correlation can be utilised to obtain fast sampling algorithms for certain graph classes that exhibit strong expanding properties; perhaps the most prominent graph example is the class of random regular graphs. Lattices (such as $\mathbb{Z}^d$) can also be treated using contour models. By contrast, it is far from clear how to apply this intuition to general graphs. In fact, sampling at low-temperatures is conjectured to be hard on general graphs, captured by so-called #BIS-hardness in the case of the Potts and independent-set examples.

Perhaps the most natural class of graphs that has so far remained elusive from this algorithmic framework is the class of sparse Erdős-Rényi random graphs like $\mathcal{G}(n, d/n)$ for $d = \Theta(1)$. This class has actually posed challenges even for high-temperature sampling [39, 16, 9, 3, 17, 43, 40] where the presence of high-degree variables typically complicates the applicability of standard sampling techniques. For low-temperature sampling however the obstacles come primarily from another side, namely from the presence of many small “sparse” parts which have weak, if any, expansion. For example, in $\mathcal{G}(n, d/n)$, this phenomenon not only causes the graph to be disconnected, but even inside the giant component there are *induced* paths of size roughly $\Theta(\log n)$ that therefore have almost no expansion; similarly, one can find various other sparse induced components which are scattered inside the giant.

Our main contribution is to develop the polymer method in the low-temperature regime for sparse random graphs. As our running example, we use the Potts and random-cluster models on $\mathcal{G}(n, d/n)$ for $d = \Theta(1)$ where we show a polynomial-time sampling algorithm for all sufficiently large q and d at all temperatures. Our approach applies more generally for models that are monotone. Key to our result is a simple polymer definition that blends easily with the connectivity properties of the graph and allows us to show that polymers have size at most $O(\log n)$.

To formally state our result, we recall a few standard definitions. The random cluster (RC) model with real parameters $q, \beta > 0$ is a weighted edge model arising from statistical physics, assigning probabilities to the edge subsets of a given graph $G = (V, E)$. For each edge subset $F \subset E$, the associated weight of the *configuration* F is given as

$$w_G(F) = w_{G;q,\beta}(F) = q^{c(F)}(e^\beta - 1)^{|F|},$$

where $c(F)$ is the number of components in the graph (V, F) . We refer to edges in F as the *in-edges* of the configuration (and edges in $E \setminus F$ as the *out-edges*). Denote by Ω_G the set of all configurations, i.e., all edge subsets of G . The Gibbs distribution $\pi_G = \pi_{G;q,\beta}$ is defined by $\pi_G(F) = w_G(F)/Z_G$ for $F \in \Omega_G$, where $Z_G = \sum_{F' \in \Omega_G} w_G(F')$ is the partition function. The random cluster model can be viewed as an equivalent edge-representation of the *Potts model* (when q is an integer), which is supported on vertex assignments $\sigma : V \rightarrow \{1, \dots, q\}$ weighted by $e^{\beta m(\sigma)}$ where $m(\sigma)$ is the number of monochromatic edges under σ .

For the random graph $\mathcal{G}(n, d/n)$ which is our focus here, the qualitative structure is conjectured to align with that on the random regular graph that has been extensively studied. In the so-called high temperature regime (low β) a typical configuration is expected to be correlated with the all-out configuration ($F = \emptyset$) and to consist of many small components – this set of configurations is known as the “disordered phase”; whereas in the so-called low temperature regime (high β) a typical configuration is expected to be correlated with

the all-in configuration ($F = E$) and to consist of a giant component (and other small components) – this set of configurations is known as the “ordered phase”. The interesting feature of the random-cluster model is that, on several classes of graphs, there is a critical threshold β_c where the two phases coexist and each have probability bounded below by a constant. This coexistence typically causes severe complications to sampling in a window around β_c , see for example [14, 26, 30] for various metastability phenomena and [29, 25, 10] for computational hardness results that build on this.

Understanding this picture on $\mathcal{G}(n, d/n)$ more precisely is quite a bit more complicated than the random regular graph, even with currently available analysis tools; the key difference is that the local neighbourhood of a vertex is given by a Poisson tree (rather than a regular tree). For example, the value of the log-partition function $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_G$ for low-temperatures is likely a rather involved expectation over Poisson trees based on understanding fixed-point equations on the underlying set of trees.² Likewise, understanding the performance of Markov chain algorithms on $\mathcal{G}(n, d/n)$ becomes extremely involved; in fact, even getting a fairly precise understanding of what happens on a Poisson tree for low temperatures is open for both the Potts and random cluster models.

Our main result is to obtain an algorithm for $\mathcal{G}(n, d/n)$ by introducing suitable developments to the polymer framework and showing how to use these together with recent MCMC techniques. As a corollary of our techniques, we obtain various tools for the RC distribution on $\mathcal{G}(n, d/n)$ that shed light on the properties of the distribution.

► **Theorem 1.** *Let d be a large enough real. Then, for all sufficiently large reals q , there is a (randomised) algorithm $\mathcal{A}$ such that the following holds whp over $G \sim \mathcal{G}(n, d/n)$, for any inverse temperature $\beta > 0$. On input G , with probability at least $3/4$, the algorithm $\mathcal{A}$ outputs in $\text{poly}(n)$ time a sample $F \in \Omega_G$ whose distribution is within TV-distance $e^{-\Omega(n)}$ from the RC distribution $\pi_{G;q,\beta}$, and an estimate $\hat{Z}$ for the RC partition function $Z_G = Z_{G;q,\beta}$ that satisfies $\hat{Z} = (1 \pm e^{-\Omega(n)})Z_G$.*

The success probability of $3/4$ in Theorem 1 can be powered in the standard way. As we will describe later in more detail (Section 4), our algorithm is based on running the *Glauber dynamics* for the random-cluster model from a suitable initial configuration, though it is a bit more involved than that since, in an interval of temperatures $[\beta_0, \beta_1]$, it needs to approximate the appropriate mixture of the ordered and disordered phases.

The running time of the algorithm in Theorem 1 is $\text{poly}(n)$ where the implicit constant in the exponent scales as $O(\log q)$; this is largely because of a crude polynomial mixing-time bound on Poisson trees with wired boundary (i.e., Poisson trees where the leaves are conditioned to belong to the same component, say by contracting them into a single vertex). It is an open question to obtain sharper bounds with the wired boundary condition, even for regular trees when q is non-integer (cf. [4, Theorem 5]).

The lower bound on q in Theorem 1 is roughly $\exp(\Omega(d \log d))$, which is essentially the “standard” bound where polymer-based techniques work (at least on bounded-degree graphs). Having q this large facilitates showing closeness of a typical sample to the extreme configurations. In the case of bounded-degree graphs, there has been some progress in lowering the value of q , mainly on random regular graphs [6, 20]. For large β these results come from first/second moment methods, though these results do not extend to non-integer q or to our $\mathcal{G}(n, d/n)$ setting.

² This is based on the fact that the model is “replica symmetric” [1]; to the best of our knowledge the formula for the log-partition function has not been established yet.

We can improve significantly upon the running time using a different (deterministic) algorithm based on estimating the probability that an edge is an in-edge (based on a certain correlation decay property known as weak spatial mixing (WSM) within the phase, see Section 2.3) and integrating the expected number of in-edges. We expect that this different approach to utilising WSM will be useful in other settings where the underlying graph has tree-like neighbourhoods.

► **Theorem 2.** *Let $R > 0$ be arbitrarily large. For all sufficiently large reals d , for all sufficiently large reals q , there is an algorithm $\mathcal{B}$ such that the following holds whp over $G \sim \mathcal{G}(n, d/n)$, for any temperature $\beta > 0$. On input G , the algorithm $\mathcal{B}$ outputs in $n^{1+\frac{1}{R}}$ time an estimate $\hat{Z}$ for the RC partition function $Z_G = Z_{G;q,\beta}$ that satisfies $\hat{Z} = (1 \pm \frac{1}{n^R})Z_G$.*

It may help to explain the parameters in Theorem 2. The parameter q controls the accuracy of the algorithm via the decay rate in Lemma 6, and d controls the running time via the length r in Lemma 6 (capturing the size of the polymer). So, for any fixed “target” (captured by $R > 0$), by first taking d large (w.r.t. R), and then taking q large (w.r.t. d), we can make both the running time and the accuracy sufficiently small in terms of the target (as per the theorem statement). See Remark 22 for further comments about the restriction that d be sufficiently large, and how this arises (via Proposition 24) in the proof of Lemma 6.

1.1 Discussion and Further Related Work

Most of the known low-temperature algorithms are based on the so-called polymer method which was developed in [32, 34]. Intuitively, a polymer represents a local part of a configuration that deviates from a certain extremal/ground state. The success of the method typically relies on the Kotecký-Preiss condition [37] that roughly controls the number of polymers and the growth of their weights. This condition guarantees that the log partition function can be approximated by truncating a relevant convergent series expansion (the cluster expansion) and then applying the interpolation techniques of [2, 41]; see also [13] for MCMC variants. The method has been very successful in the low-temperature regime [32, 34, 38, 11, 31, 15, 23, 35, 12, 36].

The main difficulty in applying the polymer method on $\mathcal{G}(n, d/n)$ is that small polymers, such as induced paths of length $\Theta(\log n)$ or other sparse induced subgraphs present in the graph, can have unusually large weight violating the growth rate required by Kotecký-Preiss type conditions. This difficulty causes undesirable restrictions; for example, [24] worked on a sparse random graph model with degrees ≥ 3 , a condition that ensures the absence of such non-expanding parts and precludes therefore $\mathcal{G}(n, d/n)$. As we will see in the following section, the main contribution of the present paper is to develop the polymer method for sparse random graphs such as $\mathcal{G}(n, d/n)$ by introducing a suitable polymer definition.

The algorithm provided in our proof of Theorem 1 is based on the Glauber dynamics Markov chain for the random-cluster model. For the case of random regular graphs, the mixing properties of Glauber dynamics are well-studied. Blanca and Gheissari [5] showed that the random cluster Glauber dynamics is mixing in time $O(n \log n)$ for all $q \geq 2$ and $\beta < \beta_u$ where β_u is the uniqueness threshold on the regular tree, and they showed that the same bound applies on $\mathcal{G}(n, d/n)$ as well [8]; see also [7] for mixing-time results on other graph classes that apply for large β . However, for β near the ordered/disordered threshold β_c (satisfying $\beta_c > \beta_u$), the chain undergoes an exponential slowdown due to metastability phenomena and phase coexistence [31, 14]. Our results suggest a similar exponential slowdown around criticality.

However, despite the worst-case mixing result, it is still possible to obtain a fast sampling algorithm based on Glauber dynamics using an appropriate initialisation from a ground state, see [27, 28, 21]. Polymer techniques are helpful in this setting since they can be used to show a notion called weak spatial mixing within a phase introduced in [27], see Section 2.3 for definitions. In the next section, we discuss more thoroughly how to adapt the polymer method for $\mathcal{G}(n, d/n)$ which is the main bottleneck of our results.

1.2 Overview: old and new polymers

As we have noted, polymers capture small, independent deviations from a certain extremal configuration called the “ground state”. For example, for the random cluster model the natural ground states are the all-in and all-out configurations, and for the ferromagnetic Potts model the ground states are the monochromatic colourings.

To apply the polymer framework, the *weight* of a polymer needs to be defined in a way that enables us to relate the weight of a configuration to the product of weights of its polymers, and thus to relate the probability of a polymer appearing in a configuration to its weight. For models with only local interactions, such as the Potts model, the characterization of these small deviations, i.e., the definition of polymers, is clear from the model. For instance, for the Potts model on low temperatures (high β), a typical configuration has the majority of vertices having the same colour. Thus the natural choice for polymers of a q -colouring σ are maximal connected sets γ which are not coloured with the majority colour. The weight w_γ measures how much weight is lost by having bichromatic edges within the polymer γ .

However, when considering how a polymer should be defined for a configuration F of the random cluster model, the answer is not as straightforward. For low-temperatures (high β), the natural ground state is the all-in configuration, thus a natural choice would be to consider (connected) sets of all-out edges. However the random cluster model has long-distance interactions, and in particular, the weight (and thus the probability) of a configuration F depends not only on the number of out-edges (or in-edges), but also on the number of components. For a component of (V, F) , it may not be the case that the set of out-edges separating it from the rest of the graph is connected. If multiple polymers are allowed to enclose a single component, it is not clear how to capture that component’s contribution to the weight of the configuration using (multiplicative) polymer weights.

One insightful solution to this problem was given for the case of the random regular graph by Helmuth, Jenssen and Perkins [31], where they iteratively added more edges to the polymers, and then used the strong expansion properties of random regular graphs to ensure the desired multiplicative properties. For completeness, we will define their polymers for a configuration F : Let $\mathcal{B}_0 = |E \setminus F|$. For $k \geq 0$, define inductively $\mathcal{B}_{k+1}$ to be the set of edges in $\mathcal{B}_k$ along with all edges incident to vertices that have at least a $\frac{5}{9}$ -fraction of their incident edges in $\mathcal{B}_k$. Then the set of *polymer edges* is $\mathcal{B}_{\text{inf}}(F) = \bigcup_{k \geq 0} \mathcal{B}_k$ and polymers are connected components of edges in $\mathcal{B}_{\text{inf}}$. Using the strong expansion of random regular graphs, they show that for any ordered configuration (V, F) , there is one giant component containing $> n/2$ vertices, and every other component has all its incident edges contained in a polymer. While the constant $\frac{5}{9}$ in the definition could be lowered to any $\frac{1}{2} + \tau$, to avoid $\mathcal{B}_{\text{inf}}(F) = E$ it has to be greater than a half.

The fact that an ordered configuration has a giant component also applies for $\mathcal{G}(n, d/n)$. However, having the constant $5/9$ (or anything larger than $1/2$) causes problems in graphs where degrees can be small. For instance, a 3-cycle of a 3-regular graph has expansion $\frac{1}{2}$. Suppose that the edges of the 3-cycle are in-edges, but the edges separating it from the rest of graph are out. Then these separating edges are not necessarily contained in a single polymer.

This makes it difficult to capture the component's contribution, as explained earlier. These problems persist even if the expansion of sufficiently large subgraphs is above $1/2$ by any margin.

In the case of $\mathcal{G}(n, d/n)$ the same problem is present in a more severe form: while large-enough connected sets can be shown to have expansion $> \frac{1}{2}$ (for d large enough), there are small subgraphs with as many as $\Theta(\log n)$ vertices with extremely low expansion. These small subgraphs can not be captured by these expansion-based polymers and it is therefore not clear how to extend the polymer definition of [31] for the case of $\mathcal{G}(n, d/n)$.

To start working towards a polymer definition, note that expanding properties of $\mathcal{G}(n, d/n)$ that hold for sets of size $\Omega(\log n)$ can be used to show that (V, F) contains a giant component and all the other components are small (provided that $|F|$ is sufficiently large relative to $|E|$). So, a natural-looking solution for the polymer-definition problem is to simply take the polymers of a configuration F to be components formed by the union of the set of out-edges $E \setminus F$ and the set of edges in E incident to vertices in small components of (V, F) (irrespective of whether they belong to F). While this definition can be endowed with a polymer-weight definition that does satisfy the desired multiplicative properties, these polymers do not capture properly the deviations from the all-in configuration. For instance, suppose there was a connected subgraph S of (V, F) with exactly one in-edge e in the cut $(S, \bar{S})$. Then it would make sense that vertices of S belonged to a polymer since the removal of a single edge e carves out S as a component, affecting significantly the marginal probability that an edge in the cut $(S, \bar{S})$ is an in-edge. Now, if the cut $(S, \bar{S})$ is large, having S as a component in a configuration (after removal of e) is a large deviation from the ordered state and is extremely unlikely to happen, and the marginal probabilities of edges in the cut should not be so sensitive to the status of a single edge. Hence, such “almost-components” must be enclosed in a polymer.

For our analysis specifically, we care about marginals of edges incident to a particular vertex v , thus we do not want the marginals of edges not in a polymer be sensitive to the state of the edges incident to v . Thus, as we show in Section 3, the natural choice of ordered polymers is to take polymer edges to be the union of the set of out-edges of F and the set of edges of G incident to vertices in subgraphs that would be disconnected by removing this vertex v . In Section 3 we show that with good probability the polymers have size $O(\log n)$ and we use this to prove *weak spatial mixing within the phase*. This is then used for bounding the mixing time from the Glauber dynamics (Theorem 1) and converting to the counting algorithm (Theorem 2).

2 Proof Outline of Theorems 1 and 2

2.1 The largest component of the graph

It is well-known that for a graph G with multiple components, the Gibbs distribution π_G is a product distribution over the individual components, and that the partition function Z_G is the product over the partition function over individual components.

For $d > 1$, $\mathcal{G}(n, d/n)$ whp contains one component of linear size while all remaining components have size $O(\log n)$ and contain at most one cycle, see for example [33, Section 5]. We can therefore brute-force $O(\log n)$ -sized components by enumerating all possible edge configurations in $\text{poly}(n)$ time; in fact, even the mixing time from *worst-case* initial configuration on such a component is going to be at most $\text{poly}(n)$ (see, e.g., [6, Lemma 6.7]).

So, we only need to focus on the largest component of $\mathcal{G}(n, d/n)$, which we denote by $C = (V_C, E_C)$. Let $n_C = |V_C|$ be the number of vertices. It is well-known (see e.g. [33, Theorem 5.4]) that n_C is determined by the conjugate $\mu \in (0, 1)$ of d which satisfies $\mu e^{-\mu} = d e^{-d}$. Using this, it is not hard to verify that for large d we have whp $n_C \geq (1 - e^{-d/3})n$ and $|E_C| = \frac{dn}{2} \pm 2ne^{-d/3}$, see Appendix A in the full version [22] for details.

For clarity of notation, we use Ω_C to denote the set of configurations, w_C for the weights of configurations, π_C for the Gibbs distribution and Z_C for the partition function for the random cluster model on C . For $F \in \Omega_C$, we use $c(F)$ to refer to the number of components in (V_C, F) .

2.2 Phases of random cluster model on $\mathcal{G}(n, d/n)$

Next, we formally introduce the notion of the *ordered* and *disordered* phases on $C = C(\mathcal{G}(n, d/n))$, as well as some necessary notation.

► **Definition 3 (Phases).** Let $\eta := 1/1000$. The ordered phase is $\Omega_C^{\text{ord}} := \{F \in \Omega_C \mid |F| \geq (1 - \eta)|E_C|\}$. The ordered distribution π_C^{ord} is defined for $F \in \Omega_C^{\text{ord}}$ by $\pi_C^{\text{ord}}(F) := w_C(F)/Z_C^{\text{ord}}$, where $Z_C^{\text{ord}} := \sum_{F \in \Omega_C^{\text{ord}}} w_C(F)$. The disordered phase is $\Omega_C^{\text{dis}} := \{F \in \Omega_C \mid |F| \leq \eta|E_C|\}$. The disordered distribution π_C^{dis} is defined for $F \in \Omega_C^{\text{dis}}$ by $\pi_C^{\text{dis}}(F) = w_C(F)/Z_C^{\text{dis}}$, where $Z_C^{\text{dis}} := \sum_{F \in \Omega_C^{\text{dis}}} w_C(F)$.

We will use the following two thresholds for β , β_0 and β_1 , defined from the following:

$$e^{\beta_0} - 1 = q^{(2-1/10)/d} \text{ and } e^{\beta_1} - 1 = q^{(2+1/10)/d}. \quad (1)$$

Note that the constant $1/10$ in the definitions above is somewhat arbitrary and could be decreased to any absolute constant $\tau > 0$. The next theorem tells us that a typical configuration on $\mathcal{G}(n, d/n)$ is either ordered or disordered and “far away” from phases’ boundaries. For the proof, see Appendix C in the full version [22].

► **Theorem 4.** Let d be sufficiently large. Then, for all q sufficiently large, the following holds whp over $G \sim \mathcal{G}(n, d/n)$, where F denotes a random configuration from the given distribution.

- (1) For all $\beta > 0$, $\pi_C\left(\frac{9\eta}{10}|E_C| \leq |F| \leq (1 - \frac{9\eta}{10})|E_C|\right) \leq e^{-n}$.
- (2) For all $\beta \geq \beta_0$, $\pi_C^{\text{ord}}\left(|F| \leq (1 - \frac{9\eta}{10})|E_C|\right) \leq e^{-n}$.
- (3) For all $\beta \leq \beta_1$, $\pi_C^{\text{dis}}\left(\frac{9\eta}{10}|E_C| \leq |F|\right) \leq e^{-n}$.
- (4) For $\beta \geq \beta_1$, $\|\pi_C - \pi_C^{\text{ord}}\|_{\text{TV}} \leq 2e^{-n}$.
- (5) For $\beta \leq \beta_0$, $\|\pi_C - \pi_C^{\text{dis}}\|_{\text{TV}} \leq 2e^{-n}$.

2.3 WSM and proof of Theorem 2 via WSM

The main challenge in order to obtain our results is showing WSM within the phase, which we define more formally here for the linear-sized component. Namely, for a vertex $v \in V_C$ and an integer r , let $B_r(v)$ be the ball of radius r around v , i.e. the induced subgraph consisting of all vertices distance $\leq r$ from v . Let $\pi_{B_r^-(v)}$ denote the Gibbs distribution on B with the *free* boundary condition (i.e., conditional on all edges outside of $B_r(v)$ being out) and $\pi_{B_r^+(v)}$ denote the Gibbs distribution on $B_r(v)$ with the *wired* boundary condition (i.e., conditional on all vertices at distance exactly r from v as being wired into a single component). For an edge $e \in E_C$, we use 1_e denote the event that e is an in-edge.

► **Definition 5.** Let $r > 0$ be an integer and $K \in (0, 1)$ be a real.

We say that C has WSM within the disordered phase at distance r with rate K if for every edge $e \in E_C$ and vertex $v \in V_C$ incident to e , it holds that $|\pi_C^{\text{dis}}(1_e) - \pi_{B_r^-(v)}(1_e)| \leq K^r$.

Similarly, C has WSM within the ordered phase at distance r with rate K if for every edge $e \in E_C$ and vertex $v \in V_C$ incident to e , it holds that $|\pi_C^{\text{ord}}(1_e) - \pi_{B_r^+(v)}(1_e)| \leq K^r$.

In Section 3, we show that for all d and q large enough, whp C has WSM within the ordered phase for all $\beta \geq \beta_0$. For the proof of WSM within the disordered phase for all $\beta \leq \beta_1$ see the Appendix B in the full version [22].

► **Lemma 6.** There is a constant $A > 0$ such that the following holds for all real d sufficiently large. For all q sufficiently large, whp over $G \sim \mathcal{G}(n, d/n)$, for all $\beta \geq \beta_0$, the largest component C of G has WSM within the ordered phase at distance $r = \lceil \frac{A}{d} \log n \rceil$ with rate $K = q^{-1/30}$.

Using this, we can prove Theorem 2. To outline briefly the main idea, consider arbitrarily large $R > 0$ and $\beta^* \geq \beta_0$; then, our goal is to approximate $Z_{C;q,\beta^*}^{\text{ord}}$ in time $n^{O(1/R)}$ within relative error n^{-R} . We view $Z_{C;q,\beta}^{\text{ord}}$ as a function of β and set $g_C(\beta) := \frac{\partial \log Z_{C;q,\beta}^{\text{ord}}}{\partial \beta}$. Note in particular that

$$\begin{aligned} g_C(\beta) &= \frac{\frac{\partial Z_{C;q,\beta}^{\text{ord}}}{\partial \beta}}{Z_{C;q,\beta}^{\text{ord}}} = \frac{e^\beta}{e^\beta - 1} \frac{\sum_{F \in \Omega_C^{\text{ord}}} |F| (e^\beta - 1)^{|F|} q^{c(F)}}{Z_{C;q,\beta}^{\text{ord}}} \\ &= \frac{e^\beta}{e^\beta - 1} \mathbf{E}_{F \sim \pi_C^{\text{ord}}} [|F|] = \frac{e^\beta}{e^\beta - 1} \sum_{e \in E_C} \pi_{C;q,\beta}^{\text{ord}}(1_e). \end{aligned} \quad (2)$$

Moreover, by setting $\beta_\infty = n^{1/R}$ and integrating, we have that

$$\frac{Z_{C;q,\beta_\infty}^{\text{ord}}}{Z_{C;q,\beta^*}^{\text{ord}}} = \exp \left(\int_{\beta^*}^{\beta_\infty} g_C(\beta) d\beta \right). \quad (3)$$

We will show later that for $\beta = \beta_\infty$ it holds that $Z_{C;q,\beta_\infty}^{\text{ord}} = (1 \pm e^{-n^{\Omega(1/R)}}) q(e^\beta - 1)^{|E_C|}$. So, to obtain the desired approximation to $Z_{C;q,\beta^*}^{\text{ord}}$, it suffices to approximate the integral $I := \int_{\beta^*}^{\beta_\infty} g_C(\beta) d\beta$. The standard way to approximate the integral would be to consider a sequence of β 's between β^* and β_∞ and, for each of them, approximate $g_C(\beta)$ using the WSM guarantee. Namely, by Lemma 6, for an edge $e \in E_C$ and a vertex v incident to it, for the ball $B_e := B_r(v)$ with $r = \lceil \frac{A}{d} \log n \rceil$, it holds that

$$|\pi_{C;q,\beta}^{\text{ord}}(1_e) - \pi_{B_e^+;q,\beta}(1_e)| \leq q^{-r/30} \leq \frac{1}{n^{R+3}}, \quad (4)$$

where the last inequality follows by taking q large enough (with respect to d). Crucially, B_e is a 1-treelike⁵ graph whose size $|V_G(B_e)|$ can be bounded by $d^r \log n = n^{O(1/R)}$; there is a fairly standard recursive way to compute $\pi_{B_e^+;q,\beta}(1_e)$ exactly (cf. Lemma 7 below). However, in order to get the desired $\frac{1}{n^R}$ relative error guarantee for the integral, we would need roughly n^R many $\hat{\beta}$'s, which would lead to a huge running time.

To obtain a faster algorithm, the key observation is that $\pi_{B_e^+;q,\beta}(1_e)$ is an explicit function of β which can be represented using an explicit rational function that can be efficiently computed (using recursions) in time polynomial in $|V_G(B_e)| = n^{O(1/R)}$. Hence we can

⁵ For a real k , a graph is called k -treelike if it becomes a tree after the removal of at most k edges.

essentially perform the integration $\int_{\beta^*}^{\beta_\infty} \pi_{B_e^+, q, \beta}(1_e)$ symbolically. A technicality that arises is that we cannot get the exact antiderivative (since in general it will be in terms of algebraic numbers) but rather a numerical approximation to the antiderivative that is within additive $1/2^{n^{O(1/R)}}$ from its true value for all $\beta \in [\beta^*, \beta_\infty]$; all of that however requires only $n^{O(1/R)}$ bits of precision and hence can be carried out in $n^{O(1/R)}$ time.

We next state more formally a few technical lemmas whose proofs can be seen in the Appendix D of the full version [22] and then show how to use them and conclude the proof of Theorem 2. We start with the computation of the marginals of edges in 1-treelike graphs.

► **Lemma 7.** *There is an algorithm that, on input an n -vertex 1-treelike graph T rooted at v , an edge e incident to v and an integer $r \geq 1$, computes in time $2^{4r} n^{O(1)}$ rational functions $g^+(q, x), g^-(q, x)$ so that $\pi_{B_r^+(v); q, \beta}(1_e) = g^+(q, e^\beta - 1)$ and $\pi_{B_r^-(v); q, \beta}(1_e) = g^-(q, e^\beta - 1)$ for all $q, \beta > 0$.*

The next lemma captures the integration part of the argument. For an integer t , its size is the number of bits needed to represent it, and for a rational $w = \frac{w_1}{w_2}$ with integers w_1, w_2 , its size is given by adding the sizes of w_1 and w_2 . Moreover, for a polynomial $P(x) = \sum_{i=0}^n c_i x^i$ of degree n and rational coefficients, its size is given by $n + \sum_{i=1}^n \text{size}(c_i)$.

► **Lemma 8.** *There is an algorithm that, on input positive rationals a, b, ε each with size $\leq n$, and integer polynomials $P(x), Q(x)$ with non-negative coefficients and size $\leq n$, computes in time $\text{poly}(n)$ a number $\hat{I}$ so that $\hat{I} = I \pm \varepsilon$ where $I = \int_a^b \frac{P(x)}{Q(x)} dx$.*

Finally, we will need the following estimate for the partition function for very large β .

► **Lemma 9.** *Let d be large enough. For all q sufficiently large, for any arbitrarily small constant $\varepsilon > 0$, whp over $G \sim \mathcal{G}(n, d/n)$ it holds that $Z_{C; q, \beta}^{\text{ord}} = (1 \pm e^{-n^\varepsilon}) q(e^\beta - 1)^{|E_C|}$ for any $\beta \geq n^{2\varepsilon}$.*

Proof of Theorem 2. We follow the argument outlined earlier in order to obtain a $\frac{1}{n^R}$ -estimate $\hat{Z}^{\text{ord}}$ for $Z_{C; q, \beta}^{\text{ord}}$ when $\beta \geq \beta_0$. An analogous argument gives an approximation $\hat{Z}^{\text{dis}}$ for $Z_{C; q, \beta}^{\text{dis}}$ for $\beta \leq \beta_1$ using WSM within the disordered phase (see Proposition 50 in the full version [22]). Then $\hat{Z}^{\text{dis}}$ is the desired approximation for $\beta < \beta_0$, $\hat{Z}^{\text{dis}} + \hat{Z}^{\text{ord}}$ for $\beta \in [\beta_0, \beta_1]$ and $\hat{Z}^{\text{ord}}$ for $\beta > \beta_1$.

So fix a target $\beta^* \geq \beta_0$ and focus on approximating $Z_{C; q, \beta^*}^{\text{ord}}$. Set $\beta_\infty = n^{1/R}$. If $\beta^* \geq \beta_\infty$, from Lemma 9 we can use $q(e^{\beta^*} - 1)^{|E_C|}$ as our estimate $\hat{Z}^{\text{ord}}$, so we can assume that $\beta^* < \beta_\infty$. From (3) we have that $\frac{Z_{C; q, \beta_\infty}^{\text{ord}}}{Z_{C; q, \beta^*}^{\text{ord}}} = \exp\left(\int_{\beta^*}^{\beta_\infty} g_C(\beta) d\beta\right)$ where $g_C(\beta) = \frac{e^\beta}{e^\beta - 1} \sum_{e \in E_C} \pi_{C; q, \beta}^{\text{ord}}(1_e)$, cf. (2). From Lemma 9 we can approximate $Z_{C; q, \beta_\infty}^{\text{ord}}$ with relative error $e^{-n^{\Omega(1/R)}}$ so to obtain the desired approximation to $Z_{C; q, \beta^*}^{\text{ord}}$, it suffices to approximate the integral $I := \int_{\beta^*}^{\beta_\infty} g_C(\beta) d\beta$ within additive error $\frac{1}{n^{R+1}}$. For an edge $e \in E_C$ and a vertex v incident to it, let $B_e = B_r(v)$ with $r = \lceil \frac{A}{d} \log n \rceil$ as in (4) where we saw that WSM gives $|\pi_{C; q, \beta}^{\text{ord}}(1_e) - \pi_{B_e^+, q, \beta}^{\text{ord}}(1_e)| \leq q^{-r/30} \leq \frac{1}{n^{R+3}}$ for q large enough w.r.t. d . The r -neighbourhoods for $G \sim \mathcal{G}(n, d/n)$ are 1-treelike whp (see Lemma 28 in the full version [22]), so by Lemma 7 we can compute in time $n^{O(1/R)}$ a rational function⁷ $g_e(x)$ so

⁷ Here we assume for convenience that q is rational; if q is irrational, the argument can be modified to use instead a rational approximation $\hat{q}$ to q satisfying $|\hat{q} - q| \leq 1/2^{n^{\Theta(1/R)}}$.

that $\pi_{B_e^+; q, \beta}(1_e) = g_e(e^\beta - 1)$. Therefore for $x = e^\beta - 1$ we have that

$$\left| g_C(\beta) - \frac{x+1}{x} \sum_{e \in E_C} g_e(x) \right| \leq \frac{|E_C|}{n^{R+3}}, \text{ so } \left| I - \sum_{e \in E_C} \int_{x^*}^{x_\infty} \frac{g_e(x)}{x} dx \right| \leq \frac{|E_C|(\beta_\infty - \beta^*)}{n^{R+3}}. \quad (5)$$

Then, using the algorithm in Lemma 8, for each $e \in E_C$ we estimate the integral $\int_{x^*}^{x_\infty} \frac{g_e(x)}{x} dx$ within additive error $1/n^{R+3}$ in time $n^{O(1/R)}$ since each of $g_e(x)/x$ and x_∞ have size $n^{O(1/R)}$. Combining the estimates gives an $\frac{|E_C|}{n^{R+3}}$ -estimate of the sum of integrals in (5), yielding a $\frac{1}{n^{R+1}}$ -estimate of the integral I . All in all, this gives the desired $\frac{1}{n^R}$ -estimate of $Z_{C; q, \beta^*}^{\text{ord}}$ in time $n^{1+O(1/R)}$. Since R can be arbitrarily large, the theorem follows. $\blacktriangleleft$

2.4 Graph properties

Before proceeding to the WSM proofs, we use the following expansion properties of $\mathcal{G}(n, d/n)$ (and its largest component). While the random graph lacks regularity and expansion from small sets, for connected subgraphs of size $\Omega(\log n)$ we can lower bound the average degree and thus obtain a weak expansion property. Since we have two graphs, G and its largest component C , for the clarity of notation, we use V_G for the vertex set of G , and E_G for its edge set. For the proofs of the following propositions see Appendices A.2 and A.3 in the full version [22].

► **Proposition 10.** *For any $\varepsilon \in (0, 1)$, there exists $A = A(\varepsilon) > 0$ such that for all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$, any connected vertex set $S_G \subseteq V_G$ with size at least $A \log(n)/d$ has average degree at least $(1 - \varepsilon)d$.*

In the following proposition, and throughout the paper, we use $\deg_G(S)$ to denote the total degree of a set $S \subseteq V_G$, that is, $\deg_G(S)$ is the sum of the degrees (in G) of vertices in S . For a subset $T \subseteq V_G$ we write $E_G(S, T)$ to denote the set of edges with one endpoint in S and the other in T . We also use $e_G(S) := |E_G(S, \bar{S})|$.

► **Proposition 11.** *There exists a constant $A > 0$, such that for all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. Let $S \subseteq V_C$ be a connected vertex set in C with $|S| \geq A \log(n)/d$ and $\deg_G(S) \leq \frac{1}{10} \deg_G(V_C)$. Then $e_G(S) \geq \frac{3}{5} \deg_G(S)$. The same holds if $S \subseteq V_C$ is a connected set in C with $A \log(n)/d \leq |S| \leq n/6$.*

We remark that the choice of $3/5$ in Proposition 11 is arbitrary, and any constant between $1/2$ and 1 could be used. Also, note that the lower bound $A \log(n)/d$ in both propositions is asymptotically optimal in n and d , as it can be shown that G contains an induced path of length $\approx \frac{\log n}{d}$.

Finally, we will use the property that after removing a constant fraction of edges, C still contains a large component. This property is well-known for expanders [42]; to prove it for C we use the fact that its *kernel* (the graph obtained from C by contracting every induced path into a single edge and removing attached trees) is an expander. The constant $1/144$ in the lemma is somewhat arbitrary and follows from the expansion constant for the kernel (see Appendix A in the full version [22]).

► **Lemma 12.** *For all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. Suppose that $\theta \in (0, \frac{1}{2})$ is a real and $E' \subseteq E_C$ is an edge subset such that $|E'| \leq \frac{|E_C|}{144} (2\theta - \frac{1}{100})$. Then $(V_C, E_C \setminus E')$ has a component with total degree $\geq \deg_G(V_C)(1 - \theta)$, and all other components of $(V_C, E_C \setminus E')$ have total degree at most $\theta \deg_G(V_C)$.*

In light of Lemma 12, we will order components of any subgraph C' of C by their total degree. So we use the phrase “largest component” to refer to the component of C' with largest total degree. A “giant” component is a component whose total degree is more than half of the total degree of C , and “small” otherwise. Thus, by applying Lemma 12 for $\theta = 1/10$, we have that for any $E' \in \Omega_C^{\text{ord}}$ the subgraph (V_C, E') contains a giant component and all the other components are small.

3 Weak spatial mixing within the ordered phase

In this section, we prove WSM within the ordered phase. We bound the difference of the marginals on e by the probability that, roughly, there is a long path avoiding the giant component of (V_C, F) . We then use the new definition of *ordered polymers* that naturally captures this event.

Formally for a vertex v and an integer $r \geq 1$, define $\mathcal{A}_{v,r}$ to be the event, that for a random configuration F , all simple length- r paths from v in G_C intersect the largest component of the graph $(V_C, F) \setminus v$, i.e., the graph (V_C, F) with v removed. Define $\mathcal{A}'_{v,r}$ to be the event that $|F \setminus E_G(B_r(v))| \geq (1 - \eta)|E_C|$. Then, we have that

$$|\pi_{B_r^+(v)}(1_e) - \pi_C^{\text{ord}}(1_e)| \leq 2\pi_C^{\text{ord}}(\neg \mathcal{A}_{v,r}) + 2\pi_C^{\text{ord}}(\neg \mathcal{A}'_{v,r}). \quad (6)$$

This follows by conditioning on the boundary configuration outside of $B_r(v)$. Namely, using the monotonicity of the RC model, this boundary condition is dominated above by the wired boundary condition on $B_r(v)$, where all vertices at the boundary of $B_r(v)$ are conditioned to belong to the same component, and hence $\pi_C^{\text{ord}}(1_e) \leq \pi_{B_r^+(v)}(1_e)$. Likewise, the event $\mathcal{A}'_{v,r}$ gives from Lemma 12 that there is a unique giant component C_F in the graph $(V_C \setminus B_r(v), F \setminus E_G(B_r(v)))$ and $\mathcal{A}_{v,r}$ that there is a set S of vertices inside $B_r(v)$ whose removal disconnects v from the vertices outside of $B_r(v)$ and every $w \in S$ belongs to C_F . Therefore, the event $\mathcal{A}_{v,r} \cap \mathcal{A}'_{v,r}$ implies that there is a set S of vertices inside $B_r(v)$ whose removal disconnects v from the vertices outside of $B_r(v)$ and every $w \in S$ belongs to C_F , i.e., S forms a wired boundary condition closer to v . From monotonicity, it follows that $\pi_{B_r^+(v)}(1_e) \leq \pi_C^{\text{ord}}(1_e \mid \mathcal{A}_{v,r} \cap \mathcal{A}'_{v,r})$. Combining these estimates on $\pi_{B_r^+(v)}(1_e)$ yields (6). (A similar argument can be found in [20, Lemma 5.7].)

It is relatively easy to bound the probability of $\neg \mathcal{A}'_{v,r}$ under π_C^{ord} . For large enough d , standard estimates for the neighbourhoods of $\mathcal{G}(n, d/n)$ (see also the Lemma 27 in the full version [22]) guarantee that whp over $G \sim \mathcal{G}(n, d/n)$, for any $r \leq \frac{A}{d} \log n$, it holds that $|E_G(B_r(v))| \leq \sqrt{n} \log n$ which is at most $\frac{\eta}{10}|E_C|$ for n large enough since $|E_C| = \Omega(n)$ whp. Hence, by Theorem 4(ii), $\pi_C^{\text{ord}}(\neg \mathcal{A}'_{v,r}) \leq e^{-n}$.

It remains to bound the probability of $\mathcal{A}_{v,r}$, for which we use *ordered polymers*.

3.1 Ordered Polymers

We want to bound for each vertex v , and for each $r \sim \log(n)/d$, the quantity $\pi_C^{\text{ord}}(\mathcal{A}_{v,r})$. To do so, we need to control the size of “polymers” with respect to a configuration F and a vertex $v \in V_C$.

We first consider non-giant components in $(V_C, F) \setminus v$; roughly, we will refer to the union of their vertices as the polymer vertices. To form polymers, we will take connected components of these vertices. The details are as follows.

► **Definition 13** (Ordered polymers). Fix $v \in V_C$. Consider $F \in \Omega_C^{\text{ord}}$. For any $w \in V_C$, define the graph $H_v(F, w)$ as follows.

$$H_v(F, w) := \begin{cases} \text{the component of } w \text{ in } (V_C, F), & \text{if } w = v \\ \text{the component of } w \text{ in } (V_C, F) \setminus v, & \text{if } w \neq v. \end{cases}$$

Let $\mathcal{V}(F, v) := \{w \in V_C \mid \deg_G(H_v(F, w)) \leq \frac{1}{2} \deg_G(V_C)\}$. The ordered polymers of F are subgraphs of C . We define two types of ordered polymers, non-singleton polymers and singleton polymers.

- For each maximal set $S \subseteq \mathcal{V}(F, v)$ that is connected in C , there is a non-singleton polymer. The edge set of this polymer consists of all edges in E_C that are incident to vertices in S . The vertex set of this polymer is the set of endpoints of these edges.
- For each edge $e \in E_C \setminus F$ that is not an edge of a non-singleton polymer of F , there is a singleton polymers of F consisting of the single edge e .

Let $\Gamma_v^{\text{ord}}(F)$ denote the set of all ordered polymers of F (with respect to v).

As we show in Lemma 16, a giant exists in (V_C, F) for all $F \in \Omega_C^{\text{ord}}$.

► **Definition 14.** Fix $v \in V_C$. Fix $F \in \Omega_C^{\text{ord}}$ and $\gamma = (U, B) \in \Gamma_v^{\text{ord}}(F)$. The inner vertices of γ are $\mathcal{V}_\gamma := U \cap \mathcal{V}(F, v)$. The out-edges of γ are the edges in $E_{\text{out}}(\gamma) := (E_C \setminus F) \cap B$ and $e_{\text{out}}(\gamma) = |E_{\text{out}}(\gamma)|$ is the number of out-edges of γ . The weight of γ is $w_C^{\text{ord}}(\gamma) := q^{c'(\gamma)}(e^\beta - 1)^{-e_{\text{out}}(\gamma)}$, where $c'(\gamma)$ is the number of components of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$ with total degree at most $(1/2) \deg_G(V_C)$.

Lemma 15 relates the definition of the polymers to the event $\mathcal{A}_{v,r}$. Later, in Lemma 17, we will also relate the weight of a configuration to the product of weights of its polymers.

► **Lemma 15.** Fix $F \in \Omega_C^{\text{ord}}$. Let r be a positive integer. Suppose that there is a simple path $v_0, v_1, \dots, v_r$ in C such that none of $v_1, \dots, v_r$ belongs to the largest component of $(V_C, F) \setminus v_0$. Then there is a polymer $\gamma \in \Gamma_{v_0}^{\text{ord}}(F)$ with $\{v_1, \dots, v_r\} \subseteq \mathcal{V}_\gamma$, hence $|\mathcal{V}_\gamma| \geq r$.

Proof. Consider $i \in [r]$. Since $v_i \neq v_0$, $H_{v_0}(F, v_i)$ is the component of v_i in $(V_C, F) \setminus v_0$. This is not the largest component in $(V_C, F) \setminus v_0$, so $\deg_G(H_{v_0}(F, v_i)) \leq \frac{1}{2} \deg_G(V_C)$ by Lemma 12. Now note that the $v_1, \dots, v_r$ form a path in C so they are contained in a single set S in Definition 13. Thus, there is $\gamma \in \Gamma_{v_0}^{\text{ord}}(F)$ with $\{v_1, \dots, v_r\} \subseteq \mathcal{V}_\gamma$. ◀

In order to prove Lemma 17, and to bound the probability that there is a large polymer, we next use Lemma 12 to get an upper bound on a size of an ordered polymer.

► **Lemma 16.** For all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$, the following holds. Fix any $v \in V_C$. For any $F \in \Omega_C^{\text{ord}}$, $(V_C, F) \setminus v$ contains a component of degree $\geq (1 - \frac{1}{10}) \deg_G(V_C)$. Hence in particular, every polymer of $\gamma \in \Gamma_v^{\text{ord}}(F)$ has $\deg_G(\mathcal{V}_\gamma) \leq \frac{1}{10} \deg_G(V_C)$.

Proof. Let $\theta = 1/10$. Whp the maximum degree of G is $O(\frac{\log n}{\log \log n})$, see e.g. [19, Theorem 3.4]. Since $|F| \geq (1 - \eta)|E_C|$ and $\eta = 1/1000$, it follows that the graph $(V_C, F) \setminus v$ contains at most $\frac{|E_C|}{144}(2\theta - \frac{1}{100})$ edges, so by Lemma 12 it has a component κ of total degree $\geq (1 - \frac{1}{10}) \deg_G(V_C)$. By Definition 13, none of the vertices in κ is in $\mathcal{V}(F, v)$. Thus $\deg_G(\mathcal{V}(F, v)) \leq \frac{1}{10} \deg_G(V_C)$, and so in particular any component of $\mathcal{V}(F, v)$ has total degree at most $\frac{1}{10} \deg_G(V_C)$.

Finally, we note that for any singleton polymer $\gamma \in \Gamma_v^{\text{ord}}(F)$, from Definition 14, $\mathcal{V}_\gamma = \emptyset$. ◀

We use Lemma 16 to relate the weight of a configuration to the product of weights of its polymers.

► **Lemma 17.** *For all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. For any $F \in \Omega_C^{\text{ord}}$ and $v \in V_C$*

$$w_C(F) = q(e^\beta - 1)^{|E_C|} \prod_{\gamma \in \Gamma_v^{\text{ord}}(F)} w_C^{\text{ord}}(\gamma).$$

Proof. Write

$$q(e^\beta - 1)^{|E_C|} \prod_{\gamma \in \Gamma_v^{\text{ord}}(F)} w_C^{\text{ord}}(\gamma) = q^{1 + \sum_{\gamma \in \Gamma_v^{\text{ord}}(F)} c'(\gamma)} (e^\beta - 1)^{|E_C| - \sum_{\gamma \in \Gamma_v^{\text{ord}}(F)} e_{\text{out}}(\gamma)}.$$

Every out-edge of F is in exactly one ordered polymer of F , thus $|E_C \setminus F| = \sum_{\gamma \in \Gamma_v^{\text{ord}}(F)} e_{\text{out}}(\gamma)$.

It remains to show that $c(F) = 1 + \sum_{\gamma \in \Gamma_v^{\text{ord}}(F)} c'(\gamma)$

By Lemma 16 whp G is such that for all $F \in \Omega_{\text{ord}}$, (V_C, F) contains a giant component with total degree $> \frac{1}{2} \deg_C(G_C)$. The 1-term in the above equality corresponds to the giant. Now consider any non-giant component κ of (V_C, F) . Its vertices must be in $\mathcal{V}(v, F)$, and thus in $\mathcal{V}_\gamma$ for some $\gamma \in \Gamma_v^{\text{ord}}(F)$, hence also $E_C(V(\kappa), \overline{V(\kappa)}) \subseteq E_{\text{out}}(\gamma)$. Since κ is connected in (V_C, F) , it must be connected in $(V_C, E_C \setminus E_{\text{out}}(\gamma))$, as $F \subseteq E_C \setminus E_{\text{out}}(\gamma)$, thus it is accounted for in $c'(\gamma)$. Therefore $1 + c(F) \leq 1 + \sum_{\gamma \in \Gamma_v^{\text{ord}}(F)} c'(\gamma)$.

Now consider κ to be a non-giant component of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$ for some $\gamma \in \Gamma_v^{\text{ord}}(F)$. Since (V_C, E_C) is connected, $E_{\text{out}}(\gamma)$ contains an out-edge e incident to some vertex w of κ . Since $F \subseteq E_C \setminus E_{\text{out}}(\gamma)$, the component of w in (V_C, F) , κ' , must be a subgraph of κ . Clearly, κ' must also be non-giant, thus all of its vertices are in $\mathcal{V}(v, F)$. Since non-singleton polymers are maximal connected subgraphs consisting of vertices in $\mathcal{V}(v, F)$, there is a unique polymer γ' containing vertices in $V(\kappa') \subseteq V(\kappa)$. Since e is an out-edge incident to w , $e \in E_{\text{out}}(\gamma')$. Thus e is not in a singleton polymer in $\Gamma_v^{\text{ord}}(F)$, so γ must be a non-singleton polymer with an endpoint of e contained in $\mathcal{V}_\gamma$. But then $\mathcal{V}_\gamma \cup \{w\}$ is a connected subset in (V_C, E_C) , and thus in particular so is $\mathcal{V}_\gamma \cup \mathcal{V}_{\gamma'}$. Since polymers of F do not intersect, it must be the case $\gamma = \gamma'$ and $\kappa = \kappa'$. Thus each non-giant component κ of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$ for any $\gamma \in \Gamma_v^{\text{ord}}(F)$ is also a non-giant component of (V_C, E_C) and for any $\gamma' \in \Gamma_v^{\text{ord}}(F)$, $\gamma' \neq \gamma$, κ is not a non-giant component of $(V_C, E_C \setminus E_{\text{out}}(\gamma'))$. This finished the second part of the equality and thus the proof. ◀

It remains to bound the probability that there is a large polymer. To do so, we show that the weights of (sufficiently large) polymers decay, and then relate the weights to the probabilities that polymers occur.

Observe that for any $F \in \Omega_C^{\text{ord}}$, any $v \in V_C$ and any non-singleton polymer $\gamma \in \Gamma_v^{\text{ord}}(F)$, all edges in $E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma})$ that are not incident to v are out-edges. To see this, note that, if there is an in-edge from $w \neq v \in \overline{\mathcal{V}_\gamma}$ to $u \in \mathcal{V}_\gamma$, then the vertices u and w would be in the same component of $H_v(F, w)$, but then by the definition of non-singleton polymers, w would be in $\mathcal{V}_\gamma$. Since, for any singleton polymer γ , $\mathcal{V}_\gamma = \emptyset$, the same statement applies. Formally we have the following.

► **Observation 18.** *Let $F \in \Omega_C^{\text{ord}}$, $v \in V_C$ and $\gamma \in \Gamma_v^{\text{ord}}(F)$. Then $E_{\text{out}}(\gamma) \supseteq E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma}) \setminus E_G(v, \mathcal{V}_\gamma)$, and hence $e_{\text{out}}(\gamma) \geq |E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma})| - |E_G(v, \mathcal{V}_\gamma)|$.*

We next use that whp G is 1-treelike, i.e. that for all d large enough, whp any radius- $\frac{\log n}{d}$ ball in G (and thus in C) can be made into a tree by removing at most a single edge, to show that $E_G(v, \mathcal{V}_\gamma)$ accounts for a small fraction of edges going from $\mathcal{V}_\gamma$.

► **Lemma 19.** *For all real d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. For any $v \in V_G$ and for any connected subset $S \subseteq V_G$ that doesn't contain v , $|E_G(v, S)| \leq 2 + \frac{d|S|}{\log n}$.*

Proof. Let r denote $\frac{\log n}{d}$. For all d sufficiently large, whp any radius- r ball in G can be made into a tree by removing at most a single edge. For the proof of this fact, see Lemma 28 in the full version [22]. Let $v \in V_G$ be any vertex. If $|E_G(v, S)| \leq 2$, we are done. So assume otherwise (which obviously implies that there is a neighbour of v in S).

Consider the radius- r ball around v in the induced graph on $S \cup \{v\}$, and denote it B . Since B is a subgraph of a radius- r ball in G , there is an edge set $\mathcal{E} \subseteq E_G(B)$ with $|\mathcal{E}| \leq 1$, such that removing $\mathcal{E}$ from B makes it a tree.

Let U be the set of neighbours of v in S . Since S is connected and does not contain v , there is a spanning tree T of $G[S]$. Note that since $B \setminus \mathcal{E}$ is a tree where all vertices of U are adjacent to v , any path between two vertices of U with length $\leq 2r$ in $G[S]$ has to contain the edge in $\mathcal{E}$. Therefore, if a component of $T \setminus \mathcal{E}$ contains $\ell \geq 2$ vertices from U , then its size is at least ℓr . Removing the edge (if any) of $\mathcal{E}$ from T disconnects it to at most 2 components. Since we assumed $|U| = |E_G(v, S)| > 2$, at least $|E_G(v, S)| - 1$ of the vertices in U are in components of $T \setminus \mathcal{E}$ together. Thus, summing up we obtain $|S| \geq (|U| - 1)r$. Rearranging we get that $|E_G(v, S)| \leq 1 + \max(1, \frac{|S|}{r}) \leq 2 + \frac{d|S|}{\log n}$. $\blacktriangleleft$

► **Corollary 20.** *For all real d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. Let $v \in V_C$, $F \in \Omega^{\text{ord}}$. Then for any $\gamma \in \Gamma_v^{\text{ord}}(F)$, it holds that $e_{\text{out}}(\gamma) \geq |E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma})| - 2 - \frac{d|\mathcal{V}_\gamma|}{\log n}$.*

Proof. By Observation 18, any in-edges in $E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma})$ are from $\mathcal{V}_\gamma$ to v . If $v \in \mathcal{V}_\gamma$, all edges in $E_G(\mathcal{V}_\gamma, \overline{\mathcal{V}_\gamma})$ are out, so the inequality follows. If $v \notin \mathcal{V}_\gamma$, recall that, for a singleton polymer γ , $\mathcal{V}_\gamma = \emptyset$ and for a non-singleton polymer γ , $\mathcal{V}_\gamma$ is connected in C . Apply Lemma 19 to show that $E_G(v, \mathcal{V}_\gamma) \leq 2 + d|\mathcal{V}_\gamma|/\log(n)$. $\blacktriangleleft$

Now we can bound weight of any ordered polymer γ , provided it is large enough, to apply expansion and average degree bounds.

► **Lemma 21.** *There exists a real $A > 0$ such that for all d large enough, and for all real $q > 1$, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds for all $\beta \geq \beta_0$. For any $F \in \Omega_C^{\text{ord}}$, $v \in V_C$ and $\gamma \in \Gamma_v^{\text{ord}}(F)$ with $|\mathcal{V}_\gamma| \geq \frac{A}{d} \log n$, it holds that $w_C^{\text{ord}}(F) \leq q^{-\deg_G(\mathcal{V}_\gamma)/(20d)}$.*

Proof. By Propositions 10 and 11, there is a real $A > 0$ such that for all d large enough, whp over G , for any connected vertex subset $S \subseteq V_C$ with $|S| \geq \frac{A \log n}{d}$, its average degree is at least $\frac{35}{36}d$. Moreover, if $\deg_G(S) \leq \frac{1}{10} \deg_G(V_C)$, then $|E_G(S, \overline{S})| \geq \frac{3}{5} \deg_G(S)$. Thus for the rest of this proof, assume that G satisfies these properties.

We first show that for any ordered polymer γ with $|\mathcal{V}_\gamma| \geq \frac{A}{d} \log n$,

$$e_{\text{out}}(\gamma) \geq \frac{4}{7} \deg_G(\mathcal{V}_\gamma). \quad (7)$$

By Lemma 16, $\deg_G(\mathcal{V}_\gamma) \leq \frac{1}{10} \deg_G(V_C)$. Thus by Corollary 20, $e_{\text{out}}(\gamma) \geq \frac{3}{5} \deg_G(\mathcal{V}_\gamma) - 2 - \frac{d|\mathcal{V}_\gamma|}{\log n}$. Finally, $\mathcal{V}_\gamma$ is a connected subset of V_C with $|\mathcal{V}_\gamma| \geq \frac{A}{d} \log n$, hence $\frac{2+d|\mathcal{V}_\gamma|/\log n}{\deg_G(\mathcal{V}_\gamma)} \leq \frac{2+d|\mathcal{V}_\gamma|/\log n}{\frac{35d|\mathcal{V}_\gamma|/36}} = \frac{2+d|\mathcal{V}_\gamma|/\log n}{35d|\mathcal{V}_\gamma|/36}$. Since $|\mathcal{V}_\gamma| = \Omega(\log n)$, this fraction is at most $1/35$ for sufficiently large n . Thus, $\frac{e_{\text{out}}(\gamma)}{\deg_G(\mathcal{V}_\gamma)} \geq \frac{3}{5} - \frac{1}{35} = \frac{4}{7}$, concluding the proof of (7).

Recall from Definition 14 that $c'(\gamma)$ is the number of components of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$ with total degree at most $(1/2) \deg_G(V_C)$, that is, the number of small components of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$.

We first argue that $c'(\gamma) \leq |\mathcal{V}_\gamma|$. Let κ be an arbitrary small component of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$, we will show that $V_G(\kappa) \subseteq \mathcal{V}_\gamma$. Since $F \subseteq E_C \setminus E_{\text{out}}(\gamma)$, all vertices in κ are contained in small components of (V_C, F) . Thus, from the polymer definition, all vertices of κ are contained in $\mathcal{V}(F, v)$. Since C is connected, the component κ of $(V_C, E_C \setminus E_{\text{out}}(\gamma))$ has

a vertex incident to some edge $e \in E_{\text{out}}(\gamma)$. Since γ is a non-singleton polymer, the edges of $E_{\text{out}}(\gamma)$ are all incident to vertices in $\mathcal{V}_\gamma$, and thus in particular there is $w \in \mathcal{V}_\gamma$ incident to e . Without loss of generality, we may assume that $w \in V_G(\kappa)$.¹⁰ We have also shown already that all vertices of κ are contained in $\mathcal{V}(F, v)$. By construction, $\mathcal{V}_\gamma$ is a maximal subset of $\mathcal{V}(F, v)$ that is connected in C . Since κ contains a vertex w of $\mathcal{V}_\gamma$, all of its vertices are contained in $\mathcal{V}_\gamma$, so $V_G(\kappa) \subseteq \mathcal{V}_\gamma$. Hence, $c'(\gamma) \leq |\mathcal{V}_\gamma|$.

Next, using (7), the average degree bound and the fact that $c'(\gamma)$ is at most $|\mathcal{V}_\gamma|$, we get that

$$\frac{c'(\gamma)}{e_{\text{out}}(\gamma)} \leq \frac{|\mathcal{V}_\gamma|}{\frac{4}{7} \deg_G(\mathcal{V}_\gamma)} \leq \frac{1}{\frac{4}{7} \frac{35}{36} d} = \frac{9}{5d} \quad (8)$$

To finish the proof, plug (8) and then (7) into the definition of a polymer weight (Definition 14) and use the fact that $e^\beta - 1 \geq q^{(2-1/10)/d}$ for $\beta \geq \beta_0$ (See (1)) to obtain:

$$w_C^{\text{ord}}(\gamma) = q^{c'(\gamma)} (e^\beta - 1)^{-e_{\text{out}}(\gamma)} \leq [q^{\frac{9}{5d} - \frac{2-1/10}{d}}]^{e_{\text{out}}(\gamma)} = q^{-e_{\text{out}}(\gamma)/10} \leq q^{-\deg_G(\mathcal{V}_\gamma)/(20d)}. \blacktriangleleft$$

► **Remark 22.** Note that Lemma 21 crucially requires d to be large enough for sufficient expansion and average degree properties to hold for all connected sets of size at least $A \log n/d$. While expansion properties hold for all $d > 1$ (see e.g. [18]), we must treat sets as large as $A \log n/d$ and for these we require large d to obtain the required average degree properties and to show that the weights of the polymers are sufficiently small.

We next relate the weight of an ordered polymer to the probability of it occurring.

► **Lemma 23.** *For all d large enough, whp over $G \sim \mathcal{G}(n, d/n)$ the following holds. For $v \in V_C$ and any $\gamma \in \bigcup_{F \in \Omega_C^{\text{ord}}} \Gamma_v^{\text{ord}}(F)$, it holds that $\pi_C^{\text{ord}}(\gamma \in \Gamma_v^{\text{ord}}(\cdot)) \leq w_C^{\text{ord}}(\gamma)$.*

Proof. Fix $v \in V_C$ and let $\gamma \in \bigcup_{F \in \Omega_C^{\text{ord}}} \Gamma_v^{\text{ord}}(F)$. If $e_{\text{out}}(\gamma) = 0$, then since C is connected, $w_C^{\text{ord}}(\gamma) = 1$, and the inequality is trivial. Thus, for the rest of the proof, assume that $E_{\text{out}}(\gamma) \neq \emptyset$. Let $\Omega_C^{\text{ord}}(\gamma) = \{F \in \Omega_C^{\text{ord}}; \gamma \in \Gamma_v^{\text{ord}}(F)\}$ and note that

$$\pi^{\text{ord}}(\gamma \in \Gamma_v^{\text{ord}}(\cdot)) = \frac{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)}{\sum_{F \in \Omega_C^{\text{ord}}} w_C(F)}.$$

For every ordered F with $\gamma \in \Gamma_v^{\text{ord}}(\gamma)$, define $F' := F \cup E_{\text{out}}(\gamma) \neq F$. We claim that $w_C(F') = w_C(F)/w_C^{\text{ord}}(\gamma)$. Clearly, $|F'| = |F| + |E_{\text{out}}(\gamma)|$ since all edges in $E_{\text{out}}(\gamma)$ are out-edges.

If γ is a singleton polymer, then neither of its endpoints are in small components of (V_C, F) , thus setting its single edge to be in does not change the number of small components, so $c(F) = c(F')$, and it follows that $w_C(F') = w_C(F)(e^\beta - 1) = w_C(F)/w_C^{\text{ord}}(\gamma)$.

Hence, assume that γ is a non-singleton polymer with $E_{\text{out}}(\gamma) \neq \emptyset$ and consider the components of (V_C, F) and (V_C, F') . From Lemma 16, there is exactly one giant component in each of (V_C, F) and (V_C, F') . We claim that the components of (V_C, F') are the giant component, together with the small components of (V_C, F) that do *not* contain vertices of $\mathcal{V}_\gamma$.

¹⁰ If not, then let u be the other endpoint of e . Since it is adjacent to e , $u \in V_G(\kappa)$ so it is in $\mathcal{V}(F, v)$. But since $u \in \mathcal{V}(F, v)$ and u is an endpoint of e , it is a vertex of γ , so by the definition of $\mathcal{V}_\gamma$, $u \in \mathcal{V}_\gamma$. In this case we can just use u as the vertex w . So we can assume $w \in V_G(\kappa)$.

To prove the claim, first suppose κ is a small component of (V_C, F) not containing vertices of $\mathcal{V}_\gamma$. We use $V_G(\kappa)$ for the set of vertices of κ . Then no edge from $E_G(V_G(\kappa), \overline{V_G(\kappa)})$ is in $E_{\text{out}}(\gamma)$, thus κ is a component of (V_C, F') .

For the other direction, let κ be a small component of (V_C, F') . The vertices of κ are in small components of (V_C, F) , so by the definition of $\mathcal{V}(F, v)$, $V_G(\kappa) \subseteq \mathcal{V}(F, v)$. We wish to show that κ is a small component of (V_C, F) that does not contain vertices of $\mathcal{V}_\gamma$.

Suppose for contradiction that $\mathcal{V}_\gamma \cap V_G(\kappa) \neq \emptyset$. Then, by the definition of non-singleton polymers, $V_G(\kappa) \subseteq \mathcal{V}_\gamma$, resulting in $E_G(V_G(\kappa), \overline{V_G(\kappa)}) \subseteq E_{\text{out}}(\gamma)$, contradicting the fact that κ is a small component of (V_C, F') . For the rest of the proof, we can therefore assume that $\mathcal{V}_\gamma \cap V_G(\kappa) = \emptyset$.

To finish the proof of the claim, we wish to show that κ is a small component of (V_C, F) . Since (V_C, F) is a subgraph of (V_C, F') , there is a non-empty subgraph κ' of κ which is a component of (V_C, F) . Suppose for contradiction that $V_G(\kappa') \subsetneq V_G(\kappa)$. Since κ is connected in F' , there exists an edge $e = \{w, u\}$ where $w \in V_G(\kappa) \setminus V_G(\kappa')$ and $u \in V_G(\kappa')$. But then e must be an in-edge for F' but not for F , hence $e \in E_{\text{out}}(\gamma)$. By the definition of non-singleton polymers, either w or u is in $\mathcal{V}_\gamma$. Both w and u are in $V_G(\kappa)$ so we have contradicted $\mathcal{V}_\gamma \cap V_G(\kappa) = \emptyset$, which we have already shown. Thus $\kappa = \kappa'$, which concludes the proof of the claim.

By the claim, $c(F) - c(F')$ is the number of small components of (V_C, F) that contain vertices of $\mathcal{V}_\gamma$. By the definition of polymers this is exactly $c'(\gamma)$, so by the definition of polymer weights $w_C(F') = q^{c(F')}(e^\beta - 1)^{|F'|} = \frac{w_C(F)(e^\beta - 1)^{e_{\text{out}}(F)}}{q^{c'(\gamma)}} = \frac{w_C(F)}{w_C^{\text{ord}}(\gamma)}$.

For every $F \in \Omega_C^{\text{ord}}(\gamma)$, both F and F' are in Ω_C^{ord} . Furthermore, $F' \notin \Omega_C^{\text{ord}}(\gamma)$ and for any distinct $F_1, F_2 \in \Omega_C^{\text{ord}}(\gamma)$, F'_1 and F'_2 are distinct. Thus

$$\pi^{\text{ord}}(\gamma \in \Gamma_v^{\text{ord}}(\cdot)) = \frac{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)}{\sum_{F \in \Omega_C^{\text{ord}}} w_C(F)} \leq \frac{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)}{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} (w_C(F) + w_C(F'))}.$$

Hence we can write

$$\pi^{\text{ord}}(\gamma \in \Gamma_v^{\text{ord}}(\cdot)) \leq \frac{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)}{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} (w_C(F) + w_C(F'))} \leq \frac{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)}{\sum_{F \in \Omega_C^{\text{ord}}(\gamma)} w_C(F)(1 + 1/w_C^{\text{ord}}(\gamma))},$$

which is $\frac{1}{1 + 1/w_C^{\text{ord}}(\gamma)} \leq w_C^{\text{ord}}(\gamma)$. $\blacktriangleleft$

With this in hand, we can show an analogue of the Kotecký-Preiss condition for our ordered polymers (but only considering those that are large enough). We will use this to bound the probability of a large polymer occurring.

► **Proposition 24.** *There is a real $A > 0$ such that for all d large enough, for all sufficiently large reals q , the following holds whp over $G \sim \mathcal{G}(n, d/n)$ for all real $\beta \geq \beta_0$. For any $v \in V_C$,*

$$\pi_C^{\text{ord}}(\exists \gamma \in \Gamma_v^{\text{ord}}(\cdot) \text{ with } |\mathcal{V}_\gamma| \geq \frac{A}{d} \log n) \leq q^{-\frac{A}{25d} \log n}.$$

Proof. By Proposition 10 and Lemma 21, there is a constant A such that whp any connected vertex set with size at least $\frac{A \log n}{d}$ has average degree at least $\frac{9}{10}d$, and any ordered polymer γ with $|\mathcal{V}_\gamma| \geq \frac{A \log n}{d}$ has $w_C^{\text{ord}}(\gamma) \leq q^{-\deg_F(\mathcal{V}_\gamma)/(20d)}$. Also whp the bound from Lemma 23 holds.

Let $\Gamma_v^{\text{ord}} = \bigcup_{F \in \Omega_C^{\text{ord}}} \Gamma_v^{\text{ord}}(F)$. By a union bound and Lemma 23 it is enough to show

$$\sum_{\gamma \in \Gamma_v^{\text{ord}}; |\mathcal{V}_\gamma| \geq A \log n / d} w_C^{\text{ord}}(\gamma) \leq q^{-A \log n / (25d)}.$$

Start by using the bound on the polymer weights from Lemma 21 to show that

$$\sum_{\gamma \in \Gamma_v^{\text{ord}}; |\mathcal{V}_\gamma| \geq A \log n/d} w_C^{\text{ord}}(\gamma) \leq \sum_{\gamma \in \Gamma_v^{\text{ord}}; |\mathcal{V}_\gamma| \geq A \log n/d} q^{-\deg_G(\mathcal{V}_\gamma)/(20d)}.$$

Then note that any γ in this sum has $|\mathcal{V}_\gamma| \geq A \log n/d$ hence by Proposition 10 it has average degree at least $\frac{9}{10}d$ hence degree at least $\frac{9A}{10} \log n$. Thus, the sum is at most

$$\sum_{D \geq \frac{9A}{10} \log n} \sum_{\gamma \in \Gamma_v^{\text{ord}}; \deg_G(\mathcal{V}_\gamma) = D} q^{-\deg_G(\mathcal{V}_\gamma)/(20d)}$$

Now we bound the number of polymers of the given total degree D . There are $\leq n(2e)^{2D-1}$ connected subgraphs of total degree D in any graph (see e.g. [24, Lemma 6]). For any connected subset $S \subseteq V_C$ of total degree D , there are at most D edges incident to its vertices, thus there are at most 2^D polymers γ with $\mathcal{V}_\gamma = S$ (each of its edges can be either in or out). Thus the sum is at most

$$\sum_{D \geq 9A \log n/10} n(2e)^{2D-1} 2^D q^{-D/(20d)}.$$

For all q sufficiently large, $8e^2 q^{-1/(20d)} \leq q^{-1/(22d)}$, so the sum is $O(nq^{-9A \log n/(220d)})$. For q sufficiently large, this is $\leq q^{-A \log n/(25d)}$. ◀

Proposition 24, Lemma 15 and equation (6) together with the bound from Theorem 4 now directly give Lemma 6, i.e. WSM within the ordered phase at the distance $r = \frac{A \log n}{d}$ (for some constant A – take twice the constant from the Proposition 24 to take into account that we want to avoid polymers of size $r - 1$).

4 Fast convergence of the RC dynamics and proof of Theorem 1

In order to prove Theorem 1, we use the RC dynamics $(X_t)_{t \geq 0}$ to obtain samples from the ordered and the disordered phases separately. For the sake of completeness, recall that a transition from X_t to X_{t+1} is done as follows:

- (1) Pick an edge $e \in E$ uniformly at random
- (2) If e is a cut edge of the graph $(V, X_t \cup \{e\})$, then with probability $\hat{p} := \frac{e^\beta}{e^\beta + q - 1}$ set $X_{t+1} := X_t \cup \{e\}$. With all remaining probability set $X_{t+1} := X_t \setminus \{e\}$
- (3) Otherwise, with probability $p := 1 - e^{-\beta}$, set $X_{t+1} := X_t \cup \{e\}$. With all remaining probability set $X_{t+1} := X_t \setminus \{e\}$.

Proving mixing of the RC dynamics (X_t) is in general a challenging task, especially since its moves depend on the current component structure. Fortunately, by now there is a well-established strategy [27, 21, 20] that we can use to show the convergence bounds of Theorems 25 and 26 based on the polymer sizes. It consists of utilising the following two main ingredients: (i) *weak spatial mixing within a phase* (WSM) that as we saw asserts that the marginal probability of an edge e being an in-edge depends only on a small neighbourhood around e , and (ii) the mixing time of the dynamics on that neighbourhood is polynomial in its size. Ingredient (ii) is fairly straightforward in sparse random graphs since small-distance neighbourhoods are typically tree-like (i.e., they contain at most a constant number of cycles) which makes poly-time bounds relatively simple to obtain.

► **Theorem 25.** *Let d be sufficiently large. Then, for all q sufficiently large, whp over $G \sim \mathcal{G}(n, d/n)$, for all $\beta \leq \beta_1$ and $\varepsilon > e^{-\Omega(n)}$, the random cluster dynamics X_t initialized from all-out satisfies $\|X_T - \pi_C^{\text{dis}}\| \leq \varepsilon$ for $T = O(n \log n \log \frac{1}{\varepsilon})$.*

► **Theorem 26.** *Let d be sufficiently large. Then, for all q sufficiently large, whp over $G \sim \mathcal{G}(n, d/n)$, for all $\beta \geq \beta_0$ and $\varepsilon > e^{-\Omega(n)}$, the random cluster dynamics X_t initialized from all-in satisfies $\|X_T - \pi_C^{\text{ord}}\| \leq \varepsilon$ for $T = \text{poly}(n, \log \frac{1}{\varepsilon})$*

Proof of Theorem 1. For $\beta \leq \beta_0$ and $\beta \geq \beta_1$, it follows from Theorems 4, 25 and 26 that we can get an $e^{-\Omega(n)}$ -sample by initializing the Glauber dynamics from all-out or all-in respectively.

Note that both theorems give mixing even for β that depends on n . We use this to obtain an approximation for Z_C and hence mixing for $\beta \in (\beta_0, \beta_1)$. By considering a sequence of β 's we can approximate Z_C^{ord} and Z_C^{dis} within $(1 \pm e^{-\Omega(n)})$ -factor using standard self-reducibility techniques (for more details see e.g. [13, 3]), and thus, by Theorem 4 get a $(1 \pm e^{-\Omega(n)})$ approximation for Z_C . For $\beta \in (\beta_0, \beta_1)$, we then obtain an $e^{-\Omega(n)}$ sample by starting in the appropriate random mixture of all-in and all-out. ◀

Given WSM within the phases and using the monotonicity of the random-cluster model, we complete the proof of Theorems 25 and 26 in Appendix E in the full version [22].

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