Formal Comparison of Outgoing Event Streams Between Compositional Performance Analysis and Real-Time Calculus

Victor Pollex
INCHRON AG, Erlangen, Germany

Frank Slomka
Institute of Embedded Systems/Real-Time Systems, Faculty of Engineering,
Computer Science and Psychology, Ulm University, Germany

Abstract

Real-time systems are required to produce correct responses which also need to be on time. Two methods to verify the timeliness of the results are the compositional performance analysis and the real-time calculus. Both methods use bounds as inputs from which they derive bounds for the timeliness. Therefore, it is preferable to have better bounds for the inputs. Any of these bounds may contain pessimism. Because both methods use a modular/compositional approach for distributed real-time systems, the pessimism in the bounds accumulate. Therefore, it is desirable to keep the pessimism as low as possible. In this work we investigate distributed real-time systems where tasks have an activation dependency, i. e., the activation of one task depends on the termination of a job of another task. Specifically, we compare the derived bound of how frequent the jobs of a task terminate and subsequently how frequent the activation dependent task activates and releases its jobs.

Our contribution is a formal comparison of these bounds that the compositional performance analysis and the real-time calculus derive by means of a mathematical proof. We show that for identical inputs the real-time calculus derives a bound that is not worse than the bound that the compositional performance analysis derives for a single module/component. The mathematical proof also removes any uncertainty that existing empirical comparisons of these bounds may contain.

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1 Introduction

For real-time systems to work correctly they must not only produce correct results, but the results must be produced on time. Two of the available methods to verify the timeliness of the results are the compositional performance analysis (CPA) and the real-time calculus (RTC). Due to their compositional/modular approach they can also be used to analyse distributed real-time systems. Figure 1 shows such a system which consists of the two processors CPU 1 and CPU 2 that have the sets $\Gamma_1 := \{\tau_1, \tau_2, \tau_3\}$ and $\Gamma_2 := \{\tau_4, \tau_5, \tau_6\}$ assigned, respectively.

For distributed systems it is common to assume that the tasks have activation dependencies as is the case for example between task τ_1 and task τ_4 and between task τ_2 and task τ_5. When a job of task τ_1 terminates, it will activate task τ_4 and release a job. Both the CPA and RTC operate on bounds of the amount of task activations, i. e., both their input and output are bounds. And due to the compositional/modular approach any pessimism that

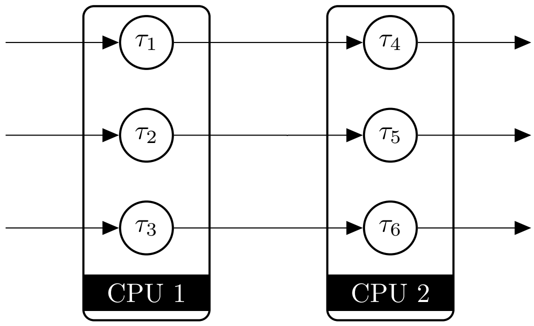


Figure 1 Example of a distributed system of two processors where each has three tasks.

the bounds may have tends to accumulate. Therefore, it is desirable to derive bounds with as little pessimism as possible. This has a direct and possibly an indirect influence on the derived bound for the timeliness of a task. For example, the bound derived for the task activation of task τ_5 directly influences the derived bound for the timeliness of task τ_5. The more frequent task τ_5 is activated, the larger the backlog of unprocessed task activations can become and subsequently the longer it might take to process all the task activations in the backlog. Thus affecting the derived bound for the timeliness of task τ_5. Depending on the scheduling that CPU 2 uses, the derived bound for the task activation of task τ_4 can indirectly influence the derived bound for the timeliness of task τ_5 due to task τ_4 interfering with task τ_5.

Our contribution is a formal comparison of the bound for the amount of task activations of an activation dependent task like τ_4. We show via a mathematical proof that the bound for the task activation that the RTC derives is not greater than the bound that the CPA derives and we can also see why that is the case. This opens up the possibility to improve CPA. Furthermore, the mathematical proof also removes any uncertainty that existing empirical comparisons of theses bounds may contain.

Structure of the Paper

The remainder of this paper has the following structure: First we describe the related work in Section 2. Then we introduce the models and the analyses of the compositional performance analyses and the real-time calculus in Section 3. Afterwards we discuss in Section 4 possible differences between the CPA and RTC that may affect the validity of the formal comparison and then we compare the bounds that both CPA and RTC derive for the activation frequency of activation dependent task. We conclude the paper with a summary in Section 5.

2 Related Work

Perathoner et al. use several small abstract systems in [7] to benchmark various formal performance analyses with these systems. The performance analyses include the compositional performance analysis and the real-time calculus and the benchmarks show that sometimes one outperforms the other but not always. It rather appears to depend on the characteristic of the system. The benchmarks are empirical, however we provide a mathematical comparison instead.

Naedele, Thiele, and Eisenring present in [6] a schedulability test with the real-time calculus for a system that uses a fixed-priority preemptive scheduling algorithm. They indicate that it is possible to derive the test in [16] from their schedulability test. Similarly, Pollex, Kollmann, and Slomka show in [8] a generalization of the response-time analysis with the help of the real-time calculus. They exemplarily derive the analysis in [16] from the real-time calculus. In [12] Schliecker presents the multiple event busy time, how to derive it for fixed-priority preemptive scheduling, and an accompanying analysis as an extension of the work in [16]. They also show how to derive a multiple event busy time from the service curves of the real-time calculus. Boyer an Roux propose in [2, 3] a model which can embed the models that the network calculus and the compositional performance analysis use, therefore making it possible to analyse a system that uses both models. However, they only look into how to interface between the different models and not how the individual analyses compare. Roux, Quinton, and Boyer use in [11] a theorem prover to show how results from the response-time analysis can be used in the network calculus and vice versa. These derivations from one method to the other are not comparisons, which is a part of our contribution.

Pollex and Slomka show in [9] that the bound for the worst-case response time, according to the compositional performance analysis, and the bound for the delay, according to the real-time calculus, are identical when the system uses a fixed priority scheduling where the tasks are independent. This comparison is for a different aspect than the one in our contribution where we compare the bounds for the amount of activations of an activation dependent task.

Slomka and Sadeghi introduce a new mathematical framework in [13, 14] which is based on mathematical tools from electric engineering to analyse real-time systems. This provides the possibility to unify the analysis were previously different approaches were necessary because of the different mathematical requirements on the models which depends on whether the real-time systems use fixed or dynamic priority scheduling. They also describe existing analyses like [16] and analyses for real-time systems with dynamic priority scheduling with the new mathematical framework. Based on this mathematical framework, Slomka and Sadeghi show in [15] preliminary work for investigating the relationship between the response-time analysis and the real-time calculus. They sketch possible similarities, however they do not express the real-time calculus with their new mathematical framework, let alone compare them.

3 Models and Analyses

In this section we describe the assumptions and notations that both the compositional performance analysis and the real-time calculus have in common followed by the common mathematical concepts that both analyses use and finally we restate the notation and the analyses themselves as presented in [17] and [12] for the real-time calculus and the compositional performance analysis, respectively.

3.1 Common Assumptions, Notations, and Definitions

For the distributed real-time system we assume that the tasks are activation dependent. Figure 2 shows an abstract excerpt of the distributed system in Figure 1 where τ_j is activation dependent on τ_i. So, whenever a job of task τ_i terminates, a job of task τ_j is released.

The common mathematical sets that we use are the positive integers $\mathbb{N} := \{1, 2, \ldots\}$, the non-negative integers $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, the real numbers $\mathbb{R}$, the extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, and the non-negative real numbers $\mathbb{R}_0^+ := \{r \in \mathbb{R} : r \geq 0\}$.

Both the RTC and CPA use in general piecewise continuous functions. However, the connection between some functions in the RTC and CPA is that they are inverse to each

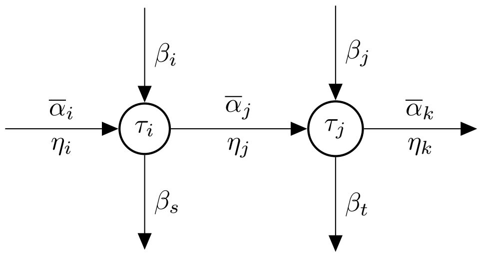


Figure 2 Excerpt of a distributed system where task τ_i connects to task τ_j.

other. But, piecewise continuous functions may contain intervals which the function maps onto the same value, which makes the function in that interval not strictly monotonic. So, for that interval no inverse exists. Therefore, we use a more general definition of an inverse function which we call pseudo-inverse.

▶ **Definition 1** (Pseudo-Inverse). *Confer [9, Definition 5]. Let $f: X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then the* pseudo-inverse *$f_{\leftarrow}^{-1}: Y \to L$ and $f_{\rightarrow}^{-1}: Y \to L$ at $y \in Y$ are*

$$f_{\leftarrow}^{-1}(y) := \inf\{x \in X : y \leq f(x)\} \text{ and } f_{\rightarrow}^{-1}(y) := \sup\{x \in X : f(x) \leq y\} \tag{1}$$

with the convention that $\inf \emptyset = \sup X$ *and* $\sup \emptyset = \inf X$.

For many of the functions that the RTC and CPA used be require that they are either left- or right-continuous.

▶ **Definition 2** (Directional Continuity). *Let $f: X \to \mathbb{R}$ be a function from a subset X of the real numbers, then f is* continuous on the left *or* right *at $x \in X$ if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \xi \in X \cap (x - \delta, x) : |f(\xi) - f(x)| < \epsilon \text{ or} \tag{2a}$$

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \xi \in X \cap (x, x + \delta) : |f(\xi) - f(x)| < \epsilon, \text{ respectively.} \tag{2b}$$

If f is continuous on the left or right at every element of X, then f is called *continuous on the left* or *continuous on the right*, respectively. Alternatively, f is called *left-continuous* or *right-continuous*, respectively. Furthermore, we denote f to be directional continuous to indicate that f is continuous on the left or right or both.

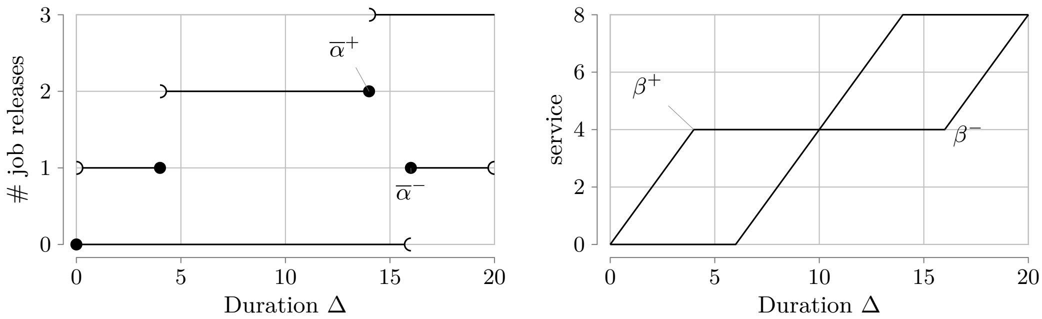
3.2 Real-Time Calculus

The RTC uses a greedy processing component (GPC) to model a task, e. g., task τ_i in Figure 2. A GPC consists of arrival curves and service curves.

▶ **Definition 3** (Event-Based Arrival Curves). *Confer [17, p. 16, Def. 1] and [17, p. 73, Def. 3]. Let $s \in \mathbb{R}$ be a point in time before $t \in \mathbb{R}$, i. e., $s \leq t$ and let $R_i[s, t)$ denote how may jobs task τ_i releases in the interval $[s, t)$. The* lower arrival curve $\overline{\alpha}_i^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *and the* upper arrival curve $\overline{\alpha}_i^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *satisfy for every duration $\Delta \in \mathbb{R}_0^+$ of an interval which starts at any point $t \in \mathbb{R}$ in time*

$$\overline{\alpha}_i^-(\Delta) \leq R_i[t, t + \Delta) \leq \overline{\alpha}_i^+(\Delta). \tag{3}$$

The event-based arrival curves denote that in *any* interval of duration $\Delta \in \mathbb{R}_0^+$, task τ_i releases at least $\overline{\alpha}_i^-(\Delta)$ and at most $\overline{\alpha}_i^+(\Delta)$ jobs. For example, a common model to specify



(a) Corresponding arrival curves for a task τ_i which releases jobs every period $p_i = 10$ and each release of a job may jitter by at most $j_i = 6$.

(b) Corresponding service curves for a TDMA policy where task τ_i is assigned a slot of length $s_i = 4$ out of a total cycle length of $c = 10$.

Figure 3 Real-Time Calculus Curves.

how often a task releases jobs is the periodic model with jitter, where task τ_i releases a job every period p_i and the release of each job may jitter by at most j_i. The corresponding arrival curves, cf. [17, p. 16, Ex. 1], are

$$\overline{\alpha}^-(\Delta) = \max\left\{\left\lfloor\frac{\Delta - j_i}{p_i}\right\rfloor, 0\right\} \text{ and } \overline{\alpha}^+(\Delta) = \begin{cases} 0 & \text{if } \Delta = 0 \\ \left\lceil\frac{\Delta + j_i}{p_i}\right\rceil & \text{if } \Delta > 0 \end{cases}.$$

For period $p_i = 10$ and jitter $j_i = 6$ Figure 3a shows the corresponding arrival curves.

▶ **Definition 4** (Resource-Based Service Curves)**.** *Confer [17, p. 19, Def. 2] and [17, p. 73, Def. 6]. Let $s \in \mathbb{R}$ be a point in time before $t \in \mathbb{R}$, i. e., $s \leq t$ and let $C_i[s, t)$ denote the amount of resources that are available to task τ_i in the interval $[s, t)$. The* lower service curve *$\beta_i^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and the* upper service curve *$\beta_i^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy for every duration $\Delta \in \mathbb{R}_0^+$ of an interval which starts at any point $t \in \mathbb{R}$ in time*

$$\beta_i^-(\Delta) \leq C_i[t, t + \Delta) \leq \beta_i^+(\Delta). \tag{4}$$

The resource-based service curves denote that in *any* interval of duration $\Delta \in \mathbb{R}_0^+$ at least $\beta_i^-(\Delta)$ and at most $\beta_i^+(\Delta)$ resources are available to task τ_i, where resources, for example, can be processor time or cycles. If the availability of the resources are distributed according to a time division multiple access (TDMA) policy with a cycle length of c and task τ_i is assigned a slot of length s_i, then the corresponding service curves, cf. [17, p. 136, (6.1)], are

$$\beta_i^-(\Delta) = \max\left\{\left\lfloor\frac{\Delta}{c}\right\rfloor \cdot s_i, \Delta - \left\lceil\frac{\Delta}{c}\right\rceil \cdot (c - s_i)\right\} \text{ and}$$
$$\beta_i^+(\Delta) = \min\left\{\left\lceil\frac{\Delta}{c}\right\rceil \cdot s_i, \Delta - \left\lfloor\frac{\Delta}{c}\right\rfloor \cdot (c - s_i)\right\}.$$

For a TDMA schedule with a cycle length of $c = 10$ where task τ_i is assigned a slot of length $s = 4$ Figure 3b shows the corresponding service curves.

The arrival curves in Definition 3 denote bounds on how many jobs a task releases, but the service curves in Definition 4 denote bounds on how much resources are available to the task to process its jobs. Since, both curves use different units, we need to transform one into the other, so that we can use them together. For the transformation we use workload curves.

▶ **Definition 5** (Workload Curves)**.** *Confer [17, p. 74, Def. 7]. Let $W(u)$ denote the amount of resources that are necessary to process u consecutively released jobs, then the* lower workload curve $\gamma^- : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *and the* upper workload curve $\gamma^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *satisfy for $u \in \mathbb{R}_0^+$ and $v \in \mathbb{R}_0^+$, where $u \le v$*

$$\gamma^-(v-u) \le W(v) - W(u) \le \gamma^+(v-u). \tag{5}$$

The workload curves denote that any k consecutively released jobs of task τ_i require at least $\gamma_i^-(k)$ and at most $\gamma_i^+(k)$ resources to process them.

One way to transform the resource-based service curves into their event-based form is to compose them with the pseudo-inverse of the workload curves. Confer [17, p. 74, (4.6) and (4.7)] and [17, p. 75, (4.11)].

$$\overline{\beta}^- = \gamma^{+\,-1}_{\rightarrow} \circ \beta^- \tag{6a}$$

$$\overline{\beta}^+ = \gamma^{-\,-1}_{\leftarrow} \circ \beta^+ \tag{6b}$$

With the arrival and service curves in their event-based form, we can describe an upper bound of the delay of a task τ. This is the longest time that the processor needs to execute a job of task τ which is from the time that the job was release until the job terminated.

▶ **Corollary 6** (Delay Bound)**.** *Confer [5, p. 28, Theorem 1.4.2] and [17, p. 26, (2.11)]. For a task τ with an upper arrival curve $\overline{\alpha}_i^+$ and lower service curve $\overline{\beta}_i^-$ the upper bound for the delay is*

$$\sup_{\lambda\in\mathbb{R}_0^+} \left\{ \inf_{\mu\in\mathbb{R}_0^+} \left\{ \mu : \overline{\alpha}_i^+(\lambda) \le \overline{\beta}_i^-(\lambda+\mu) \right\} \right\}. \tag{7}$$

The outgoing arrival curves in the RTC are expressed by means of the inf-plus convolution and deconvolution. These operate on the algebraic structure of the semiring $(\mathbb{R}\cup\{\infty\}, \wedge, +)$, where $\wedge$ is the infimum. Additionally, for the comparison we also use the related sup-plus convolution and deconvolution. The latter operate on the semiring $(\mathbb{R}\cup\{-\infty\}, \vee, +)$, where $\vee$ is the supremum.

▶ **Definition 7** (Convolution and Deconvolution)**.** *Confer [5, Definition 3.1.10, 3.2.1, 3.1.13, and 3.2.2]. Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be functions, then the convolution in inf-plus $\otimes$ and sup-plus $\overline{\otimes}$ and the deconvolution in inf-plus $\oslash$ and in sup-plus $\overline{\oslash}$ are*

$$(f \otimes g)(x) := \inf_{0\le\xi\le x} \{f(x-\xi) + g(\xi)\}, \tag{8a}$$

$$(f \,\overline{\otimes}\, g)(x) := \sup_{0\le\xi\le x} \{f(x-\xi) + g(\xi)\}, \tag{8b}$$

$$(f \oslash g)(x) := \sup_{0\le\xi} \{f(x+\xi) - g(\xi)\}, \text{ and} \tag{8c}$$

$$(f \,\overline{\oslash}\, g)(x) := \inf_{0\le\xi} \{f(x+\xi) - g(\xi)\}, \text{ respectively.} \tag{8d}$$

With the inf-plus convolution and deconvolution we can now express the outgoing arrival curves and with the sup-plus convolution and deconvolution we can express the remaining service curves.

▶ **Corollary 8** (Outgoing Arrival Curves)**.** *Confer [17, p. 22-23, Ex. 3, p. 204, (A.18), and p. 203, (A.17)] Given the arrival curves $\overline{\alpha}_i^-$ and $\overline{\alpha}_i^+$ and the service curves $\overline{\beta}_i^-$ and $\overline{\beta}_i^+$ for task τ_i, then the outgoing arrival curves are*

$$\overline{\alpha}_j^- = (\overline{\alpha}_i^- \oslash \overline{\beta}_i^+) \otimes \overline{\beta}_i^- \wedge \overline{\beta}_i^- \text{ and} \tag{9a}$$

$$\overline{\alpha}_j^+ = (\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+) \oslash \overline{\beta}_i^- \wedge \overline{\beta}_i^+. \tag{9b}$$

The outgoing arrival curves denote that in *any* interval of duration $\Delta \in \mathbb{R}_0^+$ at least $\overline{\alpha}_j^-(\Delta)$ and at most $\overline{\alpha}_j^+(\Delta)$ jobs of task τ_i terminate. Since each termination of job τ_i causes the release of a job of task τ_j the outgoing arrival curves also are the arrival curves for task τ_j.

▶ **Corollary 9** (Remaining Service Curves). *Confer [17, p. 22-23, Ex. 3, p. 201, (A.15) and (A.16)] Given the arrival curves α_i^- and α_i^+ and the service curves β_i^- and β_i^+ for task τ_i, then the remaining service curves are*

$$\beta_j^-(\Delta) = \sup_{0 \leq \lambda \leq \Delta} \left\{\beta_i^-(\lambda) - \alpha_i^+(\lambda)\right\} \text{ and } \beta_j^+(\Delta) = \inf_{\Delta \leq \lambda} \left\{\beta_i^+(\lambda) - \alpha_i^-(\lambda)\right\}. \tag{10}$$

3.3 Compositional Performance Analysis

The CPA uses two functions, the event load function (Definition 10) and the event distance function (Definition 11) to capture how many jobs a task τ releases.

▶ **Definition 10** (Event Load Functions). *Confer [12, p. 53, Definition 3.4]. The* lower (upper) event load function *of task τ_i is defined by the minimum (maximum) number of events in an event stream that may arrive in a time window of size $\Delta \in \mathbb{R}_0^+$ and is denoted by*

$$\eta_i^- : \mathbb{R}_0^+ \to \mathbb{N}_0 \ \left(\eta_i^+ : \mathbb{R}_0^+ \to \mathbb{N}_0\right). \tag{11}$$

Since the events that arrive in an event stream are those that causes the task to release jobs, the event load functions denote that in *any* interval of duration $\Delta \in \mathbb{R}_0^+$, task τ_i releases at least $\eta_i^-(\Delta)$ and at most $\eta_i^+(\Delta)$ jobs. So, the event load functions are to the CPA what the arrival curves are to the RTC. Note that in [12, p. 53, Definition 3.4] the event load functions have the domain $\mathbb{R}^+$ and the co-domain $\mathbb{N}$. In Definition 10 we extend the domain to $\mathbb{R}_0^+$ and the co-domain do $\mathbb{N}_0$. The minimum and maximum number of events that arrive in any time window of 0 is 0, so we define $\eta_i^-(0) = \eta_i^+(0) = 0$.

▶ **Definition 11** (Event Distance Functions). *Confer [12, p. 53, Definition 3.3]. The* minimum (maximum) event distance function *of task τ_i is defined by the size of the smallest (largest) time window within which k or more (less than k) events may occur and is denoted by*

$$\delta_i^- : \mathbb{N}_0 \to \mathbb{R}_0^+ \ \left(\delta_i^+ : \mathbb{N}_0 \to \mathbb{R}_0^+\right). \tag{12}$$

The event distance functions denote that n jobs of task τ_i are released in some interval with a duration of at least $\delta_i^-(n)$ and at most $\delta_i^+(n)$. Note that in [12, p. 53, Definition 3.3] the event distance functions have the domain $\mathbb{N}$. In Definition 11 we extend the domain to $\mathbb{N}_0$. The smallest and largest time window within which 0 events occur is 0, so we define $\delta_i^-(0) = \delta_i^+(0) = 0$.

Both the event load function and the event distance function are closely related, such that we can derive one function from the other.

▶ **Corollary 12** (Derivation of Event Distance Functions). *Confer [12, p. 54, (3.7) and (3.8)]. Given a lower event load function η_i^- and an upper event load function η_i^+ of task τ_i, then the minimum event distance function and the maximum event distance function are*

$$\delta_i^-(n) := \inf_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : n \leq \eta_i^+(\Delta)\right\} = \eta_i^{+\overset{-1}{\leftarrow}}(n) \text{ and} \tag{13a}$$

$$\delta_i^+(n) := \sup_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : \eta_i^-(\Delta) < n\right\}. \tag{13b}$$

Additionally, the CPA uses the multiple event busy time function (Definition 13) to capture how much time is needed to process jobs of task τ_i.

▶ **Definition 13** (Multiple Event Busy Time Function). *Confer [12, p. 63, Definition 3.6]. The* minimum (maximum) k-event busy time function *of task τ_i is given by the minimum (maximum) time it may take instances of τ_i to process k events, if all but the first of the k events arrive before the preceding is finished and are denoted by*

$$B_i^- : \mathbb{N}_0 \to \mathbb{R}_0^+ \ \left(B_i^+ : \mathbb{N}_0 \to \mathbb{R}_0^+\right). \tag{14}$$

Using the terms of the RTC, the multiple event busy time functions denote that the resources needed to process any n consecutively released jobs of task τ_i are available to task τ_i in some interval with a duration of at least $B_i^-(n)$ and at most $B_i^-(n)$. Note that in [12, p. 63, Definition 3.6] the multiple event busy time functions use $\mathbb{N}$ as domain. In Definition 13 we extend the domain to the domain $\mathbb{N}_0$. The lower and upper bound to execute 0 task activations is 0, so we define $B_i^-(0) = B_i^+(0) = 0$.

With the event distance function and the multiple event busy time functions, the CPA derives a lower bound for the duration in which task n jobs of τ_i terminate (Lemma 14).

▶ **Lemma 14** (Outgoing Minimum Event Distance Function). *Confer [12, p. 73, Theorem 3.11]. The minimum distance between n events at the output of task τ_i is*

$$\delta_j^-(n) = \max\left\{\min_{k\in\mathbb{N}_0}\left\{\delta_i^-(n+k) - B_i^+(k+1)\right\} + B_i^-(1), 0\right\}, \tag{15}$$

where B_i^+ is subadditive.

The outgoing minimum event distance function denotes that n jobs of task τ_i terminate in some interval with a duration of at least $\delta_j^-(n)$. since each termination of a job of τ_i causes the release of a job of task τ_j, the outgoing minimum event distance function is also the minimum event distance function of task τ_j. Note that in [12, p. 73, Theorem 3.11] k is from the set $K = \left\{k \in \mathbb{N} | \delta_{i,in}^-(k+1) < B_i^+(k)\right\} \subseteq \mathbb{N}_0$. In Lemma 14 we use $k \in \mathbb{N}_0$. We give a justification in Section C that K should explicitly include 0 and that (15) then remains the same whether $k \in \{0\} \cup K$ or $k \in \mathbb{N}_0$.

4 Formal Comparison

In this section we formally compare the outgoing upper event load function η_j from the CPA with the upper bound of the outgoing event-based arrival curve $\overline{\alpha}_j^+$ (9b) from the RTC (see task τ_i in Figure 2) by means of a mathematical proof (Section 4.3).

Before that, we ensure that these functions are actually comparable. So, we first consider the underlying time domain (Section 4.1) of the CPA and the RTC. The time domains can have subtle differences which may have a conceivable influence on the comparability of the functions. And second, we ensure that the bounds from which the CPA and RTC derive η_j and $\overline{\alpha}_j^+$ are the same (Section 4.2).

4.1 Underlying Time Domain

The CPA and the RTC mainly use functions that have a time duration as their domain. A time duration is the distance between two time points. So, we can treat it as a metric in the mathematical sense, which is always non-negative. Therefore, the domain of these functions is the non-negative real numbers $\mathbb{R}_0^+$. Even though the domain of these functions are the same, there is a degree of freedom regarding the underlying time domain.

Usually a system changes its state to *on* at some point in time, which is commonly defined as the origin for the time domain, i. e., the point in time is zero. Therefore, using the set of the non-negative real numbers $\mathbb{R}_0^+$ for the time domain is usually sufficient. However, whether the CPA considers the time before time zero is not clear and therefore its underlying time domain is not clear.

The RTC uses the whole set of real numbers $\mathbb{R}$ for its time domain, see Definitions 3 and 4. For the CPA there are two possibilities. First, the CPA also considers the time before time zero. Then, the time domain is the whole set of real numbers $\mathbb{R}$. Second, the CPA does not consider the time before time zero and therefore the time domain it uses is the non-negative real numbers $\mathbb{R}_0^+$. This is exactly the time domain that the Network Calculus (NC) uses. In [1, Section 9.1.3] Bouillard, Boyer, and Le Corronc discuss the effects of the NC using the non-negative real numbers and the RTC using the whole real numbers on the derived bounds for the arrival and service curves. They conclude in [1, p 217-218] that these bound are the same except for the lower bound of the outgoing arrival curve. Because the RTC is based on the NC, we can therefore infer that when limiting the time domain to the non-negative real numbers the upper bound of the outgoing arrival curve is the same as in (9b). Therefore, the underlying time domain does not affect the derived upper bound for the outgoing arrival curve of the RTC.

4.2 Input Bounds and Assumptions

Our goal is to compare the outgoing upper event load function η_j^+ from the CPA with the upper bound of the outgoing event-based arrival curve $\overline{\alpha}_j^+$ (9b) from the RTC. However, the outgoing upper event load function η_j^+ is not derived directly. Instead, we know the outgoing minimum event distance function δ_j^- (15) and we know that $\delta_j^- = \eta_j^{+\,-1}_{\;\leftarrow}$ (13a). We address this further in Section 4.3.

Nonetheless, when we inspect Equation (15), the outgoing minimum event distance function δ_j^-, we see that it derives from the minimum event distance function δ_i^- and both minimum B_i^- and maximum event busy time function B_i^+. Similarly, the outgoing upper arrival curve $\overline{\alpha}_j^+$ (9b) derives from the upper arrival curve $\overline{\alpha}_i^+$ (3) and both lower $\overline{\beta}_i^-$ and upper service curves $\overline{\beta}_i^+$ (4).

To ensure that we use the same input bounds, we assume that the upper arrival curve $\overline{\alpha}_i^+$ and both lower $\overline{\beta}_i^-$ and upper service curves $\overline{\beta}_i^+$ are given and then derive the minimum event distance function δ_i^- and both minimum B_i^- and maximum event busy time function B_i^- from them.

So, for all curves of the RTC we make the following assumption.

▶ **Assumption 1.** *Every curve ($\overline{\alpha}_i^+$, β_i^-, and γ_i^+) is increasing and not bounded above, lower curves (β^-) are right-continuous, and upper curves ($\overline{\alpha}_i^+$, β_i^+ and γ_i^+) are left-continuous.*

The assumption of increasing curves is to maintain consistency. Let there exist two durations Δ_1 and Δ_2 with $\Delta_1 < \Delta_2$ where the upper bound of the amount of task activations would decrease, i. e., $\overline{\alpha}_i^+(\Delta_1) > \overline{\alpha}_i^+(\Delta_2)$. This might suggest that by observing time intervals of longer durations, task activations could disappear. If valid lower or upper curves exist which are non-increasing, then it is possible to transform them into valid lower or upper curves that are increasing.

Assuming that the curves are not bounded above means that a system gets turned on at some point in time and it never stops. So, that there will always be new task activations and that they will always get served.

The property that curves are left- or right-continuous is important for the proofs. For example, the proof of Lemma 18 about the tightness of the transformation from event-based lower service curve $\overline{\beta}_i^-$ to the maximum multiple event busy time function B_i^+ requires that $\overline{\beta}_i^-$ is right-continuous.

▶ Remark 15. For task τ_i the upper arrival curve $\overline{\alpha}_i^+$ and the upper event load function η_i^+ describe the same, so we have

$$\eta_i^+ = \overline{\alpha}_i^+. \tag{16}$$

Wandeler uses for the upper arrival curve $\overline{\alpha}_i^+$ in [17, p. 16, (2.3)] the same expression as the upper event load function η_i^+ in [10]. From (16) follows that the minimum event distance function is the pseudo inverse of the upper arrival curve, i. e., $\delta_i^- = \overline{\alpha}_{i\ \leftarrow}^{+-1}$.

We assume that the service curves are given and derive the multiple event busy time functions from it. To avoid that the results of the comparison in Section 4.3 are tainted due to lax derivation, we show in Lemmas 16–18 that the derivations are tight.

▶ **Lemma 16** (Derivation of the Multiple Event Busy Time from Service Curve)**.** *Confer [12, p. 68, Theorem 3.6]. Let task τ_i receive an event-based service bounded from below by $\overline{\beta}_i^-$ and bounded from above by $\overline{\beta}_i^+$, then the corresponding minimum and maximum event busy time functions are*

$$B_i^- = \overline{\beta}_{i\ \leftarrow}^{+-1} \text{ and} \tag{17a}$$

$$B_i^+ = \overline{\beta}_{i\ \leftarrow}^{--1}. \tag{17b}$$

Proof. We have $\overline{\beta}_i^+(\Delta)$ ($\overline{\beta}_i^-(\Delta)$), an upper (a lower) bound of the amount of consecutive task activations of τ_i that can be serviced during Δ. For a $k \in \mathbb{N}_0$ the set $\left\{\Delta \in \mathbb{R}_0^+ : k \leq \overline{\beta}_i^+(\Delta)\right\}$ ($\left\{\Delta \in \mathbb{R}_0^+ : k \leq \overline{\beta}_i^-(\Delta)\right\}$) contains all durations in which at most (leat) k or more consecutive task activations are serviced. Therefore, the smallest of these durations, is a lower (an upper) bound for the duration in which k consecutive task activations are serviced, i. e.,

$$B_i^-(k) = \inf_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : k \leq \overline{\beta}_i^+(\Delta)\right\} = \overline{\beta}_{i\ \leftarrow}^{+-1}(k)$$

$$\left(B_i^+(k) = \inf_{\Delta \in \mathbb{R}_0^+} \left\{\Delta : k \leq \overline{\beta}_i^-(\Delta)\right\} = \overline{\beta}_{i\ \leftarrow}^{--1}(k)\right). \qquad ◀$$

▶ **Lemma 17** (Derivation of Service Curve from the Multiple Event Busy Time)**.** *Let B_i^- and B_i^+ be the minimum and maximum multiple event busy time of task τ_i, then we can derive the lower and upper bound of the service curve by*

$$\overline{\beta}_i^- = B_{i\ \rightarrow}^{+-1} \text{ and} \tag{18a}$$

$$\overline{\beta}_i^+ = B_{i\ \rightarrow}^{--1}. \tag{18b}$$

Proof. We have $B_i^+(k)$ ($B_i^-(k)$), an upper (a lower) bound of the duration in which k consecutive task activations of task τ_i are serviced. For a duration $\Delta \in \mathbb{R}_0^+$, the set $\left\{k \in \mathbb{N}_0 : B_i^+(k) \leq \Delta\right\}$ ($\left\{k \in \mathbb{N}_0 : B_i^-(k) \leq \Delta\right\}$) contains the amount of consecutive task activations that can be serviced at most (least) during Δ. Therefore, the greatest value of that set is a lower (an upper) bound for the amount of consecutive task activations that can be serviced during Δ, i. e.,

$$\overline{\beta}_i^{-}(\Delta) = \sup_{k \in \mathbb{N}_0} \left\{ k : B_i^{+}(k) \leq \Delta \right\} = {B_i^{+}}^{-1}_{\rightarrow}(k)$$

$$\left(\overline{\beta}_i^{+}(\Delta) = \sup_{k \in \mathbb{N}_0} \left\{ k : B_i^{-}(k) \leq \Delta \right\} = {B_i^{-}}^{-1}_{\rightarrow}(k) \right). \blacktriangleleft$$

▶ **Lemma 18** (Tightness of Transformation). *Let task τ_i receive an event-based service bounded from below by $\overline{\beta}_i^{-}$ and from above by $\overline{\beta}_i^{+}$, then the transformation between the service curves and of multiple busy time functions is tight.*

Proof. We have shown in Lemmas 16 and 17 how to derive the multiple event busy time function from the event-based service curves and vice versa. What remains is to show that the composition of the derivation is the identity.

On the one hand that is deriving the maximum multiple event busy time from the event-based lower service curve $B_i^{+} = {\overline{\beta}_i^{-}}^{-1}_{\leftarrow}$ and then deriving the event-based lower service curve from maximum multiple event busy time $\overline{\beta}_i^{-} = {B_i^{+}}^{-1}_{\rightarrow}$ results in the original event-based lower service curve $\overline{\beta}_i^{-}$, i. e., we need to show that $\overline{\beta}_i^{-} = {B_i^{+}}^{-1}_{\rightarrow} = {{\overline{\beta}_i^{-}}^{-1}_{\leftarrow}}^{-1}_{\rightarrow}$.

On the other hand deriving the event-based upper service curve from the minimum multiple event busy time function $\overline{\beta}_i^{+} = {B_i^{-}}^{-1}_{\rightarrow}$ and then deriving the minimum multiple event busy time from the event-based upper service curve $B_i^{-} = {\overline{\beta}_i^{+}}^{-1}_{\leftarrow}$ results in the original minimum multiple event busy time function B_i^{-}, i. e., we need to show that $B_i^{-} = {\overline{\beta}_i^{+}}^{-1}_{\leftarrow} = {{B_i^{-}}^{-1}_{\rightarrow}}^{-1}_{\leftarrow}$.

By Assumption 1 $\overline{\beta}_i^{-}$ is an increasing and right-continuous function. So, we can apply (45b) which states that $\overline{\beta}_i^{-} = {{\overline{\beta}_i^{-}}^{-1}_{\leftarrow}}^{-1}_{\rightarrow}$, therefore showing the identity. Similarly, B_i^{-} is an increasing discrete function and therefore continuous in the topological sense. So, we can apply (45a) which states that $B_i^{-} = {{B_i^{-}}^{-1}_{\rightarrow}}^{-1}_{\leftarrow}$, therefore showing the identity. ◀

▶ **Assumption 2.** *For a task τ_i, the image of the event-based service curves is a subset of the non-negative integers, i. e.,*

$$\overline{\beta}^{-}(\mathbb{R}_0^{+}) \subseteq \mathbb{N}_0 \text{ and } \overline{\beta}^{+}(\mathbb{R}_0^{+}) \subseteq \mathbb{N}_0. \tag{19}$$

In general the resource-based service curves have a subset of $\mathbb{R}_0^{+}$ as their image. To make the image of the event-based service curves a subset of the non-negative integers $\mathbb{N}_0$, we can choose $\gamma^{-} = (f^{-} \circ \lfloor \cdot \rfloor)$, where $f^{-} \colon X \to \mathbb{R}_0^{+}$ is an increasing and right-continuous function from a subset X of the non-negative real numbers $\mathbb{R}_0^{+}$, and $\gamma^{+} = (f^{+} \circ \lceil \cdot \rceil)$, where $f^{+} \colon X \to \mathbb{R}_0^{+}$ is an increasing and left-continuous function from a subset X of the non-negative real numbers $\mathbb{R}_0^{+}$, as the workload curves. Then the corresponding pseudo-inverse are

$$\gamma^{-}{}^{-1}_{\leftarrow} = (f^{-} \circ \lfloor \cdot \rfloor)^{-1}_{\leftarrow} \overset{(44b)}{=} (\lfloor \cdot \rfloor^{-1}_{\leftarrow} \circ f^{-}{}^{-1}_{\leftarrow}) \overset{(51)}{=} (\lceil \cdot \rceil \circ f^{-}{}^{-1}_{\leftarrow}) \text{ and}$$

$$\gamma^{+}{}^{-1}_{\rightarrow} = (f^{+} \circ \lceil \cdot \rceil)^{-1}_{\rightarrow} \overset{(44a)}{=} (\lceil \cdot \rceil^{-1}_{\rightarrow} \circ f^{+}{}^{-1}_{\rightarrow}) \overset{(51)}{=} (\lfloor \cdot \rfloor \circ f^{+}{}^{-1}_{\rightarrow}).$$

Both f^{-} and f^{+} have a co-domain of $\mathbb{R}_0^{+}$, therefore $\gamma^{-}{}^{-1}_{\leftarrow}$ and $\gamma^{+}{}^{-1}_{\rightarrow}$ have an image that is a subset of $\mathbb{N}_0$. So, when using these workload curves in (6a) and (6b) the resulting event-based service curves have an image that is a subset of the non-negative integers $\mathbb{N}_0$.

▶ **Assumption 3.** *For a task τ_i, the event-based lower service curve $\overline{\beta}_i^{-}$ is superadditive, i. e.,*

$$\forall \Delta_1, \Delta_2 \in \mathbb{R}_0^{+} : \overline{\beta}_i^{-}(\Delta_1 + \Delta_2) \geq \overline{\beta}_i^{-}(\Delta_1) + \overline{\beta}_i^{-}(\Delta_2). \tag{20}$$

Given an event-based lower service curve $\overline{\beta}_i^-$, it is possible to transform it into one that is also superadditive, e. g., by using its superadditive closure. See [1, p. 219, Proposition 9.3] regarding variable capacity nodes which are equivalent to the lower service curves [1, Section 9.1.3]. We require the event-based lower service curve $\overline{\beta}_i^-$ to be superadditive so that the derived maximal multiple event busy time function (17b) is subadditive (46). Then, we can use (15) in our comparison.

4.3 Upper Bounds of Outgoing Event Streams

In a distributed system we assume in general that an event releases a job for a task. When that job terminates it produces exactly one event which can in turn release a job of a different task (Section 3.1).

When we inspect the upper bound of the delay (7) we can see that a smaller event-based upper arrival curve or a larger event-based lower service curve leads to a smaller upper bound of the delay. In general when the processor uses a work-conserving scheduling algorithm the smaller the arrival curves of the tasks assigned to the processor are, the larger the lower service curve for each task will be. For example in the case when the processor uses a fixed-priority preemptive scheduling algorithm, we can apply (10) and directly see that a smaller resource-based upper arrival curve of a higher priority tasks leads to greater resource-based lower service curve for a lower priority task. So, it is desirable for an analysis of a distributed system to produce upper arrival curves that are as small as possible. With this we see that a smaller upper arrival curve is better than a larger one.

What we compare mathematically are the expressions of the bounds for the generated events that the CPA and RTC use. Therefore, we assume that the inputs to both are the same.

Under Assumptions 1–3 we supply our contribution Theorem 19. A proof that the upper bound for the generated events according the RTC (9b) is less than or equal to the upper bound for the generated events according the CPA (15).

▶ **Theorem 19.** *Let the termination of each job of task τ_i cause the release of a job of task τ_j, then under the Assumptions 1–3 the outgoing event-based upper arrival curve is less than or equal to the outgoing event load function of task τ_i, i. e.,*

$$\overline{\alpha}_j^+ \leq \eta_j^+. \tag{21}$$

We do not explicitly know the outgoing event load function $\overline{\alpha}_j^+$. However, we know the outgoing minimum event distance function δ_j^-, see (15), and we know that the minimum event distance function is the pseudo-inverse of the event load function $\delta_j^-(n) = {\eta_j^+}_{\leftarrow}^{-1}(n)$, see (13a). Furthermore, the event load functions and the event-based arrival and service curves have the non-negative integers as co-domain. So it is possible to convert from one pseudo-inverse to the other by means of Lemma 20.

▶ **Lemma 20.** *Let $f\colon I \to \mathbb{N}_0$ be an increasing function from an interval $I \subseteq \mathbb{R}$ of the real numbers $\mathbb{R}$ to the non-negative integers $\mathbb{N}_0$, then for a non-negative integer $n \in \mathbb{N}_0$*

$$f_{\rightarrow}^{-1}(n) = f_{\leftarrow}^{-1}(n+1) \tag{22}$$

Proof. Follows from the definitions of the pseudo-inverse (1) and because the co-domain of f is the non-negative integers.

$$f_{\rightarrow}^{-1}(n) = \sup_{x \in I} \{f(x) \leq n\} = \sup_{x \in I} \{f(x) < n+1\} = \inf_{x \in I} \{n+1 \leq f(x)\} = f_{\leftarrow}^{-1}(n+1) \qquad ◀$$

Instead of showing (21), we can instead infer it from comparing the pseudo-inverse of the outgoing event load function $\eta^{+\,-1}_{j\;\rightarrow}(n)$ with the pseudo-inverse of the outgoing arrival curves $\overline{\alpha}^{+\,-1}_{j\;\rightarrow}$ by means of Lemma 21.

▶ **Lemma 21.** *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions from a subset X of the real numbers, then*

$$f^{-1}_{\rightarrow} \geq g^{-1}_{\rightarrow} \Rightarrow f \leq g \text{ if } f \text{ is increasing and left-continuous.} \tag{23}$$

Proof. We show the contraposition, i. e.,

$$(\exists x \in X : g(x) < f(x)) \Rightarrow (\exists y \in Y : f^{-1}_{\rightarrow}(y) < g^{-1}_{\rightarrow}(y)).$$

Let x be an element of X, therefore $f(x)$ and $g(x)$ are elements of $\mathbb{R}$. So, (23) follows directly from (39b) and (40), i. e.,

$$g(x) < f(x) \overset{(40)}{\underset{(39b)}{\Rightarrow}} f^{-1}_{\rightarrow}(g(x)) < x \leq g^{-1}_{\rightarrow}(g(x)).$$ ◀

To be able to use (23) we need to show that the outgoing arrival curve $\overline{\alpha}^{+}_{j}$ is increasing and left-continuous. For this we first need to show that the infimum of two left-continuous functions is also left-continuous.

▶ **Lemma 22.** *Let X be a subset of $\mathbb{R}$, $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ directional continuous functions, then $(f \wedge g)$ inherits the property of directional continuity that f and g have in common.*

Proof. Let $\varepsilon > 0$, x a point in X, f and g continuous on the left (or right), and $\delta_f > 0$ and $\delta_g > 0$ such that they satisfy the property of directional continuity (2a) (or (2b)) of f at x for ε and g at x for ε respectively. Choose $\delta = \delta_f \wedge \delta_g$ and ξ as a point in X such that $0 < x - \xi < \delta$ (or $0 < \xi - x < \delta$). It has to be shown that the inequality $|(f \wedge g)(x) - (f \wedge g)(\xi)| < \varepsilon$ holds.

For the two cases ($f(x) \leq g(x)$ and $f(\xi) \leq g(\xi)$) and ($f(x) \geq g(x)$ and $f(\xi) \geq g(\xi)$) it follows that the inequality holds:

$$|f(x) - f(\xi)| < \varepsilon \quad \text{and} \quad |g(x) - g(\xi)| < \varepsilon$$

For the remaining two cases ($f(x) \leq g(x)$ and $f(\xi) \geq g(\xi)$) and ($f(x) \geq g(x)$ and $f(\xi) \leq g(\xi)$) it follows that

$$-\varepsilon < f(x) - f(\xi) \leq f(x) - g(\xi) \leq g(x) - g(\xi) < \varepsilon$$

and

$$-\varepsilon < g(x) - g(\xi) \leq g(x) - f(\xi) \leq f(x) - f(\xi) < \varepsilon$$

and therefore the inequality also holds

$$|f(x) - g(\xi)| < \varepsilon \quad \text{and} \quad |g(x) - f(\xi)| < \varepsilon$$

hence $(f \wedge g)$ inherits the property of directional continuity that f and g have in common. ◀

▶ **Lemma 23** (Monotonicity of directional continuity of outgoing arrival curve)**.** *Under Assumption 1 the outgoing event-based upper arrival curve of task τ_i is increasing and left-continuous, i. e.,*

$$\overline{\alpha}^{+}_{j} \text{ is increasing and left-continuous.} \tag{24}$$

Proof. From Assumption 1 we have that all curves, $\overline{\alpha}_i^+$, $\overline{\beta}_i^+$, and $\overline{\beta}_i^-$ are increasing. We also know that the set of increasing functions from $\mathbb{R}_0^+$ to $\overline{\mathbb{R}}_0^+$ are closed under the inf-plus convolution $\otimes$ and the minimum operator $\wedge$ [1, p. 22, Lemma 2.3] as well as under the inf-plus deconvolution $\oslash$, (47). Therefore, the event-based outgoing arrival curve $\overline{\alpha}_j^+ = (\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+) \oslash \overline{\beta}_i^- \wedge \overline{\beta}_i^+$ is also increasing.

From Assumption 1 follows by [1, Proposition 3.11] that $(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+)$ is left-continuous. Subsequently by (48a) $(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+) \oslash \overline{\beta}_i^-$ is left-continuous. And since $\overline{\beta}_i^+$ is left-continuous, so is $(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+) \oslash \overline{\beta}_i^- \wedge \overline{\beta}_i^+ = \overline{\alpha}_j^+$ (Lemma 22). ◀

Now we need to apply the pseudo-inverse operator onto the outgoing arrival curve and expand it. For this, we need to show how the pseudo-inverse operator behaves regarding the inf-plus convolution and deconvolution and the infimum. We show these in Lemmas 24–26. Additionally we need to show that the inf-plus convolution of two functions that are increasing and not bounded above is also also not bounded above Lemma 27.

▶ **Lemma 24.** *Let X be a subset of a complete lattice L, Y a totally ordered set, and $f, g\colon X \to Y$ functions, then*

$$(f \wedge g)_{\rightarrow}^{-1} = (f_{\rightarrow}^{-1} \vee g_{\rightarrow}^{-1}) \tag{25}$$

Proof. Let X be a set, Y a partially ordered set, y a point in Y, $h\colon X \to Y$ a function, and $X_{h\leq y} := \{x \in X : h(x) \leq y\}$. Observe that $h_{\rightarrow}^{-1}(y) = \sup X_{h\leq y}$.

Let x be a point in $X_{(f\wedge g)\leq y}$, then the condition $(f \wedge g)(x) \leq y$ holds, the infimum of f and g at x is less than or equal to y.

$$(f \wedge g)(x) \leq y \Rightarrow (f(x) \leq y) \parallel (g(x) \leq y)$$

Due to the totality property of Y either $f(x) \leq g(x)$ or $g(x) \leq f(x)$ must hold. Therefore the infimum of f and g at x is either $f(x)$ or $g(x)$. Since the infimum of f and g at x is less than or equal to y so is either $f(x)$ or $g(x)$.

$$(f(x) \leq y) \parallel (g(x) \leq y) \Rightarrow ((f \wedge g)(x) \leq y) \parallel ((f \wedge g)(x) \leq y)$$

The infimum of f and g at x is always less than or equal to $f(x)$ and $g(x)$. Since either $f(x)$ or $g(x)$ is less than or equal to y, so is the infimum.

Therefore the condition $(f \wedge g)(x) \leq y$ is equivalent to the condition $(f(x) \leq y) \parallel (g(x) \leq y)$, either the value of $f(x)$ is less than or equal to y or $g(x)$ is.

$$(f(x) \leq y) \parallel (g(x) \leq y) \Leftrightarrow x \in X_{f\leq y} \cup X_{g\leq y}$$

Subsequently x is either a point in $X_{f\leq y}$ or in $X_{g\leq y}$, in particular x is a point in $X_{f\leq y} \cup X_{g\leq y}$. Conversely, if x is a point in $X_{f\leq y} \cup X_{g\leq y}$ then x is a point in either $X_{f\leq y}$ or $X_{g\leq y}$. Therefore the condition $(f(x) \leq y) \parallel (g(x) \leq y)$ holds for x.

So far it has been shown that the sets $X_{(f\wedge g)\leq y}$ and $X_{f\leq y} \cup X_{g\leq y}$ are identical.

$$(f \wedge g)_{\rightarrow}^{-1}(y) \leq (f_{\rightarrow}^{-1} \vee g_{\rightarrow}^{-1})(y)$$

Let x be a point in $X_{f\leq y} \cup X_{g\leq y}$. Therefore x is a point in either $X_{f\leq y}$ or $X_{g\leq y}$ and subsequently either $x \leq f_{\rightarrow}^{-1}(y)$ or $x \leq g_{\rightarrow}^{-1}(y)$ holds. Since the supremum of $f_{\rightarrow}^{-1}(y)$ and $g_{\rightarrow}^{-1}(y)$ is always greater than or equal to either one, it follows that $x \leq (f_{\rightarrow}^{-1}(y) \vee g_{\rightarrow}^{-1}(y))$.

Hence $(f^{-1}_{\rightarrow}(y) \vee g^{-1}_{\rightarrow}(y))$ is an upper bound of $X_{f\le y} \cup X_{g\le y}$, so $(f \wedge g)^{-1}_{\rightarrow}(y) \le (f^{-1}_{\rightarrow} \vee g^{-1}_{\rightarrow})(y)$ holds.

$$(f \wedge g)^{-1}_{\rightarrow}(y) \ge (f^{-1}_{\rightarrow} \vee g^{-1}_{\rightarrow})(y)$$

$X_{f\le y}$ and $X_{g\le y}$ are clearly subsets of $X_{f\le y} \cup X_{g\le y}$. Therefore $(f \wedge g)^{-1}_{\rightarrow}(y) \ge f^{-1}_{\rightarrow}(y)$ and $(f \wedge g)^{-1}_{\rightarrow}(y) \ge g^{-1}_{\rightarrow}(y)$ hold. Hence $(f \wedge g)^{-1}_{\rightarrow}(y)$ is an upper bound of $f^{-1}_{\rightarrow}(y)$ and $g^{-1}_{\rightarrow}(y)$, so $(f \wedge g)^{-1}_{\rightarrow}(y) \ge (f^{-1}_{\rightarrow} \vee g^{-1}_{\rightarrow})(y)$ holds. ◀

▶ **Lemma 25.** *Let $f\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and $g\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be increasing functions, then*

$$(f \otimes g)^{-1}_{\rightarrow} = (f^{-1}_{\rightarrow} \,\overline{\otimes}\, g^{-1}_{\rightarrow}) \text{ if } f \text{ and } g \text{ are left-continuous and} \tag{26}$$

Proof. Let $y \in \mathbb{R}^+_0$ and let x be an element of $X_{(f\otimes g)\le y}$, then $(f \otimes g)(x) \le y$. Because f and g are increasing and left-continuous, they are lower semi-continuous. Therefore, a $\xi \in [0, x]$ exists such that $0 \le f(x - \xi) + g(\xi) = (f \otimes g)(x) \le y$. Observe, that $0 \le g(\xi) \le y$. From $f(x - \xi) + g(\xi) \le y$ we get $f(x - \xi) \le y - g(\xi)$ which implies $x - \xi \le f^{-1}_{\rightarrow}(y - g(\xi))$ (35b). From the latter we get

$$x \le f^{-1}_{\rightarrow}(y - g(\xi)) + \xi \overset{(39b)}{\le} f^{-1}_{\rightarrow}(y - g(\xi)) + g^{-1}_{\rightarrow}(g(\xi)) \overset{(8b)}{\le} (f^{-1}_{\rightarrow} \,\overline{\otimes}\, g^{-1}_{\rightarrow})(y).$$

So, $(f^{-1}_{\rightarrow} \,\overline{\otimes}\, g^{-1}_{\rightarrow})(y)$ is an upper bound of $X_{(f\otimes g)\le y}$, therefore $(f \otimes g)^{-1}_{\rightarrow}(y) = \sup X_{(f\otimes g)\le y} \le (f^{-1}_{\rightarrow} \,\overline{\otimes}\, g^{-1}_{\rightarrow})(y)$.

Let $y \in \mathbb{R}^+_0$, $\upsilon \in [0, y]$, and $x = f^{-1}_{\rightarrow}(y - \upsilon) + g^{-1}_{\rightarrow}(\upsilon)$. Observe, that $0 \le g^{-1}_{\rightarrow}(\upsilon) \le x$. From $x = f^{-1}_{\rightarrow}(y - \upsilon) + g^{-1}_{\rightarrow}(\upsilon)$ we get $x - g^{-1}_{\rightarrow}(\upsilon) = f^{-1}_{\rightarrow}(y - \upsilon)$ which implies $f(x - g^{-1}_{\rightarrow}(\upsilon)) \le y - \upsilon$ because f is increasing and left-continuous (40). From the latter we get

$$y \ge f(x - g^{-1}_{\rightarrow}(\upsilon)) + \upsilon \overset{(40)}{\ge} f(x - g^{-1}_{\rightarrow}(\upsilon)) + g(g^{-1}_{\rightarrow}(\upsilon)) \overset{(8a)}{\ge} (f \otimes g)(x).$$

Therefore, $x \le (f \otimes g)^{-1}_{\rightarrow}(y)$ (35b) and thus $(f^{-1}_{\rightarrow} \,\overline{\otimes}\, g^{-1}_{\rightarrow})(y) \le (f \otimes g)^{-1}_{\rightarrow}(y)$. ◀

▶ **Lemma 26.** *Let $f\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and $g\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be increasing functions that are not bounded above, then*

$$(f \oslash g)^{-1}_{\rightarrow} \ge (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow}). \tag{27}$$

Proof. f and g are increasing and not bounded above, so the images of $f^{-1}_{\rightarrow}$ and $g^{-1}_{\rightarrow}$ are subsets of the non-negative real numbers according to (43), i. e., $f^{-1}_{\rightarrow}(\mathbb{R}^+_0) \subseteq \mathbb{R}^+_0$ and $g^{-1}_{\rightarrow}(\mathbb{R}^+_0) \subseteq \mathbb{R}^+_0$. Therefore $(f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})$ is well-defined.

Let y be a non-negative real number ($y \in \mathbb{R}^+_0$). Let x be a non-negative real number ($x \in \mathbb{R}^+_0$) that satisfies $(f \oslash g)^{-1}_{\rightarrow}(y) < x$. According to (36b) this implies $y < (f \oslash g)(x)$ and due to (8c) a non-negative real number ξ exists ($\xi \in \mathbb{R}^+_0$) that satisfies $y < f(x + \xi) - g(\xi)$. Furthermore a non-negative real number υ exists ($\upsilon \in \mathbb{R}^+_0$) that satisfies $g(\xi) \le \upsilon < f(x + \xi) - y$, because $g(\xi)$ is a non-negative real number.

On the one hand f is increasing, therefore $y + \upsilon < f(x + \xi)$ implies $f^{-1}_{\rightarrow}(y + \upsilon) \le x + \xi$ according to (37). On the other hand $g(\xi) \le \upsilon$ implies $\xi \le g^{-1}_{\rightarrow}(\upsilon)$ according to (35b). This results altogether in $f^{-1}_{\rightarrow}(y + \upsilon) \le x + g^{-1}_{\rightarrow}(\upsilon)$ and ultimately in $x \ge f^{-1}_{\rightarrow}(y + \upsilon) - g^{-1}_{\rightarrow}(\upsilon) \ge (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})(y)$.

In conclusion, the set $X_{l,y} := \left\{x \in \mathbb{R}^+_0 : (f \oslash g)^{-1}_{\rightarrow}(y) < x\right\}$ is a subset of the set $X_{r,y} := \left\{x \in \mathbb{R}^+_0 : (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})(y) \le x\right\}$. Since $(f \oslash g)^{-1}_{\rightarrow}(y) = \inf X_{l,y}$ and $\inf X_{r,y} = (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})(y)$ this implies $(f \oslash g)^{-1}_{\rightarrow}(y) \ge (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})(y)$. Furthermore y is chosen arbitrarily, hence $(f \oslash g)^{-1}_{\rightarrow} \ge (f^{-1}_{\rightarrow} \,\overline{\oslash}\, g^{-1}_{\rightarrow})$. ◀

▶ **Lemma 27** (Unboundness of Convolution). *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be increasing functions not bounded above, then*

$$(f \otimes g) \text{ is not bounded above.} \tag{28}$$

Proof. Let y be a non-negative real number ($y \in \mathbb{R}_0^+$). f and g are not bounded above, then a $x_f \in \mathbb{R}_0^+$ and a $x_g \in \mathbb{R}_0^+$ exists such that $y < f(x_f)$ and $y < g(x_g)$. Let $x = 2 \cdot \max\{x_f, x_g\}$. Then $y < f(x-\xi)$ and $0 \le g(\xi)$ holds for every $\xi \in [0, \frac{x}{2}]$, because f is increasing. Similarly $0 \le f(x-\xi)$ and $y < g(\xi)$ holds for every $\xi \in [\frac{x}{2}, x]$, because g is increasing. Therefore $y < \inf_{0 \le \xi \le x}\{f(x-\xi)+g(\xi)\} = (f \otimes g)(x)$ and subsequently $(f \otimes g)$ is not bounded above. ◀

With Lemmas 24–27 we can now expand the pseudo-inverse of the outgoing arrival curve $\overline{\alpha}_j^+$.

▶ **Lemma 28.** *Under Assumption 1 the pseudo-inverse of the outgoing event-based upper arrival curve of task τ_i is*

$$\overline{\alpha}_{j\ \rightarrow}^{+-1} \ge \left(\overline{\alpha}_{i\ \rightarrow}^{+-1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+-1}\right) \overline{\oslash}\, \overline{\beta}_{i\ \rightarrow}^{--1} \vee \overline{\beta}_{i\ \rightarrow}^{+-1}. \tag{29}$$

Proof. Follows from
(i) the duality of $\wedge$ and $\vee$ regarding the pseudo-inverse operator (25),
(ii) the duality of $\oslash$ and $\overline{\oslash}$ regarding the pseudo-inverse operator (27),
(iii) the duality of $\otimes$ and $\overline{\otimes}$ regarding the pseudo-inverse operator (26),
(iv) $\overline{\alpha}^+$ and $\overline{\beta}^+$ being left-continuous (Assumption 1), therefore satisfying the condition of (26), and
(v) $(\overline{\alpha}^+ \otimes \overline{\beta}^+)$ being increasing [1, p. 22, Lemma 2.3] and not bounded above (28) due to all curves being increasing and not bounded above (Assumption 1), therefore satisfying the condition of (27).

So, we have

$$\begin{aligned}\overline{\alpha}_{j\ \rightarrow}^{+-1} &\overset{(9b)}{=} \left(\left(\left(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+\right) \oslash \overline{\beta}_i^-\right) \wedge \overline{\beta}_i^+\right)_{\rightarrow}^{-1} \overset{(25)}{=} \left(\left(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+\right) \oslash \overline{\beta}_i^-\right)_{\rightarrow}^{-1} \vee \overline{\beta}_{i\ \rightarrow}^{+-1} \\ &\overset{(27)}{\ge} \left(\left(\overline{\alpha}_i^+ \otimes \overline{\beta}_i^+\right)_{\rightarrow}^{-1} \overline{\oslash}\, \overline{\beta}_{i\ \rightarrow}^{--1}\right) \vee \overline{\beta}_{i\ \rightarrow}^{+-1} \overset{(26)}{=} \left(\left(\overline{\alpha}_{i\ \rightarrow}^{+-1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+-1}\right) \overline{\oslash}\, \overline{\beta}_{i\ \rightarrow}^{--1}\right) \vee \overline{\beta}_{i\ \rightarrow}^{+-1}\end{aligned}$$ ◀

We can now start comparing the pseudo-inverse of the outgoing event load function $\eta_{j\ \rightarrow}^{+-1}(n)$ with the pseudo-inverse of the outgoing arrival curves $\overline{\alpha}_{j\ \rightarrow}^{+-1}$. For this we first note that the image of pseudo-inverse of the event-based upper service curve is the non-negative reals Remark 29.

▶ Remark 29. The pseudo-inverse of the event-based upper service curve maps to the non-negative real numbers

$$0 \le \overline{\beta}_{i\ \rightarrow}^{+-1} \tag{30}$$

Proof. By definition (Definition 4) the domain of the resource-based upper service curve is the non-negative real-numbers. The event-based upper service curve is a composition of functions (Equation (6b)), therefore the domain does not change. Furthermore, by definition of the pseudo-inverse of a function (Equation (1)), it cannot map into values that are less than the values in the domain of the function. Therefore, the pseudo-inverse of the event-based upper service curve maps to the non-negative real numbers. ◀

▶ **Lemma 30.** *The pseudo-inverse of the outgoing event load function of task τ_i is*

$$\eta_{j\,\rightarrow}^{+\,-1}(n) = \delta_j^-(n+1). \tag{31}$$

Proof. Follows directly from Equations (13a) and (22)

$$\eta_{j\,\rightarrow}^{+\,-1}(n) \overset{(22)}{=} \eta_{j\,\leftarrow}^{+\,-1}(n+1) \overset{(13a)}{=} \delta_j^-(n+1)$$ ◀

▶ **Lemma 31.** *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g\colon \mathbb{R}_0^+ \to \mathbb{N}_0$, then*

$$\min_{n\in\mathbb{N}_0} \left\{ f_\rightarrow^{-1}(n) - g_\rightarrow^{-1}(n) \right\} = \inf_{r\in\mathbb{R}_0^+} \left\{ f_\rightarrow^{-1}(r) - g_\rightarrow^{-1}(r) \right\} \tag{32}$$

Proof. Let $h := f_\rightarrow^{-1} - g_\rightarrow^{-1}$,$b_X := \inf_{x\in X}\{h(x)\}$, r a non-negative real number and $n := \lfloor r \rfloor$ a non-negative integer.

(i) $b_{\mathbb{R}_0^+} \le b_{\mathbb{N}_0}$: Follows from $\mathbb{N}_0$ being a subset of $\mathbb{R}_0^+$.

(ii) $f_\rightarrow^{-1}(n) \le f_\rightarrow^{-1}(r)$: Follows from $n \le r$ and Equation (42a).

(iii) $g_\rightarrow^{-1}(r) = g_\rightarrow^{-1}(n)$: Follows from Definition 1 and because the co-domain of g are the non-negative integers

$$g_\rightarrow^{-1}(r) = \sup_{x\in\mathbb{R}_0^+} \{x : g(x) \le r\} = \sup_{x\in\mathbb{R}_0^+} \{x : g(x) \le \lfloor r \rfloor\} = g_\rightarrow^{-1}(\lfloor r \rfloor) = g_\rightarrow^{-1}(n)$$

(iv) $h(n) \le h(r)$: Follows from Items ii and iii

(v) $b_{\mathbb{N}_0} \le b_{\mathbb{R}_0^+}$: Follows because for every $r \in \mathbb{R}_0^+$ an $n \in \mathbb{N}_0$ exists such that $h(n) \le h(r)$.

This Lemma follows from i and v. ◀

▶ **Lemma 32.** *Let τ_i be a task that triggers task τ_j, then*

$$\min_{k\in\mathbb{N}_0} \left\{ \delta_i^-(n+1+k) - B_i^+(k+1) \right\} + B_i^-(1) \le \left(\overline{\alpha}_{i\,\rightarrow}^{+\,-1} \,\overline{\otimes}\, \overline{\beta}_{i\,\rightarrow}^{+\,-1} \right) \overline{\oslash}\, \overline{\beta}_{i\,\rightarrow}^{-\,-1} \tag{33}$$

Proof. Follows from

(i) the derivation of the pseudo-inverse of the upper event load function from the event distance function (31),

(ii) the event load function being equal to the event-based upper arrival curve (16),

(iii) the pseudo-inverse of the event-based lower service curve being equal to the maximum multiple busy event time (17b),

(iv) the minimum multiple busy event time being equal to the pseudo-inverse of the event-based upper service curve (17a),

(v) the relation between the pseudo-inverses when the domain of the functions are integers (22),

(vi) the definition of the supremum-plus convolution (8b),

(vii) the equality (32), and

(viii) the definition of the supremum-plus deconvolution (8d), i. e.,

$$\begin{aligned}
&\min_{k\in\mathbb{N}_0}\left\{\delta_i^-(n+1+k) - B_i^+(k+1)\right\} + B_i^-(1)\\
&\overset{(31)}{=} \min_{k\in\mathbb{N}_0}\left\{\eta_{i\ \rightarrow}^{+\ -1}(n+k) - B_i^+(k+1)\right\} + B_i^-(1)\\
&\overset{(16)}{=} \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) - B_i^+(k+1)\right\} + B_i^-(1)\\
&\overset{(17b)}{=} \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) - \overline{\beta}_{i\ \leftarrow}^{-\ -1}(k+1)\right\} + B_i^-(1)\\
&\overset{(17a)}{=} \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) - \overline{\beta}_{i\ \leftarrow}^{-\ -1}(k+1)\right\} + \overline{\beta}_{i\ \leftarrow}^{+\ -1}(1)\\
&\overset{(22)}{=} \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) - \overline{\beta}_{i\ \rightarrow}^{-\ -1}(k)\right\} + \overline{\beta}_{i\ \leftarrow}^{+\ -1}(1)\\
&\overset{(22)}{=} \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) - \overline{\beta}_{i\ \rightarrow}^{-\ -1}(k)\right\} + \overline{\beta}_{i\ \rightarrow}^{+\ -1}(0)\\
&= \min_{k\in\mathbb{N}_0}\left\{\overline{\alpha}_{i\ \rightarrow}^{+\ -1}(n+k) + \overline{\beta}_{i\ \rightarrow}^{+\ -1}(0) - \overline{\beta}_{i\ \rightarrow}^{-\ -1}(k)\right\}\\
&\overset{(8b)}{\leq} \min_{k\in\mathbb{N}_0}\left\{(\overline{\alpha}_{i\ \rightarrow}^{+\ -1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+\ -1})(n+k) - \overline{\beta}_{i\ \rightarrow}^{-\ -1}(k)\right\}\\
&\overset{(32)}{=} \inf_{k\in\mathbb{R}_0^+}\left\{(\overline{\alpha}_{i\ \rightarrow}^{+\ -1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+\ -1})(n+k) - \overline{\beta}_{i\ \rightarrow}^{-\ -1}(k)\right\}\\
&\overset{(8d)}{=} ((\overline{\alpha}_{i\ \rightarrow}^{+\ -1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+\ -1}) \overline{\oslash}\, \overline{\beta}_{i\ \rightarrow}^{-\ -1})(n)
\end{aligned}$$ ◀

▶ **Lemma 33.** *Under the Assumptions 1 and 2 the pseudo-inverse of the outgoing event load function is less than or equal to the pseudo-inverse of the outgoing event-based upper arrival curve for task τ_i,*

$$\eta_{j\ \rightarrow}^{+\ -1} \leq \overline{\alpha}_{j\ \rightarrow}^{+\ -1}. \tag{34}$$

Proof. Follows from

(i) the derivation of the pseudo-inverse of the event load function (31),
(ii) the definition of the outgoing event distance function (15),
(iii) the co-domain of the pseudo-inverse of the event-based upper service curve is the non-negative real numbers, and (30)
(iv) the inequality (33), i. e.,

$$\begin{aligned}
\eta_{j\ \rightarrow}^{+\ -1}(n) &\overset{(31)}{=} \delta_j^-(n+1) \overset{(15)}{=} \max\left\{\min_{k\in\mathbb{N}_0}\left\{\delta_i^-(n+1+k) - B_i^+(k+1)\right\} + B_i^-(1), 0\right\}\\
&\overset{(30)}{\leq} \max\left\{\min_{k\in\mathbb{N}_0}\left\{\delta_i^-(n+1+k) - B_i^+(k+1)\right\} + B_i^-(1), \overline{\beta}_{i\ \rightarrow}^{+\ -1}(n)\right\}\\
&\overset{(33)}{\leq} \max\left\{\left(\left(\overline{\alpha}_{i\ \rightarrow}^{+\ -1} \overline{\otimes}\, \overline{\beta}_{i\ \rightarrow}^{+\ -1}\right) \overline{\oslash}\, \overline{\beta}_{i\ \rightarrow}^{-\ -1}\right)(n), \overline{\beta}_{i\ \rightarrow}^{+\ -1}(n)\right\} \overset{(29)}{\leq} \overline{\alpha}_{j\ \rightarrow}^{+\ -1}(n).
\end{aligned}$$ ◀

With all these preparations we can now finally prove our contribution.

Proof of Theorem 19. (21) follows from

(i) Lemma 21, Equation (23), where we show that we can imply the order of functions from their pseudo-inverse, i. e., $f_{\rightarrow}^{-1} \geq g_{\rightarrow}^{-1} \Rightarrow f \leq g$,
(ii) the outgoing event-based upper arrival curve $\overline{\alpha}_j^+$ being increasing (Lemma 23) and left-continuous (Lemma 23), therefore satisfying the conditions of Lemma 21, Equation (23),

(iii) the derivation of the pseudo-inverse of the upper load function for the generated events $\eta_{j\rightarrow}^{+-1}$ from its event distance function δ_j^- (Lemma 30)

(iv) the derivation of the pseudo-inverse of the event-based upper arrival curve for the generated events $\overline{\alpha}_{j\rightarrow}^{+-1}$ from its event-based upper arrival curve $\overline{\alpha}_j^+$ (Lemma 28, Equation (29)), and

(v) that the pseudo-inverse of the upper load function for the generated events $\eta_{j\rightarrow}^{+-1}$ is less than or equal to the pseudo-inverse of the event-based upper arrival curve for the generated events $\overline{\alpha}_{j\rightarrow}^{+-1}$ (Lemma 33), i. e.,

$$\eta_{j\rightarrow}^{-1} \leq \overline{\alpha}_{j\rightarrow}^{+-1} \overset{(23)}{\Rightarrow} \overline{\alpha}_j^+ \leq \eta_j. \qquad \blacktriangleleft$$

We proved in Theorem 19 that the outgoing event-based upper arrival curve $\overline{\alpha}_j^+$ is less than or equal to the outgoing event load function η_j^+. In the proof to Lemma 33 we can see why this is the case. The pseudo-inverse of the event-based upper service curve $\overline{\beta}_{i\rightarrow}^{+-1}$ maps to the non-negative real numbers, Remark 29, and is therefore always greater than or equal to 0. And a part of the pseudo-inverse of the outgoing event-based upper arrival curve $\left(\overline{\alpha}_{i\rightarrow}^{+-1} \overline{\otimes}\, \overline{\beta}_{i\rightarrow}^{+-1}\right) \overline{\oslash}\, \overline{\beta}_{i\rightarrow}^{--1}$ is greater than or equal to a part of the outgoing minimum event distance function $\min_{k\in\mathbb{N}_0}\left\{\delta_i^-(n+1+k) - B_i^+(k+1)\right\} + B_i^-(1)$, (33). In its proof we can see that one reason for that is due to the sup-plus convolution $\overline{\otimes}$.

5 Summary

We took the bounds for the generated events as presented by [12] for the compositional performance analysis and by [17] for the real-time calculus. By means of a proof we showed that the bound of the real-time calculus is not greater than the bound of the compositional performance analysis. Additionally we identified the mathematical reason why this is the case. Which opens the possibility to adapt the method that the real-time calculus uses to improve the method that the compositional performance analysis uses.

References

1 Anne Bouillard, Marc Boyer, and Euriell Le Corronc. *Deterministic Network Calculus*. John Wiley & Sons, Ltd, 2018. `doi:10.1002/9781119440284`.

2 Marc Boyer and Pierre Roux. A common framework embedding network calculus and event stream theory. Technical report, ONERA - The french aerospace lab, May 2016. Preprint. URL: `https://hal.archives-ouvertes.fr/hal-01311502`.

3 Marc Boyer and Pierre Roux. Embedding network calculus and event stream theory in a common model. In *21st IEEE International Conference on Emerging Technologies and Factory Automation, ETFA 2016, Berlin, Germany, September 6-9, 2016*, pages 1–8. IEEE, 2016. `doi:10.1109/ETFA.2016.7733565`.

4 Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics: A Foundation for Computer Science, 2nd Ed.* Addison-Wesley, 1994. URL: `https://www-cs-faculty.stanford.edu/%7Eknuth/gkp.html`.

5 Jean-Yves Le Boudec and Patrick Thiran. *Network Calculus: A Theory of Deterministic Queuing Systems for the Internet*, volume 2050 of *Lecture Notes in Computer Science*. Springer, 2001. `doi:10.1007/3-540-45318-0`.

6 Martin Naedele, Lothar Thiele, and Michael Eisenring. Characterizing Variable Task Releases and Processor Capacities. Technical report, ETH Zürich, 1999. `doi:10.3929/ethz-a-004289034`.

7 Simon Perathoner, Ernesto Wandeler, Lothar Thiele, Arne Hamann, Simon Schliecker, Rafik Henia, Razvan Racu, Rolf Ernst, and Michael González Harbour. Influence of Different System Abstractions on the Performance Analysis of Distributed Real-Time Systems. In Christoph M. Kirsch and Reinhard Wilhelm, editors, *Proceedings of the 7th ACM & IEEE International conference on Embedded software, EMSOFT 2007, September 30 - October 3, 2007, Salzburg, Austria*, pages 193–202. ACM, 2007. doi:10.1145/1289927.1289959.

8 Victor Pollex, Steffen Kollmann, and Frank Slomka. Generalizing Response-Time Analysis. In *16th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications, RTCSA 2010, Macau, SAR, China, 23-25 August 2010*, pages 203–211. IEEE Computer Society, 2010. doi:10.1109/RTCSA.2010.36.

9 Victor Pollex and Frank Slomka. A Mathematical Comparison Between Response-Time Analysis and Real-Time Calculus for Fixed-Priority Preemptive Scheduling. In Martina Maggio, editor, *34th Euromicro Conference on Real-Time Systems (ECRTS 2022)*, volume 231 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 7:1–7:25, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ECRTS.2022.7.

10 Kai Richter. *Compositional Scheduling Analysis Using Standard Event Models*. PhD thesis, Technical University Carolo-Wilhemina of Braunschweig, 2005. doi:10.24355/dbbs.084-200511080100-362.

11 Pierre Roux, Sophie Quinton, and Marc Boyer. A Formal Link Between Response Time Analysis and Network Calculus. In Martina Maggio, editor, *34th Euromicro Conference on Real-Time Systems (ECRTS 2022)*, volume 231 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:22, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ECRTS.2022.5.

12 Simon Schliecker. *Performance Analysis of Multiprocessor Real-Time Systems with Shared Resources*. PhD thesis, Technical University Carolo-Wilhemina of Braunschweig, 2011. doi:10.24355/dbbs.084-201111210932-0.

13 Frank Slomka and Mohammadreza Sadeghi. HeRTA: Heaviside Real-Time Analysis, 2020. Preprint. doi:10.48550/arXiv.2007.12112.

14 Frank Slomka and Mohammadreza Sadeghi. Beyond the limitations of real-time scheduling theory: a unified scheduling theory for the analysis of real-time systems. *SICS Software-Intensive Cyber Physical Systems*, 35(3-4):201–236, 2021. doi:10.1007/s00450-021-00429-1.

15 Frank Slomka and Mohammadreza Sadeghi. Work-in-Progress Abstract: On the relationship between scheduling theory and real-time calculus. In *27th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications, RTCSA 2021, Houston, TX, USA, August 18-20, 2021*, pages 195–197. IEEE, 2021. doi:10.1109/RTCSA52859.2021.00030.

16 Ken W. Tindell, Alan Burns, and Andy J. Wellings. An Extendible Approach for Analyzing Fixed Priority Hard Real-Time Tasks. *Real-Time Systems*, 6(2):133–151, March 1994. doi:10.1007/BF01088593.

17 Ernesto Wandeler. *Modular Performance Analysis and Interface-Based Design for Embedded Real-Time Systems*. PhD thesis, Swiss Federal Institute of Technology Zurich, 2006. doi:10.3929/ethz-a-005328667.

A Properties of Pseudo-Inverse Functions

▶ **Lemma 34** ([9, Lemma 37], [5, p. 131, Theorem 3.1.2, (3.4)] and [2, Proposition 6, (7) and (6)])**.** *Let $f\colon X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$y \leq f(x) \Rightarrow f_{\leftarrow}^{-1}(y) \leq x \text{ and} \tag{35a}$$

$$f(x) \leq y \Rightarrow x \leq f_{\rightarrow}^{-1}(y). \tag{35b}$$

▶ **Lemma 35** ([9, Lemma 38], [2, Proposition 6, (14) and (13)]). *Let $f\colon X \to Y$ be a mapping from a subset X of a complete lattice L to a totally ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$x < f_{\leftarrow}^{-1}(y) \Rightarrow f(x) < y \text{ and} \tag{36a}$$
$$f_{\rightarrow}^{-1}(y) < x \Rightarrow y < f(x). \tag{36b}$$

▶ **Lemma 36** ([9, Lemma 39], [5, p. 131, Theorem 3.1.2, (3.5)] and [2, Proposition 6, (9))]). *Let $f\colon X \to Y$ be an isotone mapping from a totally ordered subset X of a complete lattice L to a partially ordered set Y, then for an element $x \in X$ and an element $y \in Y$*

$$y < f(x) \Rightarrow f_{\rightarrow}^{-1}(y) \leq x \text{ and} \tag{37}$$
$$x < f_{\rightarrow}^{-1}(y) \Rightarrow f(x) \leq y. \tag{38}$$

Proof of (37). See [9, Lemma 39]. ◀

Proof of (38). Let $x < f_{\rightarrow}^{-1}(y) = \sup X_{f\leq y}$, then an element ξ of $X_{f\leq y}$ must exist such that $x < \xi \leq f_{\rightarrow}^{-1}(y)$, due to the total order of X. This implies $f(x) \leq f(\xi)$, because f is isotone and $f(\xi) \leq y$, so $f(x) \leq y$. ◀

▶ **Lemma 37.** *Confer [2, Proposition 7, (19)]. Let $f\colon X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y and let 1_X be the identity map on X, then*

$$(f_{\leftarrow}^{-1} \circ f) \leq 1_X \text{ and} \tag{39a}$$
$$1_X \leq (f_{\rightarrow}^{-1} \circ f). \tag{39b}$$

▶ **Lemma 38** ([9, Lemma 41], [2, Proposition 6, (17), and (18)]). *Let $f\colon X \to \mathbb{R}$ be an increasing and left-continuous function from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$y < f(x) \Rightarrow f_{\rightarrow}^{-1}(y) < x \text{ and } x \leq f_{\rightarrow}^{-1}(y) \Rightarrow f(x) \leq y. \tag{40}$$

▶ **Lemma 39** ([9, Lemma 42], [2, Proposition 6, (15), and (16)]). *Let $f\colon X \to \mathbb{R}$ be an increasing and right-continuous function from a subset X of the real numbers $\mathbb{R}$, then for an element $x \in X$ and an element $y \in \mathbb{R}$*

$$f(x) < y \Rightarrow x < f_{\leftarrow}^{-1}(y) \text{ and } f_{\leftarrow}^{-1}(y) \leq x \Rightarrow y \leq f(x). \tag{41}$$

▶ **Lemma 40** (Isotone pseudo-inverse). *[9, Lemma 45]. Let $f\colon X \to Y$ be a mapping from a subset X of a complete lattice L to a partially ordered set Y, then*

$$f_{\rightarrow}^{-1} \text{ is isotone.} \tag{42a}$$

▶ **Lemma 41.** *Let $f\colon X \to \mathbb{R}_0^+$ be an increasing function from a subset X of the non-negative real numbers $\mathbb{R}_0^+$ that is not bounded above, then the image of $f_{\rightarrow}^{-1}$ is a subset of the non-negative real numbers, i. e.,*

$$f_{\rightarrow}^{-1}(\mathbb{R}_0^+) \subseteq \mathbb{R}_0^+. \tag{43}$$

Proof. Let y be a non-negative real number, then $0 = \sup \emptyset \leq \sup X_{f\leq y} = f_{\rightarrow}^{-1}(y)$, because the empty set $\emptyset$ is a subset of $X_{f\leq y}$. Furthermore, an $x \in X$ exists such that $y < f(x)$, because f is not bounded above. Therefore, (37) implies $f_{\rightarrow}^{-1}(y) \leq x$. So, $f_{\rightarrow}^{-1}(y)$ is a non-negative real number. ◀

▶ **Lemma 42** (Pseudo-inverse of a composition). *[9, Lemma 48], [2, Proposition 6, (25) and (24)]. Let $f\colon X \to Y$ be a function from a subset X of a complete lattice L to a subset Y of the real numbers and let $g\colon Y \to \mathbb{R}$ be an increasing function, then*

$$(g \circ f)^{-1}_{\rightarrow} = (f^{-1}_{\rightarrow} \circ g^{-1}_{\rightarrow}) \text{ if } g \text{ is left-continuous and} \tag{44a}$$

$$(g \circ f)^{-1}_{\leftarrow} = (f^{-1}_{\leftarrow} \circ g^{-1}_{\leftarrow}) \text{ if } g \text{ is right-continuous.} \tag{44b}$$

▶ **Lemma 43** (The Pseudo-Inverse Operators are inverse to each other). *[9, Lemma 49] Let $f\colon X \to \mathbb{R}$ be an increasing function from a subset X of $\overline{\mathbb{R}}$, then*

$$f^{-1}{}_{\rightarrow}^{-1}{}_{\leftarrow} = f \text{ if } f \text{ is left-continuous and} \tag{45a}$$

$$f^{-1}{}_{\leftarrow}^{-1}{}_{\rightarrow} = f \text{ if } f \text{ is right-continuous.} \tag{45b}$$

▶ **Lemma 44.** *Let $f\colon X \to \mathbb{R}$ be a function not bounded above from a subset X of a complete lattice closed under addition, then*

$$f^{-1}_{\leftarrow} \text{is subadditive if } f \text{ is superadditive.} \tag{46}$$

Proof. Let $y_1, y_2 \in \mathbb{R}$. We have to show that $f^{-1}_{\leftarrow}(y_1 + y_2) \leq f^{-1}_{\leftarrow}(y_1) + f^{-1}_{\leftarrow}(y_2)$.

(i) The set $X_{y \leq f} := \{x \in X : y \leq f(x)\}$ is not empty for any $y \in \mathbb{R}$, because f is not bounded above. Therefore, we can combine the sum of the infima of two of those sets, i. e., $\inf X_{y_1 \leq f} + \inf X_{y_2 \leq f} = \inf_{x_1, x_2 \in X} \{x_1 + x_2 : y_1 \leq f(x_1) \text{ and } y_2 \leq f(x_2)\}$.

(ii) Let $x_1, x_2 \in X$ such that $y_1 \leq f(x_1)$ and $y_2 \leq f(x_2)$, then $y_1 + y_2 \leq f(x_1) + f(x_2)$.

(iii) f is superadditive by assumption. So, $\forall x_1, x_2 \in X : f(x_1 + x_2) \geq f(x_1) + f(x_2)$.

(iv) X is closed under addition, so we have $X + X := \{x_1 + x_2 : x_1, x_2 \in X\} \subseteq X$.

$$\begin{aligned}
f^{-1}_{\leftarrow}(y_1) + f^{-1}_{\leftarrow}(y_2) &\overset{(1)}{=} \inf_{x_1 \in X} \{x_1 : y_1 \leq f(x_1)\} + \inf_{x_2 \in X} \{x_2 : y_2 \leq f(x_2)\} \\
&\overset{(i)}{=} \inf_{x_1, x_2 \in X} \{x_1 + x_2 : y_1 \leq f(x_1) \text{ and } y_2 \leq f(x_2)\} \\
&\overset{(ii)}{\geq} \inf_{x_1, x_2 \in X} \{x_1 + x_2 : y_1 + y_2 \leq f(x_1) + f(x_2)\} \\
&\overset{(iii)}{\geq} \inf_{x_1, x_2 \in X} \{x_1 + x_2 : y_1 + y_2 \leq f(x_1 + x_2)\} \\
&\overset{(iv)}{\geq} \inf_{x \in X} \{x : y_1 + y_2 \leq f(x)\} \overset{(1)}{=} f^{-1}_{\leftarrow}(y_1 + y_2).
\end{aligned}$$ ◀

▶ **Lemma 45** (Monotonicity). *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing function and $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a function, then*

$$(f \oslash g) \text{ is increasing.} \tag{47}$$

Proof. From $x_1 \leq x_2$ follows that $x_1 + \xi \leq x_2 + \xi$ holds for every $\xi \in \mathbb{R}_0^+$ and subsequently $f(x_1 + \xi) \leq f(x_2 + \xi)$, since f is increasing. Furthermore $f(x_1 + \xi) - g(\xi) \leq f(x_2 + \xi) - g(\xi)$ holds for every $\xi \in \mathbb{R}_0^+$, therefore

$$(f \oslash g)(x_1) = \sup_{\xi \in \mathbb{R}_0^+} \{f(x_1 + \xi) - g(\xi)\} \leq \sup_{\xi \in \mathbb{R}_0^+} \{f(x_2 + \xi) - g(\xi)\} = (f \oslash g)(x_2)$$ ◀

▶ **Lemma 46** (Directional Continuity of Deconvolution). *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an increasing function and $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a function, then*

$$(f \oslash g) \text{ is left-continuous if } f \text{ is left-continuous and} \tag{48a}$$

$$(f \mathbin{\overline{\oslash}} g) \text{ is right-continuous if } f \text{ is right-continuous.} \tag{48b}$$

Proof. Let $\varepsilon > 0$.

Let f be left-continuous. If $x = 0$, then for any $\delta > 0$ it follows that $|(f \oslash g)(x) - (f \oslash g)(\xi)| < \varepsilon$ holds for all $\xi \in \mathbb{R}_0^+$ with $x - \delta < \xi < x$, since no such ξ exists. Therefore $(f \oslash g)$ is trivially left-continuous at $x = 0$.

Let $x \in \mathbb{R}^+$, then $(f \oslash g)(x) - \frac{\varepsilon}{2} < (f \oslash g)(x) = \sup_{\xi \in \mathbb{R}_0^+} \{f(x+\xi) - g(\xi)\}$ holds. Therefore a $s \in \mathbb{R}_0^+$ exists such that $(f \oslash g)(x) - \frac{\varepsilon}{2} < f(x+s) - g(s)$. Since f is left-continuous, there exists a $\delta > 0$ such that $\delta < x$ and $|f(x+s) - f(\xi+s)| < \frac{\varepsilon}{2}$ holds for all $x - \delta < \xi < x$.

$$\begin{aligned}(f \oslash g)(x) &< f(x+s) - g(s) + \frac{\varepsilon}{2} < f(\xi+s) + \frac{\varepsilon}{2} - g(s) + \frac{\varepsilon}{2} \\ &= f(\xi+s) - g(s) + \varepsilon \leq (f \oslash g)(\xi) + \varepsilon\end{aligned}$$

Note that $(f \oslash g)$ is increasing since f is increasing. Thus $(f \oslash g)(\xi) \leq (f \oslash g)(x) < (f \oslash g)(x) + \varepsilon$ for any $\xi \in \mathbb{R}_0^+$ with $\xi < x$.

Therefore $(f \oslash g)$ is left-continuous.

Let f be right-continuous.

Let $x \in \mathbb{R}_0^+$, then $\inf_{\xi \in \mathbb{R}_0^+} \{f(x+\xi) - g(\xi)\} = (f \overline{\oslash} g)(x) < (f \overline{\oslash} g)(x) + \frac{\varepsilon}{2}$ holds. Therefore a $s \in \mathbb{R}_0^+$ exists such that $f(x+s) - g(s) < (f \overline{\oslash} g)(x) + \frac{\varepsilon}{2}$. Since f is right-continuous, there exists a $\delta > 0$ such that $|f(x+s) - f(\xi+s)| < \frac{\varepsilon}{2}$ holds for all $x < \xi < x + \delta$.

$$\begin{aligned}(f \overline{\oslash} g)(\xi) &\leq f(\xi+s) - g(s) < f(x+s) + \frac{\varepsilon}{2} - g(s) \\ &< (f \overline{\oslash} g)(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = (f \overline{\oslash} g)(x) + \varepsilon\end{aligned}$$

Note that $(f \overline{\oslash} g)$ is increasing since f is increasing. Thus $(f \overline{\oslash} g)(x) \leq (f \overline{\oslash} g)(\xi) < (f \overline{\oslash} g)(\xi) + \varepsilon$ for any $\xi \in \mathbb{R}_0^+$ with $x < \xi$.

Therefore $(f \overline{\oslash} g)$ is right-continuous. ◀

B Properties of Floor and Ceiling Functions

▶ **Lemma 47** ([4, p. 69, (3.7)]). *Let $x \in \mathbb{R}$ and let $z \in \mathbb{Z}$, then*

$$x < z \Leftrightarrow \lfloor x \rfloor < z \text{ and } z \leq x \Leftrightarrow z \leq \lfloor x \rfloor, \text{ and} \tag{49}$$
$$z < x \Leftrightarrow z < \lceil x \rceil \text{ and } x \leq z \Leftrightarrow \lceil x \rceil \leq z. \tag{50}$$

▶ **Lemma 48** (Duality of pseudo-inverse). *The pseudo-inverse of the floor function is the ceiling function and vice versa, i. e.,*

$$\lfloor \cdot \rfloor_{\leftarrow}^{-1} = \lceil \cdot \rceil \text{ and } \lceil \cdot \rceil_{\rightarrow}^{-1} = \lfloor \cdot \rfloor. \tag{51}$$

Proof. Let $y \in \mathbb{R}$, then

$$\begin{aligned}&\lfloor \cdot \rfloor_{\leftarrow}^{-1}(y) = \inf_{x \in \mathbb{R}} \{x : y \leq \lfloor x \rfloor\} \overset{(50)}{=} \inf_{x \in \mathbb{R}} \{x : \lceil y \rceil \leq \lfloor x \rfloor\} \overset{(49)}{=} \inf_{x \in \mathbb{R}} \{x : \lceil y \rceil \leq x\} = \lceil y \rceil \text{ and} \\ &\lceil \cdot \rceil_{\rightarrow}^{-1}(y) = \sup_{x \in \mathbb{R}} \{x : \lceil x \rceil \leq y\} \overset{(49)}{=} \sup_{x \in \mathbb{R}} \{x : \lceil x \rceil \leq \lfloor y \rfloor\} \overset{(50)}{=} \sup_{x \in \mathbb{R}} \{x : x \leq \lfloor y \rfloor\} = \lfloor y \rfloor.\end{aligned}$$ ◀

▶ **Lemma 49.** *Let $y \in \mathbb{R}$, then*

$$\lfloor \cdot \rfloor_{\rightarrow}^{-1}(y) = \lfloor y + 1 \rfloor \text{ and } \lceil \cdot \rceil_{\leftarrow}^{-1}(y) = \lceil y - 1 \rceil. \tag{52}$$

Proof. Let $y \in \mathbb{R}$, then

$$\begin{aligned}
\lfloor\cdot\rfloor_{\rightarrow}^{-1}(y) &= \sup_{x\in\mathbb{R}}\{x : \lfloor x\rfloor \le y\} \overset{(49)}{=} \sup_{x\in\mathbb{R}}\{x : \lfloor x\rfloor \le \lfloor y\rfloor\} = \sup_{x\in\mathbb{R}}\{x : \lfloor x\rfloor < \lfloor y\rfloor + 1\} \\
&\overset{(49)}{=} \sup_{x\in\mathbb{R}}\{x : x < \lfloor y+1\rfloor\} = \lfloor y+1\rfloor \text{ and} \\
\lceil\cdot\rceil_{\leftarrow}^{-1}(y) &= \inf_{x\in\mathbb{R}}\{x : y \le \lceil x\rceil\} \overset{(50)}{=} \inf_{x\in\mathbb{R}}\{x : \lceil y\rceil \le \lceil x\rceil\} = \inf_{x\in\mathbb{R}}\{x : \lceil y\rceil - 1 < \lceil x\rceil\} \\
&\overset{(50)}{=} \inf_{x\in\mathbb{R}}\{x : \lceil y-1\rceil < x\} = \lceil y-1\rceil\,.
\end{aligned}$$

◀

C CPA Deriving Outgoing Minimum Event Distances

To derive the Minimum Event Distances, the CPA first bounds the exit time of an event n produced by task τ_i.

▶ **Lemma 50** ([12, p. 70, Lemma 3.7]). *The exit time $e_i(n)$ of any event n produced by task τ_i is bounded by*

$$e_i(n) \le \max_{k\ge 0}\left\{a_i(n-k) + B_i^+(k+1)\right\} \tag{53}$$

where $a_i(n-k)$ is the arrival time of the k-th event before event n. $B_i^+(k+1)$ is the maximum multiple event busy time for $k+1$ events to be processed by τ_i.

Equation (53) uses $\mathbb{N}_0$ as the set of events which could have been backlogged. To constrain the set $\mathbb{N}_0$ the CPA uses an approximation of when events arrive relative to each other.

▶ **Lemma 51** ([12, p. 71, Lemma 3.8]). *With respect to the arrival of an event p at task τ_i, the event q, arrives at τ_i no later than*

$$a_i(p) \le \begin{cases} a_i(q) - \delta_{i,in}^-(q-p+1), & \text{if } p < q \\ a_i(q) + \delta_{i,in}^+(p-q+1), & \text{if } p > q \end{cases}. \tag{54}$$

With Lemma 51 the CPA can constrain the set of events that could be unfinished at the time when an event arrives at the input.

▶ **Lemma 52** ([12, p. 72, Lemma 3.9]). *When a new event arrives at the input of a task τ_i, the set of the number of events that may be unfinished at this time is bounded by*

$$K = \left\{k \in \mathbb{N} | \delta_{i,in}^-(k+1) < B_i^+(k)\right\} \tag{55}$$

where $\delta_{i,in}^-$ is the minimum distance between events that activate τ_i and B_i^+ is the multiple event busy time.

Therefore, the exit time of an event n produced by task τ_i is also bounded when using the constrained set (55) unified with 0 instead of $\mathbb{N}_0$.

▶ **Lemma 53** ([12, p. 73, Corollary 3.10]). *The exit time $e_i(n)$ of any event n produced by task τ_i is bounded by*

$$e_i(n) \le \max_{k\in K}\left\{a_i(n-k) + B_i^+(k+1)\right\} \tag{56}$$

$$K = \{0\} \cup \left\{k \in \mathbb{N} | \delta_{i,in}^-(k+1) < B_i^+(k)\right\} \tag{57}$$

Note that in [12, p. 73, Corollary 3.10] the set K does not include 0 and therefore K can be empty, which we consider an oversight. In that case, there are no unfinished events at the time when event n arrives. Subsequently, the exit time of event n is then no later than $a_i(n) + B_i^+(1)$, which we get when we include 0 into K.

This constrained set K then leads to the derived outgoing minimum event distance function.

▶ **Lemma 54** ([12, p. 73, Theorem 3.11])**.** *Given a task τ_i with maximum busy time function $B_i^+(n)$ and minimum busy time $B_i^-(1)$. When the minimum distance between n arriving events is bounded by $\delta_{i,in}^-(n)$ then the minimum distance between n events at the output of this task is given by $\delta_{i,out}^-(n)$ as follows:*

$$\delta_{i,out}^-(n) = \max\left\{0, \min_{k\in K}\left\{\delta_{i,in}^-(n+k) - B_i^+(k+1)\right\} + B_i^-(1)\right\} \tag{58}$$

$$K = \{0\} \cup \left\{k \in \mathbb{N} | \delta_{i,in}^-(k+1) < B_i^+(k)\right\}. \tag{59}$$

The proof of Lemma 54 uses Lemma 53 for the bound of the exit time of an event n. We show that the bounds Lemma 50 and Lemma 53 for the exit time of an event n are the same. Then we can use Lemma 50 in the proof of Lemma 54 and obtain the same bound for the outgoing minimum busy time function $\delta_{i,out}^-(n)$, but using $K = \mathbb{N}_0$ instead.

▶ **Lemma 55.** *The bound for the exit time in Lemma 50 and Lemma 53 are the same, if the maximum busy time function $B_i^+(n)$ is subadditive.*

Proof. Let $K_1 = \mathbb{N}_0$ and $K_2 = \{0\} \cup \left\{k \in \mathbb{N} | \delta_{i,in}^-(k+1) < B_i^+(k)\right\}$. Then, we need to show that $\max_{k\in K_1}\left\{a_i(n-k) + B_i^+(k+1)\right\} = \max_{k\in K_2}\left\{a_i(n-k) + B_i^+(k+1)\right\}$.

K_1 is a superset of K_2, so let $k \in K_1 \setminus K_2$. Because the maximum busy time function $B_i^+(n)$ is subadditive, we get

$$a_i(n-k) + B_i^+(k+1) \le a_i(n-k) + B_i^+(k) + B_i^+(1).$$

k is not in K_2, so $B_i^+(k) \le \delta_{i,in}^-(k+1)$ holds. Therefore we get

$$a_i(n-k) + B_i^+(k) + B_i^+(1) \le a_i(n-k) + \delta_{i,in}^-(k+1) + B_i^+(1).$$

Because $0 \in K_2$, we have $k > 0$ and subsequently $n - k < n$. Therefore, according to Lemma 51, $a_i(n-k) \le a_i(n) - \delta_{i,in}^-(n-(n-k)+1)$. Thus we get

$$a_i(n-k) + \delta_{i,in}^-(k+1) + B_i^+(1) \le a_i(n) + B_i^+(1).$$

Therefore, $a_i(n-k) + B_i^+(k+1) \le a_i(n) + B_i^+(1)$. So, any $k \in K_1 \setminus K_2$ will not change $\max_{k\in K_2}\left\{a_i(n-k) + B_i^+(k+1)\right\}$ even if it were included in K_2. ◀

▶ **Lemma 56** ([12, p. 73, Theorem 3.11])**.** *Given a task τ_i with maximum busy time function $B_i^+(n)$ and minimum busy time $B_i^-(1)$. When the minimum distance between n arriving events is bounded by $\delta_{i,in}^-(n)$ then the minimum distance between n events at the output of this task is given by $\delta_{i,out}^-(n)$ as follows:*

$$\delta_{i,out}^-(n) = \max\left\{0, \min_{k\in\mathbb{N}_0}\left\{\delta_{i,in}^-(n+k) - B_i^+(k+1)\right\} + B_i^-(1)\right\} \tag{60}$$

Proof. In Lemma 55 we show that it does not matter whether we use for $K = \mathbb{N}_0$ or $K = \left\{k \in \mathbb{N}_0 | \delta_{i,in}^-(k+1) \le B_i^+(k)\right\}$. Therefore the proof is analogous to [12, p. 73, Theorem 3.11] but using $K = \mathbb{N}_0$ instead. ◀