Effective Kan Fibrations for W-Types in Homotopy Type Theory

Shinichiro Tanaka 🄳

Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands

─── **Abstract** ───

We investigate effective Kan fibrations in the context of the semantics of Homotopy Type Theory (HoTT). Effective Kan fibrations were proposed by Benno van den Berg and Eric Faber as constructive alternatives to classical Kan fibrations for modeling HoTT. Our work specifically explores their interaction with W-types in HoTT, which are inductive types representing well-founded trees, and extends this exploration to variants such as M-types. By using the categorical properties of W-types, we show that effective Kan fibrations model them. Additionally, we examine the behavior of quotient maps and discuss that certain cases can also be classified as effective Kan fibrations.

1 Introduction

1.1 Background and motivation

Homotopy Type Theory (HoTT) started with the homotopy-theoretic interpretation of Martin-Löf's dependent type theory and the discovery of the univalence axiom by Voevodsky in the 2000s [11] [12]. Awodey and Warren, and independently Voevodsky, provided this homotopy interpretation. A certain form of Martin-Löf type theory can be modeled in any Quillen model category [1], giving HoTT a semantics with topological intuition. The category of simplicial sets, one of the most significant model categories, models types as Kan complexes and dependent types as Kan fibrations, as proposed by Voevodsky.

Kan fibrations, a key construct in this model, possess a lifting property analogous to the homotopy lifting property in classical homotopy theory. However, the initial Kan fibration model is non-constructive, as closure under pushforward of Kan fibrations is unprovable [2].

To develop constructive models, cubical sets were proposed as an alternative to simplicial sets. Nevertheless, the original Kan fibration concept is still appealing, and there have been endeavors to create a constructive version that maintains Voevodsky's ideas. Gambino and Sattler proposed a model treating Kan fibrations as structure rather than property [6]. Then, to overcome difficulties in this approach, van den Berg and Faber introduced effective Kan fibrations [2], which is the primary focus of our paper.

Verification of effective Kan fibrations modeling HoTT includes modeling all type formers, such as Π-types, and a primary focus of this thesis is W-types, which represent well-founded trees (e.g., natural numbers) introduced by Martin-Löf in [10]. W-types, already modeled by classical HoTT semantics using Kan fibrations [3], are examined here using effective Kan fibrations.

Prior research regarding this concept includes [5] and [7]. The latter defined a special case of effective Kan fibrations, called symmetric effective Kan fibrations. The primary difference between effective Kan fibrations and symmetric effective Kan fibrations lies in the fact that,

except for certain exceptions, each horn problem in effective Kan fibrations has two solutions – positive and negative – whereas this distinction does not exist in symmetric effective Kan fibrations. This paper focuses on effective Kan fibrations and employs a slightly modified version of the framework introduced in [7] to incorporate the distinction between positive and negative solutions.

As will be mentioned, constructing W-types categorically involves transfinite induction. To ensure the entire argument is fully constructive, we must deal with this constructively, which is currently an open problem.

1.2   Outline

The first section introduces preliminary information about simplicial sets, providing essential foundational knowledge for the subsequent sections.

Then, based on the paper [3] on W-types in HoTT in the classical setting, we will extend this study using a constructive approach. We define the categories of effective Kan fibrations and their variants. This categorical framework is crucial for describing W-types and M-types as certain kinds of colimits and limits, respectively.

Next, we explore whether effective Kan fibrations can model W-types and M-types by examining the existence and behavior of filtered colimits and limits in these categories. We then explain how W-types are generated by polynomial functors, and investigate the role of these functors as maps in the categorical structures defined in previous sections.

Finally, we proceed to the main theorem: Effective Kan fibrations can indeed model W-types and their variants, providing a comprehensive conclusion to the theoretical framework developed.

Additionally, for future work, to explore potential applications of W-types in constructive mathematics, we will examine their role in defining quotients and discuss related challenges. Specifically, we address issues involving epimorphisms by introducing additional structures on effective Kan fibrations. A potential goal of this approach is to show that such structures could enable effective Kan complexes and fibrations to model quotient types and quotient maps, respectively, in a constructive framework.

2   Effective Kan fibrations/complexes

2.1   Simplicial sets

▶ **Definition 1.** *The simplex category Δ is a category whose objects are the finite, non-empty, linearly ordered sets $[n] = \{0, \ldots, n\}$ for each $n \geq 0$, and whose morphisms $[n] \to [m]$ are nondecreasing maps.*

For each $n \geq 0$, there are two important types of morphisms in Δ. An injection $d_i \colon [n-1] \to [n]$ that omits the i-th element is called the i-th *face map*. A surjection $s_i \colon [n+1] \to [n]$ that maps two distinct elements to i is called the i-th *degeneracy map*. Specifically,

$$d_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases} \quad s_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

These maps satisfy the *simplicial identities*:

$$s_j \circ d_k = \begin{cases} d_{k-1} \circ s_j & \text{if } k > j+1, \\ \text{id} & \text{if } k \in \{j, j+1\}, \\ d_k \circ s_{j-1} & \text{if } k < j, \end{cases}$$

$d_j \circ d_k = d_{k+1} \circ d_j$ (if $k \geq j$), and $s_j \circ s_k = s_k \circ s_{j+1}$ (if $j \geq k$).

▶ **Definition 2.** *A contravariant functor* $\Delta \to \mathbf{Set}$, *namely a presheaf on* Δ, *is called a simplicial set. The category of simplicial sets is denoted by* $\mathbf{sSet}$.

Given a simplicial set X, we write $X_n := X([n])$, making X a graded set $\{X_n\}_{n \in \mathbb{N}}$. Since X is contravariant, the face maps and degeneracy maps induce actions $X(d_i) \colon X_n \to X_{n-1}$ and $X(s_i) \colon X_n \to X_{n+1}$, respectively. For any locally small category $\mathcal{C}$ and $C \in \mathrm{ob}(\mathcal{C})$, the Yoneda lemma identifies the natural bijection $\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}}(y_C, F) \cong FC$ for every presheaf F with the representable presheaf $y_C = \mathrm{Hom}_{\mathcal{C}}(-, C)$. Setting $\mathcal{C} = \Delta$, we have:

▶ **Definition 3.** *In* $\mathbf{sSet}$, *for each* $[n]$, *the representable functor is called the* n*-standard simplex and denoted by* Δ^n *instead of* $y_{[n]}$, *i.e.,* $\Delta^n([m]) = \mathrm{Hom}_{\Delta}([m], [n])$.

A face map $d_k \in \mathrm{Hom}_{\Delta}([n-1], [n])$. This corresponds to a face map $d_k \colon \Delta^{n-1} \to \Delta^n$ in $\mathbf{sSet}$. Similarly, we have degeneracy maps in $\mathbf{sSet}$.

▶ **Definition 4.** *The boundary* $\partial\Delta^n$ *of* Δ^n *is the union of all its* $(n-1)$*-dimensional faces:*

$$\partial\Delta^n = \bigcup_{i=0}^{n} d_i(\Delta^{n-1}).$$

▶ **Definition 5.** *Let* $n \geq 1$ *and* $0 \leq m \leq n$. *The* m*-th horn of* Δ^n *is the simplicial subset:*

$$\Lambda_m^n = \bigcup_{\substack{0 \leq i \leq n \\ i \neq m}} d_i(\Delta^{n-1})$$

If $0 < m < n$, Λ_m^n *is called an inner horn, and if* $m \in \{0, n\}$, *it is called an outer horn.*

By the density theorem, every presheaf is a colimit of representable presheaves. That is, a horn Λ_m^n is a colimit of standard simplices. This is expressed as follows [8]:

▶ **Lemma 6.** *A horn* Λ_m^n *can be expressed by the following coequalizer in the following commutative "fork".*

Besides this lemma, we will also see another perspective of constructing horns in the next section. The following lemma will be used frequently, so we state it here for reference. The proof follows from the universal property of the coequalizer and coproduct in **Lemma 6**.

▶ **Lemma 7.** *In* $\mathbf{sSet}$, *given* $f, g \colon \Lambda_m^n \to S$, *if* $f \circ d_k = g \circ d_k$ *for all* k *with* $0 \leq k \leq n$ *and* $k \neq m$, *then* $f = g$, *i.e.,* $(d_k)_{k \neq m}$ *are jointly epic.*

2.2   Preliminaries on effective Kan fibrations/complexes

Now, let us see an alternative construction of a horn described in [7].

▶ **Construction 8.** *Let $\epsilon \in \{0,1\}$. In the commutative squares (1) each of which is a pullback square, a horn $\Lambda^{n+1}_{i+1-\epsilon}$ is alternatively constructed as the pushout of the left square.*

$$
\begin{array}{ccccc}
\partial\Delta^n & \longrightarrow & s_i^*(\partial\Delta^n) & \longrightarrow & \partial\Delta^n \\
\big\uparrow & \lrcorner & \big\uparrow & \lrcorner & \big\downarrow \\
\Delta^n & \xrightarrow{d_{i+\epsilon}} & \Delta^{n+1} & \xrightarrow{s_i} & \Delta^n
\end{array}
\tag{1}
$$

Notice that except for the edge cases ($m = 0$ or $m = n + 1$), there are two ways to obtain Λ^{n+1}_m, since two pairs $(i, \epsilon) = (m - 1, 0)$ and $(m, 1)$ satisfy $i + 1 - \epsilon = m$ for $0 < m < n + 1$. At the edge cases, however, only one pair satisfies the condition: $(i, \epsilon) = (0, 1)$ when $m = 0$ and $(i, \epsilon) = (n, 0)$ when $m = n + 1$. Thus Λ^{n+1}_m for $0 < m < n + 1$ is created by d_{m-1} (when $\epsilon = 0$) or d_{m+1} (when $\epsilon = 1$) in (1). We call a horn *negative* if $\epsilon = 0$ and *positive* if $\epsilon = 1$, and denote it by $(\Lambda^{n+1}_m, \pm)$. Note that at the edges, $(\Lambda^{n+1}_0, +)$ and $(\Lambda^{n+1}_{n+1}, -)$ do not exist.

▶ **Definition 9.** *In **sSet**, let p be a Kan fibration, i.e., for every horn inclusion $i\colon (\Lambda^n_m, \pm) \to \Delta^n$, every pair of maps x and y with $px = yi$ as in the right square in (4), there exists a lift. For each such p, we can define a function lift_p that assigns a lift $\mathsf{lift}_p(x, y)$ for each lifting problem which is represented as a pair (x, y).*

As mentioned in the introduction, this paper uses a framework from [7], which is described below, with a modification to distinguish between positive and negative solutions discussed above.

▶ **Definition 10.** *Let p be a Kan fibration equipped with lift_p. For every lifting problem (x, y) and $s_j\colon \Delta^{n+1} \to \Delta^n$ with $0 \le j \le n$ as in (4), a horn map $s_j^*(x)\colon (\Lambda^{n+1}_{m^*}, \pm) \to X$, where m^* is as given in (2), is defined by (3).*

$$
m^* \in \begin{cases} \{m\} & \text{if } m < j \\ \{m, m+1\} & \text{if } m = j \\ \{m+1\} & \text{if } m > j, \end{cases} \quad (2) \qquad s_j^*(x) \circ d_k = \begin{cases} x \circ s_j \circ d_k & \text{if } k \neq j, j+1, m^* \\ \mathsf{lift}_p(x, y) & \text{if } \begin{array}{l} k \in \{j, j+1\} \\ \text{and } k \neq m^*. \end{array} \end{cases} \quad (3)
$$

▶ **Definition 11.** *A map $p\colon X \to Y$ in **sSet** is an effective Kan fibration if it comes equipped with lift_p, and satisfies the compatibility (or uniformity) condition: $\mathsf{lift}_p(s_j^*(x), ys_j) = \mathsf{lift}_p(x, y) \circ s_j$ for every $0 \le j \le n$, m^*, and $s_j^*(x)$. In (4), two horns have the same $\pm$ signs.*

$$
\begin{array}{ccccc}
& & \xrightarrow{\quad s_j^*(x) \quad} & & \\
\Lambda^{n+1}_{m^*} & \longrightarrow & \Lambda^n_m \xrightarrow{\ x\ } & & X \\
i' \big\downarrow & \underset{\mathsf{lift}_p(s_j^*(x), ys_j)}{\nearrow} & \big\uparrow i \quad \underset{\mathsf{lift}_p(x,y)}{\nearrow} & & \big\downarrow p \\
\Delta^{n+1} & \xrightarrow{\ s_j\ } & \Delta^n & \xrightarrow{\ y\ } & Y
\end{array}
\tag{4}
$$

When $Y = \{*\}$, the one-point simplicial set, the lower triangle is trivial and X is called an effective Kan complex.

The $\pm$ distinction is important for bookkeeping since mere existence of a lift is not sufficient anymore in the effective setting. However, for simplicity, we will mostly drop $+$ and $-$ signs from horns throughout the rest of this paper.

▶ Remark 12. Formally, an effective Kan fibration is a pair (p, lift_p). Also, in the case of an effective Kan complex, the fibration is the unique map $!_X \colon X \to \{*\}$ but we write it (X, lift_X) instead of $(!_X, \mathsf{lift}_{!_X})$. Also, we just denote $\mathsf{lift}_X(x)$ instead of $\mathsf{lift}_{!_X}(x, !_{\Delta^n})$.

By a slight abuse of notation, we sometimes refer to X in the pair (X, lift_X) simply as an effective Kan complex, and we say that X is in **EffKanCplx**. Similarly, the same convention applies in the case of fibrations.

▶ **Definition 13. EffKanFib** *is the category of effective Kan fibrations whose objects and morphisms are as follows.*

> **Objects:** *Effective Kan fibrations (p, lift_p).*
> **Morphisms:** *A morphism $(\varphi, \psi) \colon p \to q$ is a pair of maps $\varphi \colon X \to Y$ and $\psi \colon A \to B$ such that $q \circ \varphi = \psi \circ p$, and for an arbitrary horn $(\Lambda^n_m, \pm)$, a map $x \colon (\Lambda^n_m, \pm) \to X$ and a map $a \colon \Delta^n \to A$ such that $px = a\iota$, each triangle of the following diagram commutes. In particular, $\varphi \circ \mathsf{lift}_p(x, a) = \mathsf{lift}_q(\varphi x, \psi a)$.*

$$
\begin{array}{ccccc}
\Lambda^n_m & \xrightarrow{\ x\ } & X & \xrightarrow{\ \varphi\ } & Y \\
{\scriptstyle \iota}\Big\downarrow & {\scriptstyle \mathsf{lift}_p(x,a)} \nearrow & \Big\downarrow {\scriptstyle p} & {\scriptstyle \mathsf{lift}_q(\varphi x, \psi a)} \nearrow & \Big\downarrow {\scriptstyle q} \\
\Delta^n & \xrightarrow{\ a\ } & A & \xrightarrow{\ \psi\ } & B
\end{array}
$$

Note that the classical Kan fibrations are maps in **sSet**. This **EffKanFib** is further endowed with additional structures.

▶ **Definition 14.** *The category $\mathbf{EffKan}/A$ of effective Kan fibrations whose codomain is a simplicial set A is defined in a similar way to* **Definition 13**. *In that definition, let $B := A$ and $\psi := 1_A$.*

▶ **Definition 15.** *The category* **EffKanCplx** *of effective Kan complexes is defined as* $\mathbf{EffKan}/1$.

As mentioned in **Remark 12**, we just regard the objects in **EffKanCplx** as (X, lift_X) instead of $(!_X, \mathsf{lift}_{!_X})$.

2.3 Properties of effective Kan fibrations/complexes

In this section, we will examine filtered colimits and limits in the categories in the previous section, as they play important roles in the construction of W-types and the variants later. Recall that a *filtered diagram* $F \colon \mathcal{I} \to \mathcal{C}$ satisfies that (1) $\mathcal{I}$ has at least one object, (2) for each pair $i, j \in \mathcal{I}$, there is $k \in \mathcal{I}$ with a pair of morphisms $i \to k$ and $j \to k$ of $\mathcal{I}$, (3) for each pair $i, j \in \mathcal{I}$ and each pair $a, b \colon i \to j$, there is a morphism $c \colon j \to k$ of $\mathcal{I}$ such that $F(c \circ a) = F(c \circ b)$ in $\mathcal{C}$. And a *filtered colimit* is the colimit of a filtered diagram. Also, in **Set**, filtered colimits can be explicitly calculated: For a filtered diagram $F \colon \mathcal{I} \to \mathbf{Set}$, we have $\mathrm{colim}_{i \in \mathcal{I}} F_i = \left(\coprod_{i \in \mathcal{I}} F_i \right) / \sim$, where $x_i \sim x_j$ if and only if there exist $k \in \mathcal{I}, \varphi \colon i \to k$ and $\psi \colon j \to k$ such that $F(\varphi)(x_i) = F(\psi)(x_j)$, for $i, j \in I$, $x_i \in F_i$ and $x_j \in F_j$.

Applying the above calculation to hom-sets, we can obtain useful facts for **sSet**. First, more generally, recall that we have $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n$ by the Yoneda lemma, and colimits in **sSet** are computed degree-wise. Thus, every evaluation functor $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, -) \colon \mathbf{sSet} \to \mathbf{Set}$ preserves all small colimits, i.e., for any small diagram $F \colon \mathcal{I} \to \mathbf{sSet}$, we have

$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{colim}_{i \in \mathcal{I}} F_i) \cong \mathrm{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, F_i)$. Moreover, suppose F is also filtered. A horn Λ_m^n is a finite colimit of standard n-simplices, namely finitely presentable, and thus the functor $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda_m^n, -)\colon \mathbf{sSet} \to \mathbf{Set}$ preserves all filtered colimits. In particular, we have the following.

▶ **Lemma 16.** *Let* $F\colon \mathcal{I} \to \mathbf{sSet}$ *be a filtered diagram with filtered colimit* X. *Denote* $F(i) = X_i$ *for* $i \in \mathcal{I}$ *and write* $\beta_i\colon X_i \to X$ *for the coprojections. For each* $n \geq 1$ *and each* $0 \leq m \leq n$, *every morphism* $x\colon \Lambda_m^n \longrightarrow X$ *into the colimit factors through a coprojection: there exists* $i \in \mathcal{I}$ *and a map* $x_i\colon \Lambda_m^n \to X_i$ *such that* $x = \beta_i \circ x_i$.

▶ **Lemma 17.** *With the same notation, suppose a map* $x\colon \Lambda_m^n \to X$ *admits two factorizations* $x = \beta_i \circ x_i$ *and* $x = \beta_j \circ x_j$ *through* X_i *and* X_j. *Then there exists* $k \in \mathcal{I}$ *and morphisms* $f\colon i \to k$ *and* $g\colon j \to k$ *in* $\mathcal{I}$ *such that* $x_k = F(f) \circ x_i = F(g) \circ x_j$ *and* $x = \beta_k \circ x_k$.

Let us rewrite the maps of the form $F(f)$ or $F(g)$ in the **Lemma 17** as φ_{ik} or φ_{jk}, respectively. Also, let us call them *mediating maps*.

The next proposition is an important tool for our main theorem. Similar results hold for **EffKan**$/A$ and for **EffKanFib** in place of **EffKanCplx**. While the result for **EffKanFib** would immediately imply the results for **EffKanCplx** and **EffKan**$/A$, since the core idea behind the proofs of all three versions is essentially the same, we choose to focus on the simplest case for **EffKanCplx**.

▶ **Proposition 18.** *The forgetful functor* **EffKanCplx** $\to$ **sSet** *creates small filtered colimits.*

Proof. Let $F\colon \mathcal{I} \to \mathbf{EffKanCplx}$ be a small filtered diagram. Let $F(i) := (X_i, \mathrm{lift}_{X_i})$. Let $U\colon \mathbf{EffKanCplx} \to \mathbf{sSet}$ denote the forgetful functor that sends $(X_i, \mathrm{lift}_{X_i})$ to X_i. Suppose X is the filtered colimit of $(X_i\colon i \in \mathcal{I})$ in **sSet** (under $U \circ F$). Our goal is to equip X with a horn-filler operation lift_X, thereby obtaining an effective Kan complex (X, lift_X) and showing that it is a filtered colimit in **EffKanCplx**.

Fix an arbitrary horn Λ_m^n and a map $x\colon \Lambda_m^n \to X$. First, let us define $\mathrm{lift}_X(x)$ for x. By **Lemma 16**, we observed that x factors through some X_i with the coprojection β_i. We define

$$\mathrm{lift}_X(x) := \beta_i \circ \mathrm{lift}_{X_i}(x_i). \tag{5}$$

As discussed earlier, the factorization is not unique. However, this is well-defined, since we can show that it does not depend on the particular choice. For, suppose there are two different factorizations via X_i and X_j. By **Lemma 17**, there is X_k with $x_k = x_i \circ \varphi_{ik} = x_j \circ \varphi_{jk}$ as follows:
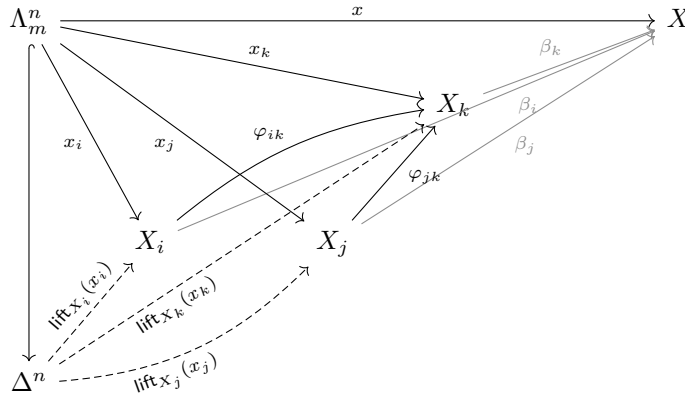
Then, since φ_{ik} and φ_{jk} are morphisms in **EffKanCplx**, we have $\varphi_{ik} \circ \mathsf{lift}_{X_i}(x_i) = \mathsf{lift}_{X_k}(x_k) = \varphi_{jk} \circ \mathsf{lift}_{X_j}(x_j)$. From this, we can observe that $\beta_i \circ \mathsf{lift}_{X_i}(x_i) = \beta_k \circ \varphi_{ik} \circ \mathsf{lift}_{X_i}(x_i) = \beta_k \circ \varphi_{jk} \circ \mathsf{lift}_{X_j}(x_j) = \beta_j \circ \mathsf{lift}_{X_j}(x_j)$. That is, the definition of $\mathsf{lift}_X(x)$ is not dependent on the particular factorization.

To verify the compatibility condition, we pull back the horn along s_p for any $0 \le p \le n$, and we also consider $s_p^*(x) \colon \Lambda_{m^*}^{n+1} \to X$ defined in **Definition 10**, and we will show $\mathsf{lift}_X(s_p^*(x)) = \mathsf{lift}_X(x) \circ s_p$ as in **Definition 11**. For any $i \in \mathcal{I}$, consider the map $s_p^*(x_i)$ of x_i along s_p as shown in the diagram below. We claim that $s_p^*(x)$ factors through X_i with $s_p^*(x) = \beta_i \circ s_p^*(x_i)$. To see this, it suffices to show that $s_p^*(x) \circ d_q = \beta_i \circ s_p^*(x_i) \circ d_q$ for all $q \in [n] \setminus \{m^*\}$ by **Lemma 7**. By **Definition 10**,

$$s_p^*(x) \circ d_q = \begin{cases} x \circ s_p \circ d_q & \text{if } q \ne p, p+1, m^* \\ \mathsf{lift}_X(x) & \text{if } q \in \{p, p+1\} \setminus \{m^*\}, \end{cases}$$

and also

$$\beta_i \circ s_p^*(x_i) \circ d_q = \begin{cases} \beta_i \circ x_i \circ s_p \circ d_q & \text{if } q \ne p, p+1, m^* \\ \beta_i \circ \mathsf{lift}_{X_i}(x_i) & \text{if } q \in \{p, p+1\} \setminus \{m^*\}. \end{cases}$$

Now, comparing both cases for q, we see that $x \circ s_p \circ d_q = \beta_i \circ x_i \circ s_p \circ d_q$ for the first case, since $x = \beta_i \circ x_i$. Also, $\mathsf{lift}_X(x) = \beta_i \circ \mathsf{lift}_{X_i}(x_i)$ for the second case by the definition of $\mathsf{lift}_X(x)$ provided earlier in (5). Hence, this proves the claim about the factorization.
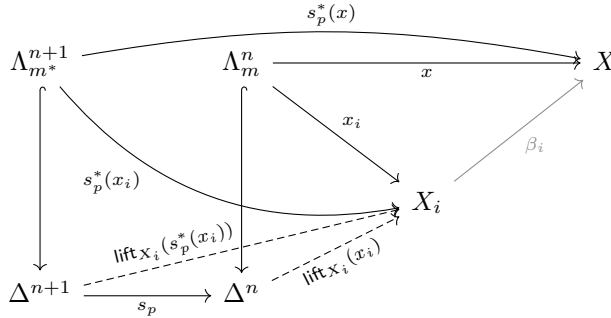
Due to this factorization, we can use (5) to define $\mathsf{lift}_X(s_p^*(x)) = \beta_i \circ \mathsf{lift}_{X_i}(s_p^*(x_i))$ independently of i. Then, using the compatibility for X_i, we have $\mathsf{lift}_X(s_p^*(x)) = \beta_i \circ \mathsf{lift}_{X_i}(s_p^*(x_i)) = \beta_i \circ \mathsf{lift}_{X_i}((x_i) \circ s_p) = \mathsf{lift}_X(x) \circ s_p$, which is the compatibility for X, and thus (X, lift_X) is an effective Kan complex.

Next, we check the coprojections are maps in **EffKanCplx**. For each β_i and each $\overline{x_i} \colon \Lambda_m^n \to X_i$, we have $\mathsf{lift}_X(\beta_i \circ \overline{x_i}) = \beta_i \circ \mathsf{lift}_{X_i} \bar{x}_i$ by definition, meaning exactly that β_i is in **EffKanCplx**.

Finally, we show that the cocone $\big((X, \mathsf{lift}_X), \{\beta_i \colon X_i \to X\}_{i \in I}\big)$ is colimiting. Let $\big((Y, \mathsf{lift}_Y), \{\gamma_i \colon X_i \to Y\}_{i \in I}\big)$ be any cocone of the morphisms in **EffKanCplx** over F. Forgetting the data regarding the lifting structure, the maps γ_i form a cocone of simplicial-set maps over $U \circ F$. Since $X = \mathrm{colim}_{i \in I} X_i$ in **sSet**, the universal property of the colimit gives a unique map $\theta \colon X \to Y$ such that $\theta \circ \beta_i = \gamma_i$ for all $i \in \mathcal{I}$. Then θ preserves lifts as follows: $\theta \circ \mathsf{lift}_X(x) = \theta \circ \beta_i \circ \mathsf{lift}_{X_i}(x_i) = \gamma_i \circ \mathsf{lift}_{X_i}(x_i) = \mathsf{lift}_Y(\gamma_i \circ x_i) = \mathsf{lift}_Y(\theta \circ \beta_i \circ x_i) = \mathsf{lift}_Y(\theta \circ x)$. Hence θ is a morphism in **EffKanCplx**. For the uniqueness, suppose $\psi \colon X \to Y$ is another morphism with $\psi \circ \beta_i = \gamma_i$ for all $i \in \mathcal{I}$. But then $U(\psi) = U(\theta)$ implies $\psi = \theta$ by the faithfulness of U.

Consequently the cocone $\big((X, \mathsf{lift}_X), \{\beta_i\}_{i \in I}\big)$ is colimiting in **EffKanCplx**, and thus (X, lift_X) is a filtered colimit in **EffKanCplx**.   ◄

▶ Remark 19. As mentioned before, using analogous reasoning, this proposition can be extended to the category **EffKan**$/A$ easily, and further generalized to **EffKanFib**, provided that similar assumptions hold for the mediating maps in both cases, although we only require the cases for **EffKanCplx** and **EffKan**$/A$ in this paper.

Let us also look at limits.

▶ **Proposition 20.** *The forgetful functor* **EffKanCplx** → **sSet** *creates small limits.*

Proof. Let $F \colon \mathcal{I} \to$ **EffKanCplx** be a diagram for a small category $\mathcal{I}$. Denote $F(i) = (X_i, \mathsf{lift}_{X_i})$. Using the forgetful functor $U \colon$ **EffKanCplx** → **sSet** that sends $(X_i, \mathsf{lift}_{X_i})$ to X_i, we want to show that if X is the limit of $(X_i \colon i \in \mathcal{I})$ in **sSet** (under $U \circ F$), and each X_i has the structure of an effective Kan complex $(X_i, \mathsf{lift}_{X_i})$, then so does X.

For each $i \in \mathcal{I}$, let $\pi_i \colon X \to X_i$ denote the projection. Fix an arbitrary horn problem $x \colon \Lambda_m^n \to X$. For each X_i, the composition $\pi_i \circ x \colon \Lambda_m^n \to X_i$ admits a lift $\mathsf{lift}_{X_i}(\pi_i \circ x) \colon \Delta^n \to X_i$. Now, since each $\varphi_{ii'} \colon X_i \to X_i'$ is a morphism in **EffKanCplx**, we have $\varphi_{ii'} \circ \mathsf{lift}_{X_i}(\pi_i \circ x) = \mathsf{lift}_{X_i'}(\varphi_{ii'} \circ \pi_i \circ x) = \mathsf{lift}_{X_i'}(\pi_{i'} \circ x)$ for all $i, i' \in \mathcal{I}$. Hence, the collection of maps $\mathsf{lift}_{X_k}(\pi_k \circ x)$ over $k \in \mathcal{I}$ forms a cone over the diagram F. By the universal property of X, there exists the unique map $\langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}} \colon \Delta^n \to X$ such that $\pi_j \circ \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}} = \mathsf{lift}_{X_j}(\pi_j \circ x)$ for all $j \in \mathcal{I}$.
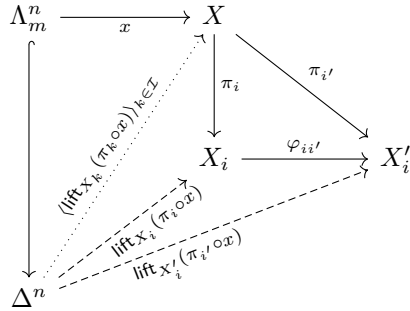
We claim that $\mathsf{lift}_X(x) := \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}}$ gives the desired structure. We need to show its compatibility. Consider $\Lambda_{m^*}^{n+1}$ and an arbitrary s_p. We need to show $\mathsf{lift}_X(s_p^*(x)) = \mathsf{lift}_X(x) \circ s_p$, i.e., using the definition of lift_X that we have just given, we need to show $\langle \mathsf{lift}_{X_k}(\pi_k \circ s_p^*(x)) \rangle_{k \in \mathcal{I}} = \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}} \circ s_p$.
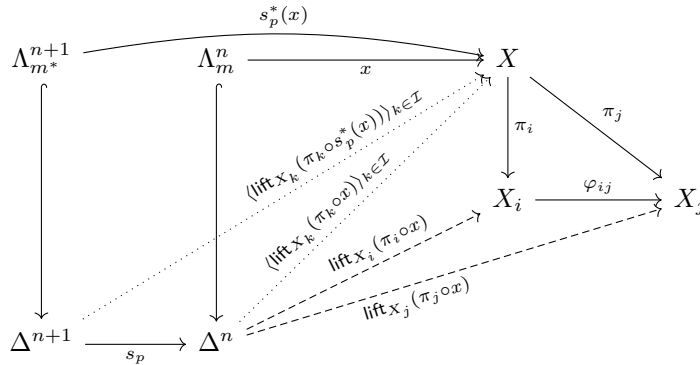
First, we claim $s_p^*(\pi_j \circ x) = \pi_j \circ s_p^*(x)$ for each $j \in \mathcal{I}$. As before, we examine all possible faces d_q to see $s_p^*(\pi_j \circ x) \circ d_q = \pi_j \circ s_p^*(x) \circ d_q$.

$$s_p^*(\pi_j \circ x) \circ d_q = \begin{cases} \pi_j \circ x \circ s_p \circ d_q & \text{if } q \neq p, p+1, m^* \\ \mathsf{lift}_X(\pi_j \circ x) & \text{if } q \in \{p, p+1\} \setminus \{m^*\}. \end{cases}$$

$$\pi_j \circ s_p^*(x) \circ d_q = \begin{cases} \pi_j \circ x \circ s_p \circ d_q & \text{if } q \neq p, p+1, m^* \\ \pi_j \circ \mathsf{lift}_{X_j}(x) & \text{if } q \in \{p, p+1\} \setminus \{m^*\}. \end{cases}$$

In the first case they coincide. In the second case, using lift_X defined earlier, we have $\pi_j \circ \mathsf{lift}_X(x) = \pi_j \circ \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in I} = \mathsf{lift}_X(\pi_j \circ x)$. Thus, both sides are equal for all possible faces and the claim follows.

Now, because of this claim, our goal is to show $\langle \mathsf{lift}_{X_k}(s_p^*(\pi_j \circ x)) \rangle_{k \in \mathcal{I}} = \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}} \circ s_p$. By the compatibility condition for X_i, we have $\mathsf{lift}_{X_i}(s_p^*(\pi_i \circ x)) = \mathsf{lift}_{X_i}(\pi_i \circ x) \circ s_p$. Then by the universal property of X, we see that $\langle \mathsf{lift}_{X_k}(s_p^*(\pi_j \circ x)) \rangle_{k \in \mathcal{I}} = \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in \mathcal{I}} \circ s_p$, as desired. Thus, (X, lift_X) is an effective Kan complex.

It is immediate that the projections are morphisms in **EffKanCplx**, as $\pi_i \circ \mathsf{lift}_X(x) = \pi_i \circ \langle \mathsf{lift}_{X_k}(\pi_k \circ x) \rangle_{k \in I} = \mathsf{lift}_{X_i}(\pi_i \circ x)$ by the definition of our lift lift_X.

Let $\big((Y, \mathsf{lift}_Y), \{\tau_i : Y \to X_i\}_{i \in I}\big)$ be any other cone in **EffKanCplx** over F. Forgetting fillers, the maps $\{\tau_i\}$ form a cone in **sSet** over $U \circ F$, so the universal property of the limit gives a unique map $\theta : Y \to X$ such that $\pi_i \circ \theta = \tau_i$ for all $i \in \mathcal{I}$. To show that θ preserves lifts, we have $\pi_i \circ \theta \circ \mathsf{lift}_Y(x) = \tau_i \circ \mathsf{lift}_Y(x) = \mathsf{lift}_{X_i}(\tau_i \circ x) = \mathsf{lift}_{X_i}(\pi_i \circ \theta \circ x) = \pi_i \circ \mathsf{lift}_X(\theta \circ x)$. Since the projections are jointly monic in **sSet**, we have $\theta \circ \mathsf{lift}_Y(x) = \mathsf{lift}_X(\theta \circ x)$. Thus, θ is a morphism in **EffKanCplx**. For the uniqueness, if $\psi : Y \to X$ is another morphism in **EffKanCplx** with $\pi_i \circ \psi = \tau_i$, then $U(\psi) = U(\theta)$ by the uniqueness of the limit in **sSet**, and by the faithfulness of U, we have $\psi = \theta$. Therefore, the cone $\big((X, \mathsf{lift}_X), \{\pi_i\}_{i \in I}\big)$ is limiting in **EffKanCplx**, and thus (X, lift_X) is a limit in **EffKanCplx**. ◄

3 W-types and variations

We introduce the categorical definition of W-types to show that effective Kan complexes model them. While our primary goal is this categorical perspective, a brief note about the type-theoretic development is as follows: W-types were introduced by Martin-Löf in the late 1970s in [10], and most recently their type-theoretic aspects are summarized in "the HoTT book" [12]. These types generalize structures such as natural numbers, lists, and binary trees, encapsulating the recursive properties of inductive types.

3.1 W-types

We summarize the categorical formulation of W-types as described in [3]. For an endofunctor $F : \mathcal{E} \to \mathcal{E}$, an *algebra* is a pair (X, α) with $\alpha : FX \to X$, and morphisms $\varphi : (X, \alpha) \to (Y, \beta)$ satisfy $\varphi \circ \alpha = \beta \circ F\varphi$. The *initial algebra* is the initial object in this category of F-algebras, if it exists.

In a locally cartesian closed category $\mathcal{E}$, for every $f : A \to B$, the pullback functor $f^* : \mathcal{E}/B \to \mathcal{E}/A$ has left and right adjoints, Σ_f and Π_f, forming $\Sigma_f \dashv f^* \dashv \Pi_f$. In particular, for the unique morphism $! : A \to 1$, the adjunction $\Sigma_! \dashv !^* \dashv \Pi_!$ is written as $\Sigma_A \dashv A^* \dashv \Pi_A$. Note that A^* is also written as $- \times A$.

▶ **Definition 21.** *In a locally cartesian closed category $\mathcal{E}$, given $f\colon A \to B$, the polynomial functor P_f is the composite $\Sigma_A \circ \Pi_f \circ B^*$.*

$$P_f\colon \mathcal{E} \xrightarrow{B^*} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}, \tag{6}$$

The W-type $W(f)$ is the initial algebra of P_f, if it exists.

To describe W-types in **sSet**, it suffices to describe them in presheaves.

3.1.1   W-types in Set

Given a polynomial functor $P_f(X) = \sum_{a \in A} X^{B_a}$, where $B_a = f^{-1}(a)$, a W-type consists of labeled, well-founded trees with edges directed towards the root. Each $a \in A$ is a node, and $b \in B_a$ is an edge leading to the node a. Each $\ell \in A$ with $B_\ell = \varnothing$ is the end of a branch (namely, a leaf node) in the tree. Thus, for example, a branch would look like $\ell \xrightarrow{b} a \xrightarrow{b'} a' \xrightarrow{b} a \cdots \to r$ with the root r being some non-leaf node.

For a P_f-algebra (X, μ) with structure map $\mu\colon P_f(X) \to X$, restricting μ to the a-th summand induces the component map $\mu_a\colon X^{B_a} \to X$. With this in mind, we see how an important structure map called sup is obtained. The collection $W(f)$ of well-founded trees is inductively constructed as follows: Each tree is of the form $\sup_a(t)$, where $a \in A$ and $t\colon B_a \to W(f)$ assigns subtrees to branches labeled by B_a. The base case includes trees with no branches ($B_a = \emptyset$). Larger trees are constructed by attaching $t(b) \in W(f)$ for each $b \in B_a$. The map $\sup\colon P_f(W(f)) \to W(f)$ given by $\sup_a\colon W(f)^{B_a} \to W(f)$ equips $W(f)$ with a P_f-algebra structure. Then, inductively we can see that for any other P_f-algebra (X, μ), there is a unique map $\varphi\colon W(f) \to X$ with $\varphi(\sup_a(t)) = \mu_a(\varphi \circ t)$ for any $a \in A$ and $t \in W(f)^{B_a}$, which shows that $W(f)$ is the W-type associated to f. Concretely, the rank function $\mathrm{rk}\colon W(f) \to Ord$ assigns ordinals to trees based on their well-foundedness: $\mathrm{rk}(\sup_a(t)) = \sup\{\mathrm{rk}(t(b)) + 1 \mid b \in B_a\}$. Furthermore, we define $W(f)_{<\alpha} := \{w \in W(f) : \mathrm{rk}(w) < \alpha\}$. Then, $W(f)_{<0} = \emptyset$, $W(f)_{<\alpha+1} \cong P_f(W(f)_{<\alpha})$, and $W(f)_{<\lambda} = \mathrm{colim}_{\alpha<\lambda} W(f)_{<\alpha}$, for limit ordinals λ. By transfinite induction on $w \in W(f)$, we can show that $\mathrm{rk}(w) < \kappa$ for some sufficiently large regular cardinal κ, implying $W(f) = W(f)_{<\kappa}$. This transfinite chain of sets:

$$0 \xrightarrow{!} P_f(0) \xrightarrow{P_f(!)} P_f(P_f(0)) \xrightarrow{P_f(P_f(!))} P_f(P_f(P_f(0))) \xrightarrow{P_f(P_f(P_f(!)))} \cdots \tag{7}$$

converges to $W(f)$. Here, 0 denotes the initial object, which is $\varnothing$ in **Set**. Note that $P_f^\alpha(0) = W(f)_{<\alpha}$.

3.1.2   W-types in Set$^{\mathcal{C}^{\mathrm{op}}}$

We show that the category of presheaves **Set**$^{\mathcal{C}^{\mathrm{op}}}$, where $\mathcal{C}$ is a small category, has all W-types. Given a map $f\colon B \to A$ in **Set**$^{\mathcal{C}^{\mathrm{op}}}$, define $\hat{A} = \{(C, a) \mid C \in \mathcal{C},\ a \in A(C)\}$ and $\hat{B}_{(C,a)} = \{(\alpha\colon D \to C,\ b \in B(D)) \mid f_D(b) = \alpha^*(a)\}$, where $\alpha^*(a) := A\alpha(a)$. Let $\hat{f}\colon \sum_{(C,a) \in \hat{A}} \hat{B}_{(C,a)} \to \hat{A}$ be the projection. Since $\hat{A}$ and $\hat{B}_{(C,a)}$ are sets, we consider $W(\hat{f})$. Since $(W(\hat{f}), \sup)$ is the initial $P_{\hat{f}}$-algebra, by Lambek's lemma, the structure map $\sup\colon P_{\hat{f}} W(\hat{f}) \xrightarrow{\cong} W(\hat{f})$ is an isomorphism, so every $w \in W(\hat{f})$ is uniquely of the form $\sup_{(C,a)}(t)$ with $t \in W(\hat{f})^{\hat{B}_{(C,a)}}$. We give $W(\hat{f})$ a presheaf structure over $\mathcal{C}$. On objects, $W(\hat{f})(C)$ consists of trees rooted at C. On morphisms, for $\alpha\colon D \to C$, define $\alpha^*(\sup_{(C,a)}(t)) = \sup_{(D,\alpha^*(a))}(\alpha^*t)$, where α^*t is defined by $(\alpha^*t)(\beta, b) = t(\alpha\beta, b)$. Then, we can see that this is functorial, so $W(\hat{f})$ is a presheaf. Assign ranks to elements by $\mathrm{rk}(\sup_{(C,a)}(t)) = \sup\{\mathrm{rk}(t(\beta, b)) + 1 \mid (\beta, b) \in \hat{B}_{(C,a)}\}$. Define subpresheaves as $W(\hat{f})_{<\alpha} = \{w \in W(\hat{f}) \mid \mathrm{rk}(w) < \alpha\}$. To obtain $W(f)$, select hereditarily natural trees: A tree

$\sup_{(C,a)}(t)$ is *natural* if for all $(\alpha\colon D \to C,\ b) \in \hat{B}_{(C,a)}$ and $\beta\colon E \to D$, it lives in the fiber over $\mathrm{dom}(\alpha)$, and $t(\alpha\beta,\ \beta^*(b)) = \beta^*(t(\alpha, b))$. A tree is *hereditarily natural* if all its subtrees are natural. These hereditarily natural trees form a subpresheaf $W(f) \subseteq W(\hat{f})$, and $W(f)_{<\alpha} := W(\hat{f})_{<\alpha} \cap W(f)$ is also a presheaf. Then, again, $W(f)_{<0} = 0$, $W(f)_{<\alpha+1} = P_f(W(f)_{<\alpha})$ and $W(f)_{<\lambda} = \mathrm{colim}_{\alpha<\lambda} W(f)_{<\alpha}$ (for limit λ). This chain, the same as (7), converges to $W(f)$, since $W(f) = W(f)_{<\kappa}$ for large enough κ.

Setting $\mathcal{C} = \Delta$, we view W-types $W(f)$ in **sSet** as filtered colimits, which motivated our prior investigation.

▶ **Remark 22.** The categorical construction of W-types matches their definition in HoTT [12]. Given a type A and a type family $B\colon A \to \mathcal{U}$, the inductive type W-type $\mathsf{W}_{(a:A)}B(a)$ is defined by the constructor sup and induction principle. The constructor is specified as follows: For each $a : A$ and function $f\colon B(a) \to \mathsf{W}_{(a:A)}B(a)$, there exists a term $\mathsf{sup}(a, f) : \mathsf{W}_{(a:A)}B(a)$.

3.2 Initial algebras for dependent polynomial functors

Now let us introduce the first variation. As seen above, a W-type is the initial algebra for a polynomial functor. We can generalize the notion of polynomial functors as follows:

▶ **Definition 23.** *Suppose we are given a diagram of the form*

$$
\begin{array}{ccc}
B & \xrightarrow{\ f\ } & A \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
C & & C
\end{array}
$$

in a locally cartesian closed category $\mathcal{E}$. Then this diagram determines an endofunctor on $\mathcal{E}/C$, the dependent polynomial functor

$$D_f\colon \mathcal{E}/C \xrightarrow{\ h^*\ } \mathcal{E}/B \xrightarrow{\ \Pi_f\ } \mathcal{E}/A \xrightarrow{\ \Sigma_g\ } \mathcal{E}/C.$$

In **Set**, it behaves as $D_f(X)_c = \sum_{a \in A_c} \prod_{b \in B_a} X_{h(b)}$. The initial algebra for D_f is a subset of $W(f)$, defined by the condition that an edge labeled by $b \in B$ with source node labeled by $a \in A$ satisfies $g(a) = h(b)$. The concept of rank extends naturally from W-types, and this initial algebra generalizes from **Set** to presheaves, in particular **sSet**.

3.3 M-types

As a dual to a W-type, an *M-type* is defined as the final coalgebra of a polynomial functor associated with a morphism $f : B \to A$ in a locally cartesian closed category $\mathcal{E}$.

In both **Set** and **sSet**, this final coalgebra, denoted $M(f)$, captures both well-founded and non-well-founded trees labeled according to f. The construction of $M(f)$ can be understood as the limit of the following chain:

$$\cdots \longrightarrow P_f(P_f(P_f(1))) \longrightarrow P_f(P_f(1)) \longrightarrow P_f(1) \longrightarrow 1,$$

where P_f is the polynomial functor associated with f, and 1 denotes the terminal object. Notably, this sequence stabilizes at the ordinal ω, leading to the isomorphism $M(f) \cong \lim_{n \in \mathbb{N}} P_f^n(1)$. Thus, unlike certain W-types which require transfinite recursion to generate new well-founded trees, M-types do not require it.

4 P_f on effective Kan fibrations

Before stating the main theorem, we lift each functor in the composite D_f from a functor on the slice categories of **sSet** to a functor on the categories of **EffKan**$/(-)$. In particular, we obtain the result for P_f. Recall that D_f is of the form $\Sigma_g \Pi_f h^*$. First let us look at the second component, Π_f. The result was established by van den Berg and Faber [2]:

▶ **Lemma 24.** *The functor* $\Pi_f \colon \mathbf{sSet}/B \to \mathbf{sSet}/A$ *lifts to* $\Pi_f \colon \mathbf{EffKan}/B \to \mathbf{EffKan}/A$ *with an effective Kan fibration* $f \colon B \to A$.

For the first component (the pullback functor) of D_f, we first

▶ **Lemma 25.** *Effective Kan fibrations are stable under pullbacks along any map.*

Proof. Let (p, lift_p) with $p \colon X \to A$ be an effective Kan fibration, and $a \colon B \to A$ be any simplicial map. Consider an arbitrary horn problem (x, b) against the map a^*p. Then, we have the lift $\mathsf{lift}_p(\varphi \circ x, a \circ b)$, which we denote by γ.

In the inner left commutative square below, by the universal property of $B \times_A X$, there exists a unique morphism $\langle \gamma, b \rangle$, as shown in the diagram. We claim $\mathsf{lift}_{a^*p}(x, b) := \langle \gamma, b \rangle$ is well-defined.

$$
\begin{array}{ccccc}
\Lambda_m^n & \xrightarrow{\ x\ } & B \times_A X & \xrightarrow{\ \varphi\ } & X \\
\Big\downarrow{\iota} & {\langle \gamma,b \rangle}\ \ {a^*p}\Big\downarrow & & {}\lrcorner & \Big\downarrow{p} \\
\Delta^n & \xrightarrow[\ b\ ]{} & B & \xrightarrow[\ a\ ]{} & A
\end{array}
$$

First, to see it is a lift, clearly the lower triangle commutes ($a^*p \circ \langle \gamma, b \rangle = b$). For the upper triangle, we have $\varphi \circ \langle \gamma, b \rangle \circ \iota = \gamma \circ \iota = \varphi \circ x$, and $a^*p \circ \langle \gamma, b \rangle \circ \iota = b \circ \iota = a^*p \circ x$. Then, by universal property of $B \times_A X$, we obtain $\langle \gamma, b \rangle \circ \iota = x$.

Next, we need to show this lift satisfies the compatibility condition. For every possible j and m^*, we have the following diagram.

$$
\begin{array}{cccccc}
\Lambda_{m^*}^{n+1} & \xrightarrow{s_j^*(x)} & \Lambda_m^n & \xrightarrow{\ x\ } & B \times_A X & \xrightarrow{\ \varphi\ } & X \\
\Big\downarrow & {\langle \gamma \circ s_j, b \circ s_j \rangle}\ {\iota}\Big\downarrow & {\gamma \circ s_j}\ \ {a^*p}\ {\gamma} & & {}\lrcorner & & \Big\downarrow{p} \\
\Delta^{n+1} & \xrightarrow[\ s_j\ ]{} & \Delta^n & \xrightarrow[\ b\ ]{} & B & \xrightarrow[\ a\ ]{} & A
\end{array}
$$

By the compatibility of p, $\mathsf{lift}_p(\varphi \circ s_j^*(x), a \circ b \circ s_j) = \gamma \circ s_j$. Then, by the universal property of $B \times_A X$, we obtain $\langle \gamma \circ s_j, b \circ s_j \rangle$, and $\mathsf{lift}_{a^*p}(s_j^*(x), b \circ s_j) = \langle \gamma \circ s_j, b \circ s_j \rangle$ for the same reason as above. Now, since $\varphi \circ \langle \gamma \circ s_j, b \circ s_j \rangle = \gamma \circ s_j = \varphi \circ \langle \gamma, b \rangle \circ s_j$ and $a^*p \circ \langle \gamma \circ s_j, b \circ s_j \rangle = b \circ s_j = a^*p \circ \langle \gamma, b \rangle \circ s_j$, by the universal property, we obtain $\langle \gamma \circ s_j, b \circ s_j \rangle = \langle \gamma, b \rangle \circ s_j$, which shows the compatibility of a^*p. ◀

▶ **Lemma 26.** *For a simplicial-set map* $h \colon B \to C$, *the functor* $h^* \colon \mathbf{sSet}/C \to \mathbf{sSet}/B$ *lifts to* $h^* \colon \mathbf{EffKan}/C \to \mathbf{EffKan}/B$.

Proof. First, let us see how it is defined on objects. Given an effective Kan fibration (k, lift_k) with $k\colon X \to C$, let us define $h^*\big((k, \mathsf{lift}_k)\big) := (h^*k, \mathsf{lift}_{h^*k})$, where lift_{h^*k} is described in **Lemma 25**, by extending $h^*\colon \mathbf{sSet}/C \to \mathbf{sSet}/B$. This is well-defined: The first component is inherited from the underlying simplicial sets, and the second component is uniquely determined by the universal property of the pullback.

Next, we see that it is well-defined on morphisms. Take $(k, \mathsf{lift}_k), (l, \mathsf{lift}_l) \in \mathbf{EffKan}/C$. Suppose we are given a pair of maps $\varphi\colon X \to Y$ and $1_C\colon C \to C$ which together form a morphism $(k, \mathsf{lift}_k) \to (l, \mathsf{lift}_l)$ in $\mathbf{EffKan}/C$. We need to show that a pair of maps $h^*\varphi$ and 1_B is a morphism in $\mathbf{EffKan}/B$. That is, for an arbitrary horn problem as in the following, we need to show $h^*\varphi \circ L_1 = L_2$, where $L_1 = \mathsf{lift}_{h^*k}(x, b)$ and $L_2 = \mathsf{lift}_{h^*l}(h^*\varphi \circ x, b)$.
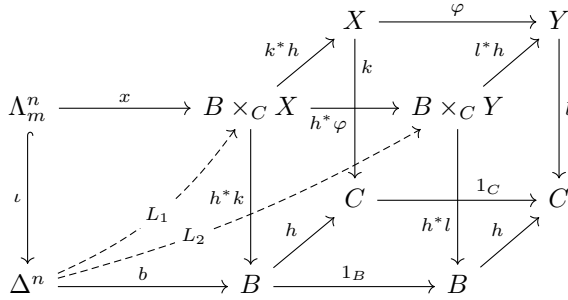
As a prism diagram, each face of this cube is commutative. We see that

$$
\begin{aligned}
l^*h \circ h^*\varphi \circ \mathsf{lift}_{h^*k}(x, b) &= \varphi \circ k^*h \circ \mathsf{lift}_{h^*k}(x, b) && \text{(top face of the cube)} \\
&= \varphi \circ \mathsf{lift}_k(k^*h \circ x, h \circ b) && \text{(by \textbf{Lemma 25})} \\
&= \mathsf{lift}_l(\varphi \circ k^*h \circ x, h \circ b) && (\varphi \text{ is a map in } \mathbf{EffKan}/C) \\
&= \mathsf{lift}_l(l^*h \circ h^*\varphi \circ x, h \circ b) && \text{(top face of the cube)} \\
&= l^*h \circ \mathsf{lift}_{h^*l}(h^*\varphi \circ x, b) && \text{(by \textbf{Lemma 25}).}
\end{aligned}
$$

Also,

$$
\begin{aligned}
h^*l \circ h^*\varphi \circ \mathsf{lift}_{h^*k}(x, b) &= h^*k \circ \mathsf{lift}_{h^*k}(x, b) && \text{(front face of the cube)} \\
&= b && \text{(by \textbf{Lemma 25})} \\
&= h^*l \circ \mathsf{lift}_{h^*l}(h^*\varphi \circ x, b). && \text{(by \textbf{Lemma 25})}
\end{aligned}
$$

Thus, by the universal property of $B \times_C Y$, we have $h^*\varphi \circ \mathsf{lift}_{h^*k}(x, b) = \mathsf{lift}_{h^*l}(h^*\varphi \circ x, b)$ as desired, and h^* preserves a morphism in $\mathbf{EffKan}/B$. It obviously preserves the identity. Also, for $\psi\colon Y \to Z$, we have $h^*\big((\varphi, 1_C) \circ (\psi, 1_C)\big) = h^*(\varphi, 1_C) \circ h^*(\psi, 1_C)$, especially because φ and ψ are morphisms in $\mathbf{EffKan}/C$, and they respect lifts. ◄

Setting $C = 1$, we have the following.

▶ **Corollary 27.** *We obtain the functor* $B^*\colon \mathbf{EffKanCplx} \to \mathbf{EffKan}/B$.

For the last component $\Sigma_{(-)}$ of D_f, we first show this lemma.

▶ **Lemma 28.** *Effective Kan fibrations are stable under composition.*

Proof. Let (p, lift_p) with $p \colon X \to A$ be an effective Kan fibration, and $g \colon A \to C$ be an effective Kan fibration. Consider an arbitrary horn problem (x, c) against the map $g \circ p$. Using lift_g we have $\mathsf{lift}_g(p \circ x, c)$. Then we obtain the lift $\mathsf{lift}_p(x, \mathsf{lift}_g(p \circ x, c))$, which we call α in the diagram.

$$
\begin{array}{ccc}
\Lambda^n_m & \xrightarrow{\ \ x\ \ } & X \\
& \nearrow^{\alpha} & \downarrow{\scriptstyle p} \\
{\scriptstyle \iota}\big\downarrow & \xrightarrow{\ \mathsf{lift}_g(p \circ x, c)\ } & A \\
& & \downarrow{\scriptstyle g} \\
\Delta^n & \xrightarrow{\ \ c\ \ } & C
\end{array}
$$

Then, this α is also a lift for $g \circ p$, since the lower triangle commutes: $g \circ p \circ \alpha = g \circ \mathsf{lift}_g(p \circ x, c) = x$. Thus, we can define $\mathsf{lift}_{g \circ p}(x, c) := \alpha$. The compatibility follows from the compatibility of lift_p and lift_q: for every possible j and m^*, we have $\mathsf{lift}_{g \circ p}(x, c) \circ s_j = \mathsf{lift}_p(x, \mathsf{lift}_g(p \circ x, c)) \circ s_j = \mathsf{lift}_p(s_j^*(x), \mathsf{lift}_g(p \circ x, c) \circ s_j) = \mathsf{lift}_p(s_j^*(x), \mathsf{lift}_g(p \circ s_j^*(x), c \circ s_j)) = \mathsf{lift}_{g \circ p}(s_j^*(x), c \circ s_j)$. The last equation is by our definition of $\mathsf{lift}_{g \circ p}$. ◀

▶ **Corollary 29.** *Given an effective Kan fibration $p \colon X \to A$, if A is an effective Kan complex, then so is X.*

Proof. By **Lemma 28**, the unique map $!_A \circ p = !_X$ is effective Kan, i.e., X is effective Kan. ◀

▶ **Lemma 30.** $\Sigma_g \colon \mathbf{EffKan}/A \to \mathbf{EffKan}/C$ *for an effective Kan fibration $g \colon A \to C$ is a functor.*

Proof. For objects in $\mathbf{EffKan}/A$, Σ_g is defined by postcomposition with g. Thus, Σ_g is well-defined on objects as seen in **Lemma 28**.

For morphisms, let $(p, \mathsf{lift}_p), (p', \mathsf{lift}_{p'}) \in \mathbf{EffKan}/A$. Suppose we have maps $\varphi \colon X \to Y$ and $1_A \colon A \to A$ such that $(\varphi, 1_A)$ is a morphism in $\mathbf{EffKan}/A$ from (p, lift_p) to $(p', \mathsf{lift}_{p'})$. We need to show that $\Sigma_g(\varphi, 1_A)$ is a morphism in $\mathbf{EffKan}/C$. This follows immediately. By construction, as established in **Lemma 28**, the lifts associated with $g \circ p$ and $g \circ p'$ are created by p and p', respectively. Also it is immediate that Σ_g preserves the identities and respects composition. ◀

Setting $C = 1$, we have the following.

▶ **Corollary 31.** *We obtain the functor $\Sigma_A \colon \mathbf{EffKan}/A \to \mathbf{EffKanCplx}$ for an effective Kan complex A.*

Due to the functoriality of each component of P_f in **Lemma 24**, **Corollary 27**, and **Corollary 31**, a polynomial functor P_f on $\mathbf{sSet}$ is extended to a functor on $\mathbf{EffKanCplx}$, preserving morphisms in $\mathbf{EffKanCplx}$ in particular.

▶ **Corollary 32.** *We have a functor $P_f \colon \mathbf{EffKanCplx} \to \mathbf{EffKanCplx}$ for an effective Kan fibration $f \colon B \to A$.*

Let us observe the following as well.

▶ **Lemma 33.** *The map $P_f(X) \to A$, where $f \colon B \to A$ is an effective Kan fibration and X and A are effective Kan complexes, is an effective Kan fibration.*

Proof. The composite $\Pi_f !_B^*$ preserves effective Kan fibrations, and $\Pi_f !_B^*(X \to 1) = P_f(X) \to A$. ◄

Then, by **Corollary 29**, the following holds.

▶ **Corollary 34.** *For an effective Kan fibration $f\colon B \to A$ between effective Kan complexes, if X is an effective Kan complex, so is $P_f(X)$.*

5 Main results

5.1 W-types

Now, we present the main theorem in this paper. However, readers are advised to refer to **Remark 36** following the proof, which offers additional considerations on the limitations of the argument presented.

▶ **Theorem 35.** *If $f\colon B \to A$ is an effective Kan fibration between effective Kan complexes, then, for any ordinal α, the map $W(f)_{<\alpha} \to A$ is also an effective Kan fibration with $W(f)_{<\alpha}$ being an effective Kan complex; in particular, $W(f) \to A$ is an effective Kan fibration with $W(f)$ being an effective Kan complex.*

Proof. We argue by transfinite induction. For the base case $\alpha = 0$, the claim vacuously holds. Also, the unique mediating map $!\colon 0 \to P_f(0)$, which is trivially a morphism in **EffKanCplx**, is carried to the morphism $\Pi_f B^*(!)$ in **EffKan**$/A$.

For a successor ordinal $\alpha + 1$, suppose $P_f^\alpha(0) \to A$ is effective Kan, and $P_f^\alpha(!)\colon P_f^\alpha(0) \to P_f^{\alpha+1}(0)$ is a morphism in **EffKanCplx**. Then, by **Lemma 33** and **Corollary 34**, $P_f(P_f^\alpha(0)) \to A$, namely $W(f)_{<\alpha+1} \to A$, is an effective Kan fibration with $W(f)_{<\alpha+1}$ being an effective Kan complex. Also $P_f^{\alpha+1}(!)$ is a morphism in **EffKanCplx** by **Corollary 32**, so the mediating map $\Pi_f B^*\big(P_f^{\alpha+1}(!)\big)$ is in **EffKan**$/A$.

For a limit ordinal α, suppose $P_f^\lambda(0) \to A$ is effective Kan for all $\lambda < \alpha$, and all the relevant mediating maps are in **EffKan**$/A$. Then, by **Remark 19**, the filtered colimit $W(f)_{<\alpha} \to A$ in **EffKan**$/A$ is also an effective Kan fibration, with $W(f)_{<\alpha}$ being an effective Kan complex.

As a result of this transfinite induction, since $W(f) = W(f)_{<\kappa}$ for a large enough κ as seen in the construction of W-types, $W(f) \to A$ is an effective Kan fibration and $W(f)$ is an effective Kan complex, as desired. ◄

▶ Remark 36. As will be discussed in the conclusion, the transfinite induction employed here relies on the classical concept of ordinals. Therefore, a future challenge is to develop a constructive approach in this context.

5.2 Initial algebras for dependent polynomial functors

▶ **Proposition 37.** *If we have a diagram*

$$
\begin{array}{ccc}
B & \xrightarrow{\ f\ } & A \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
C & & C
\end{array}
$$

of Kan fibrations in **EffKan**$/C$, *then the initial D_f-algebra is in* **EffKan**$/C$.

Proof. We saw that this endofunctor D_f is well-defined on **EffKan**$/C$ in the previous section (each component of the composite D_f preserves morphisms in its category, which is by **Lemma 24**, **26** and **30**). **EffKan**$/C$ has a filtered colimit by **Remark 19**, and it has an initial algebra which can be built as the filtered colimit of a sufficiently long chain of $D_f(0)$. Thus, this initial algebra is in **EffKan**$/C$. ◀

5.3 M-types

▶ **Theorem 38.** *If $f\colon B \to A$ is an effective Kan fibration between effective Kan complexes, then $M(f)$ is an effective Kan complex as well.*

Proof. We have $P_f(1) \cong A$, which is in **EffKanCplx** by assumption. We know $M(f) \cong \lim_{n \in \mathbb{N}} P_f^n(1)$ as mentioned in section 3.3, and P_f is an endofunctor on **EffKanCplx**. Thus, by **Proposition 20** this $M(f)$ is in **EffKanCplx**, i.e., it is an effective Kan complex. ◀

6 Future work

Constructively extending the scope of the original research on W-types [3], we focused on the modeling of W-types using Kan fibrations. As an application of W-types, the original research discussed quotients. Specifically, it addressed the universal Kan fibration $\pi\colon E \to U$, characterized by the property that any other Kan fibration can be obtained as a pullback of π. Then, it explored the application of W-types associated with π, namely $W(\pi)$. As discussed in [3], quotients are required in this application (under certain conditions, quotient types and maps are modeled by fibrant objects and fibrations, respectively). Inspired by this, our potential application would be to develop similar arguments within a constructive framework. Also, regardless of the context, considering quotient types still remain significant. However, this poses challenges, because dealing with epimorphisms in **sSet** requires selecting individual elements from their fibers, a process that depends on the axiom of choice. To avoid this, one must establish constructive rules for such selections. These rules, however, need to be carefully designed to maintain consistency across all dimensions while remaining compatible with the constructive model we adopt.

The most challenging part is to show the constructive version of the following proposition.

▶ **Proposition 39.** *In* **sSet***, suppose in a commutative triangle*

where p is an epi, and both p and g are Kan fibrations. Then f is also a Kan fibration.

The proof for this can be found in [3]. However, in a constructive setting, unless we impose additional structures on the fibrations, it is likely that this cannot even be proven for effective Kan fibrations. As mentioned above, this p needs to be equipped with some structured way to have a section. For this, [5] and [7] formulated the following version of effective Kan fibrations with additional structures:

▶ **Definition 40.** *Let (p, lift_p) be an effective Kan fibration. We say that lift_p gives the structure of a degenerate-preferring Kan fibration, if for all $n \in \mathbb{N}$, all $g \in X_n$, all possible j in s_j and m in Λ_m^{n+1} that makes the following diagram commute, the lift satisfies $\mathsf{lift}_p(gs_j\iota, pgs_j) = gs_j$.*

*We say a fibration with such lifting structure is in **D-pkf** (for the case of fibrant objects X we say X is in **D-pkf**).*

$$
\begin{array}{ccc}
\Lambda^{n+1}_m & \xrightarrow{\;gs_j\iota\;} & X \\
{\scriptstyle\iota}\Big\uparrow & {\scriptstyle g}\nearrow & \Big\downarrow{\scriptstyle p} \\
\Delta^{n+1} & \xrightarrow[s_j]{} \Delta^n \xrightarrow[pg]{} & Y
\end{array}
$$

If we give the maps in **Proposition 39 D-pkf** structure, we can prove the constructive version of this proposition [5].

▶ **Proposition 41.** *Suppose in a commutative triangle*

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;p\;\;} & Y \\
{\scriptstyle g}\searrow & & \swarrow{\scriptstyle f} \\
& A &
\end{array}
$$

*where p is an epi, and both p and g are **D-pkf**. Also, suppose p has a section $\tilde{p}_0 \colon Y_0 \to X_0$ at the vertices level. Then f is also **D-pkf**.*

Since we need more results besides **Proposition 39** for quotient types, first we need to define the following (by [7]).

▶ **Definition 42.** *Let $q \colon X \to Y$ be a map of simplicial sets. Let $\tilde{q} = (\tilde{q}_n \colon Y_n \to X_n)_{n \in \mathbb{N}}$ be a collection of functions. We call $\tilde{q}$ a degeneracy-section of q if and only if*

- *q has a right inverse $\tilde{q}$ such that $q \circ \tilde{q} = 1_Y$.*
- *for all $n \in \mathbb{N}$ and all $0 \le j \le n$, we have $\tilde{q}_{n+1}(s_j^*(y)) = s_j^*\big(\tilde{q}_n(y)\big)$.*

Here, y is a map $y \colon \Delta^n \to Y$ identified as $y \in Y_n$ by Yoneda lemma (and by abuse of notation).

Note that the second condition can be also viewed just as $\tilde{q}_{n+1}(y \circ s_j) = \tilde{q}_n(y) \circ s_j$.

▶ **Definition 43.** *If an effective Kan fibration p has this degeneraracy-section, we say that p is in **DS**.*

First of all, even with the assumption of degeneracy-sections alone, we can at least prove that quotient maps are modeled by effective Kan fibrations.

▶ **Proposition 44.** *If R is an equivalence relation on X, $q \colon X \to X/R$ is in **DS** with a degeneracy-section $\tilde{q}$ and both projections $\pi_i \colon R \to X$ $(i = 1, 2)$ are effective Kan fibrations (with structures lift_{π_1} and lift_{π_2}, respectively), then so is $X \to X/R$.*

For the proof, we can modify the proof of the classical version in [3]. This extra assumption of degeneracy-sections allows us to check the compatibility condition of effective Kan fibrations.

However, to see that the quotient type X/R is modeled by a fibrant object, we need the constructive version of **Proposition 39**, and as mentioned earlier, at least **D-pkf** proves it. Thus, let us first consider the following.

▶ **Definition 45.** *In **Definition 40**, if the **D-pkf** map p has a degeneracy-section $\tilde{p}$, we call it **D-pkf** ∩ **DS** (also we use this ∩ notation for other instances).*

Using this, we could attempt to prove the constructive version of the corollary for X/R (originally it is "Corollary 4.4." in [3]).

First recall that a pseudo-equivalence relation $(s,t) : R \to X \times X$ satisfies:

- Reflexivity: There exists $\rho : X \to R$ such that $(s,t)\rho$ is the diagonal map Δ_X.
- Symmetry: There exists $\sigma : R \to R$ with $s\sigma = t$ and $t\sigma = s$.
- Transitivity: In the pullback diagram below, there exists $\tau : P \to R$ such that $s\tau = sp_{12}$ and $t\tau = tp_{23}$.

$$
\begin{array}{ccc}
P & \xrightarrow{\ p_{12}\ } & R \\
{\scriptstyle p_{23}}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\
R & \xrightarrow[\ s\ ]{} & X
\end{array}
$$

The constructive version of "Corollary 4.4." in [3] would then be formulated as follows:

[Quotient : D-pkf $\cap$ DS] *"Suppose R is a pseudo-equivalence relation on X, the quotient map $q \colon X \to X/R$ has a degeneracy-section $\tilde{q}$, and $R \hookrightarrow X \times X$ is an effective Kan fibration. Moreover, suppose that X is in **D-pkf**. Then the quotient map $X \xrightarrow{q} X/R$ is in **D-pkf** $\cap$ **DS**, and X/R is in **D-pkf**."*

Once we can show that q is in **D-pkf** $\cap$ **DS** in this statement, the last part about X/R being in **D-pkf** is immediate by setting $A = 1$ in **Proposition 41**. However, it may not be possible to show q is in **D-pkf** (while showing that it is effective Kan is immediate by **Proposition 44**, showing that it is degenerate-preferring is not). Thus, to achieve more ideal conditions, we try to weaken the assumptions of **Proposition 41**, thereby establishing a more comprehensive result.

Then, we propose yet another version of effective Kan fibrations with additional structure as follows:

▶ **Definition 46.** *Assume (p, lift_p) is an effective Kan fibration with a degeneracy-section $\tilde{p}$. (p, lift_p) is said to be degeneracy-section-preferring, if $\mathsf{lift}_p(\tilde{p}_{n+1}(ys_j)\iota, ys_j) = \tilde{p}_{n+1}(ys_j)$ for a horn problem as in the following commutative diagram for all $n \in \mathbb{N}$, all possible j in s_j and m in Λ_m^{n+1}. We call this **DS-pkf**.*

$$
\begin{array}{ccc}
\Lambda_m^{n+1} & \xrightarrow{\ \tilde{p}_{n+1}(ys_j)\iota\ } & X \\
{\scriptstyle \iota}\downarrow & {\scriptstyle \tilde{p}_n(y)}\nearrow \quad & \downarrow{\scriptstyle p} \quad \uparrow{\scriptstyle \tilde{p}} \\
\Delta^{n+1} & \xrightarrow[\ s_j\ ]{} \Delta^n \xrightarrow[\ y\ ]{} & Y
\end{array}
$$

Note that it is straightforward to check that this lift_p indeed satisfies the compatibility condition of effective Kan fibrations. It is also straightforward to see that any **D-pkf** $\cap$ **DS** is **DS-pkf**, although **D-pkf** $\not\subset$ **DS-pkf** and **DS-pkf** $\not\subset$ **D-pkf**.

We have checked that if we assume a similar statement to **Proposition 41** where we replace **D-pkf** with **DS-pkf** instead of holds, then we are able to show the constructive version of "Corollary 4.4." in [3], which would be formulated as follows:

[Quotient : DS-pkf] *"Suppose R is a pseudo-equivalence relation on X, the quotient map $q \colon X \to X/R$ has a degeneracy-section $\tilde{q}$, and $R \xrightarrow{\pi} X \times X$ is an effective Kan fibration. Moreover, suppose that X is in **DS-pkf**. Then the quotient map $X \xrightarrow{q} X/R$ is in **DS-pkf**, and X/R is in **DS-pkf**."*

We have also defined even more general structures which do not depend on degeneracy-sections as follows:

▶ **Definition 47.** *Assume (p, lift_p) is an effective Kan fibration with a section $\tilde{p}_0 \colon Y_0 \to X_0$ at vertices. (p, lift_p) is called vertex-section-preferring, if $\text{lift}_p(\tilde{p}_0(y), ys_0) = \tilde{p}_0(y)s_0$ for a horn problem of the following commutative diagram with $m = 0, 1$. We call this **VS-pkf** (and **VS-pkc** for the case of fibrant objects).*

$$
\begin{array}{ccc}
\Lambda^1_m & \xrightarrow{\ \bar{p}_0(y)\ } & X \\
\downarrow{\scriptstyle \iota} & {\scriptstyle \tilde{p}_0(y)}\nearrow & \downarrow{\scriptstyle p} \\
\Delta^1 & \xrightarrow[s_0]{} \Delta^0 \xrightarrow[y]{} & Y
\end{array}
$$

Note that although $\Lambda^1_m = \Delta^0$ for both $m = 0, 1$, the inclusion ι depends on m. Also, it is immediate that any **DS-pkf** is **VS-pkf**.

In this way, we can broaden our search for the candidate in the constructive version of "Corollary 4.4." in [3]. However, it is also required that those maps have to satisfy **Proposition 41** too.

To sum up, we have

$$
\begin{array}{ccc}
\textbf{VS-pkf} & \supset & \textbf{VS-pkf} \cap \textbf{DS} \\
& & \cup \\
& & \textbf{DS-pkf} \\
& & \cup \\
\textbf{D-pkf} & \supset & \textbf{D-pkf} \cap \textbf{DS}
\end{array}
$$

Under the condition of **D-pkf**, we can prove **Proposition 41**. However, this degenerate-preferring structure is quite strong, making it hard to solve [Quotient : D-pkf ∩ DS]. On the other hand, **VS-pkf** might be too general for the condition in **Proposition 41**. The condition of **D-pkf** with a section on vertices in **Proposition 41** is vastly generalized, and that would make the proof more combinatorially complicated. However, it is still expected to be achievable: it has a section only at the vertices level, but a lift construction algorithm is expected to work from the vertices level to higher levels using the compatibility condition of the fibrations and the universal properties of horn construction. It would then improve [Quotient : DS-pkf] by replacing **DS-pkf** by **VS-pkf**.

7 Conclusion

In this paper, we have demonstrated that effective Kan fibrations indeed model W-types as expected. The transition from classical to constructive proofs was generally smooth, though certain challenges arose. In particular, constructively, choices can be subtle. In **Proposition 18**, we addressed the issue of arbitrariness in the choice of factorizations by imposing conditions such that the mediating maps are in **EffKanCplx** or **EffKan**/A. Fortunately, these conditions posed no issues for W-types, as their categorical interpretation ensures that all the relevant mediating maps in W-types are such maps.

One significant challenge is that as mentioned in **Remark 36**, the W-types employed here are non-constructive due to their reliance on linearly ordered transfinite ordinals, invoking the Law of Excluded Middle (LEM). Addressing this issue was beyond the scope of this paper, leading us to adopt a classical transfinite argument. Constructive alternatives to ordinals

remain a promising avenue for future work. Various attempts, such as those in [9] and more recently in [4], focus on countable ordinals. However, there are difficulties representing other limit ordinals. Nevertheless, our strategy of interpreting W-types as certain filtered colimits suggests that it may apply to constructive ordinals, which, though not linearly ordered, are anticipated to form a filtered poset. In this context, **Proposition 18** and the result that polynomial functors can be set on the category of effective Kan complexes are expected to remain relevant.

Also, for future work, we discussed various additional structures on effective Kan fibrations, such as **D-pkf**, **DS-pkf**, and **VS-pkf**, to constructively model quotient types and establish key propositions in constructive settings. This direction is beneficial not only for applications of W-types but also for broadening the applicability of these methods within the related fields of constructive semantics for HoTT.

References

1   Steve Awodey and Michael A. Warren. *Homotopy theoretic models of identity types*. Math. Proc. Cambridge Philos. Soc., 146(1): pp. 45–55, 2009. `doi:10.1017/S0305004108001783`.

2   Benno van den Berg and Eric Faber. *Effective Kan Fibrations in Simplicial Sets*. Springer, 2022. `doi:10.1007/978-3-031-18900-5`.

3   Benno van den Berg and Ieke Moerdijk. *W-types in homotopy type theory*. Mathematical Structures in Computer Science, 25(5): pp. 1100–1115, 2015. `doi:10.1017/S0960129514000516`.

4   Thierry Coquand, Henri Lombardi, and Stefan Neuwirth. *Constructive theory of ordinals*. arXiv, 2022. arXiv preprint arXiv:2201.04352. `doi:10.48550/arXiv.2201.04352`.

5   Storm Diephuis. Effective kan fibrations for simplicial groupoids, semisimplicial sets and ex$^\infty$, 2023. https://eprints.illc.uva.nl/id/eprint/2247/1/MoL-2023-06.text.pdf.

6   Nicola Gambino and Christian Sattler. *The Frobenius Condition, Right Properness, and Uniform Fibrations*. Journal of Pure and Applied Algebra, 221(12): pp. 3027–3068, 2017. `doi:10.1016/j.jpaa.2017.02.013`.

7   Freek Geerligs. Symmetric effective and degenerate-preferring kan complexes, 2023. https://studenttheses.uu.nl/handle/20.500.12932/43910.

8   Paul Goerss and John F. Jardine. *Simplicial homotopy theory*. Birkhäuser, 1999. `doi:10.1007/978-3-0348-8707-6`.

9   Per Martin-Löf. *Notes on constructive mathematics*. Almqvist & Wiksell, Stockholm, 1970. URL: `https://worldcat.org/oclc/472257944`.

10 Per Martin-Löf. *Constructive mathematics and computer programming*. In *Logic, Methodology and Philosophy of Science VI*, volume VI, pages pp. 153–175. Elsevier, 1982. `doi:10.1016/S0049-237X(09)70189-2`.

11 Egbert Rijke. *Introduction to homotopy type theory*. Cambridge University Press, 2022. `doi:10.48550/arXiv.2212.11082`.

12 Univalent Foundations Project. Homotopy type theory -– univalent foundations of mathematics, 2013. Accessed: 2025-05-02. URL: `https://homotopytypetheory.org/book/`.