

# Algorithmic Aspects of Semistability of Quiver Representations

Yuni Iwamasa  

Graduate School of Informatics, Kyoto University, Japan

Taihei Oki  

Institute for Chemical Reaction Design and Discovery (ICReDD), Hokkaido University, Sapporo, Japan

Center for Advanced Intelligence Project, RIKEN, Tokyo, Japan

Tasuku Soma  

The Institute of Statistical Mathematics, Tokyo, Japan

Center for Advanced Intelligence Project, RIKEN, Tokyo, Japan

---

## Abstract

We study the semistability of quiver representations from an algorithmic perspective. We present efficient algorithms for several fundamental computational problems on the semistability of quiver representations: deciding the semistability and  $\sigma$ -semistability, finding the maximizers of King's criterion, and computing the Harder–Narasimhan filtration. We also investigate a class of polyhedral cones defined by the linear system in King's criterion, which we refer to as King cones. For rank-one representations, we demonstrate that these King cones can be encoded by submodular flow polytopes, enabling us to decide the  $\sigma$ -semistability in strongly polynomial time. Our approach employs submodularity in quiver representations, which may be of independent interest.

**2012 ACM Subject Classification** Computing methodologies → Algebraic algorithms; Mathematics of computing → Submodular optimization and polymatroids

**Keywords and phrases** quivers,  $\sigma$ -semistability, King's criterion, operator scaling, submodular flow

**Digital Object Identifier** 10.4230/LIPIcs.ICALP.2025.99

**Category** Track A: Algorithms, Complexity and Games

**Related Version** *Full Version*: <https://arxiv.org/abs/2407.06493>

**Funding** This work was supported by JSPS KAKENHI Grant Number JP24K21315, Japan.

*Yuni Iwamasa*: Supported by JSPS KAKENHI Grant Numbers JP22K17854, JP24K02901, Japan.

*Taihei Oki*: Supported by JST ERATO Grant Number JPMJER1903, JST CREST Grant Number JPMJCR24Q2, JST FOREST Grant Number JPMJFR232L, JSPS KAKENHI Grant Number JP22K17853, and Start-up Research Funds in ICReDD, Hokkaido University.

*Tasuku Soma*: Supported by JPSP KAKENHI Grant Number JP19K20212 and JST, PRESTO Grant Number JPMJPR24K5, Japan.

**Acknowledgements** The authors thank Hiroshi Hirai and Keiya Sakabe for their valuable comments on an earlier version of this paper. The last author thanks Cole Franks for bringing the submodularity of quiver representations to his attention. The authors thank an anonymous reviewer for pointing out the reference [35].

## 1 Introduction and our contribution

*Quiver representation* is a simple generalization of matrices that has led to surprisingly deep extensions of various results in linear algebra [14]. In this paper, we study the *semistability* of quiver representations, which is a central concept in the *geometric invariant theory* (GIT), from an algorithmic perspective. The semistability of quiver representations appears in *oper-*



© Yuni Iwamasa, Taihei Oki, and Tasuku Soma;  
licensed under Creative Commons License CC-BY 4.0

52nd International Colloquium on Automata, Languages, and Programming (ICALP 2025).

Editors: Keren Censor-Hillel, Fabrizio Grandoni, Joël Ouaknine, and Gabriele Puppis

Article No. 99; pp. 99:1–99:18



Leibniz International Proceedings in Informatics

LIPIcs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



ator scaling [25, 24, 17, 5, 19], Brascamp–Lieb (BL) inequality [23], Tyler’s M-estimator [20], and scatter estimation of structured normal models [1], which have attracted considerable attention in theoretical computer science owing to their connection to the noncommutative Edmonds’ problem, algebraic complexity theory, and submodular optimization [35, 31, 6, 26]. The goal of this paper is to provide efficient algorithms for various fundamental computational problems on the semistability of quiver representations. In the following, we describe the problems more formally and present our results.

### 1.1 Semistability of quiver representations

Here, we present the formal definition of a quiver representation. We follow the terminologies in [14, 6]. Let  $Q = (Q_0, Q_1)$  be a quiver with a vertex set  $Q_0$  and an arc set  $Q_1$ . In this paper, we consider only acyclic quivers except otherwise stated. For each arc  $a \in Q_1$ , we denote the tail and head of  $a$  by  $ta$  and  $ha$ , respectively. A *representation*  $V$  of  $Q$  consists of complex vector spaces  $V(i)$  for vertex  $i \in Q_0$  and linear maps  $V(a) : V(ta) \rightarrow V(ha)$  for arc  $a \in Q_1$ . A *subrepresentation*  $W$  of  $V$  is a representation of the same quiver such that  $W(i) \leq V(i)$  for  $i \in Q_0$ , and  $W(a) = V(a)|_{W(ta)}$  and  $\text{im } W(a) \leq W(ha)$  for  $a \in Q_1$ . The vector of dimensions of  $V(i)$  is called the *dimension vector* of the representation, denoted by  $\underline{\dim} V$ . We call the vector space of all representations of  $Q$  with the dimension vector  $\alpha$  the *representation space* of  $Q$  with dimension vector  $\alpha$ , which we denote by  $\text{Rep}(Q, \alpha)$ . After fixing the dimension vector  $\alpha$  and a basis of each  $V(i)$ , we can represent  $V(a)$  as an  $\alpha(ha) \times \alpha(ta)$  matrix. Therefore, the representation space can be identified as

$$\text{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \text{Mat}(\alpha(ha), \alpha(ta)),$$

where  $\text{Mat}(m, n)$  denotes the space of  $m \times n$  complex matrices.

Fix a dimension vector  $\alpha$ . Let

$$\text{GL}(Q, \alpha) := \prod_{i \in Q_0} \text{GL}(\alpha(i)),$$

where  $\text{GL}(n)$  denotes the general linear group of degree  $n$ . Then,  $\text{GL}(Q, \alpha)$  acts on the representation space by a change of basis:

$$g \cdot V := (g_{ha} V(a) g_{ta}^{-1})_{a \in Q_1}.$$

Note that this is a left action, i.e.,  $(gh) \cdot V = g \cdot (h \cdot V)$  for  $g, h \in \text{GL}(Q, \alpha)$ . We say that a representation  $V$  is *semistable* under the  $\text{GL}(Q, \alpha)$ -action if the orbit closure of  $V$  does not contain the origin, i.e.,

$$\inf_{g \in \text{GL}(Q, \alpha)} \sum_{a \in Q_1} \|g_{ha} V(a) g_{ta}^{-1}\|_{\text{F}}^2 > 0.$$

Otherwise,  $V$  is said to be *unstable*. The set of all unstable representations is called the *null-cone* of the  $\text{GL}(Q, \alpha)$ -action. It is easy to see that any representation is unstable under the  $\text{GL}(Q, \alpha)$ -action if  $Q$  is acyclic.<sup>1</sup>

---

<sup>1</sup> The readers may wonder whether we can check the semistability (under  $\text{GL}(Q, \alpha)$ -action) of quiver representations of *cyclic* quivers. We will address this point later.

However, the semistability of quivers under subgroups of  $\mathrm{GL}(Q, \alpha)$  turns out to be more intricate. Let  $\sigma \in \mathbb{Z}^{Q_0}$  be an integer vector on  $Q_0$ , which we call a *weight*. Let  $\chi_\sigma$  be the corresponding multiplicative character of  $\mathrm{GL}(Q, \alpha)$ , i.e.,

$$\chi_\sigma(g) = \prod_{i \in Q_0} \det(g_i)^{\sigma(i)}.$$

Note that  $\chi_\sigma$  is a one-dimensional representation of  $\mathrm{GL}(Q, \alpha)$ ;  $\mathrm{GL}(Q, \alpha)$  acts on  $\mathbb{C}$  by  $g \cdot x := \chi_\sigma(g)x$ . A representation  $V$  is said to be  $\sigma$ -semistable if the orbit closure of  $(V, 1) \in \mathrm{Rep}(Q, \alpha) \oplus \mathbb{C}$  under the  $\mathrm{GL}(Q, \alpha)$  action does not contain the origin, i.e.,

$$\inf_{g \in \mathrm{GL}(Q, \alpha)} \left( \sum_{a \in Q_1} \|g_{ha} V(a) g_{ta}^{-1}\|_F^2 + |\chi_\sigma(g)|^2 \right) > 0.$$

It turns out that checking the  $\sigma$ -semistability of a quiver representation includes operator scaling (noncommutative rank computation) and the membership problem of the BL polytopes. We will see these examples in the following sections.

Our first result is a deterministic algorithm that, given a quiver representation  $V$  and weight  $\sigma$ , decides whether the representation is  $\sigma$ -semistable in time polynomial in the *bit complexity* of  $V$  and absolute values of the entries of  $\sigma$ . Let  $\alpha(Q_0) := \sum_{i \in Q_0} \alpha(i)$ .

► **Theorem 1.1** (informal version of Theorem 3.4). *Let  $Q$  be an acyclic quiver,  $V$  a representation of  $Q$ , and  $\sigma$  a weight. There is a deterministic algorithm that decides the  $\sigma$ -semistability of  $V$  in time polynomial in the size of  $Q$ ,  $\alpha(Q_0)$ , bit complexity of  $V$ , and absolute values of the entries of  $\sigma$ .*

This improves the previous result [30] which runs in time polynomial in the number of *paths* in  $Q$ , which can be exponential in the size of  $Q$ . Furthermore, if the absolute value of the entries of  $\sigma$  is constant, our algorithm runs in polynomial time. This includes the known result for operator scaling [24].

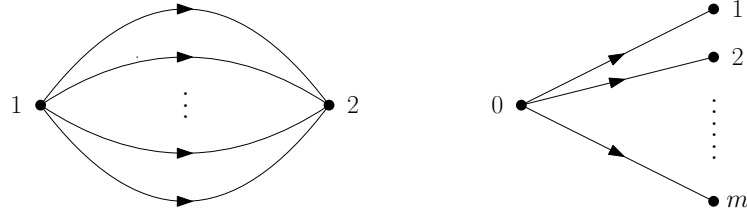
## 1.2 King's criterion

King [32] showed the following characterization of  $\sigma$ -semistability, which is known as *King's criterion*. Let  $\sigma(\alpha) := \sum_{i \in Q_0} \sigma(i)\alpha(i)$  for a dimension vector  $\alpha$ . Then, a representation  $V$  is  $\sigma$ -semistable if and only if  $\sigma(\dim V) = 0$  and  $\sigma(\dim W) \leq 0$  for any subrepresentation  $W$  of  $V$ .

King's criterion is a common generalization of the noncommutative rank (nc-rank) computation and membership problem of BL polytopes.

► **Example 1.2** (nc-rank). Let  $Q$  be the generalized Kronecker quiver with  $m$  parallel arcs,  $\alpha = (n, n)$ , and  $\sigma = (1, -1)$ ; see Figure 1. Any representation  $V$  of  $Q$  with the dimension vector  $\alpha$  can be regarded as an  $n \times n$  linear matrix  $A = \sum_{a=1}^m x_a V(a)$ , where  $x_a$  is an indeterminate. A subrepresentation  $W$  of  $V$  is determined by a pair of subspaces  $(W(1), W(2))$  such that  $\sum_{a=1}^m V(a)W(1) \leq W(2)$ . Then, King's criterion reads that  $V$  is  $\sigma$ -semistable if and only if  $\dim U - \dim(\sum_{a=1}^m V(a)U) \leq 0$  for any subspace  $U \leq \mathbb{C}^n$ , which is equivalent to that  $A$  is *nc-nonsingular*. More generally, the nc-rank of  $A$  is equal to the minimum of  $n + \dim U - \dim(\sum_{a=1}^m V(a)U)$  over all subspaces  $U \leq \mathbb{C}^n$  [15].

► **Example 1.3** (BL polytope). Let  $Q$  be a star quiver with  $m$  leaves. We assume that  $Q_0 = \{0, 1, \dots, m\}$  and 0 is the root; see Figure 1. Let  $\alpha = (n, n_1, \dots, n_m)$  and  $\sigma = (d, -c_1, \dots, -c_m)$  for positive integers  $d, c_1, \dots, c_m$ . A real representation  $V$  of  $Q$  with the



■ **Figure 1** Generalized Kronecker quiver (left) and star quiver (right).

dimension vector  $\alpha$  can be regarded as a tuple of the matrices  $(B_1, \dots, B_m)$ , where  $B_i$  is an  $n_i \times n$  matrix. Again, a subrepresentation  $W$  is an  $(m+1)$ -tuple of the subspaces  $(W(0), W(1), \dots, W(m))$  such that  $B_i W(0) \leq W(i)$  for  $i \in [m]$ . King's criterion reads that  $V$  is  $\sigma$ -semistable if and only if  $dn = \sum_{i=1}^m c_i n_i$  and  $dn \dim W(0) - \sum_{i=1}^m c_i n_i \dim(B_i W(0)) \leq 0$  for any subspace  $W(0)$ . This is equivalent to that  $p = (c_1/d, \dots, c_m/d)$  is in the BL polytope of linear operators  $B_1, \dots, B_m$  [3].

We study the following optimization problem: given a quiver representation  $V$  and weight  $\sigma$ , find a subrepresentation  $W$  of  $V$  that maximizes  $\sigma(\underline{\dim} W)$ . In the case of the nc-rank, such a subrepresentation corresponds to a subspace  $U$  that maximizes  $\dim U - \dim(\sum_{a=1}^m A_i U)$ . Such a subspace is called a *shrunk subspace* and can be regarded as a certificate of the nc-rank [31, 19]. In the case of the BL polytopes, the problem corresponds to *separation* for the BL polytope [23].

King's criterion can be regarded as *maximizing a modular function over the modular lattice of subrepresentations*. For any subrepresentations  $W_1, W_2$  of  $V$ , define the subrepresentations  $W_1 + W_2$  and  $W_1 \cap W_2$  as follows. For each  $i \in Q_0$ ,

$$(W_1 + W_2)(i) := W_1(i) + W_2(i), \quad (W_1 \cap W_2)(i) := W_1(i) \cap W_2(i),$$

where the addition and intersection on the right-hand side are those of the vector spaces. Furthermore, the linear map of  $a \in Q_1$  in  $W_1 + W_2$  (resp.  $W_1 \cap W_2$ ) is defined as the restriction of  $V(a)$  to  $(W_1 + W_2)(ta)$  (resp.  $(W_1 \cap W_2)(ta)$ ). Then,  $W_1 + W_2$  and  $W_1 \cap W_2$  are indeed subrepresentations of  $V$ . Thus, the subrepresentations of  $V$  form a modular lattice. Furthermore, the function  $f(W) := \sigma(\underline{\dim} W)$  is a modular function, i.e., for any subrepresentations  $W_1, W_2$  of  $V$ ,

$$f(W_1) + f(W_2) = f(W_1 + W_2) + f(W_1 \cap W_2).$$

Thus, subrepresentations maximizing  $f$  form a sublattice, and there is a unique inclusion-wise minimum maximizer. Our second result is a deterministic algorithm to find such a maximizer of King's criterion.

► **Theorem 1.4** (informal version of Theorem 3.5). *Let  $Q$  be an acyclic quiver,  $V$  a representation of  $Q$ , and  $\sigma$  a weight. There is a deterministic algorithm that finds the inclusion-wise minimum maximizer  $W$  of King's criterion in time polynomial in the size of  $Q$ ,  $\alpha(Q_0)$ , bit complexity of  $V$ , and absolute values of the entries of  $\sigma$ .*

King's criterion was originally proved using the *Hilbert-Mumford criterion* (see, e.g., [14, Sections 9.6 and 9.8]), a fundamental result in the GIT.

### 1.3 Harder-Narasimhan filtration

We use the algorithm for finding the maximizers of King's criterion to devise an algorithm for finding the *Harder-Narasimhan (HN) filtration* [27, 28] of a quiver representation. Roughly speaking, the HN-filtration decomposes a quiver representation into the direct sum of smaller representations.

More precisely, let  $\sigma \in \mathbb{Z}^{Q_0}$  be a weight and  $\tau \in \mathbb{Z}_+^{Q_0}$  a strictly monotone weight, i.e., a nonnegative weight such that  $\tau(\underline{\dim} W) > 0$  if  $W \neq \{0\}$ . Here,  $\{0\}$  denotes the *zero representation*, which is the representation whose dimension vector is the zero vector. We define the *slope* of a quiver nonzero representation  $V$  as  $\mu(V) = \sigma(\underline{\dim} V) / \tau(\underline{\dim} V)$ . We say that  $V$  is  $\mu$ -semistable<sup>2</sup> if  $\mu(W) \leq \mu(V)$  for any nonzero subrepresentation  $W$  of  $V$ . The HN-filtration theorem states that for any quiver representation  $V$ , there exists a unique filtration  $\{0\} = W_0 < W_1 < \dots < W_k = V$  such that (i)  $\mu(W_i/W_{i-1}) > \mu(W_{i+1}/W_i)$  for  $i \in [k-1]$  and (ii)  $W_i/W_{i-1}$  is  $\mu$ -semistable. Here,  $W_i < W_{i+1}$  means that  $W_i$  is a subrepresentation of  $W_{i+1}$  with  $W_i \neq W_{i+1}$ , and  $W_i/W_{i-1}$  is a representation of  $Q$  such that  $(W_i/W_{i-1})(j)$  is the quotient space  $W_i(j)/W_{i-1}(j)$  for  $j \in Q_0$  and  $(W_i/W_{i-1})(a)$  is the corresponding quotient linear map of  $W_i(a)$  for  $a \in Q_1$ . We note that semistability with respect to a slope can be reduced to that for a weight.

Our third result is a deterministic algorithm for finding the HN-filtration.

► **Theorem 1.5 (informal).** *Let  $Q$  be an acyclic quiver,  $V$  a representation of  $Q$ , and  $\mu = \sigma/\tau$  a slope. There is a deterministic algorithm that finds the HN-filtration of  $V$  with respect to  $\mu$  in time polynomial in the size of  $Q$ ,  $\alpha(Q_0)$ , bit complexity of  $V$ , and absolute values of the entries of  $\sigma$  and  $\tau$ .*

This result improves a recent result [7] which runs in time polynomial in the number of paths in  $Q$ .

Recently, Hirai and Sakabe [29] introduced the *coarse Dulmage-Mendelsohn (DM) decomposition* of a linear matrix, generalizing the classic DM-decomposition of a bipartite graph. They showed that a natural gradient flow of operator scaling converges to the coarse DM-decomposition. However, their result did not provide an efficient algorithm to compute the coarse DM-decomposition because the gradient flow may take exponential time to converge. We show that the coarse DM-decomposition is indeed a special case of the HN-filtration for the generalized Kronecker quiver. Since the absolute values of the weights involved for this special case are polynomially bounded, our algorithm finds the coarse DM-decomposition in polynomial time.

### 1.4 King's polyhedral cone, rank-one representations, and submodular flow

Motivated by King's criterion, we investigate a polyhedral cone that is the set of  $\sigma \in \mathbb{R}^{Q_0}$  satisfying  $\sigma(\underline{\dim} V) = 0$  and  $\sigma(\underline{\dim} W) \leq 0$  for any subrepresentation  $W$  of  $V$ . Since the number of distinct  $\underline{\dim} W$  is finite, the above linear system is also finite and hence defines a polyhedral cone. We call the polyhedral cone the *King cone* of a quiver representation  $V$ . Interestingly, the King cone is related to the cone of feasible flows in network-flow problems.

<sup>2</sup> It is also called  $(\sigma : \tau)$ -semistability in the literature.

Let us first consider the easiest case. If  $V$  is a quiver representation with  $\dim V(i) = 1$  for all  $i \in Q_0$ , then King's criterion characterizes the existence of a nonnegative flow  $\varphi$  on the support quiver with the boundary condition  $\partial\varphi = \sigma$ . To state it more precisely, we introduce notation on flows. Let  $Q = (Q_0, Q_1)$  be a quiver (or a directed graph). For a vertex subset  $X \subseteq Q_0$ , let  $\text{Out}(X)$  denote the set of outgoing arcs from  $X$ , i.e.,  $\text{Out}(X) := \{a = (i, j) : i \in X, j \in Q_0 \setminus X\}$ . Similarly, let  $\text{In}(X)$  denote the set of incoming arcs to  $X$ . If  $X = \{i\}$ , we abbreviate  $\text{Out}(\{i\})$  and  $\text{In}(\{i\})$  as  $\text{Out}(i)$  and  $\text{In}(i)$ , respectively. If  $\text{Out}(X) = \emptyset$ , then  $X$  is called a *lower set* of  $Q$ . For a flow  $\varphi \in \mathbb{R}^{Q_1}$  on  $Q$ , its *boundary*  $\partial\varphi \in \mathbb{R}^{Q_0}$  is defined by  $\partial\varphi(i) := \sum_{a \in \text{Out}(i)} \varphi(a) - \sum_{a' \in \text{In}(i)} \varphi(a')$  for  $i \in Q_0$ .

Let us return to the  $\sigma$ -semistability of a representation  $V$  with  $\dim V(i) = 1$  for all  $i \in Q_0$ . In this case,  $V(a) \in \mathbb{C}$  for each arc  $a \in Q_1$ , and a subrepresentation  $W$  of  $V$  can be identified with a vertex subset  $X \subseteq Q_0$ . By the definition of a subrepresentation, if  $i \in X$  and  $W(a) \neq 0$  for  $a = (i, j) \in Q_1$ , then  $j \in X$ . This implies that  $X$  is a lower set in the support quiver of  $V$ , namely, the subquiver of  $Q$  whose arcs are  $a \in Q_1$  with  $V(a) \neq 0$ . Therefore, King's criterion is equivalent to the purely combinatorial condition that  $\sigma(Q_0) = 0$  and  $\sigma(X) \leq 0$  for each lower set  $X$  of  $Q$ , which characterizes the existence of a nonnegative flow  $\varphi$  on the support quiver with the boundary condition  $\partial\varphi = \sigma$  by Gale's theorem [22] (see, e.g., [33, Theorem 9.2]).

By generalizing the above observation, we show that if  $V$  is a *rank-one* representation of  $Q$ , i.e.,  $V(a)$  is a rank-one matrix for each  $a \in Q_1$ , then

- King's criterion can be rephrased as a purely combinatorial condition with respect to the linear matroids arising from the rank-one matrices  $V(a)$  of  $Q$ , and
- the rephrased condition above can be further viewed as the feasibility condition of a network flow-type problem called *submodular flow*.

That is, the King cone is representable as the feasibility of a certain instance of the submodular flow problem. This enables us to decide the  $\sigma$ -semistability for rank-one representations in strongly polynomial time.

► **Theorem 1.6 (informal).** *Let  $Q$  be an acyclic quiver,  $V$  a rank-one representation of  $Q$ , and  $\sigma$  a weight. Then,  $\sigma$  is in the King cone if and only if there is a feasible flow in the instance of submodular flow constructed from  $V$ ,  $\sigma$ . Therefore, using standard submodular flow algorithms, we can decide the  $\sigma$ -semistability of rank-one representations in strongly polynomial time.*

This theorem recovers the following well-known results when applied to the generalized Kronecker quiver and a star quiver.

- A rank-one linear matrix  $\sum_{k=1}^m x_k v_k f_k$  is (nc-)nonsingular (where  $v_k$  is a column vector and  $f_k$  a row vector) if and only if the linear matroids of  $(f_k : k \in [m])$  and  $(v_k : k \in [m])$  have a common base [34].
- If each linear operator  $B_i = f_i$  is of rank-one for  $i \in [m]$  (where  $f_i$  is a row vector), the BL polytope coincides with the base polytope of the linear matroid of  $(f_i : i \in [m])$  [2].

## 1.5 Semistability of general quivers

Thus far, we have considered the  $\sigma$ -semistability of acyclic quivers. As a complementary result, we show that the semistability of cyclic quivers under the  $\text{GL}(Q, \alpha)$ -action can be efficiently reduced to *noncommutative polynomial identity testing*. In particular, we show that the polynomial can be represented by an *algebraic branching program* (ABP). This yields a deterministic algorithm for deciding the semistability of general quivers because noncommutative polynomial identity testing for ABP can be conducted in deterministic

polynomial time [36]. Note that [6] devised another deterministic algorithm for the problem with their framework of noncommutative optimization, which is built upon deep results in various areas of mathematics. See also the discussion in related work.<sup>3</sup>

## 1.6 Our techniques

In this subsection, we outline our techniques.

**$\sigma$ -semistability.** Our starting point is a reduction of general acyclic quivers to the generalized Kronecker quiver [13, 30]. We decompose the weight  $\sigma = \sigma^+ - \sigma^-$ , where  $\sigma^+(i) := \max\{\sigma(i), 0\}$  and  $\sigma^-(i) := \max\{-\sigma(i), 0\}$ . Let  $Q_0^+$  and  $Q_0^-$  be the sets of vertices  $i$  such that  $\sigma(i) > 0$  and  $\sigma(i) < 0$ , respectively. Derksen and Makam [13] showed that the  $\sigma$ -semistability of a representation  $V$  with the dimension vector  $\alpha$  is equivalent to the nc-nonsingularity of a partitioned linear matrix

$$A : \bigoplus_{i \in Q_0^+} (\mathbb{C}^{\alpha(i)})^{\oplus \sigma^+(i)} \rightarrow \bigoplus_{i \in Q_0^-} (\mathbb{C}^{\alpha(i)})^{\oplus \sigma^-(i)},$$

where the  $(s, p; t, q)$ -block ( $s \in Q_0^+$ ,  $p \in [\sigma^+(s)]$ ,  $t \in Q_0^-$ ,  $q \in [\sigma^-(t)]$ ) of  $A$  is given by a linear matrix

$$\sum_{P: s \rightarrow t \text{ path}} x_{P,p,q} V(P).$$

Here,  $x_{P,p,q}$  is an indeterminate and  $V(P)$  is the linear map corresponding to the path  $P$ , i.e.,  $V(P) := V(a_k) \cdots V(a_1)$  for  $P = (a_1, \dots, a_k)$  as a sequence of arcs. However, because the number of indeterminates is exponential, applying the known nc-rank computation algorithms in a black box manner does not yield the desired time complexity.

Inspired by the above reduction, we introduce the following scaling problem for quiver representations. We define a scaling  $V_{g,h}$  of the quiver representation  $V$  by the block matrices  $g = \bigoplus (g_t : t \in Q_0^-)$  and  $h = \bigoplus (h_s : s \in Q_0^+)$  as

$$V_{g,h}(a) := \begin{cases} g_{ha} V(a) h_{ta}^\dagger & \text{if } a \in \text{Out}(Q_0^+) \cap \text{In}(Q_0^-), \\ V(a) h_{ta}^\dagger & \text{if } a \in \text{In}(Q_0^-) \setminus \text{Out}(Q_0^+), \\ g_{ha} V(a) & \text{if } a \in \text{Out}(Q_0^+) \setminus \text{In}(Q_0^-), \\ V(a) & \text{otherwise.} \end{cases}$$

Furthermore, let  $(b^+, b^-) \in \bigoplus_{i \in Q_0^+} \mathbb{Q}^{\alpha(s)} \times \bigoplus_{t \in Q_0^-} \mathbb{Q}^{\alpha(t)}$  be vectors such that  $b^+(s) = \frac{\sigma^+(s)}{N} \mathbf{1}_{\alpha(s)}$  and  $b^-(t) = \frac{\sigma^-(t)}{N} \mathbf{1}_{\alpha(t)}$ , where  $N := \sigma^+(\alpha) = \sigma^-(\alpha)$  and  $\mathbf{1}$  denotes the all-one vector. We say that  $V$  is *approximately scalable* (to the marginals  $(b^+, b^-)$ ) if for any  $\varepsilon > 0$ ,

<sup>3</sup> After submitting the first version of this paper, an anonymous reviewer pointed out that this result for general quivers was sketched in [35]. See Theorem 10.8 and the last paragraph of Section 10.2 in [35]. At a very high level, our algorithm and Mulmuley's results follow a similar strategy, although Mulmuley's result requires several deep algebro-geometric backgrounds. We believe that our proof is more explicit and elementary, and hence, is worthy to be presented here for completeness.

there exist block matrices  $g$  and  $h$  such that  $V_{g,h}$  satisfies

$$\left\| \sum_{s \in Q_0^+} \sum_{P: s \rightarrow t \text{ path}} V_{g,h}(P) V_{g,h}(P)^\dagger - \text{Diag}(b^-(t)) \right\|_{\text{tr}} < \varepsilon \quad (t \in Q_0^-),$$

$$\left\| \sum_{t \in Q_0^-} \sum_{P: s \rightarrow t \text{ path}} V_{g,h}(P)^\dagger V_{g,h}(P) - \text{Diag}(b^+(s)) \right\|_{\text{tr}} < \varepsilon \quad (s \in Q_0^+),$$

where the norm is the trace norm.<sup>4</sup> This is an instance of *operator scaling with specified marginals* [17, 5]. The crucial observation is that even though there may exist exponentially many  $s \rightarrow t$  paths, the above matrix sum can be computed efficiently by exploiting the underlying quiver structure. Therefore, we can use a simple iterative algorithm in [5] to check  $V$  is approximately scalable for a fixed  $\varepsilon > 0$ . We can show that it runs in  $O(\varepsilon^{-2} \text{poly}(|Q|, \alpha(Q_0), b))$  time, where  $b$  is the bit complexity of  $V$ . Furthermore, we show that it is sufficient to consider  $\varepsilon = O(1/N)$  to decide the  $\sigma$ -semistability of  $V$ . This yields our algorithm for checking the  $\sigma$ -semistability of quiver representations.

**Maximizers in King's criterion.** We follow a similar approach to find a maximizer in King's criterion. We use the above linear matrix of the reduction [13] directly and show that the shrunk subspaces of the above linear matrix correspond to the maximizers of King's criterion. Then, we show that the necessary operations in the recent shrunk subspace algorithm [19] can be performed efficiently, enabling us to find the inclusion-wise minimum maximizer of King's criterion efficiently. Note that the correspondence between the shrunk subspaces and maximizers of King's criterion is shown in [30] using abstract algebra; we provide a more direct and elementary proof using submodularity.

**HN-filtration.** Our HN-filtration algorithm is based on *principal partitions* of submodular systems [21]. For a slope  $\mu = \sigma/\tau$ , we consider a parametric modular function

$$f_\lambda(W) := \lambda \tau(\dim W) - \sigma(\dim W)$$

on the subrepresentations  $W$  of  $V$ , where  $\lambda \in \mathbb{R}$  is a parameter. Let  $\mathcal{L}(\lambda)$  denote the modular lattice of the minimizers of  $f_\lambda$  and let  $W^-(\lambda)$  and  $W^+(\lambda)$  be the minimum and maximum minimizers of  $f_\lambda$ , respectively. By the standard argument in principal partition, we show that  $W^+(\lambda) \leq W^-(\lambda')$  for  $\lambda > \lambda'$ . Furthermore, there must be a finite set of  $\lambda$  such that  $\mathcal{L}(\lambda)$  consists of more than one element. We call such a value of  $\lambda$  a *critical value*. Let  $\lambda_1 > \dots > \lambda_k$  be the critical values. Then, they induce the filtration

$$\{0\} = W^-(\lambda_1) < W^+(\lambda_1) = W^-(\lambda_2) < W^+(\lambda_2) = \dots = W^-(\lambda_k) < W^+(\lambda_k) = V.$$

We show that this coincides with the HN-filtration. Each  $W^-(\lambda)$  and  $W^+(\lambda)$  can be found by our algorithm for maximizers of King's criterion for fixed  $\lambda$ . The possible candidates for critical values can be easily enumerated, enabling us to find the HN-filtration efficiently.

**Strongly polynomial-time algorithm for rank-one representations.** When  $V$  is a rank-one representation, each rank-one matrix  $V(a)$  is representable as  $v_a f_a$  for some nonzero vector  $v_a \in V(ha)$  and nonzero dual vector  $f_a \in V(ta)^*$ . Based on this representation, we

<sup>4</sup> The choice of the trace norm is not important here; we can use any unitary invariant norm.

assign each vertex  $i \in Q_0$  to two linear matroids  $\mathbf{M}_i^+$  and  $\mathbf{M}_i^-$ , where the first is generated by  $\{f_a : f_a \in \text{Out}(i)\}$  and the second by  $\{v_a : v_a \in \text{In}(i)\}$ . Then, we can simulate a subrepresentation  $W$  of  $V$  as a lower set  $X$  of the directed graph  $D[V]$  constructed from  $V$ ; its vertex set is the (disjoint) union of the ground sets  $\{f_a : f_a \in \text{Out}(i)\}$  and  $\{v_a : v_a \in \text{In}(i)\}$  of the matroids  $\mathbf{M}_i^+$  and  $\mathbf{M}_i^-$  for  $i \in Q_0$ ; its arc set represents  $Q_1$  and the dependencies as “if  $W(ha) = W(tb)$  contains  $v_a$  then  $W(tb) \not\subseteq \ker f_b$ ; hence,  $W(hb)$  must contain  $v_b$  for  $W$  to be a subrepresentation of  $V$ .” This enables us to rephrase King’s criterion as a combinatorial condition on the lower sets  $X$  of  $D[V]$  as

$$\sum_{i \in Q_0} (\sigma^+(i) (\dim V(i) - r_i^+(\{f_a : f_a \in \text{Out}(i)\} \setminus X)) - \sigma^-(i) r_i^-(\{v_a : v_a \in \text{In}(i)\} \cap X)) \leq 0,$$

where  $r_i^+$  and  $r_i^-$  denote the rank functions of  $\mathbf{M}_i^+$  and  $\mathbf{M}_i^-$ , respectively. We further rephrase the above combinatorial condition as the feasibility characterization of a certain instance of the submodular flow problem by Frank [16]. Thus, we can check the  $\sigma$ -semistability of a rank-one representation  $V$  by checking the feasibility of the instance generated by  $V$  of the submodular flow problem.

**Semistability of general quivers.** For the semistability of general quivers, we use an invariant polynomial characterization of the null-cone. By the general theory of GIT, a representation  $V$  is semistable if and only if there exists a  $\text{GL}(Q, \alpha)$ -invariant homogeneous polynomial  $p$  on the representation space  $\text{Rep}(Q, \alpha)$  such that  $p(V) \neq 0$ . The Le Bruyn-Procesi theorem [4] stated that the ring of invariant polynomials is generated by polynomials in the form of

$$\text{tr}[V(a_k) \cdots V(a_2)V(a_1)],$$

for a closed path<sup>5</sup>  $(a_1, a_2, \dots, a_k)$  in  $Q$  with length  $k \geq 1$ . Furthermore, closed paths with length  $1 \leq k \leq \alpha(Q_0)^2$  generate the invariant ring, where  $\alpha = \dim V$ .

Therefore, we can decide the semistability by checking whether the above polynomial is nonzero at some vertex  $i$  and closed path  $C$ . The obstacle is that the number of closed paths can be exponential. To this end, we consider another polynomial in *noncommutative* indeterminate  $x_a$  ( $a \in Q_1$ ) defined as

$$\sum_{C: \text{closed path starting at } i} x^C \text{tr } V(C)$$

for each vertex  $i \in Q_0$ , where  $x^C = x_{a_k} \cdots x_{a_1}$  for  $C = (a_1, \dots, a_k)$ . Then,  $V$  is semistable if and only if this noncommutative polynomial is nonzero at some vertex  $i$ . Note that the noncommutativity is essential to distinguish closed paths with the same arc sets. For example, consider a quiver with a single vertex and two self-loops (say,  $a$  and  $b$ ). Then, closed walks  $C = abab \cdots ab$  and  $C' = a^k b^k$  of length  $2k$  have the same number of  $a$  and  $b$ , but their trace  $\text{tr } V(C)$  and  $\text{tr } V(C')$  are different in general for  $k \geq 2$ . So if we use commutative indeterminates, multiple closed walks correspond to a single monomial, and we cannot decide whether  $\text{tr } V(C) \neq 0$  for some  $C$  or not by checking whether the polynomial is zero or not.

Yet, we need to show how to perform noncommutative polynomial identity testing for this polynomial in deterministic polynomial time. Using the underlying quiver structure, we can show that this polynomial can be written as an ABP of polynomial size. Applying the algorithm of [36], we obtain our algorithm for the semistability of general quivers.

<sup>5</sup> Here, a closed path means a sequence  $(a_1, \dots, a_k)$  of arcs such that  $ha_l = ta_{l+1}$  ( $l \in [k]$ ), where  $a_{k+1} := a_1$ . In graph theory, it is usually called a *closed walk*. In this paper, we follow the standard terminologies in quiver representation.

We remark that this is the only place where a nontrivial algebraic result from the GIT machinery is needed. The other algorithms and analysis can be understood with elementary linear algebra (assuming the known analysis of operator scaling algorithms, which involves some abstract algebra).

## 1.7 Related work

Existing studies on algorithms for quiver semistability have focused on *bipartite* quivers [11, 9, 10, 18]. We remark that semistability in bipartite quivers is essentially operator scaling with a block structure. In bipartite quivers, the number of paths is equal to the number of arcs; hence, a weak running time was sufficient in the previous studies. [30, 7] studied general acyclic quivers. They first used the reduction of [13] to the generalized Kronecker quiver and applied the nc-rank algorithm [31] in a black box manner. Hence, their algorithm runs in time polynomial in the number of paths, which can be exponential.

Several polyhedral cones associated with quiver representations have been studied in the literature [8, 37]. The *moment cone* of a quiver  $Q$  and a dimension vector  $\alpha$  is the polyhedral cone generated by the highest weights of the representations of  $Q$  with the dimension vector  $\alpha$ . The membership of the moment cone can be decided in strongly polynomial time for bipartite quivers [8] and even general acyclic quivers [37]. Another polyhedral cone is the conic hull of weight  $\sigma \in \mathbb{Z}^{Q_0}$  such that there exists a nonzero semi-invariant polynomial in  $\text{Rep}(Q, \alpha)$  with weight  $\sigma$ . The membership problem of this cone is called the *generic semistability problem* [8]. By definition,  $\sigma$  is in this cone if and only if there exists a *generic*  $\sigma$ -semistable representation of  $Q$ , hence the name. To the best of our knowledge, the generic semistability problem remains open for general acyclic quivers.

The semistability of quiver representations is a special case of semistability in the GIT. In the most abstract setting, GIT studies group actions on algebraic varieties. We say that a point in the variety is semistable if its orbit closure does not contain the origin. Bürgisser et al. [6] proposed a framework of *noncommutative optimization* to devise algorithms for GIT problems in the general setting. Although noncommutative optimization is a broad and general framework, it does not always provide efficient algorithms for all GIT problems. Currently, most of the known tractable problems originate from a family of operator scaling problems, which are also contained in the semistability of quiver representations. Furthermore, it is built upon deep results in various areas of mathematics, such as algebraic geometry, Lie algebra, and representation theory, rendering it difficult for non-experts to understand. Another conceptual contribution of this paper is identifying the semistability of quiver representations as a useful subclass of GIT. The semistability of quiver representations is rich enough to capture various interesting problems in the literature while also supporting the design efficient algorithms using elementary techniques.

## 1.8 Organization of this paper

The rest of this paper is organized as follows. Section 2 introduces the necessary background and notation of quiver representations, operator scaling, and noncommutative rank. Section 3 presents our algorithms for deciding the  $\sigma$ -semistability and finding maximizers of King's criterion. Due to the space limitation, the other results are placed in the full version.

## 2 Preliminaries

We denote the set of nonnegative integers, rational, and real numbers by  $\mathbb{Z}_+$ ,  $\mathbb{Q}_+$ , and  $\mathbb{R}_+$ , respectively. We let  $[m, n] := \{m, m+1, \dots, n-1, n\}$  for  $m, n \in \mathbb{Z}$  with  $m \leq n$  and  $[n] := [1, n] = \{1, \dots, n\}$  for a positive integer  $n$ . We denote the set of  $m \times n$  complex matrices by  $\text{Mat}(m, n)$ . We simply denote  $\text{Mat}(n, n)$  by  $\text{Mat}(n)$ . The conjugate transpose of a matrix  $A$  is denoted by  $A^\dagger$ . The subgroup of the upper triangular matrices in  $\text{GL}(n)$  (i.e., the Borel subgroup) is denoted by  $\text{B}(n)$ . For two vector spaces  $U$  and  $V$ , we mean by  $U \leq V$  that  $U$  is a subspace of  $V$ . Let  $\langle S \rangle$  denote the vector space spanned by a multiset  $S$  of the vectors. For a vector  $b \in \mathbb{C}^n$ , we denote by  $\text{Diag}(b)$  the  $n \times n$  diagonal matrix such that the entries of  $b$  are on the diagonal.

### 2.1 Operator scaling, matrix space, and noncommutative rank

A linear map  $\Phi : \text{Mat}(n) \rightarrow \text{Mat}(m)$  is said to be *completely positive*, or *CP* for short, if  $\Phi(X) = \sum_{\ell=1}^k A_\ell X A_\ell^\dagger$  for some  $A_\ell \in \text{Mat}(m, n)$ . These  $A_\ell$  are called the *Kraus operators* of  $\Phi$ . The *dual* map  $\Phi^* : \text{Mat}(m) \rightarrow \text{Mat}(n)$  of  $\Phi$  is defined by  $\Phi^*(X) = \sum_{\ell=1}^k A_\ell^\dagger X A_\ell$ . For  $(g, h) \in \text{GL}(m) \times \text{GL}(n)$ , we define the scaling  $\Phi_{g,h}$  of  $\Phi$  by

$$\Phi_{g,h}(X) := g\Phi(h^\dagger X h)g^\dagger = \sum_{\ell=1}^k (gA_\ell h^\dagger)X(gA_\ell h^\dagger)^\dagger.$$

If  $m = n$ , the CP map is said to be *square*.

Let  $\Phi : \text{Mat}(n) \rightarrow \text{Mat}(n)$  be a square CP map. Let  $\text{ds}(\Phi) := \|\Phi(I) - I\|_F^2 + \|\Phi^*(I) - I\|_F^2$ , where  $\|\cdot\|_F$  denotes the Frobenius norm. Then,  $\Phi$  is said to be *approximately scalable* if for any  $\varepsilon > 0$ , there exists  $(g, h) \in \text{GL}(n) \times \text{GL}(n)$  such that  $\text{ds}(\Phi_{g,h}) \leq \varepsilon$ . The goal of the *operator scaling* problem is to decide whether a given CP map is approximately scalable or not.

Operator scaling is closely related to the noncommutative rank (nc-rank) of linear matrices. An  $m \times n$  symbolic matrix  $A$  is called a *linear matrix* if  $A = \sum_{\ell=1}^k x_\ell A_\ell$  for indeterminates  $x_\ell$  and matrices  $A_\ell \in \text{Mat}(m, n)$ . Sometimes, it is more convenient to see a linear matrix as a *matrix space* spanned by  $A_1, \dots, A_k$ . We denote the corresponding matrix space of a linear matrix  $A$  by  $\mathcal{A}$ . For a subspace  $U \leq \mathbb{C}^n$ , let  $\mathcal{A}U := \langle \{Au : A \in \mathcal{A}, u \in U\} \rangle$ , which is a subspace of  $\mathbb{C}^m$ . The *nc-rank* of a linear matrix  $A$  (denoted by  $\text{nc-rank } A$ ) is defined by

$$\text{nc-rank } A := \min\{n + \dim \mathcal{A}U - \dim U : U \leq \mathbb{C}^n\}.$$

A square linear matrix  $A$  is said to be *nc-nonsingular* if  $\text{nc-rank } A = n$ . Informally speaking,  $\text{nc-rank } A$  is the rank of  $A$ , where the indeterminates  $x_i$  are pairwise noncommutative, i.e.,  $x_i x_j \neq x_j x_i$  for  $i \neq j$ . See [12, 15] for more details. A pair  $(L, R)$  of subspaces  $L \leq \mathbb{C}^m$  and  $R \leq \mathbb{C}^n$  is called an *independent subspace* if  $L \cap \mathcal{A}R = \{0\}$ . Over the complex field,  $(L, R)$  is independent if and only if  $\text{tr}(\Pi_L \Phi(\Pi_R)) = 0$ , where  $\Pi_L$  denotes the orthogonal projection matrix onto  $L$ . Then,

$$\text{nc-rank } A = m + n - \max\{\dim L + \dim R : (L, R) \text{ an independent subspace}\}.$$

An independent subspace  $(L, R)$  is said to be *maximum* if  $\dim L + \dim R$  is maximum. In particular, a square linear matrix  $A$  is nc-nonsingular if and only if  $\dim L + \dim R \leq n$  for any independent subspace  $(L, R)$ . Gurvits' theorem [25] states that a square CP map  $\Phi$  with the Kraus operator  $A_\ell$  ( $\ell \in [k]$ ) is approximately scalable if and only if the linear matrix  $\sum_{\ell=1}^k x_\ell A_\ell$  is nc-nonsingular.

## 2.2 Operator scaling with specified marginals

*Operator scaling with specified marginals* is a generalization of operator scaling. Let  $(b^+, b^-) \in \mathbb{R}_+^n \times \mathbb{R}_+^m$  be a pair of nonincreasing nonnegative vectors, which we call the *target marginals*. We say that a CP map  $\Phi : \text{Mat}(n) \rightarrow \text{Mat}(m)$  is *approximately scalable to the target marginals*  $(b^+, b^-)$  if there exist nonsingular upper triangular matrices  $(g, h) \in \text{B}(m) \times \text{B}(n)$  such that

$$\|\Phi_{g,h}(I) - \text{Diag}(b^-)\|_{\text{tr}} \leq \varepsilon, \quad \|\Phi_{g,h}^*(I) - \text{Diag}(b^+)\|_{\text{tr}} \leq \varepsilon.$$

Such target marginals are said to be *feasible*. We define  $\Delta b^+ \in \mathbb{R}_+^n$  as  $\Delta b_j^+ = b_j^+ - b_{j+1}^+$  for  $j \in [n]$ , where we conventionally define  $b_{n+1}^+ := 0$ . Similarly, we define  $\Delta b^- \in \mathbb{R}_+^m$ . Let  $F_j^+ = \langle e_1, \dots, e_j \rangle$  be the standard flag of  $\mathbb{C}^n$  for  $j \in [n]$ . Similarly, we define  $F_i^-$  for  $i \in [m]$ . The following theorem characterizes the set of feasible marginals by a certain linear system.

► **Theorem 2.1** ([17, Theorem 18]). *Let  $\Phi : \text{Mat}(n) \rightarrow \text{Mat}(m)$  be a CP map and  $(b^+, b^-) \in \mathbb{R}_+^n \times \mathbb{R}_+^m$  a pair of nonincreasing nonnegative vectors. Then,  $\Phi$  is approximately scalable to the marginals  $(b^+, b^-)$  if and only if  $\sum_{j=1}^n b_j^+ = \sum_{i=1}^m b_i^- =: B$  and*

$$\sum_{i=1}^m \Delta b_i^- \dim(L \cap F_i^-) + \sum_{j=1}^n \Delta b_j^+ \dim(R \cap F_j^+) \leq B$$

for any independent subspace  $(L, R)$  of  $\Phi$ .

Let  $(b^+, b^-)$  be a feasible marginal with rational entries. There is an efficient algorithm that finds a scaling of a given CP map  $\Phi$  whose marginal is  $\varepsilon$ -close to  $(b^+, b^-)$  [17, 5].

► **Theorem 2.2** (Theorem 1.13 in [5], specialized for operator scaling). *Let  $\varepsilon > 0$  be an accuracy parameter and  $\Phi : \text{Mat}(n) \rightarrow \text{Mat}(m)$  a CP map with Gaussian integer Kraus operators. Let  $(b^+, b^-) \in \mathbb{Q}^n \times \mathbb{Q}^m$  be target marginals such that  $b_1^+ \geq \dots \geq b_n^+ \geq 0$ ,  $b_1^- \geq \dots \geq b_m^- \geq 0$ , and  $\sum_{j=1}^n b_j^+ = \sum_{i=1}^m b_i^- = 1$ . Then, Algorithm 1 finds upper triangular  $g, h$  such that  $\|\Phi_{g,h}(I) - \text{Diag}(b^-)\|_{\text{tr}} \leq \varepsilon$  and  $\|\Phi_{g,h}^*(I) - \text{Diag}(b^+)\|_{\text{tr}} \leq \varepsilon$  in  $T = O(\varepsilon^{-2}(b + N \log(\ell N)))$  iterations, where  $b$  is the maximum bit length of the target marginals  $(b^+, b^-)$ ,  $N := \max\{m, n\}$ , and  $\ell$  is the smallest positive integer such that  $\ell(b^+, b^-)$  is an integer. Furthermore, each iteration can be executed in time  $O(N^3)$ .*

■ **Algorithm 1** Operator Sinkhorn iteration for specified marginals [17, 5].

- 
- 1: **for**  $t = 1, \dots, T$  **do**
  - 2:   ▷ *Left normalization*
  - 3:   Compute the Cholesky decomposition  $CC^\dagger = \Phi(I)$ . Set  $g = \text{Diag}(b^-)^{1/2}C^{-1}$  and  $\Phi \leftarrow \Phi_{g,I}$ .
  - 4:   ▷ *Right normalization*
  - 5:   Compute the Cholesky decomposition  $CC^\dagger = \Phi^*(I)$ . Set  $h = \text{Diag}(b^+)^{1/2}C^{-1}$  and  $\Phi \leftarrow \Phi_{I,h}$ .
- 

## 3 Reduction to nc-rank and algorithms for semistability

In this section, we present our algorithms for deciding  $\sigma$ -semistability and finding maximizers of King's criterion.

### 3.1 Reduction from semistability to nc-rank

Here, we recall the reduction of the  $\sigma$ -semistability of an acyclic quiver to nc-nonsingularity testing [13].

Let  $Q$  be an acyclic quiver and  $V$  a representation of  $Q$  with the dimension vector  $\alpha$ . Let  $\sigma \in \mathbb{Z}^{Q_0}$  be a weight.<sup>6</sup> Let  $Q_0^+$  and  $Q_0^-$  be the sets of vertices  $i$  such that  $\sigma(i) > 0$  and  $\sigma(i) < 0$ , respectively. Let  $\sigma^+(i) := \max\{\sigma(i), 0\}$  and  $\sigma^-(i) := \max\{-\sigma(i), 0\}$  for each  $i \in Q_0$ . Note that  $\sigma = \sigma^+ - \sigma^-$ . Let  $N := \sigma^+(\alpha) = \sigma^-(\alpha)$ . We define an  $N \times N$  partitioned linear matrix  $A$  as follows. As a linear map,

$$A : \bigoplus_{s \in Q_0^+} (\mathbb{C}^{\alpha(s)})^{\oplus \sigma^+(s)} \rightarrow \bigoplus_{t \in Q_0^-} (\mathbb{C}^{\alpha(t)})^{\oplus \sigma^-(t)}. \quad (1)$$

The  $(s, p; t, q)$ -block ( $s \in Q_0^+$ ,  $p \in [\sigma^+(s)]$ ,  $t \in Q_0^-$ ,  $q \in [\sigma^-(t)]$ ) of  $A$  is given by a linear matrix

$$\sum_{P: s \rightarrow t \text{ path}} x_{P,p,q} V(P),$$

where  $x_{P,p,q}$  is an indeterminate and  $V(P)$  is the linear map corresponding to the path  $P$ , i.e.,  $V(P) := V(a_k) \cdots V(a_1)$  for  $P = (a_1, \dots, a_k)$  as a sequence of arcs. The number of indeterminates in  $A$  is equal to

$$\sum_{s \in Q_0^+} \sum_{t \in Q_0^-} \sigma^+(s) \sigma^-(t) m(s, t),$$

where  $m(s, t)$  denotes the number of  $s \rightarrow t$  paths in  $Q$ . Thus, the number of indeterminates may be exponential in general.

The following lemma connects the nc-rank of  $A$  with King's criterion, which is shown in [30, Theorem 3.3] using abstract algebra.

► **Lemma 3.1.** *The minimal maximizer of  $\sigma(\dim W)$  for the subrepresentations  $W$  of  $V$  corresponds to the minimal shrunk subspace of  $A$ .*

By the lemma, one can check the  $\sigma$ -semistability of a quiver representation by checking whether the corresponding linear matrix is nc-nonsingular or not. However, the naive reduction does not give a polynomial time algorithm, as the number of indeterminates in the linear matrix may be exponential.

### 3.2 Scaling algorithm for $\sigma$ -semistability

We present a scaling algorithm for deciding the  $\sigma$ -semistability. The idea is to reduce the problem to operator scaling with specified marginals.

Let us define a CP map  $\Phi_V$  corresponding to a quiver representation  $V$ . As a linear map,

$$\Phi_V : \bigoplus_{s \in Q_0^+} \text{Mat}(\alpha(s)) \rightarrow \bigoplus_{t \in Q_0^-} \text{Mat}(\alpha(t)).$$

<sup>6</sup> We do not assume  $\sigma(\alpha) = 0$  here because the reduction does not need it. This is useful for finding a maximizer in King's criterion.

## 99:14 Algorithmic Aspects of Semistability of Quiver Representations

Let  $X = \bigoplus (X_s : s \in Q_0^+)$  be an input block matrix. Then,

$$(\Phi_V(X))_t := \sum_{s \in Q_0^+} \sum_{P: s \rightarrow t \text{ path}} V(P) X_s V(P)^\dagger$$

for  $t \in Q_0^-$ . The dual map

$$\Phi_V^* : \bigoplus_{t \in Q_0^-} \text{Mat}(\alpha(t)) \rightarrow \bigoplus_{s \in Q_0^+} \text{Mat}(\alpha(s))$$

is given by

$$(\Phi_V^*(Y))_s := \sum_{t \in Q_0^-} \sum_{P: s \leftarrow t \text{ path}} V(P)^\dagger Y_t V(P)$$

for  $s \in Q_0^+$ . Analogous to the scaling of CP maps, we define a scaling  $V_{g,h}$  of the quiver representation  $V$  by block matrices  $g = (g_t : t \in Q_0^-)$  and  $h = (h_s : s \in Q_0^+)$  as

$$V_{g,h}(a) := \begin{cases} g_{ha} V(a) h_{ta}^\dagger & \text{if } a \in \text{Out}(Q_0^+) \cap \text{In}(Q_0^-), \\ V(a) h_{ta}^\dagger & \text{if } a \in \text{In}(Q_0^-) \setminus \text{Out}(Q_0^+), \\ g_{ha} V(a) & \text{if } a \in \text{Out}(Q_0^+) \setminus \text{In}(Q_0^-), \\ V(a) & \text{otherwise.} \end{cases}$$

Here is the key lemma that relates the  $\sigma$ -semistability of a quiver representation to the feasibility of a specific marginal of  $\Phi_V$ .

► **Lemma 3.2.** *Let  $V$  be a representation of an acyclic quiver  $Q$  with the dimension vector  $\alpha$  and  $\sigma$  a weight with  $\sigma(\alpha) = 0$ . Let  $(b^+, b^-)$  be the target marginals such that  $b^+(s) = \frac{\sigma^+(s)}{N} \mathbf{1}_{\alpha(s)}$  and  $b^-(t) = \frac{\sigma^-(t)}{N} \mathbf{1}_{\alpha(t)}$ , where  $N := \sigma^+(\alpha) = \sigma^-(\alpha)$ . Then,  $\Phi_V$  is approximately scalable to the marginals  $(b^+, b^-)$  if and only if  $V$  is  $\sigma$ -semistable.*

To check the feasibility of the marginal, one can use the scaling algorithm for operator scaling with the specified marginals (Algorithm 1). This yields Algorithm 2.

■ **Algorithm 2** Scaling algorithm for  $\sigma$ -semistability.

---

**Require:** a representation  $V$  of an acyclic quiver  $Q$  and a weight  $\sigma$

- 1: Let  $b^+(s) := \frac{\sigma^+(s)}{N} \mathbf{1}_{\alpha(s)}$  and  $b^-(t) := \frac{\sigma^-(t)}{N} \mathbf{1}_{\alpha(t)}$ , where  $N = \max\{\sigma^+(\alpha), \sigma^-(\alpha)\}$ .
  - 2: Set  $\varepsilon = \frac{1}{6N}$  and  $T := O(\varepsilon^{-2}(b + d \log(Nd)))$ , where  $d = \max\{\alpha(Q_0^+), \alpha(Q_0^-)\}$ , and  $b$  is the maximum bit length of  $\sigma$ .
  - 3: **for**  $t = 1, \dots, T$  **do**
  - 4:   ▷ *Left normalization*
  - 5:   If  $\|\Phi_V(I) - \text{Diag}(b^-)\|_{\text{tr}} \leq \varepsilon$  then **return Yes**.
  - 6:   Compute the Cholesky decomposition  $CC^\dagger = \Phi_V(I)$ . Set  $g = \text{Diag}(b^-)^{1/2} C^{-1}$  and update  $V \leftarrow V_{g,I}$ .
  - 7:   ▷ *Right normalization*
  - 8:   If  $\|\Phi_V^*(I) - \text{Diag}(b^+)\|_{\text{tr}} \leq \varepsilon$  then **return Yes**.
  - 9:   Compute the Cholesky decomposition  $CC^\dagger = \Phi_V^*(I)$ . Set  $h = \text{Diag}(b^+)^{1/2} C^{-1}$  and update  $V \leftarrow V_{I,h}$ .
  - 10: **return No**.
-

To run the algorithm, we first need to show how to compute the value of  $\Phi_V$  in polynomial time. Note that the naive computation of  $\Phi_V$  requires exponential time, as the number of terms in the sum is exponential. However, we can compute  $\Phi_V$  in polynomial time using the underlying quiver structure. We show an algorithm in Algorithm 3. The algorithm for the dual map is similar and therefore omitted.

■ **Algorithm 3** Algorithm for computing  $\Phi_V(X)$ .

**Require:** a representation  $V$  of an acyclic quiver  $Q$  and a block matrix  $X = (X_s : s \in Q_0^+)$

- 1: Let  $X_i := O$  for  $i \in Q_0 \setminus Q_0^+$ .
- 2: **for**  $a \in Q_1$  in a topological order **do**
- 3:    $X_{ha} \leftarrow X_{ha} + V(a)X_{ta}V(a)^\dagger$ .
- 4: **return**  $(X_i : i \in Q_0^-)$ .

Next, we show an upper bound of the accuracy parameter  $\varepsilon$  that is sufficient to decide the  $\sigma$ -semistability.

► **Lemma 3.3.** *Let  $(b^+, b^-)$  be as above. Let  $0 < \varepsilon \leq \frac{1}{6N}$ . If there exist  $g, h$  such that  $\|(\Phi_V)_{g,h}(I) - \text{Diag}(b^-)\|_{\text{tr}} \leq \varepsilon$  and  $\|(\Phi_V)_{g,h}^*(I) - \text{Diag}(b^+)\|_{\text{tr}} \leq \varepsilon$ , then  $V$  is  $\sigma$ -semistable.*

► **Theorem 3.4.** *Let  $V$  be a representation of an acyclic quiver  $Q$  with Gaussian integer entries and  $\sigma$  be a weight with  $\sigma(\alpha) = 0$ . Algorithm 2 correctly decides the  $\sigma$ -semistability of  $V$  in  $O(N^2(b + d \log(Nd)))$  iterations, where  $N = \sigma^+(\alpha) = \sigma^-(\alpha)$ ,  $d = \max\{\alpha(Q_0^+), \alpha(Q_0^-)\}$ , and  $b$  is the maximum bit length of  $\sigma$ . Each iteration can be executed in  $O(|Q_1|\alpha_{\max}^3 + \sum_{s \in Q_0^+} \alpha(s)^3 + \sum_{t \in Q_0^-} \alpha(t)^3)$  time, where  $\alpha_{\max} = \max_{i \in Q_0} \alpha(i)$ .*

**Proof.** If  $V$  is  $\sigma$ -semistable, then  $\Phi_V$  is approximately scalable to the marginals  $(b^+, b^-)$ . By Theorem 2.1, the algorithm must find such a scaling within the stated iterations. Consequently, the algorithm outputs **Yes**. If  $V$  is not  $\sigma$ -semistable, then there is no scaling  $g, h$  such that  $\|(\Phi_V)_{g,h}(I) - \text{Diag}(b^-)\|_{\text{tr}} \leq \varepsilon$  and  $\|(\Phi_V)_{g,h}^*(I) - \text{Diag}(b^+)\|_{\text{tr}} \leq \varepsilon$  for  $\varepsilon = 1/6N$  by Lemma 3.3. Thus, the algorithm outputs **No**. This proves the correctness of the algorithm.

The number of iterations is immediate from the algorithm. The time complexity of each iteration is dominated by the computations of  $\Phi_V(I)$  and  $\Phi_V^*(I)$  and that of the block Cholesky decomposition. The former takes  $O(|Q_1|\alpha_{\max}^3)$  time and the latter takes  $O(\sum_{t \in Q_0^-} \alpha(t)^3 + \sum_{s \in Q_0^+} \alpha(s)^3)$  time. ◀

### 3.3 Finding the extreme maximizer in King's criterion

In this subsection, we extend the result from the previous section to find the extreme maximizer in King's criterion. The idea is to use the shrunk subspace algorithm for the linear matrix (1).

Let

$$\Phi_V^\sigma : \bigoplus_{i \in Q_0^+} \text{Mat}(\alpha(i))^{\sigma^+(i)} \rightarrow \bigoplus_{i \in Q_0^-} \text{Mat}(\alpha(i))^{\sigma^-(i)}$$

be a CP map that maps  $X = \bigoplus (X_{s,p} : s \in Q_0^+, p \in [\sigma^+(s)])$  to

$$(\Phi_V^\sigma(X))_{t,q} = \sum_{s \in Q_0^+} \sum_{p \in [\sigma^+(s)]} \sum_{P: s \rightarrow t \text{ path}} V(P)X_{s,p}V(P)^\dagger$$

for  $t \in Q_0^-$  and  $q \in [\sigma^-(t)]$ . Let  $J_\sigma^+ = \bigoplus_{s \in Q_0^+} \sigma^+(s) I_{\alpha(s)}$  and  $J_\sigma^- = \bigoplus_{t \in Q_0^-} \sigma^-(t) I_{\alpha(t)}$ . Note that

$$\begin{aligned} (\Phi_V^\sigma(I))_{t,q} &= \sum_{s \in Q_0^+} \sigma^+(s) \sum_{P: s \rightarrow t \text{ path}} V(P) V(P)^\dagger = \Phi_V(J_\sigma^+)_t, \\ ((\Phi_V^\sigma)^*(I))_{s,p} &= \sum_{t \in Q_0^-} \sigma^-(t) \sum_{P: s \rightarrow t \text{ path}} V(P)^\dagger V(P) = \Phi_V^*(J_\sigma^-)_s. \end{aligned}$$

Thus, one can compute  $\Phi_V^\sigma(I)$  and  $(\Phi_V^\sigma)^*(I)$  in  $\text{poly}(|Q|, |\alpha|, |\sigma|)$  time using Algorithm 3. By Lemma 3.1, the minimum shrunk subspace of  $\Phi_V^\sigma$  corresponds to the minimum maximizer in King's criterion. To find the minimum shrunk subspace of  $\Phi_V^\sigma$ , one can simply use the algorithm of [19], which runs in  $\text{poly}(|Q|, |\alpha|, |\sigma|, b)$  time, because one can compute  $\Phi_V^\sigma(I)$  and  $(\Phi_V^\sigma)^*(I)$  in  $\text{poly}(|Q|, |\alpha|, |\sigma|, b)$  time.

The *maximum* maximizer can also be found by considering  $\Phi^*$  instead of  $\Phi$ . To see this, observe that the maximum shrunk subspace corresponds to the maximum independent subspace  $(L, R)$  such that  $\dim L$  is the smallest. Since  $\text{tr}(\Pi_L \Phi(\Pi_R)) = \text{tr}(\Phi^*(\Pi_L) \Pi_R)$ ,  $L$  is the minimum shrunk subspace of  $\Phi^*$ .

Therefore, we obtain the following theorem.

► **Theorem 3.5.** *For a quiver representation  $V$  of an acyclic quiver  $Q$  with the dimension vector  $\alpha$  with Gaussian integer entries and a weight  $\sigma$ , the minimum and maximum maximizers of King's criterion can be found in  $\text{poly}(|Q|, |\alpha|, |\sigma|, b)$  time, where  $b$  denotes the bit complexity of  $V$ .*

---

## References

- 1 Carlos Améndola, Kathlén Kohn, Philipp Reichenbach, and Anna Seigal. Invariant theory and scaling algorithms for maximum likelihood estimation. *SIAM Journal on Applied Algebra and Geometry*, 5(2):304–337, 2021. doi:10.1137/20M1328932.
- 2 Franck Barthe. On a reverse form of the Brascamp-Lieb inequality. *Inventiones Mathematicae*, 134(2):335–361, 1998. doi:10.1007/s002220050267.
- 3 Jonathan Bennett, Anthony Carbery, Michael Christ, and Terence Tao. The Brascamp-Lieb inequalities: Finiteness, structure and extremals. *Geometric and Functional Analysis*, 17(5):1343–1415, 2008. doi:10.1007/s00039-007-0619-6.
- 4 Lieven Le Bruyn and Claudio Procesi. Semisimple representations of quivers. *Transactions of the American Mathematical Society*, 317(2):585, 1990. doi:10.2307/2001477.
- 5 Peter Burgisser, Cole Franks, Ankit Garg, Rafael Oliveira, Michael Walter, and Avi Wigderson. Efficient algorithms for tensor scaling, quantum marginals, and moment polytopes. In *Proceedings of the IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 883–897, 2018. doi:10.1109/FOCS.2018.00088.
- 6 Peter Bürgisser, Cole Franks, Ankit Garg, Rafael Oliveira, Michael Walter, and Avi Wigderson. Towards a theory of non-commutative optimization: Geodesic 1st and 2nd order methods for moment maps and polytopes. In *Proceedings of the 60th IEEE Annual Symposium on Foundations of Computer Science (FOCS)*, pages 845–861, 2019. doi:10.1109/FOCS.2019.00055.
- 7 Chi-Yu Cheng. A deterministic algorithm for Harder-Narasimhan filtrations for representations of acyclic quivers. *Algebra & Number Theory*, 18(2):319–347, 2024. doi:10.2140/ant.2024.18.319.
- 8 Calin Chindris, Brett Collins, and Daniel Kline. Membership in moment cones and quiver semi-invariants for bipartite quivers. *arXiv*, 2022.

- 9 Calin Chindris and Harm Derksen. The capacity of quiver representations and Brascamp-Lieb constants. *International Mathematics Research Notices*, 2022(24):19399–19430, 2021. doi:10.1093/imrn/rnab064.
- 10 Calin Chindris and Jasim Ismaeel. A quiver invariant theoretic approach to radial isotropy and the paulsen problem for matrix frames. *SIAM Journal on Applied Algebra and Geometry*, 6(4):536–562, 2022. doi:10.1137/21M141470X.
- 11 Calin Chindris and Daniel Kline. Simultaneous robust subspace recovery and semi-stability of quiver representations. *Journal of Algebra*, 577:210–236, 2021. doi:10.1016/j.jalgebra.2021.03.005.
- 12 P. M. Cohn. *Skew Fields: Theory of General Division Rings*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1995.
- 13 Harm Derksen and Visu Makam. Polynomial degree bounds for matrix semi-invariants. *Advances in Mathematics*, 310:44–63, 2017. doi:10.1016/j.aim.2017.01.018.
- 14 Harm Derksen and Jerzy Weyman. *An Introduction to Quiver Representations*, volume 184. American Mathematical Soc., 2017.
- 15 Marc Fortin and Christophe Reutenauer. Commutative/noncommutative rank of linear matrices and subspaces of matrices of low rank. *Séminaire Lotharingien de Combinatoire [electronic only]*, 52:B52f–12, 2004.
- 16 András Frank. Finding feasible vectors of Edmonds-Giles polyhedra. *Journal of Combinatorial Theory, Series B*, 36(3):221–239, 1984. doi:10.1016/0095-8956(84)90029-7.
- 17 Cole Franks. Operator scaling with specified marginals. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 190–203, 2018. doi:10.1145/3188745.3188932.
- 18 Cole Franks and Visu Makam. iPCA and stability of star quivers. *arXiv*, 2023. doi:10.48550/arXiv.2302.09658.
- 19 Cole Franks, Tasuku Soma, and Michel X Goemans. Shrunk subspaces via operator sinkhorn iteration. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1655–1668, 2023. doi:10.1137/1.9781611977554.CH62.
- 20 William Cole Franks and Ankur Moitra. Rigorous guarantees for Tyler’s M-estimator via quantum expansion. In *Proceedings of the 33rd Conference on Learning Theory (COLT)*, volume 125, pages 1601–1632. PMLR, 2020. URL: <http://proceedings.mlr.press/v125/franks20a.html>.
- 21 Satoru Fujishige. *Theory of Principal Partitions Revisited*, pages 127–162. Springer Berlin Heidelberg, 2009. doi:10.1007/978-3-540-76796-1\_7.
- 22 David Gale. A theorem on flows in networks. *Pacific Journal of Mathematics*, 7(2):1073–1082, 1957.
- 23 Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson. Algorithmic and optimization aspects of Brascamp-Lieb inequalities, via operator scaling. *Geometric and Functional Analysis*, 28(1):100–145, 2018. doi:10.1007/s00039-018-0434-2.
- 24 Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson. Operator scaling: Theory and applications. *Foundations of Computational Mathematics*, 2019. doi:10.1007/s10208-019-09417-z.
- 25 Leonid Gurvits. Classical complexity and quantum entanglement. *Journal of Computer and System Sciences*, 69(3):448–484, 2004. doi:10.1016/j.jcss.2004.06.003.
- 26 Masaki Hamada and Hiroshi Hirai. Computing the nc-rank via discrete convex optimization on CAT(0) spaces. *SIAM Journal on Applied Algebra and Geometry*, 5(3):455–478, 2021. doi:10.1137/20M138836X.
- 27 G Harder and M S Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen*, 212(3):215–248, 1975.
- 28 Lutz Hille and José Antonio de la Peña. Stable representations of quivers. *Journal of Pure and Applied Algebra*, 172(2):205–224, 2002. doi:10.1016/S0022-4049(01)00167-0.

- 29   Hiroshi Hirai and Keiya Sakabe. Gradient descent for unbounded convex functions on hadamard manifolds and its applications to scaling problems. In *Proceedings of the IEEE 65th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 2387–2402, 2024. doi:10.1109/FOCS61266.2024.00139.
- 30   Alana Huszar. Non-commutative rank and semi-stability of quiver representations. *arXiv*, 2021. doi:10.48550/arXiv.2111.00039.
- 31   Gábor Ivanyos, Youming Qiao, and K V Subrahmanyam. Constructive non-commutative rank computation is in deterministic polynomial time. *Computational Complexity*, 27(4):561–593, 2018. doi:10.1007/s00037-018-0165-7.
- 32   Alastair D King. Moduli of representations of finite dimensional algebras. *The Quarterly Journal of Mathematics*, 45(4):515–530, 1994.
- 33   Bernhard Korte and Jens Vygen. *Combinatorial Optimization: Theory and Algorithms*. Springer-Verlag, Berlin, 6 edition, 2018. doi:10.1007/978-3-662-56039-6.
- 34   László Lovász. Singular spaces of matrices and their application in combinatorics. *Bulletin of the Brazilian Mathematical Society*, 20:87–99, 1989.
- 35   Ketan Mulmuley. Geometric complexity theory V: Efficient algorithms for Noether normalization. *Journal of the American Mathematical Society*, 30(1):225–309, 2017.
- 36   Ran Raz and Amir Shpilka. Deterministic polynomial identity testing in non-commutative models. *Computational Complexity*, 14(1):1–19, 2005. doi:10.1007/s00037-005-0188-8.
- 37   Michèle Vergne and Michael Walter. Moment cone membership for quivers in strongly polynomial time. *arXiv*, 2023.